

ON APPROXIMATIONS BY TRIGONOMETRIC POLYNOMIALS OF CLASSES OF FUNCTIONS DEFINED BY MODULI OF SMOOTHNESS

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ABSTRACT. In this paper, we give a characterization of Nikol'skiĭ-Besov type classes of functions, given by integral representations of moduli of smoothness, in terms of series over the moduli of smoothness. Also, necessary and sufficient conditions in terms of monotone or lacunary Fourier coefficients for a function to belong to a such a class are given. In order to prove our results, we make use of certain recent reverse Copson- and Leindler-type inequalities.

1. INTRODUCTION

Let $f \in L_p[0, 2\pi]$, $1 < p < \infty$, be a 2π -periodic function. We say that the function f has monotone Fourier coefficients if it has a cosine Fourier series with

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos nx, \quad a_n \downarrow 0.$$

We say that the function f has lacunary Fourier coefficients if

$$f(x) \sim \sum_{\nu=1}^{\infty} \lambda_{\nu} \cos \nu x,$$

where

$$\lambda_{\nu} = \begin{cases} a_{\mu} \geq 0 & \text{for } \nu = 2^{\mu}, \\ 0 & \text{for } \nu \neq 2^{\mu}, \end{cases}$$

that is

$$f(x) \sim \sum_{\mu=0}^{\infty} a_{\mu} \cos 2^{\mu} x, \quad a_{\mu} \geq 0.$$

By $\omega_k(f, t)_p$ we denote the modulus of smoothness of order k in L_p metrics of a function $f \in L_p$, $1 < p < \infty$:

$$\omega_k(f, t)_p = \sup_{|h| \leq t} \|\Delta_h^k f\|_p,$$

where

$$\Delta_h^k f(x) = \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} f(x + \nu h)$$

is the k -th order shift operator.

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By $E_n(f)_p$ we denote the best approximation in L_p metrics of a function $f \in L_p$, $1 < p < \infty$, by means of trigonometric polynomials whose degree is not greater than $n - 1$, i.e.

$$E_n(f)_p = \inf_{T_{n-1}} \|f - T_{n-1}\|_p,$$

where $T_{n-1} = \sum_{\nu=0}^{n-1} (\alpha_\nu \cos \nu x + \beta_\nu \sin \nu x)$, α_ν and β_ν are arbitrary real numbers.

We say that a 2π -periodic function f belongs to the Nikol'skiĭ-Besov class $N(p, \theta, r, \lambda, \varphi)$, $1 < p < \infty$, if the following conditions are satisfied

- (1) $f \in L_p[0, 2\pi]$;
- (2) Numbers θ, r, λ belong to the interval $(0, \infty)$, and k is an integer satisfying $k > r + \lambda$;
- (3) The following inequality holds true

$$\left(\int_0^\delta t^{-r\theta-1} \omega_k(f, t)_p^\theta dt + \delta^{\lambda\theta} \int_\delta^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^\theta dt \right)^{1/\theta} \leq C\varphi(\delta),$$

while the function φ satisfies the conditions

- (4) φ is a non-negative continuous function on $(0, 1)$ and $\varphi \neq 0$;
- (5) For every δ_1, δ_2 such that $0 \leq \delta_1 \leq \delta_2 \leq 1$ holds $\varphi(\delta_1) \leq C_1 \varphi(\delta_2)$;
- (6) For every δ such that $0 \leq \delta \leq \frac{1}{2}$ holds $\varphi(2\delta) \leq C_2 \varphi(\delta)$,

where constants² C, C_1 and C_2 do not depend on δ_1, δ_2 and δ .

A more detailed approach to the classes $N(p, \theta, r, \lambda, \varphi)$ is given in [6] and [12] (see also [2]). In the paper, we give a characterization of $N(p, \theta, r, \lambda, \varphi)$ classes of functions in terms of series over their moduli of smoothness. Then we give the necessary and sufficient conditions in terms of monotone or lacunary Fourier coefficients for a function $f \in L_p[0, 2\pi]$ to belong to a class $N(p, \theta, r, \lambda, \varphi)$. In the process of proving the results, we make use of certain recent reverse l_p -type inequalities [10], closely related to Copson's and Leindler's inequalities.

Finally, by making use of our results, we construct an example of a function having a lacunary Fourier series, which shows that $N(p, \theta, r, \lambda, \varphi)$ classes are properly embedded between the appropriate Nikol'skiĭ classes and Besov classes.

2. STATEMENT OF RESULTS

Now we formulate our results.

Theorem 2.1. *A function f belongs to the class $N(p, \theta, r, \lambda, \varphi)$ if and only if³*

$$\left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^\theta \nu^{(r+\lambda)\theta-1} \right)^{1/\theta} \leq C\varphi \left(\frac{1}{n} \right), \quad (1)$$

where constant C does not depend on n .

Theorem 2.2. *For a function $f \in L_p[0, 2\pi]$, $1 < p < \infty$, such that*

$$f(x) \sim \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad a_\nu \downarrow 0, \quad (2)$$

²Without mentioning it explicitly, we will consider all the constants positive.

³Here and below we assume that the parameters θ, r, λ and k satisfy the condition 2, and the function φ satisfies the conditions 4–6 of the definition of the class $N(p, \theta, r, \lambda, \varphi)$.

to belong to the class $N(p, \theta, r, \lambda, \varphi)$ it is necessary and sufficient that its Fourier coefficients satisfy the condition

$$\left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{r\theta+\lambda\theta+\theta-\theta/p-1} \right)^{1/\theta} \leq C \varphi \left(\frac{1}{n} \right),$$

where constant C does not depend on n .

Regarding Theorem 2.1, a very interesting open question remains its analogue for functions with general monotone Fourier coefficients, generalized in the sense of [13, 9].

Corollary 2.1. Put $\varphi(\delta) = \delta^{\alpha}$, $0 < \alpha < \lambda$, in the definition of the class $N(p, \theta, r, \lambda, \varphi)$, we obtain [6] the Nikol'skiĭ class $H_p^{r+\alpha}$. Thus Theorems 2.1 and 2.2 give the single coefficient condition

$$a_{\nu} \leq \frac{C}{\nu^{r+\alpha+1-\frac{1}{p}}},$$

for $f \in H_p^{r+\alpha}$, given in [5], where the function f is given by (2).

Corollary 2.2. If $\varphi(\delta) \geq C$, then we obtain [6] the Besov class $B_p^{\theta r}$. Thus Theorems 2.1 and 2.2 give the necessary and sufficient condition

$$\sum_{\nu=1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} < \infty$$

for $f \in B_p^{\theta r}$, given in [11], where the function f is given by (2).

Theorem 2.3. For a function $f \in L_p$, $1 < p < \infty$, such that

$$f(x) \sim \sum_{\nu=1}^{\infty} \lambda_{\nu} \cos \nu x, \quad (3)$$

and

$$\lambda_{\nu} = \begin{cases} a_{\mu} \geq 0 & \text{for } \nu = 2^{\mu}, \\ 0 & \text{for } \nu \neq 2^{\mu}, \end{cases}$$

to belong to the class $N(p, \theta, r, \lambda, \varphi)$ it is necessary and sufficient that its Fourier coefficients satisfy the condition⁴

$$\left(\sum_{\nu=m+1}^{\infty} \lambda_{\nu}^{\theta} \nu^{r\theta} + m^{-\lambda\theta} \sum_{\nu=1}^m \lambda_{\nu}^{\theta} \nu^{(r+\lambda)\theta} \right)^{1/\theta} \leq C \varphi \left(\frac{1}{m} \right),$$

where constant C does not depend on m .

Corollary 2.3. Putting $\varphi(\delta) = \delta^{\alpha}$, $0 < \alpha < \lambda$, in the definition of the class $N(p, \theta, r, \lambda, \varphi)$, we obtain [6] the Nikol'skiĭ class $H_p^{r+\alpha}$. Thus Theorem 2.3 gives the single coefficient condition

$$a_{\mu} \leq C 2^{-\mu(r+\alpha)}$$

for $f \in H_p^{r+\alpha}$, where the function f is given by (3).

Corollary 2.4. If $\varphi(\delta) = C$, then we obtain [6] the Besov class $B_p^{\theta r}$. Thus Theorem 2.3 gives the necessary and sufficient condition

$$\sum_{\mu=1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta} < \infty$$

for $f \in B_p^{\theta r}$, given in [11], where the function f is given by (3).

⁴Here and below we assume that the parameters θ , r , λ and k satisfy the condition 2, and the function φ satisfies the conditions 4–6.

Example 2.1. Let

$$f(x) \sim \sum_{\mu=0}^{\infty} a_{\mu} \cos 2^{\mu} x,$$

where are

$$a_{\mu} = 2^{-\mu r} (\mu + 1)^{-(\alpha+1/\theta)}, \quad \alpha > 0.$$

Then, we have

$$C_1 n^{-\alpha} \leq \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta} \right)^{1/\theta} \leq C_2 n^{-\alpha}$$

and

$$C_3 n^{-(\alpha+1/\theta)} \leq \left(2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} 2^{\mu(r+\lambda)\theta} \right)^{1/\theta} \leq C_4 n^{-(\alpha+1/\theta)},$$

thus implying (see the proof of Theorem 2.3) $f \in N(p, \theta, r, \lambda, \varphi)$ for $\varphi(\delta) = (\ln \frac{1}{\delta})^{-\alpha}$. This means that classes N are classes of embedding between classes H and B .

3. AUXILIARY STATEMENTS

In order to establish our results, we use the following lemmas.

Lemma 3.1. *Let $0 < \alpha < \beta < \infty$ and $a_{\nu} \geq 0$. The following inequality holds true*

$$\left(\sum_{\nu=1}^n a_{\nu}^{\beta} \right)^{1/\beta} \leq \left(\sum_{\nu=1}^n a_{\nu}^{\alpha} \right)^{1/\alpha}.$$

Proof of the lemma is due to Jensen [4, p. 43].

Lemma 3.2. *Let $\{a_{\nu}\}_{\nu=1}^{\infty}$ be a sequence of non-negative numbers, $\alpha > 0$, λ a real number, m and n positive integers such that $m < n$. Then*

(1) *for $1 \leq p < \infty$ the following equalities hold*

$$\begin{aligned} \sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_{\nu} \nu^{\lambda} \right)^p &\leq C_1 \sum_{\mu=m}^n \mu^{\alpha-1} (a_{\mu} \mu^{\lambda+1})^p, \\ \sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^{\mu} a_{\nu} \nu^{\lambda} \right)^p &\leq C_2 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^p; \end{aligned}$$

(2) *for $0 < p \leq 1$ the following equalities hold*

$$\begin{aligned} \sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_{\nu} \nu^{\lambda} \right)^p &\geq C_3 \sum_{\mu=m}^n \mu^{\alpha-1} (a_{\mu} \mu^{\lambda+1})^p, \\ \sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^{\mu} a_{\nu} \nu^{\lambda} \right)^p &\geq C_4 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_{\mu} \mu^{\lambda+1})^p, \end{aligned}$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_{\nu}\}_{\nu=1}^{\infty}$.

Proof of the lemma is given in [4, p. 308].

Lemmas 3.3 and 3.4 that follow state certain l_p -type inequalities which are reversed to the ones given in Lemma 3.2 and closely related to Copson's and Leindler's inequalities (see, e.g., [3, 7, 8, 14]).

We write $a_{\nu} \downarrow$ if $\{a_{\nu}\}_{\nu=1}^{\infty}$ is a monotone-decreasing sequence of non-negative numbers, i.e. if $a_{\nu} \geq a_{\nu+1} \geq 0$ ($\nu = 1, 2, \dots$).

Lemma 3.3. *Let $a_{\nu} \downarrow$, $\alpha > 0$, λ a real number, m and n positive integers. Then*

(1) for $1 \leq p < \infty$, $n \geq 16m$ the following equalities hold

$$\begin{aligned} \sum_{\mu=m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p &\geq C_1 \sum_{\mu=8m}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\ \sum_{\mu=m}^n \mu^{-\alpha-1} \left(\sum_{\nu=m}^\mu a_\nu \nu^\lambda \right)^p &\geq C_2 \sum_{\mu=4m}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p; \end{aligned}$$

(2) for $0 < p \leq 1$, $n \geq 4m$ the following equalities hold

$$\begin{aligned} \sum_{\mu=4m}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p &\leq C_3 \sum_{\mu=m}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\ \sum_{\mu=4m}^n \mu^{-\alpha-1} \left(\sum_{\nu=4m}^\mu a_\nu \nu^\lambda \right)^p &\leq C_4 \sum_{\mu=m}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p, \end{aligned}$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_\nu\}_{\nu=1}^\infty$.

Proof of the lemma is given in [10].

Lemma 3.4. Let $a_\nu \downarrow$, $\alpha > 0$, λ a real number, m and n positive integers. For $0 < p < \infty$ the following inequalities hold

$$\begin{aligned} C_1 \sum_{\mu=1}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p &\leq \sum_{\mu=1}^n \mu^{\alpha-1} \left(\sum_{\nu=\mu}^n a_\nu \nu^\lambda \right)^p \leq C_2 \sum_{\mu=1}^n \mu^{\alpha-1} (a_\mu \mu^{\lambda+1})^p, \\ C_3 \sum_{\mu=1}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p &\leq \sum_{\mu=1}^n \mu^{-\alpha-1} \left(\sum_{\nu=1}^\mu a_\nu \nu^\lambda \right)^p \leq C_4 \sum_{\mu=1}^n \mu^{-\alpha-1} (a_\mu \mu^{\lambda+1})^p, \end{aligned}$$

where constants C_1 , C_2 , C_3 and C_4 depend only on numbers α , λ and p , and do not depend on m , n as well as on the sequence $\{a_\nu\}_{\nu=1}^\infty$.

The lemma is also proved in [10].

Lemma 3.5. Let $f \in L_p[0, 2\pi]$ for a fixed p from the interval $1 < p < \infty$ and let

$$f(x) \sim \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad a_\nu \downarrow 0.$$

The following inequalities hold

$$\begin{aligned} C_1 \frac{1}{n^k} \left(\sum_{\nu=1}^n a_\nu^p \nu^{(k+1)p-2} \right)^{1/p} + \left(\sum_{\nu=n+1}^{\infty} a_\nu^p \nu^{p-2} \right)^{1/p} &\leq \omega_k \left(f, \frac{1}{n} \right)_p \\ &\leq C_2 \frac{1}{n^k} \left(\sum_{\nu=1}^n a_\nu^p \nu^{(k+1)p-2} \right)^{1/p} + \left(\sum_{\nu=n+1}^{\infty} a_\nu^p \nu^{p-2} \right)^{1/p}, \end{aligned}$$

where constants C_1 and C_2 do not depend on n and f .

The lemma is proved in [11].

Lemma 3.6. A function f belongs to the class $N(p, \theta, r, \lambda, \varphi)$ if and only if

$$\left(\sum_{\mu=n+1}^{\infty} 2^{\mu\theta} E_{2^\mu}(f)_p^\theta + 2^{-n\lambda\theta} \sum_{\mu=0}^n 2^{\mu(r+\lambda)\theta} E_{2^\mu}(f)_p^\theta \right)^{1/\theta} \leq C \varphi \left(\frac{1}{2^n} \right),$$

where constant C does not depend on n .

Proof of the lemma is given in [6].

Lemma 3.7. *Let $f \in L_p$, $1 < p < \infty$, and*

$$f(x) \sim \sum_{\mu=0}^{\infty} a_{\mu} \cos 2^{\mu} x, \quad a_{\mu} \geq 0.$$

The following inequalities hold

$$C_1 \left(\sum_{\mu=0}^{\infty} a_{\mu}^2 \right)^{1/2} \leq \|f\|_p \leq C_2 \left(\sum_{\mu=0}^{\infty} a_{\mu}^2 \right)^{1/2},$$

where constants C_2 and C_1 do not depend on f .

Proof of the lemma is due to Zygmund [16, vol. I, p. 326].

Corollary 3.1. *Lemma 3.7 yields the following estimate*

$$C_1 \left(\sum_{\mu=n}^{\infty} a_{\mu}^2 \right)^{1/2} \leq E_{2^n}(f)_p \leq C_2 \left(\sum_{\mu=n}^{\infty} a_{\mu}^2 \right)^{1/2},$$

where constants C_2 and C_1 do not depend on n and f .

4. PROOFS

Now we prove our results.

Proof of Theorem 2.1. Put

$$I_1 = \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^{\theta} dt, \quad I_2 = \int_{\frac{1}{n+1}}^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^{\theta} dt.$$

We have [4, p. 55]

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^{\theta} dt = \sum_{\nu=n+1}^{\infty} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} \omega_k(f, t)_p^{\theta} dt \\ &\leq \sum_{\nu=n+1}^{\infty} \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} dt \leq C_1 \sum_{\nu=n+1}^{\infty} \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{r\theta-1} \end{aligned}$$

and, taking into account properties of modulus of smoothness [15],

$$I_1 \geq \sum_{\nu=n+1}^{\infty} \omega_k\left(f, \frac{1}{\nu+1}\right)_p^{\theta} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-r\theta-1} dt \geq C_2 \sum_{\nu=n+1}^{\infty} \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{r\theta-1}.$$

In an analogous way we estimate

$$I_2 \leq \sum_{\nu=1}^n \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-(r+\lambda)\theta-1} dt \leq C_3 \sum_{\nu=1}^n \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{(r+\lambda)\theta-1}$$

and

$$I_2 \geq \sum_{\nu=1}^n \omega_k\left(f, \frac{1}{\nu+1}\right)_p^{\theta} \int_{\frac{1}{\nu+1}}^{\frac{1}{\nu}} t^{-(r+\lambda)\theta-1} dt \geq C_4 \sum_{\nu=1}^n \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{(r+\lambda)\theta-1}.$$

Let $f \in N(p, \theta, r, \lambda, \varphi)$. For a positive integer n we put $\delta = \frac{1}{n+1}$. Then we have

$$\begin{aligned} I^{\theta} &= I_1 + \delta^{\lambda\theta} I_2 \\ &\geq C_5 \left(\sum_{\nu=n+1}^{\infty} \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k\left(f, \frac{1}{\nu}\right)_p^{\theta} \nu^{(r+\lambda)\theta-1} \right). \end{aligned}$$

Hence we obtain

$$\begin{aligned} J &= \left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{(r+\lambda)\theta-1} \right)^{1/\theta} \\ &\leq C_6 I \leq C_7 \varphi(\delta) = C_7 \varphi \left(\frac{1}{n+1} \right) \leq C_8 \varphi \left(\frac{1}{n} \right), \end{aligned}$$

which proves inequality (1).

Now we suppose that inequality (1) holds. For $\delta \in (0, 1)$ we choose the positive integer n satisfying $\frac{1}{n+1} < \delta \leq \frac{1}{n}$. Then, taking into consideration the estimates from above for I_1 and I_2 we have

$$\begin{aligned} I^{\theta} &= \int_0^{\frac{1}{n+1}} t^{-r\theta-1} \omega_k(f, t)_p^{\theta} dt + \int_{\frac{1}{n+1}}^{\delta} t^{-r\theta-1} \omega_k(f, t)_p^{\theta} dt \\ &\quad + \delta^{\lambda\theta} \int_{\delta}^1 t^{-(r+\lambda)\theta-1} \omega_k(f, t)_p^{\theta} dt \leq I_1 + \delta^{\lambda\theta} I_2 \\ &\leq C_9 \left(\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{(r+\lambda)\theta-1} \right). \end{aligned}$$

Hence

$$I \leq C_{10} J \leq C_{11} \varphi \left(\frac{1}{n} \right) \leq C_{12} \varphi \left(\frac{1}{2n} \right) \leq C_{13} \varphi(\delta),$$

implying $f \in N(p, \theta, r, \lambda, \varphi)$.

Proof of Theorem 2.1 is completed. \square

Proof of Theorem 2.2. Theorem 2.1 implies that the condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$\sum_{\nu=n+1}^{\infty} \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{r\theta-1} + n^{-\lambda\theta} \sum_{\nu=1}^n \omega_k \left(f, \frac{1}{\nu} \right)_p^{\theta} \nu^{(r+\lambda)\theta-1} \leq C_1 \varphi \left(\frac{1}{n} \right)^{\theta},$$

where constant C_1 does not depend on n . Lemma 3.5 yields that the last estimate is equivalent to the estimate

$$\begin{aligned} &\sum_{\nu=n+1}^{\infty} \nu^{(r-k)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} + \sum_{\nu=n+1}^{\infty} \nu^{r\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \\ &\quad + n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{(r+\lambda-k)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\quad + n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \leq C_2 \varphi \left(\frac{1}{n} \right)^{\theta}, \end{aligned}$$

where constant C_2 does not depend on n . Hence, if we denote the terms on the left-hand side of the inequality by J_1 , J_2 , J_3 and J_4 respectively, then condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$J_1 + J_2 + J_3 + J_4 \leq C_2 \varphi \left(\frac{1}{n} \right)^{\theta}. \quad (4)$$

Now we estimate the terms J_1 , J_2 , J_3 and J_4 from below and above by means of expression taking part in the condition of the theorem.

First we estimate J_1 and J_2 from below. We have

$$\begin{aligned} J_1 &= \sum_{\nu=n+1}^{\infty} \nu^{(r-k)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\geq \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left(\sum_{\mu=n+1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}. \end{aligned}$$

For $k-r > 0$, making use of Lemmas 3.2 and 3.3 we obtain

$$\begin{aligned} J_1 &\geq C_3 \sum_{\nu=4(n+1)}^{\infty} \nu^{-(k-r)\theta-1} (a_{\nu}^p \nu^{(k+1)p-2\nu})^{\theta/p} \\ &= C_3 \sum_{\nu=4(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}. \end{aligned} \quad (5)$$

In an analogous way, for $r\theta > 0$ we get

$$J_2 = \sum_{\nu=n+1}^{\infty} \nu^{r\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \geq C_4 \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}. \quad (6)$$

We estimate the term J_2 from above:

$$J_2 \leq C_5 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} \nu^{r\theta-1} (a_{\nu}^p \nu^{p-2\nu})^{\theta/p} = C_5 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}. \quad (7)$$

For J_1 we have

$$\begin{aligned} J_1 &\leq C_6 \left(\sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left(\sum_{\mu=n+1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \right. \\ &\quad \left. + \sum_{\nu=n+1}^{\infty} \nu^{-(k-r)\theta-1} \left(\sum_{\mu=1}^n a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \right), \end{aligned}$$

and applying once more Lemmas 3.2 and 3.3 we obtain

$$J_1 \leq C_7 \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-(k-r)\theta} \left(\sum_{\mu=1}^n a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}. \quad (8)$$

Put

$$I_1 = n^{-(k-r)\theta} \sum_{\mu=1}^n a_{\mu}^p \mu^{(k+1)p-2}.$$

Then for

$$I_2 = I_1 n^{(k-r)\theta},$$

taking into account that $(k+1)p-2 \geq 0$ and $a_{\nu} \downarrow 0$ we get

$$\begin{aligned} I_2 &= \sum_{\mu=1}^n a_{\mu}^p \mu^{(k+1)p-2} \leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_{\mu}^p \mu^{(k+1)p-2} + a_{\lceil \frac{n}{2} \rceil+1}^p \sum_{\mu=\lceil \frac{n}{2} \rceil+1}^n \mu^{(k+1)p-2} \\ &\leq \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_{\mu}^p \mu^{(k+1)p-2} + C_8 n^{(k+1)p-1} a_{\lceil \frac{n}{2} \rceil+1}^p \leq C_9 \sum_{\mu=1}^{\lceil \frac{n}{2} \rceil} a_{\mu}^p \mu^{(k+1)p-2}. \end{aligned}$$

Since $k - r - \lambda > 0$, we have

$$\begin{aligned} I_1^{\theta/p} &\leq C_{10} n^{-(k-r)\theta} \left(\sum_{\mu=1}^{\lfloor \frac{n}{2} \rfloor} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\leq C_{11} n^{-\lambda\theta} \sum_{\nu=\lfloor \frac{n}{2} \rfloor}^n \nu^{-(k-r-\lambda)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p} \\ &\leq C_{11} n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{-(k-r-\lambda)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}. \end{aligned}$$

Applying Lemma 3.4 we obtain

$$\begin{aligned} I_1^{\theta/p} &\leq C_{12} n^{-\lambda\theta} \sum_{\nu=1}^n \nu^{-(k-r-\lambda)\theta-1} (a_{\nu}^p \nu^{(k+1)p-2} \nu)^{\theta/p} \\ &= C_{12} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \end{aligned}$$

From (8) it follows that

$$J_1 \leq C_{13} \left(\sum_{\nu=\lfloor \frac{n+1}{4} \rfloor}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \quad (9)$$

This way, inequalities (5), (6), (7) and (9) yield

$$\begin{aligned} C_{14} \sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} &\leq J_1 + J_2 \\ &\leq C_{15} \left(\sum_{\nu=\lfloor \frac{n+1}{4} \rfloor}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (10)$$

Now we estimate J_3 and J_4 . Put

$$A_1 = n^{\lambda\theta} J_3 = \sum_{\nu=1}^n \nu^{(r+\lambda-k)\theta-1} \left(\sum_{\mu=1}^{\nu} a_{\mu}^p \mu^{(k+1)p-2} \right)^{\theta/p}$$

and

$$A_2 = n^{\lambda\theta} J_4 = \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p},$$

applying Lemma 3.4 for $r + \lambda - k < 0$ we get

$$A_1 \leq C_{16} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \quad (11)$$

We estimate A_2 in an analogous way:

$$\begin{aligned} A_2 &\leq C_{17} \left(\sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^n a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right. \\ &\quad \left. + \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right) \\ &\leq C_{18} \left(\sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + n^{(r+\lambda)\theta} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \right). \end{aligned} \quad (12)$$

We estimate the series

$$B = \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p}.$$

First let $\frac{\theta}{p} > 1$. Applying Hölder inequality we have

$$\begin{aligned} \sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} &\leq \left(\sum_{\mu=n+1}^{\infty} (a_{\mu}^p \mu^{p-1+rp-p/\theta})^{\theta/p} \right)^{p/\theta} \\ &\quad \times \left(\sum_{\mu=n+1}^{\infty} \mu^{-(rp-p/\theta+1)\theta/(\theta-p)} \right)^{(\theta-p)/\theta}. \end{aligned}$$

Since $(rp - \frac{p}{\theta} + 1)\frac{\theta}{\theta-p} = rp\frac{\theta}{\theta-p} + 1 > 1$, we get

$$\sum_{\mu=n+1}^{\infty} a_{\mu}^p \mu^{p-2} \leq C_{19} n^{-rp} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \mu^{\theta-\theta/p+rp-1} \right)^{p/\theta}.$$

So, for $\frac{\theta}{p} > 1$ we have proved that

$$B \leq C_{20} n^{-r\theta} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \mu^{r\theta+\theta-\theta/p-1}.$$

Let $\frac{\theta}{p} \leq 1$. For given n we choose the positive integer N such that $2^N \leq n+1 < 2^{N+1}$. Then we have

$$\begin{aligned} B &\leq \left(\sum_{\mu=2^N}^{\infty} a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \leq \left(\sum_{\nu=N}^{\infty} a_{2^{\nu}}^p \sum_{\mu=2^{\nu}}^{2^{\nu+1}-1} \mu^{p-2} \right)^{\theta/p} \\ &\leq C_{21} \left(\sum_{\nu=N}^{\infty} a_{2^{\nu}}^p 2^{\nu(p-1)} \right)^{\theta/p}. \end{aligned}$$

Making use of Lemma 3.1 we obtain

$$\begin{aligned} B &\leq C_{21} \sum_{\nu=N}^{\infty} a_{2^{\nu}}^{\theta} 2^{\nu(\theta-\theta/p)} \leq C_{22} \sum_{\nu=N}^{\infty} \sum_{\mu=2^{\nu-1}}^{2^{\nu}-1} a_{\mu}^{\theta} \mu^{\theta-\theta/p-1} \\ &= C_{22} \sum_{\nu=2^{N-1}}^{\infty} a_{\nu}^{\theta} \nu^{\theta-\theta/p-1} \leq C_{22} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{\theta-\theta/p-1} \\ &\leq C_{22} \left[\frac{n+1}{4} \right]^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}. \end{aligned}$$

Since for $n \geq 3$ holds $\lceil \frac{n+1}{4} \rceil \geq \frac{n}{12}$, we get

$$B \leq C_{23} n^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}.$$

This way, for $0 < \frac{\theta}{p} < \infty$ we proved that

$$B \leq C_{24} n^{-r\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1}.$$

Hence (12) yields

$$A_2 \leq C_{25} \left(\sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + n^{\lambda\theta} \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right).$$

Now, from (11) it follows that

$$\begin{aligned} J_3 + J_4 &= n^{-\lambda\theta} (A_1 + A_2) \\ &\leq C_{26} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (13)$$

Further, we estimate the series

$$A_3 = \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} = A_4 + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1},$$

where is

$$\begin{aligned} A_4 &= \sum_{\nu=\lceil \frac{n+1}{4} \rceil}^n a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \leq C_{27} a_{\lceil \frac{n+1}{4} \rceil}^{\theta} n^{r\theta+\theta-\theta/p} \\ &\leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^{\lceil \frac{n+1}{4} \rceil} a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \leq C_{28} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}. \end{aligned}$$

Hence

$$A_3 \leq C_{29} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \quad (14)$$

Making use of (14) and (13) we have

$$J_3 + J_4 \leq C_{30} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right).$$

Hence, applying (14) in (10) we obtain

$$\begin{aligned} J_1 + J_2 + J_3 + J_4 &\leq C_{31} \left(n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} + \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} \right). \end{aligned} \quad (15)$$

Now we estimate A_1 and A_2 from below. Making use of Lemma 3.4 we get

$$A_1 \geq C_{32} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1},$$

and in an analogous way

$$A_2 \geq \sum_{\nu=1}^n \nu^{(r+\lambda)\theta-1} \left(\sum_{\mu=\nu}^n a_{\mu}^p \mu^{p-2} \right)^{\theta/p} \geq C_{33} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

Hence

$$A_1 + A_2 \geq C_{34} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

This way the following inequality holds

$$J_3 + J_4 \geq C_{35} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1}.$$

From (10) it follows that

$$J_1 + J_2 + J_3 + J_4 \geq C_{36} \left(\sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \quad (16)$$

Since

$$\begin{aligned} \sum_{\nu=n+1}^{\nu=8(n+1)-1} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} &\leq C_{37} a_n^{\theta} n^{r\theta+\theta-\theta/p} \\ &\leq C_{38} n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \end{aligned}$$

holds, we have

$$\begin{aligned} \sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \\ \leq C_{39} \left(\sum_{\nu=8(n+1)}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned}$$

Now, estimates (16) and (15) imply

$$\begin{aligned} C_{40} \left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right) \\ \leq J_1 + J_2 + J_3 + J_4 \\ \leq C_{41} \left(\sum_{\nu=n+1}^{\infty} a_{\nu}^{\theta} \nu^{r\theta+\theta-\theta/p-1} + n^{-\lambda\theta} \sum_{\nu=1}^n a_{\nu}^{\theta} \nu^{(r+\lambda)\theta+\theta-\theta/p-1} \right). \end{aligned}$$

This way we proved that condition (1) is equivalent to the condition of the theorem. Since condition (1) is equivalent to the condition $f \in N(p, \theta, r, \lambda, \varphi)$, proof of Theorem 2.2 is completed. \square

Proof of Theorem 2.3. Considering Lemma 3.6, condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$\sum_{\nu=n+1}^{\infty} 2^{\nu r\theta} E_{2^{\nu}}(f)_p^{\theta} + 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} E_{2^{\nu}}(f)_p^{\theta} \leq C_{42} \varphi \left(\frac{1}{2^n} \right)^{\theta},$$

where constant C does not depend on n . Corollary 3.1 yields that the last estimate is equivalent to the estimate

$$\sum_{\nu=n+1}^{\infty} 2^{\nu r\theta} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2} + 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2} \leq C_{43} \varphi \left(\frac{1}{2^n} \right)^{\theta}, \quad (17)$$

where constant C_{43} does not depend on n .

Put

$$J_1 = \sum_{\nu=n+1}^{\infty} 2^{\nu r\theta} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2}, \quad J_2 = 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2},$$

we estimate J_1 and J_2 from below and above.

Let $0 < \frac{\theta}{2} \leq 1$. Using Lemma 3.1, changing the order of summation we get

$$J_1 \leq \sum_{\nu=n+1}^{\infty} 2^{\nu r \theta} \sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} = \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \sum_{\nu=n+1}^{\mu} 2^{\nu r \theta}.$$

Therefrom, taking into consideration that $r\theta > 0$ while computing the second sum we obtain

$$J_1 \leq C_{44} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta}.$$

Let $1 \leq \frac{\theta}{2} < \infty$ and $0 < \varepsilon < r$. Applying Hölder inequality we have

$$A = \sum_{\mu=\nu}^{\infty} a_{\mu}^2 \leq \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} 2^{\mu \varepsilon \theta} \right)^{2/\theta} \left(\sum_{\mu=\nu}^{\infty} 2^{-2\mu \varepsilon \theta'} \right)^{1/\theta'},$$

where is $\frac{2}{\theta} + \frac{1}{\theta'} = 1$. Computing the second sum we obtain

$$A \leq \frac{C_{45}}{2^{2\varepsilon\nu}} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} 2^{\mu \varepsilon \theta} \right)^{2/\theta}.$$

Now we have

$$\begin{aligned} J_1 &\leq C_{46} \sum_{\nu=n+1}^{\infty} 2^{\nu(r-\varepsilon)\theta} \sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} 2^{\mu \varepsilon \theta} \\ &= C_{46} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu \varepsilon \theta} \sum_{\nu=n+1}^{\mu} 2^{\nu(r-\varepsilon)\theta} \leq C_{47} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta}. \end{aligned}$$

This way, for $0 < \theta < \infty$ we have

$$J_1 \leq C_{48} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta},$$

where constant C_{48} does not depend on n .

Now we estimate J_1 from below.

Let $1 \leq \frac{\theta}{2} < \infty$. Making use of Lemma 3.1 we get

$$J_1 \geq \sum_{\nu=n+1}^{\infty} 2^{\nu r \theta} \sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} = \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} \sum_{\nu=n+1}^{\mu} 2^{\nu r \theta}.$$

Computing the second sum we get

$$J_1 \geq C_{49} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta}.$$

Let $0 < \frac{\theta}{2} \leq 1$ and $\varepsilon > 0$. Applying Hölder inequality we have

$$\sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} 2^{-\mu \varepsilon \theta} \leq \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2} \left(\sum_{\mu=\nu}^{\infty} 2^{-\mu \varepsilon \theta \theta'} \right)^{1/\theta'} \leq \frac{C_{50}}{2^{\nu \varepsilon \theta}} \left(\sum_{\mu=\nu}^{\infty} a_{\mu}^2 \right)^{\theta/2},$$

where is $\frac{\theta}{2} + \frac{1}{\theta'} = 1$. The last estimate implies

$$J_1 \geq C_{51} \sum_{\nu=n+1}^{\infty} 2^{\nu(r+\varepsilon)\theta} \sum_{\mu=\nu}^{\infty} a_{\mu}^{\theta} 2^{-\mu \varepsilon \theta}.$$

Changing the order of summation and then computing the second sum we obtain

$$J_1 \geq C_{51} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{-\mu \varepsilon \theta} \sum_{\nu=n+1}^{\mu} 2^{\nu(r+\varepsilon)\theta} \geq C_{52} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta},$$

where constant C_{52} does not depend on n .

Consequently, for every $0 < \theta < \infty$ the following estimate holds

$$C_{53} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta} \leq J_1 \leq C_{54} \sum_{\mu=n+1}^{\infty} a_{\mu}^{\theta} 2^{\mu r \theta}, \quad (18)$$

where constants C_{53} and C_{54} do not depend on n .

Now we estimate J_2 . Obviously

$$J_2 \geq 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2}.$$

Let $1 \leq \frac{\theta}{2} < \infty$. Applying Lemma 3.1, changing the order of summation, and then computing the second sum we obtain

$$\begin{aligned} J_2 &\geq 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \sum_{\mu=\nu}^n a_{\mu}^{\theta} \\ &= 2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} \sum_{\nu=0}^{\mu} 2^{\nu(r+\lambda)\theta} \geq C_{55} 2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} 2^{\mu(r+\lambda)\theta}. \end{aligned}$$

Let $0 < \frac{\theta}{2} \leq 1$ and $\varepsilon > 0$. Applying Hölder inequality we get

$$\sum_{\mu=\nu}^n a_{\mu}^{\theta} 2^{-\mu\varepsilon\theta} \leq \left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2} \left(\sum_{\mu=\nu}^n 2^{-\mu\varepsilon\theta'} \right)^{1/\theta'} \leq \frac{C_{56}}{2^{\nu\varepsilon\theta}} \left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2},$$

where is $\frac{\theta}{2} + \frac{1}{\theta'} = 1$. The last estimate implies

$$J_2 \geq C_{57} 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda+\varepsilon)\theta} \sum_{\mu=\nu}^n a_{\mu}^{\theta} 2^{-\mu\varepsilon\theta}.$$

Changing the order of summation and computing the second sum we have

$$J_2 \geq C_{57} 2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} 2^{-\mu\varepsilon\theta} \sum_{\nu=0}^{\mu} 2^{\nu(r+\lambda+\varepsilon)\theta} \geq C_{58} 2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} 2^{\mu(r+\lambda)\theta}.$$

Thus, for every $0 < \theta < \infty$ holds

$$J_2 \geq C_{59} 2^{-n\lambda\theta} \sum_{\mu=0}^n a_{\mu}^{\theta} 2^{\mu(r+\lambda)\theta}. \quad (19)$$

Now we estimate J_2 from above. Taking into consideration that $(r+\lambda)\theta > 0$, we have

$$\begin{aligned} J_2 &\leq C_{60} 2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2} + \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^2 \right)^{\theta/2} \right) \\ &\leq C_{61} \left(2^{-n\lambda\theta} \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2} + 2^{nr\theta} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^2 \right)^{\theta/2} \right). \quad (20) \end{aligned}$$

Since

$$2^{nr\theta} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^2 \right)^{\theta/2} \leq \sum_{\mu=n+1}^{\infty} 2^{\nu r \theta} \left(\sum_{\mu=n+1}^{\infty} a_{\mu}^2 \right)^{\theta/2} = J_1$$

holds and an upper bound for J_1 is already found, we estimate from above the expression

$$J_3 = \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^n a_{\mu}^2 \right)^{\theta/2}.$$

Let $0 < \frac{\theta}{2} \leq 1$. Applying Lemma 3.1 we obtain

$$J_3 \leq \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \sum_{\mu=\nu}^n a_\mu^\theta = \sum_{\mu=0}^n a_\mu^\theta \sum_{\nu=0}^{\mu} 2^{\nu(r+\lambda)\theta} \leq C_{62} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta}.$$

Let $1 \leq \frac{\theta}{2} < \infty$ and $0 < \varepsilon < r + \lambda$. Then applying Hölder inequality we have

$$\sum_{\mu=\nu}^n a_\mu^2 \leq \left(\sum_{\mu=\nu}^n a_\mu^\theta 2^{\mu\varepsilon\theta} \right)^{2/\theta} \left(\sum_{\mu=\nu}^n 2^{-2\mu\varepsilon\theta'} \right)^{1/\theta'},$$

where is $\frac{2}{\theta} + \frac{1}{\theta'} = 1$. Using the last estimate we get

$$J_3 \leq \sum_{\nu=0}^n 2^{\nu(r+\lambda)\theta} \left(\sum_{\mu=\nu}^{\infty} 2^{-2\mu\varepsilon\theta'} \right)^{\frac{\theta}{2\theta'}} \sum_{\mu=\nu}^n a_\mu^\theta 2^{\mu\varepsilon\theta} \leq C_{63} \sum_{\nu=0}^n 2^{\nu(r+\lambda-\varepsilon)\theta} \sum_{\mu=\nu}^n a_\mu^\theta 2^{\mu\varepsilon\theta}.$$

Changing the order of summation and computing the second sum we obtain

$$J_3 \leq C_{63} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu\varepsilon\theta} \sum_{\nu=0}^{\mu} 2^{\nu(r+\lambda-\varepsilon)\theta} \leq C_{64} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta}.$$

Therefore, for every $0 < \theta < \infty$ the following estimate holds

$$J_3 \leq C_{65} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta}.$$

Now making use of inequalities (20) and (18) we have

$$J_2 \leq C_{66} \left(2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} + \sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} \right).$$

This way, inequalities (18), (19) and the last inequality imply the estimate

$$\begin{aligned} C_{67} \left(\sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} \right) &\leq J_1 + J_2 \\ &\leq C_{68} \left(\sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} \right), \end{aligned}$$

where constants C_{67} and C_{68} do not depend on n . Hence, considering the condition (17) we conclude that condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$A_n = \sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} \leq C_{69} \varphi \left(\frac{1}{2^n} \right)^\theta, \quad (21)$$

where constant C_{69} does not depend on n .

We put

$$D_m = \sum_{\nu=m+1}^{\infty} \lambda_\nu^\theta \nu^{r\theta} + m^{-\lambda\theta} \sum_{\nu=1}^m \lambda_\nu^\theta \nu^{(r+\lambda)\theta}.$$

For given m we choose the positive integer n such that $2^n \leq m+1 < 2^{n+1}$.

First we consider the case $2^n < m+1 < 2^{n+1}$. We have

$$\begin{aligned} D_m &= \sum_{\nu=2^{n+1}}^{\infty} \lambda_\nu^\theta \nu^{r\theta} + \sum_{\nu=m+1}^{2^{n+1}-1} \lambda_\nu^\theta \nu^{r\theta} + m^{-\lambda\theta} \sum_{\nu=1}^{2^n-1} \lambda_\nu^\theta \nu^{(r+\lambda)\theta} \\ &\quad + m^{-\lambda\theta} \sum_{\nu=2^n}^m \lambda_\nu^\theta \nu^{(r+\lambda)\theta}. \end{aligned}$$

Since $\lambda_\nu = 0$ for $\nu \neq 2^\mu$, we get

$$\begin{aligned} D_m &= \sum_{\nu=2^{n+1}}^{\infty} \lambda_\nu^\theta \nu^{r\theta} + m^{-\lambda\theta} \sum_{\nu=1}^{2^n-1} \lambda_\nu^\theta \nu^{(r+\lambda)\theta} + m^{-\lambda\theta} \lambda_{2^n}^\theta 2^{n(r+\lambda)\theta} \\ &= \sum_{\mu=n+1}^{\infty} \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \lambda_\nu^\theta \nu^{r\theta} + m^{-\lambda\theta} \sum_{\mu=0}^{n-1} \sum_{\nu=2^\mu}^{2^{\mu+1}-1} \lambda_\nu^\theta \nu^{(r+\lambda)\theta} + m^{-\lambda\theta} \lambda_{2^n}^\theta 2^{n(r+\lambda)\theta} \\ &= \sum_{\mu=n+1}^{\infty} \lambda_{2^\mu}^\theta 2^{\mu r\theta} + m^{-\lambda\theta} \sum_{\mu=0}^{n-1} \lambda_{2^\mu}^\theta 2^{\mu(r+\lambda)\theta} + m^{-\lambda\theta} \lambda_{2^n}^\theta 2^{n(r+\lambda)\theta}. \end{aligned}$$

Further, since $\lambda_{2^\mu} = a_\mu$, we get

$$D_m = \sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + m^{-\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta}.$$

Hence, for $2^n < m+1 < 2^{n+1}$ we obtain

$$\begin{aligned} C_{70} \left(\sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} \right) &\leq D_m \\ &\leq C_{71} \left(\sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} \right), \end{aligned}$$

where constants C_{70} and C_{71} do not depend on m and n .

Let us assume now that $m+1 = 2^n$. In an analogous way we have

$$\begin{aligned} D_m &= \sum_{\nu=2^n}^{\infty} \lambda_\nu^\theta \nu^{r\theta} + 2^{-n\lambda\theta} \sum_{\nu=1}^{2^n-1} \lambda_\nu^\theta \nu^{(r+\lambda)\theta} \\ &= \sum_{\mu=n}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^{n-1} a_\mu^\theta 2^{\mu(r+\lambda)\theta} \\ &= \sum_{\mu=n+1}^{\infty} a_\mu^\theta 2^{\mu r\theta} + 2^{-n\lambda\theta} \sum_{\mu=0}^n a_\mu^\theta 2^{\mu(r+\lambda)\theta} = A_n. \end{aligned}$$

Thus, for $2^n \leq m+1 < 2^{n+1}$ the following estimate holds

$$C_{72} A_n \leq D_m \leq C_{73} A_n,$$

where constants C_{72} and C_{73} do not depend on m and n . Hence, considering the condition (21) we conclude that condition $f \in N(p, \theta, r, \lambda, \varphi)$ is equivalent to the condition

$$D_m \leq C_{74} \varphi \left(\frac{1}{2^n} \right)^\theta, \quad (22)$$

where constant C_{74} does not depend on m and n .

Since $\frac{1}{2^n} < \frac{2}{m+1} < \frac{2}{m}$, we get

$$\varphi \left(\frac{1}{2^n} \right) \leq C_{75} \varphi \left(\frac{2}{m+1} \right) \leq C_{76} \varphi \left(\frac{2}{m} \right),$$

where constant C_{76} does not depend on m and n ; and since $\frac{1}{2^n} \geq \frac{1}{m+1} \geq \frac{1}{2m}$ we get

$$\varphi \left(\frac{1}{2^n} \right) \geq C_{77} \varphi \left(\frac{1}{2m} \right) \geq C_{78} \varphi \left(\frac{1}{m} \right),$$

where constant C_{78} does not depend on m and n . This way, condition (22) is equivalent to the condition

$$D_m \leq C_{79} \varphi \left(\frac{1}{m} \right)^\theta,$$

where constant C_{79} does not depend on m .

This completes the proof of Theorem 2.3. \square

Remark 4.1. Notice that another way of proving Theorems 2.2 and 2.3 is presented in [12]. Our approach here is similar to that used in [1].

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