Chapter One

A LIBRARY OF FUNCTIONS

Contents

1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In January, 2007, the temperatures in Fresno, California were unusually low for the winter and much of the orange crop was lost. The daily high temperatures for January 9–18 are given in Table 1.1.

Table 1.1 *Daily high temperature in Fresno, January 9–18, 2007*

Date (January 2007)	9	10	11	12	13	14	15	16	17	18
High temperature (°F)	32	32	39	25	23	25	24	25	28	29

Although you may not have thought of something so unpredictable as temperature as being a function, the temperature *is* a function of date, because each day gives rise to one and only one high temperature. There is no formula for temperature (otherwise we would not need the weather bureau), but nevertheless the temperature does satisfy the definition of a function: Each date, t, has a unique high temperature, H, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the temperature example, the domain is the set of dates $\{9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ and the range is the set of temperatures $\{23, 24, 25, 28, 29, 32, 39\}$. We call the function f and write $H = f(t)$. Notice that a function may have identical outputs for different inputs (January 12, 14, and 16, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \leq t \leq b$ is written $[a, b]$.

The set of numbers t such that $a < t < b$ is written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily high temperatures in Fresno can be represented by the graph in Figure 1.1, as well as by Table 1.1.

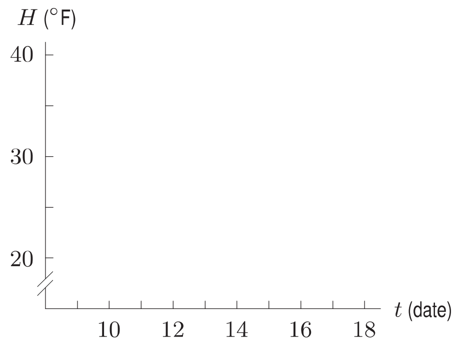


Figure 1.1: Fresno temperatures, January 2007

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair degree of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f. The graph of this function is in Figure 1.2.

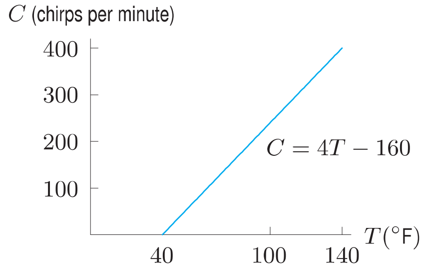


Figure 1.2: Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we restrict the domain to nonnegative values of x.

Example 1 The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 136°F. What is the domain of this function f?

Solution If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T, any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40$°F, we see that T cannot be less than 40°F. (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above $136°$. Thus, for the function $C = f(T)$ we have

$$\begin{aligned} \text{Domain} &= \text{All } T \text{ values between } 40^\circ F \text{ and } 136^\circ F \\ &= \text{All } T \text{ values with } 40 \le T \le 136 \\ &= [40, 136]. \end{aligned}$$

Example 2 Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40$°F) and 384 (at $T = 136$°F). Hence,

$$\begin{aligned} \text{Range} &= \text{All } C \text{ values from } 0 \text{ to } 384 \\ &= \text{All } C \text{ values with } 0 \le C \le 384 \\ &= [0, 384]. \end{aligned}$$

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we say that f is an *increasing function*. The coefficient 2 tells us the rate, in inches per year, at which the height increases.

Table 1.2 *Men's Olympic pole vault winning height (approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year}.$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the *intercept* on the vertical axis.

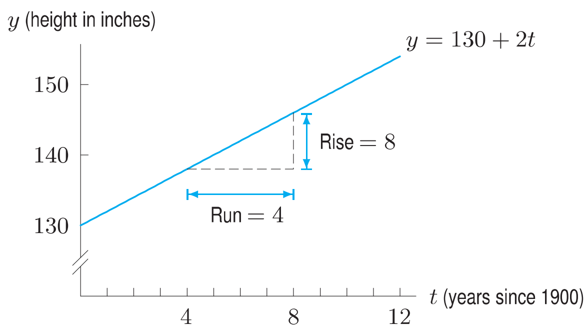


Figure 1.3: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130+2t$ predicts that the height in the 2004 Olympics would be 338 inches or 28 feet 2 inches, which is considerably higher than the actual value of 19 feet 6.25 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.39t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.39, tells us that the world record decreased by about 0.39 seconds per year. We say that g is a *decreasing function*.

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4). Since $m = \Delta y/\Delta x$, the units of m are y-units over x-units.

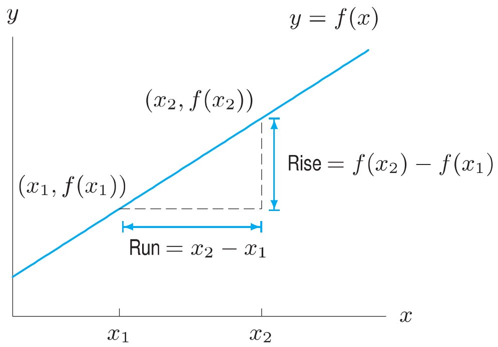


Figure 1.4: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y-values that are constant for equally spaced x-values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the graphs are straight lines. The constants m and b are called *parameters*; their meaning is shown in Figures 1.5 and 1.6. Notice the greater the magnitude of m, the steeper the line.

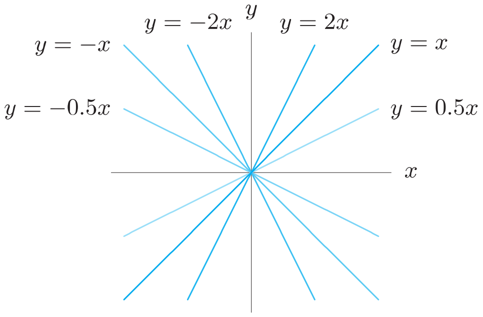


Figure 1.5: The family $y = mx$ (with $b = 0$)

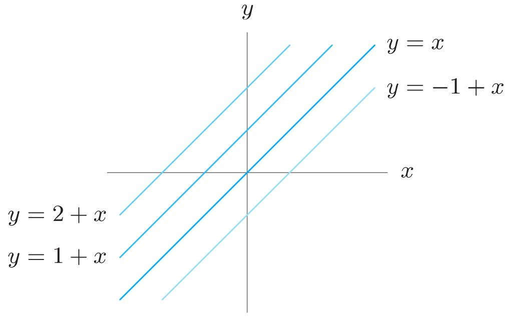


Figure 1.6: The family $y = b + x$ (with $m = 1$)

Increasing versus Decreasing Functions

The terms increasing and decreasing can be applied to other functions, not just linear ones. See Figure 1.7. In general,

> A function f is **increasing** if the values of $f(x)$ increase as x increases.
> A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
>
> The graph of an *increasing* function *climbs* as we move from left to right.
> The graph of a *decreasing* function *falls* as we move from left to right.
>
> A function $f(x)$ is **monotonic** if it increases for all x or decreases for all x.

Figure 1.7: Increasing and decreasing functions

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, the area, A, of a circle is proportional to the square of the radius, r, because

$$A = f(r) = \pi r^2.$$

> We say y is (directly) **proportional** to x if there is a nonzero constant k such that
>
> $$y = kx.$$
>
> This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Exercises and Problems for Section 1.1

Exercises

1. The population of a city, P, in millions, is a function of t, the number of years since 1970, so $P = f(t)$. Explain the meaning of the statement $f(35) = 12$ in terms of the population of this city.

2. When a patient with a rapid heart rate takes a drug, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch the heart rate against time from the moment the drug is administered.

3. Describe what Figure 1.8 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

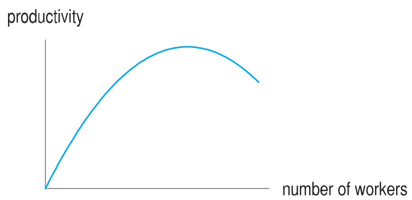


Figure 1.8

For Exercises 4–7, find an equation for the line that passes through the given points.

4. $(0, 0)$ and $(1, 1)$
5. $(0, 2)$ and $(2, 3)$
6. $(-2, 1)$ and $(2, 3)$
7. $(-1, 0)$ and $(2, 6)$

For Exercises 8–11, determine the slope and the y-intercept of the line whose equation is given.

8. $2y + 5x - 8 = 0$
9. $7y + 12x - 2 = 0$
10. $-4y + 2x + 8 = 0$
11. $12x = 6y + 4$

12. Match the graphs in Figure 1.9 with the following equations. (Note that the x and y scales may be unequal.)

    (a) $y = x - 5$ **(b)** $-3x + 4 = y$
    (c) $5 = y$ **(d)** $y = -4x - 5$
    (e) $y = x + 6$ **(f)** $y = x/2$

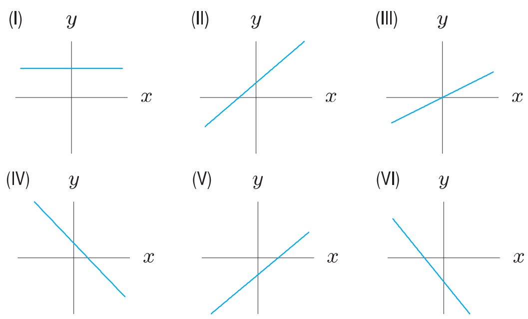


Figure 1.9

13. Match the graphs in Figure 1.10 with the following equations. (Note that the x and y scales may be unequal.)

    (a) $y = -2.72x$ **(b)** $y = 0.01 + 0.001x$
    (c) $y = 27.9 - 0.1x$ **(d)** $y = 0.1x - 27.9$
    (e) $y = -5.7 - 200x$ **(f)** $y = x/3.14$

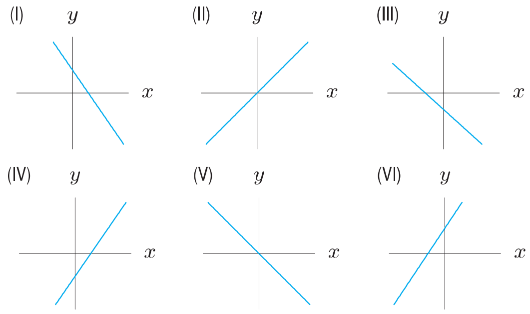


Figure 1.10

14. Estimate the slope and the equation of the line in Figure 1.11.

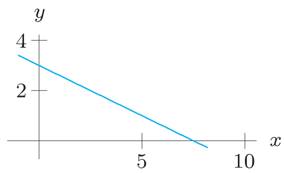


Figure 1.11

15. Find an equation for the line with slope m through the point (a, c).

16. Find a linear function that generates the values in Table 1.3.

Table 1.3

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

For Exercises 17–19, use the facts that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals of one another.

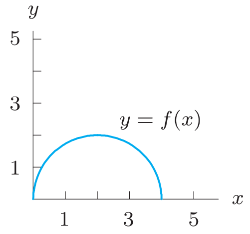
17. Find an equation for the line through the point $(2, 1)$ which is perpendicular to the line $y = 5x - 3$.

18. Find equations for the lines through the point $(1, 5)$ that are parallel to and perpendicular to the line with equation $y + 4x = 7$.

19. Find equations for the lines through the point (a, b) that are parallel and perpendicular to the line $y = mx + c$, assuming $m \neq 0$.

For Exercises 20–23, give the approximate domain and range of each function. Assume the entire graph is shown.

20.



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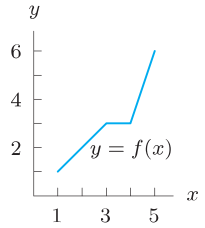

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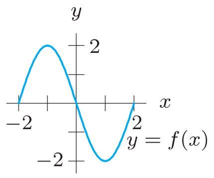

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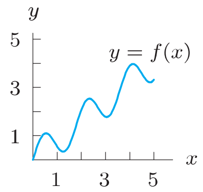


Find domain and range in Exercises 24–25.

24. $y = x^2 + 2$

25. $y = \dfrac{1}{x^2 + 2}$

26. If $f(t) = \sqrt{t^2 - 16}$, find all values of t for which $f(t)$ is a real number. Solve $f(t) = 3$.

27. If $g(x) = (4 - x^2)/(x^2 + x)$, find the domain of $g(x)$. Solve $g(x) = 0$.

In Exercises 28–32, write a formula representing the function.

28. The volume of a sphere is proportional to the cube of its radius, r.

29. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t.

30. The strength, S, of a beam is proportional to the square of its thickness, h.

31. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin.

32. The number of animal species, N, of a certain body length, l, is inversely proportional to the square of l.

Problems

33. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a, in years.

- **(a)** Interpret the statement $f(5) = 6$
- **(b)** Sketch a possible graph of V against a. Is f an increasing or decreasing function? Explain.
- **(c)** Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.

34. Which graph in Figure 1.12 best matches each of the following stories?[1] Write a story for the remaining graph.

- **(a)** I had just left home when I realized I had forgotten my books, and so I went back to pick them up.
- **(b)** Things went fine until I had a flat tire.
- **(c)** I started out calmly but sped up when I realized I was going to be late.

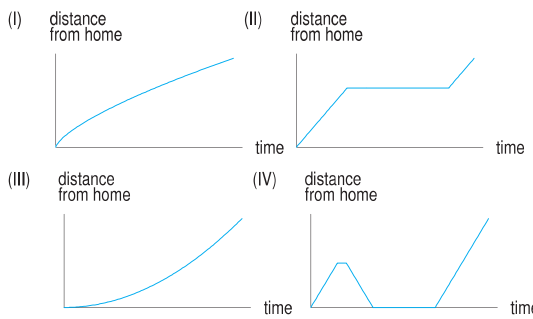


Figure 1.12

35. A company rents cars at $40 a day and 15 cents a mile. Its competitor's cars are $50 a day and 10 cents a mile.

- **(a)** For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.
- **(b)** On the same axes, graph both functions.
- **(c)** How should you decide which company is cheaper?

36. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed $40, while one using 1600 cubic feet was billed $55.

- **(a)** What is the charge per cubic foot?
- **(b)** Write an equation for the total cost of a resident's water as a function of cubic feet of water used.
- **(c)** How many cubic feet of water used would lead to a bill of $100?

37. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.13.

- **(a)** What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.
- **(b)** Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

[1] Adapted from Jan Terwel, "Real Math in Cooperative Groups in Secondary Education." *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234, (Reading: Addison Wesley, 1990).

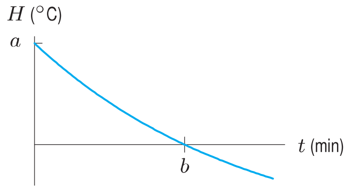


Figure 1.13

38. The force, F, between two atoms depends on the distance r separating them. See Figure 1.14. A positive F represents a repulsive force; a negative F represents an attractive force.

(a) What happens to the force if the atoms start with $r = a$ and are

(i) Pulled slightly further apart?

(ii) Pushed slightly closer together?

(b) The atoms are said to be in *stable equilibrium* if the force between them is zero and the atoms tend to return to the equilibrium after a minor disturbance. Does $r = a$ represent a stable equilibrium? Explain.

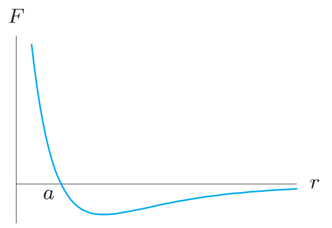


Figure 1.14

39. A controversial 1992 Danish study[2] reported that men's average sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.

(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

40. The graph of Fahrenheit temperature, °F, as a function of Celsius temperature, °C, is a line. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C both represent water's freezing point.

(a) What is the slope of the graph?

(b) What is the equation of the line?

(c) Use the equation to find what Fahrenheit temperature corresponds to 20°C.

(d) What temperature is the same number of degrees in both Celsius and Fahrenheit?

41. The demand function for a certain product, $q = D(p)$, is linear, where p is the price per item in dollars and q is the quantity demanded. If p increases by \$5, market research shows that q drops by two items. In addition, 100 items are purchased if the price is \$550.

(a) Find a formula for

(i) q as a linear function of p

(ii) p as a linear function of q

(b) Draw a graph with q on the horizontal axis.

42. The cost of planting seed is usually a function of the number of acres sown. The cost of the equipment is a *fixed cost* because it must be paid regardless of the number of acres planted. The cost of supplies and labor varies with the number of acres planted and are called *variable costs*. Suppose the fixed costs are \$10,000 and the variable costs are \$200 per acre. Let C be the total cost, measured in thousands of dollars, and let x be the number of acres planted.

(a) Find a formula for C as a function of x.

(b) Graph C against x.

(c) Which feature of the graph represents the fixed costs? Which represents the variable costs?

43. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

44. A flight from Dulles Airport in Washington, DC, to LaGuardia Airport in New York City has to circle LaGuardia several times before being allowed to land. Plot a graph of the distance of the plane from Washington, DC, against time, from the moment of takeoff until landing.

45. **(a)** Consider the functions graphed in Figure 1.15(a). Find the coordinates of C.

(b) Consider the functions in Figure 1.15(b). Find the coordinates of C in terms of b.

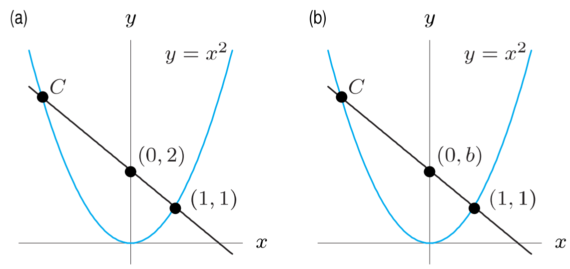


Figure 1.15

[2] "Investigating the Next Silent Spring," *US News and World Report*, p. 50-52, (March 11, 1996).

46. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen. What do the experimental data in Table 1.4 tell you about Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.4 and Table 1.5?

Table 1.4

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table 1.5

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

1.2 EXPONENTIAL FUNCTIONS

Population Growth

The population of Nevada[3] from 2000 to 2006 is given in Table 1.6. To see how the population is growing, we look at the increase in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same.

Table 1.6 *Population of Nevada (estimated), 2000–2006*

Year	Population (millions)	Change in population (millions)
2000	2.020	
		0.073
2001	2.093	
		0.075
2002	2.168	
		0.078
2003	2.246	
		0.081
2004	2.327	
		0.084
2005	2.411	
		0.087
2006	2.498	

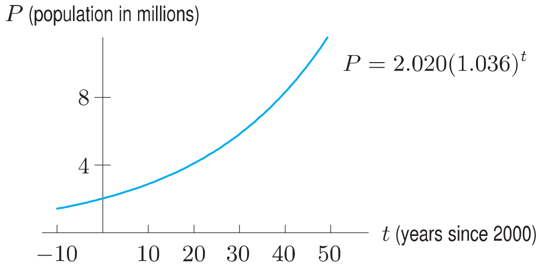


Figure 1.16: Population of Nevada (estimated): Exponential growth

Suppose we divide each year's population by the previous year's population. For example,

$$\frac{\text{Population in 2001}}{\text{Population in 2000}} = \frac{2.093 \text{ million}}{2.020 \text{ million}} = 1.036$$

$$\frac{\text{Population in 2002}}{\text{Population in 2001}} = \frac{2.168 \text{ million}}{2.093 \text{ million}} = 1.036.$$

The fact that both calculations give 1.036 shows the population grew by about 3.6% between 2000 and 2001 *and* between 2001 and 2002. Similar calculations for other years show that the population grew by a factor of about 1.036, or 3.6%, every year. Whenever we have a constant growth factor (here 1.036), we have exponential growth. The population t years after 2000 is given by the *exponential* function

$$P = 2.020(1.036)^t.$$

If we assume that the formula holds for 50 years, the population graph has the shape shown in Figure 1.16. Since the population is growing faster and faster as time goes on, the graph is bending upward; we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, eventually climb extremely quickly.

> To recognize that a table of t and P values comes from an exponential function $P = P_0 a^t$, look for ratios of P values that are constant for equally spaced t values.

[3]www.census.gov, accessed May 14, 2007.

Concavity

We have used the term concave up[4] to describe the graph in Figure 1.16. In words:

> The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward. (See Figure 1.17 for four possible shapes.) A line is neither concave up nor concave down.

Figure 1.17: Concavity of a graph

Elimination of a Drug from the Body

Now we look at a quantity which is decreasing exponentially instead of increasing. When a patient is given medication, the drug enters the bloodstream. As the drug passes through the liver and kidneys, it is metabolized and eliminated at a rate that depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since every hour the amount remaining is 60% of the previous amount, we have

$$\begin{aligned} f(0) &= 250 \\ f(1) &= 250(0.6) \\ f(2) &= (250(0.6))(0.6) = 250(0.6)^2, \end{aligned}$$

and after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This is an *exponential decay function*. Some values of the function are in Table 1.7; its graph is in Figure 1.18.

Notice the way the function in Figure 1.18 is decreasing. Each hour a smaller quantity of the drug is removed than in the previous hour. This is because as time passes, there is less of the drug in the body to be removed. Compare this to the exponential growth in Figure 1.16, where each step upward is larger than the previous one. Notice, however, that both graphs are concave up.

Table 1.7

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

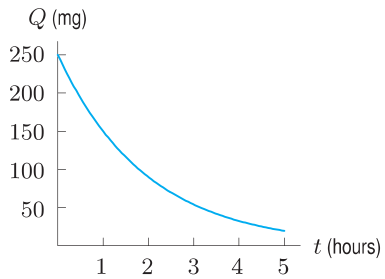


Figure 1.18: Drug elimination: Exponential decay

[4]In Chapter 2 we consider concavity in more depth.

The General Exponential Function

We say P is an **exponential function** of t with base a if

$$P = P_0 a^t,$$

where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.
If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 1^t = P_0$ is then a constant function.

The value of a is closely related to the percent growth (or decay) rate. For example, if $a = 1.03$, then P is growing at 3%; if $a = 0.94$, then P is decaying at 6%.

Example 1 Suppose that $Q = f(t)$ is an exponential function of t. If $f(20) = 88.2$ and $f(23) = 91.4$:

(a) Find the base. (b) Find the growth rate. (c) Evaluate $f(25)$.

Solution

(a) Let

$$Q = Q_0 a^t.$$

Substituting $t = 20, Q = 88.2$ and $t = 23, Q = 91.4$ gives two equations for Q_0 and a:

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0:

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a, gives

$$a = \left(\frac{91.4}{88.2}\right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the growth rate is $0.012 = 1.2\%$.

(c) We want to evaluate $f(25) = Q_0 a^{25} = Q_0(1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0(1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

Half-Life and Doubling Time

Radioactive substances, such as uranium, decay exponentially. A certain percentage of the mass disintegrates in a given unit of time; the time it takes for half the mass to decay is called the *half-life* of the substance.

A well-known radioactive substance is carbon-14, which is used to date organic objects. When a piece of wood or bone was part of a living organism, it accumulated small amounts of radioactive carbon-14. Once the organism dies, it no longer picks up carbon-14. Using the half-life of carbon-14 (about 5730 years), we can estimate the age of the object. We use the following definitions:

The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.

The Family of Exponential Functions

The formula $P = P_0 a^t$ gives a family of exponential functions with positive parameters P_0 (the initial quantity) and a (the base, or growth/decay factor). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.19 and 1.20.) All members of the family $P = P_0 a^t$ are concave up.

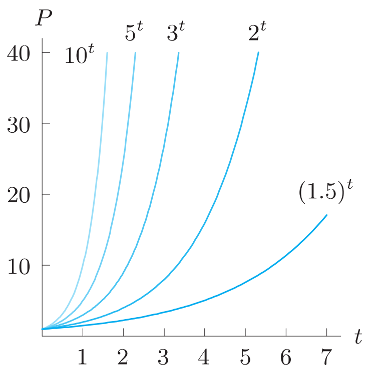


Figure 1.19: Exponential growth: $P = a^t$, for $a > 1$

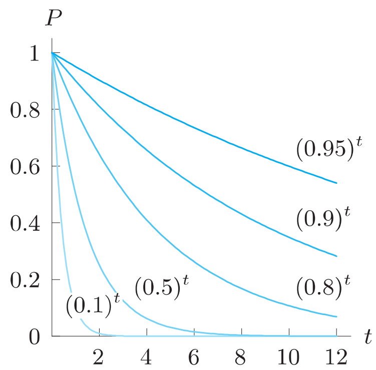


Figure 1.20: Exponential decay: $P = a^t$, for $0 < a < 1$

Example 2 Figure 1.21 is the graph of three exponential functions. What can you say about the values of the six constants, a, b, c, d, p, q?

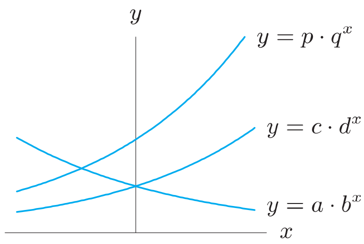


Figure 1.21

Solution All the constants are positive. Since a, c, p represent y-intercepts, we see that $a = c$ because these graphs intersect on the y-axis. In addition, $a = c < p$, since $y = p \cdot q^x$ crosses the y-axis above the other two.

Since $y = a \cdot b^x$ is decreasing, we have $0 < b < 1$. The other functions are increasing, so $1 < d$ and $1 < q$.

Exponential Functions with Base e

The most frequently used base for an exponential function is the famous number $e = 2.71828\ldots$. This base is used so often that you will find an e^x button on most scientific calculators. At first glance, this is all somewhat mysterious. Why is it convenient to use the base $2.71828\ldots$? The full answer to that question must wait until Chapter 3, where we show that many calculus formulas come out neatly when e is used as the base. We often use the following result:

Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form

$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}$$

and any **exponential decay** function can be written, for some $0 < a < 1$ and $k > 0$, as

$$Q = Q_0 a^t \quad \text{or} \quad Q = Q_0 e^{-kt},$$

where P_0 and Q_0 are the initial quantities.

We say that P and Q are growing or decaying at a *continuous*[5] *rate* of k. (For example, $k = 0.02$ corresponds to a continuous rate of 2%.)

Example 3 Convert the functions $P = e^{0.5t}$ and $Q = 5e^{-0.2t}$ into the form $y = y_0 a^t$. Use the results to explain the shape of the graphs in Figures 1.22 and 1.23.

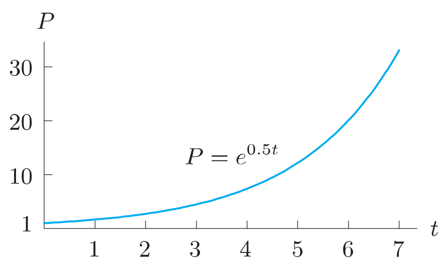


Figure 1.22: An exponential growth function

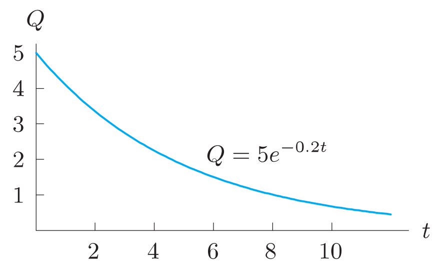


Figure 1.23: An exponential decay function

Solution We have

$$P = e^{0.5t} = (e^{0.5})^t = (1.65)^t.$$

Thus, P is an exponential growth function with $P_0 = 1$ and $a = 1.65$. The function is increasing and its graph is concave up, similar to those in Figure 1.19. Also,

$$Q = 5e^{-0.2t} = 5(e^{-0.2})^t = 5(0.819)^t,$$

so Q is an exponential decay function with $Q_0 = 5$ and $a = 0.819$. The function is decreasing and its graph is concave up, similar to those in Figure 1.20.

Example 4 The quantity, Q, of a drug in a patient's body at time t is represented for positive constants S and k by the function $Q = S(1 - e^{-kt})$. For $t \geq 0$, describe how Q changes with time. What does S represent?

Solution The graph of Q is shown in Figure 1.24. Initially none of the drug is present, but the quantity increases with time. Since the graph is concave down, the quantity increases at a decreasing rate. This is realistic because as the quantity of the drug in the body increases, so does the rate at which the body excretes the drug. Thus, we expect the quantity to level off. Figure 1.24 shows that S is the saturation level. The line $Q = S$ is called a *horizontal asymptote*.

[5] The reason that k is called the continuous rate is explored in detail in Chapter 11.

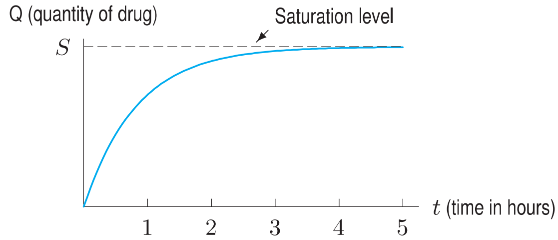


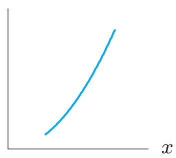
Figure 1.24: Buildup of the quantity of a drug in body

Exercises and Problems for Section 1.2

Exercises

In Exercises 1–4, decide whether the graph is concave up, concave down, or neither.

1.



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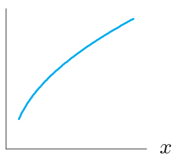

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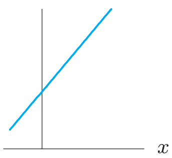

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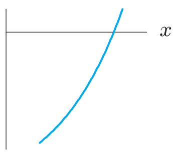


The functions in Exercises 5–8 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

5. $P = 5(1.07)^t$

6. $P = 7.7(0.92)^t$

7. $P = 3.2e^{0.03t}$

8. $P = 15e^{-0.06t}$

Write the functions in Problems 9–12 in the form $P = P_0a^t$. Which represent exponential growth and which represent exponential decay?

9. $P = 15e^{0.25t}$

10. $P = 2e^{-0.5t}$

11. $P = P_0e^{0.2t}$

12. $P = 7e^{-\pi t}$

In Problems 13–14, let $f(t) = Q_0a^t = Q_0(1+r)^t$.

(a) Find the base, a.

(b) Find the percentage growth rate, r.

13. $f(5) = 75.94$ and $f(7) = 170.86$

14. $f(0.02) = 25.02$ and $f(0.05) = 25.06$

15. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

(a) The population increases by 50 people a year.
(b) The population increases by 5% a year.

16. Identify the x-intervals on which the function graphed in Figure 1.25 is:

(a) Increasing and concave up
(b) Increasing and concave down
(c) Decreasing and concave up
(d) Decreasing and concave down

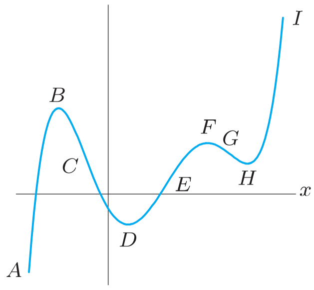


Figure 1.25

Problems

17. An air-freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:

(a) 2 grams a day **(b)** 12% a day

18. In 2007, the world's population reached 6.7 billion and was increasing at a rate of 1.2% per year. Assume that this growth rate remains constant. (In fact, the growth rate has decreased since 1987.)

(a) Write a formula for the world population (in billions) as a function of the number of years since 2007.
(b) Use your formula to estimate the population of the world in the year 2020.
(c) Sketch a graph of world population as a function of years since 2007. Use the graph to estimate the doubling time of the population of the world.

19. A photocopy machine can reduce copies to 80% of their original size. By copying an already reduced copy, further reductions can be made.

(a) If a page is reduced to 80%, what percent enlargement is needed to return it to its original size?
(b) Estimate the number of times in succession that a page must be copied to make the final copy less than 15% of the size of the original.

20. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

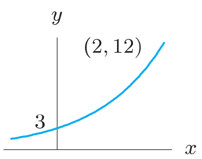
(a) Graph the total number of people who have tried such a product against time.
(b) What do you know about the concavity of the graph?

21. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

(a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.
(b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

Give a possible formula for the functions in Problems 22–25.

22.



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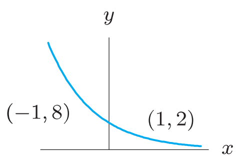

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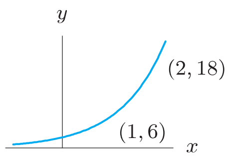
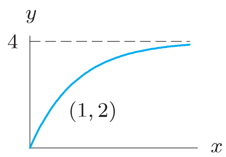


25.



26. **(a)** A population, P, grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0e^{kt}$, with P_0, k constants.
(b) Plot the population in part (a) against time.

27. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

28. During April 2006, Zimbabwe's inflation rate averaged 0.67% a day. This means that, on average, prices went up by 0.67% from one day to the next.

(a) By what percentage did prices in Zimbabwe increase in April of 2006?
(b) Assuming the same rate all year, what was Zimbabwe's annual inflation rate during 2006?

29. **(a)** The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is Q_0.
(b) What percentage of the original amount of radium is left after 500 years?

30. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 1990?

31. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by guessing and checking.

32. Aircraft require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

33. Each of the functions g, h, k in Table 1.8 is increasing, but each increases in a different way. Which of the graphs in Figure 1.26 best fits each function?

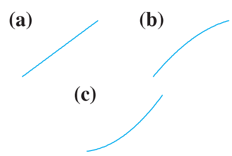


Figure 1.26

Table 1.8

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

34. Each of the functions in Table 1.9 decreases, but each decreases in a different way. Which of the graphs in Figure 1.27 best fits each function?

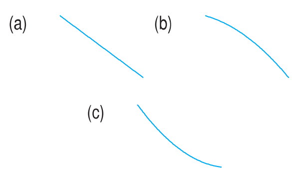


Figure 1.27

Table 1.9

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

35. **(a)** Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

36. The median price, P, of a home rose from \$60,000 in 1980 to \$180,000 in 2000. Let t be the number of years since 1980.

(a) Assume the increase in housing prices has been linear. Give an equation for the line representing price, P, in terms of t. Use this equation to complete column (a) of Table 1.10. Use units of \$1000.
(b) If instead the housing prices have been rising exponentially, find an equation of the form $P = P_0 a^t$ to represent housing prices. Complete column (b) of Table 1.10.
(c) On the same set of axes, sketch the functions represented in column (a) and column (b) of Table 1.10.
(d) Which model for the price growth do you think is more realistic?

Table 1.10

t	(a) Linear growth price in \$1000 units	(b) Exponential growth price in \$1000 units
0	60	60
10		
20	180	180
30		
40		

37. Estimate graphically the doubling time of the exponentially growing population shown in Figure 1.28. Check that the doubling time is independent of where you start on the graph. Show algebraically that if $P = P_0 a^t$ doubles between time t and time $t + d$, then d is the same number for any t.

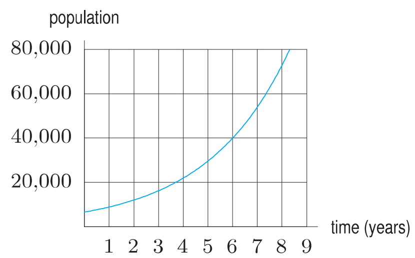


Figure 1.28

1.3 NEW FUNCTIONS FROM OLD

Shifts and Stretches

The graph of a constant multiple of a given function is easy to visualize: each y-value is stretched or shrunk by that multiple. For example, consider the function $f(x)$ and its multiples $y = 3f(x)$ and $y = -2f(x)$. Their graphs are shown in Figure 1.29. The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3; the factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflects it about the x-axis. You can think of the multiples of a given function as a family of functions.

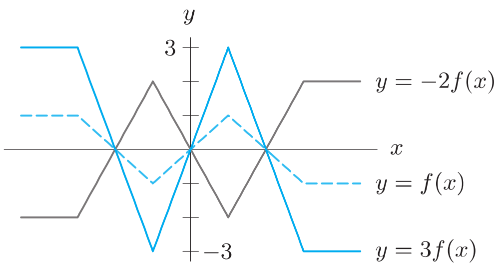


Figure 1.29: Multiples of the function $f(x)$

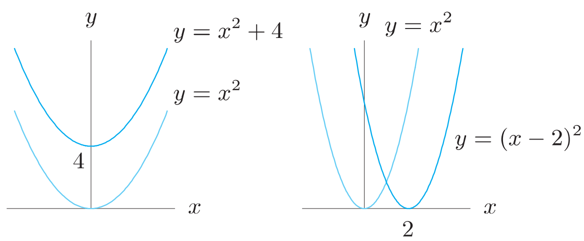


Figure 1.30: Graphs of $y = x^2$ with $y = x^2 + 4$ and $y = (x - 2)^2$

It is also easy to create families of functions by shifting graphs. For example, $y - 4 = x^2$ is the same as $y = x^2 + 4$, which is the graph of $y = x^2$ shifted up by 4. Similarly, $y = (x - 2)^2$ is the graph of $y = x^2$ shifted right by 2. (See Figure 1.30.)

- Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.
- Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).

Composite Functions

If oil is spilled from a tanker, the area of the oil slick grows with time. Suppose that the oil slick is always a perfect circle. Then the area, A, of the oil slick is a function of its radius, r:

$$A = f(r) = \pi r^2.$$

The radius is also a function of time, because the radius increases as more oil spills. Thus, the area, being a function of the radius, is also a function of time. If, for example, the radius is given by

$$r = g(t) = 1 + t,$$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1 + t)^2.$$

We are thinking of A as a *composite function* or a "function of a function," which is written

$$A = \underbrace{f(g(t))}_{\substack{\text{Composite function;} \\ f \text{ is outside function,} \\ g \text{ is inside function}}} = \pi(g(t))^2 = \pi(1 + t)^2.$$

To calculate A using the formula $\pi(1 + t)^2$, the first step is to find $1 + t$, and the second step is to square and multiply by π. The first step corresponds to the inside function $g(t) = 1 + t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1 If $f(x) = x^2$ and $g(x) = x + 1$, find each of the following:

(a) $f(g(2))$ (b) $g(f(2))$ (c) $f(g(x))$ (d) $g(f(x))$

Solution

(a) Since $g(2) = 3$, we have $f(g(2)) = f(3) = 9$.
(b) Since $f(2) = 4$, we have $g(f(2)) = g(4) = 5$. Notice that $f(g(2)) \neq g(f(2))$.
(c) $f(g(x)) = f(x + 1) = (x + 1)^2$.
(d) $g(f(x)) = g(x^2) = x^2 + 1$. Again, notice that $f(g(x)) \neq g(f(x))$.

Example 2 Express each of the following functions as a composition:

(a) $h(t) = (1 + t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

(a) For $(1 + t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1 + t^3$. The second stage is taking the 27^{th} power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1 + t^3) = (1 + t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1+y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

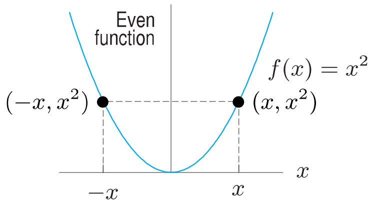
(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.31. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g, the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y-axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y-axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.



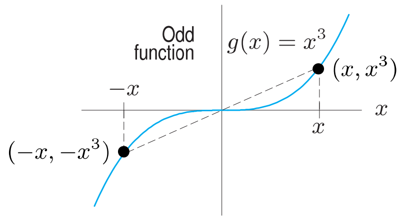


Figure 1.31: Symmetry of even and odd functions

> For any function f,
> f is an **even** function if $f(-x) = f(x)$ for all x.
> f is an **odd** function if $f(-x) = -f(x)$ for all x.

For example, $g(x) = e^{x^2}$ is even and $h(x) = x^{1/3}$ is odd. However, many functions do not have any symmetry and are neither even nor odd.

Inverse Functions

On August 26, 2005, the runner Kenenisa Bekele of Ethiopia set a world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.11, where $t = f(d)$ is the number of seconds Bekele took to complete the first d meters of the race. For example, Bekele ran the first 4000 meters in 629.98 seconds, so $f(4000) = 629.98$. The function f was useful to athletes planning to compete with Bekele.

Let us now change our point of view and ask for distances rather than times. If we ask how far Bekele ran during the first 629.98 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1}, the *inverse function*[6] of f. We write $f^{-1}(629.98) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Bekele ran during the first t seconds of his race. See Table 1.12 which contains values of f^{-1}.

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \leq d \leq 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \leq t \leq 1577.53$, which is the domain of f^{-1}.

[6]The notation f^{-1} represents the inverse function, which is not the same as the reciprocal, $1/f$.

Table 1.11 *Bekele's running time*

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

Table 1.12 *Distance run by Bekele*

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
315.63	2000
629.98	4000
944.66	6000
1264.63	8000
1577.53	10000

Which Functions Have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.32.)

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Bekele time function did have an inverse is that each running time, t, corresponds to a unique distance, d.

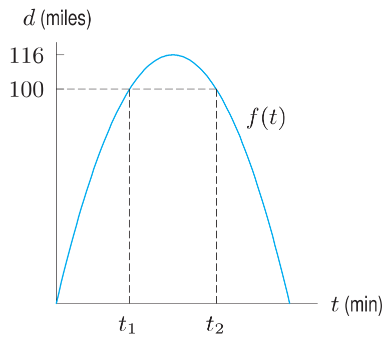


Figure 1.32: Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

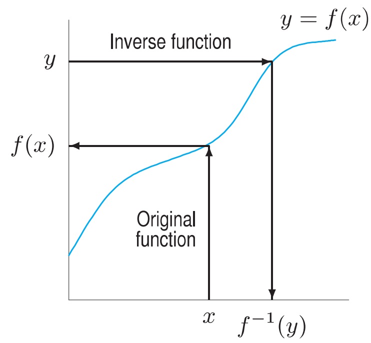


Figure 1.33: A function which has an inverse

Figure 1.33 suggests when an inverse exists. The original function, f, takes us from an x-value to a y-value, as shown in Figure 1.33. Since having an inverse means there is a function going from a y-value to an x-value, the crucial question is whether we can get back. In other words, does each y-value correspond to a unique x-value? If so, there's an inverse; if not, there is not. This principle may be stated geometrically, as follows:

> A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

For example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice.

Definition of an Inverse Function

If the function f is invertible, its inverse is defined as follows:

$$f^{-1}(y) = x \quad \text{means} \quad y = f(x).$$

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function. In Section 1.1, we looked at the snow tree cricket, whose chirp rate, C, in chirps per minute, is approximated at the temperature, T, in degrees Fahrenheit, by the formula

$$C = f(T) = 4T - 160.$$

So far we have used this formula to predict the chirp rate from the temperature. But it is also possible to use this formula backward to calculate the temperature from the chirp rate.

Example 3 Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution Since C is an increasing function, f is invertible. We know $C = 4T - 160$. We solve for T, giving

$$T = \frac{C}{4} + 40,$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

Graphs of Inverse Functions

The function $f(x) = x^3$ is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x, giving

$$x = y^{1/3}.$$

The inverse function is

$$f^{-1}(y) = y^{1/3}$$

or, if we want to call the independent variable x,

$$f^{-1}(x) = x^{1/3}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.34. Notice that these graphs are the reflections of one another about the line $y = x$. For example, $(8, 2)$ is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and $(2, 8)$ is on the graph of $y = x^3$ because $8 = 2^3$. The points $(8, 2)$ and $(2, 8)$ are reflections of one another about the line $y = x$. In general, if the x- and y-axes have the same scales:

> The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

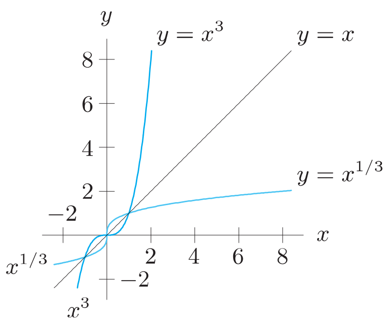


Figure 1.34: Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections about the line $y = x$

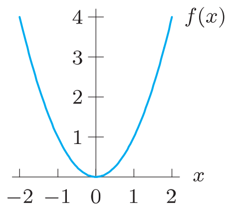
Exercises and Problems for Section 1.3

Exercises

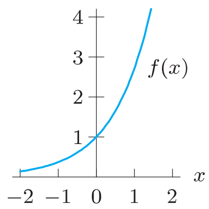
For the functions f in Exercises 1–3, graph:

(a) $f(x+2)$ **(b)** $f(x-1)$ **(c)** $f(x)-4$
(d) $f(x+1)+3$ **(e)** $3f(x)$ **(f)** $-f(x)+1$

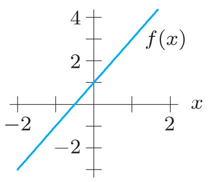
1.



2.



3.



In Exercises 4–7, use Figure 1.35 to graph the functions.

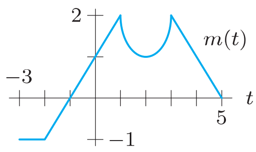


Figure 1.35

4. $n(t)=m(t)+2$

5. $p(t)=m(t-1)$

6. $k(t)=m(t+1.5)$

7. $w(t)=m(t-0.5)-2.5$

For the functions f and g in Exercises 8–11, find

(a) $f(g(1))$ **(b)** $g(f(1))$ **(c)** $f(g(x))$
(d) $g(f(x))$ **(e)** $f(t)g(t)$

8. $f(x)=x^2$, $g(x)=x+1$

9. $f(x)=\sqrt{x+4}$, $g(x)=x^2$

10. $f(x)=e^x$, $g(x)=x^2$

11. $f(x)=1/x$, $g(x)=3x+4$

12. For $g(x)=x^2+2x+3$, find and simplify:

(a) $g(2+h)$ **(b)** $g(2)$
(c) $g(2+h)-g(2)$

13. If $f(x)=x^2+1$, find and simplify:

(a) $f(t+1)$ **(b)** $f(t^2+1)$ **(c)** $f(2)$
(d) $2f(t)$ **(e)** $[f(t)]^2+1$

14. For $f(n)=3n^2-2$ and $g(n)=n+1$, find and simplify:

(a) $f(n)+g(n)$
(b) $f(n)g(n)$
(c) The domain of $f(n)/g(n)$
(d) $f(g(n))$
(e) $g(f(n))$

Simplify the quantities in Exercises 15–18 using $m(z)=z^2$.

15. $m(z+1)-m(z)$

16. $m(z+h)-m(z)$

17. $m(z)-m(z-h)$

18. $m(z+h)-m(z-h)$

19. Let p be the price of an item and q be the number of items sold at that price, where $q=f(p)$. What do the following quantities mean in terms of prices and quantities sold?

(a) $f(25)$ **(b)** $f^{-1}(30)$

20. Let $C=f(A)$ be the cost, in dollars, of building a store of area A square feet. In terms of cost and square feet, what do the following quantities represent?

(a) $f(10{,}000)$ **(b)** $f^{-1}(20{,}000)$

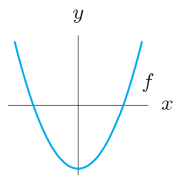
21. Let $f(x)$ be the temperature (°F) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?

22. Let $m=f(A)$ be the minimum annual gross income, in thousands of dollars, needed to obtain a 30-year home mortgage loan of A thousand dollars at an interest rate of 6%. What do the following quantities represent in terms of the income needed for a loan?

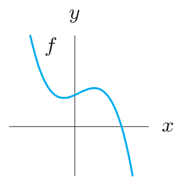
(a) $f(100)$ **(b)** $f^{-1}(75)$

For Exercises 23–24, decide if the function $y=f(x)$ is invertible.

23.



24.



For Exercises 25–27, use a graph of the function to decide whether or not it is invertible.

25. $f(x)=x^2+3x+2$

26. $f(x)=x^3-5x+10$

27. $f(x)=x^3+5x+10$

Are the functions in Exercises 28–35 even, odd, or neither?

28. $f(x)=x^6+x^3+1$

29. $f(x)=x^3+x^2+x$

30. $f(x)=x^4-x^2+3$

31. $f(x)=x^3+1$

32. $f(x)=2x$

33. $f(x)=e^{x^2-1}$

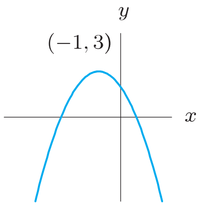
34. $f(x)=x(x^2-1)$

35. $f(x)=e^x-x$

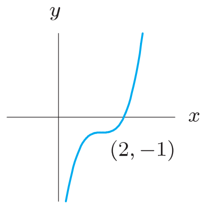
Problems

Find possible formulas for the graphs in Exercises 36–37 using shifts of x^2 or x^3.

36.



37.



38. **(a)** Use Figure 1.36 to estimate $f^{-1}(2)$.
(b) Sketch a graph of f^{-1} on the same axes.

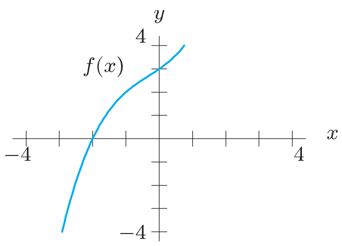


Figure 1.36

39. How does the graph of $Q = S(1 - e^{-kt})$ in Example 4 on page 14 relate to the graph of the exponential decay function, $y = Se^{-kt}$?

40. Write a table of values for f^{-1}, where f is as given below. The domain of f is the integers from 1 to 7. State the domain of f^{-1}.

x	1	2	3	4	5	6	7
$f(x)$	3	−7	19	4	178	2	1

For Problems 41–44, decide if the function f is invertible.

41. $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.

42. $f(t)$ is the number of customers in Macy's department store at t minutes past noon on December 18, 2008.

43. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.

44. $f(w)$ is the cost of mailing a letter weighing w grams.

For Problems 45–50, use the graphs in Figure 1.37.

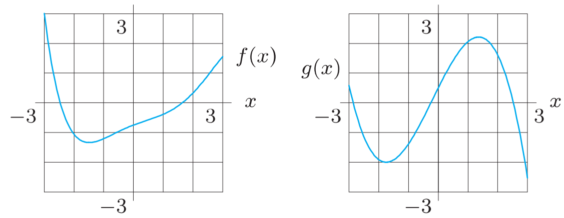


Figure 1.37

45. Estimate $f(g(1))$.

46. Estimate $g(f(2))$.

47. Estimate $f(f(1))$.

48. Graph $f(g(x))$.

49. Graph $g(f(x))$.

50. Graph $f(f(x))$.

In Problems 51–54, use Figure 1.38 to estimate the function value or explain why it cannot be done.

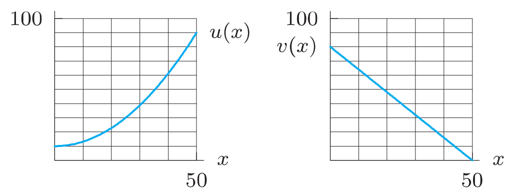


Figure 1.38

51. $u(v(10))$

52. $u(v(40))$

53. $v(u(10))$

54. $v(u(40))$

For Problems 55–58, determine functions f and g such that $h(x) = f(g(x))$. [Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.]

55. $h(x) = (x + 1)^3$

56. $h(x) = x^3 + 1$

57. $h(x) = \sqrt{x^2 + 4}$

58. $h(x) = e^{2x}$

59. A spherical balloon is growing with radius $r = 3t + 1$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.

60. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves where $n = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

61. The cost of producing q articles is given by the function $C = f(q) = 100 + 2q$.

(a) Find a formula for the inverse function.
(b) Explain in practical terms what the inverse function tells you.

62. A kilogram weighs about 2.2 pounds.

(a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p.

(b) Find a formula for the inverse function of f. What does this inverse function tell you, in practical terms?

63. The graph of $f(x)$ is a parabola that opens upward and the graph of $g(x)$ is a line with negative slope. Describe the graph of $g(f(x))$ in words.

64. Figure 1.39 is a graph of the function $f(t)$. Here $f(t)$ is the depth in meters below the Atlantic Ocean floor where t million-year-old rock can be found.[7]

(a) Evaluate $f(15)$, and say what it means in practical terms.

(b) Is f invertible? Explain.

(c) Evaluate $f^{-1}(120)$, and say what it means in practical terms.

(d) Sketch a graph of f^{-1}.

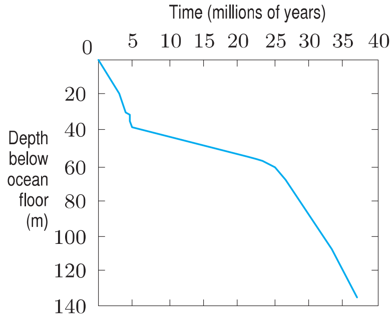


Figure 1.39

1.4 LOGARITHMIC FUNCTIONS

In Section 1.2, we approximated the population of Nevada (in millions) by the function

$$P = f(t) = 2.020(1.036)^t,$$

where t is the number of years since 2000. Now suppose that instead of calculating the population at time t, we ask when the population will reach 10 million. We want to find the value of t for which

$$10 = f(t) = 2.020(1.036)^t.$$

We use logarithms to solve for a variable in an exponent.

Logarithms to Base 10 and to Base e

We define the *logarithm* function, $\log_{10} x$, to be the inverse of the exponential function, 10^x, as follows:

> The **logarithm** to base 10 of x, written $\mathbf{\log_{10} x}$, is the power of 10 we need to get x. In other words,
>
> $$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$
>
> We often write $\log x$ in place of $\log_{10} x$.

The other frequently used base is e. The logarithm to base e is called the *natural logarithm* of x, written $\ln x$ and defined to be the inverse function of e^x, as follows:

> The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$

[7]Data of Dr. Murlene Clark based on core samples drilled by the research ship *Glomar Challenger*, taken from *Initial Reports of the Deep Sea Drilling Project*.

Values of $\log x$ are in Table 1.13. Because no power of 10 gives 0, $\log 0$ is undefined. The graph of $y = \log x$ is shown in Figure 1.40. The domain of $y = \log x$ is positive real numbers; the range is all real numbers. In contrast, the inverse function $y = 10^x$ has domain all real numbers and range all positive real numbers. The graph of $y = \log x$ has a vertical asymptote at $x = 0$, whereas $y = 10^x$ has a horizontal asymptote at $y = 0$.

One big difference between $y = 10^x$ and $y = \log x$ is that the exponential function grows extremely quickly whereas the log function grows extremely slowly. However, $\log x$ does go to infinity, albeit slowly, as x increases. Since $y = \log x$ and $y = 10^x$ are inverse functions, the graphs of the two functions are reflections of one another about the line $y = x$, provided the scales along the x- and y-axes are equal.

Table 1.13 *Values for* $\log x$ *and* 10^x

x	$\log x$
0	undefined
1	0
2	0.3
3	0.5
4	0.6
$\vdots$	$\vdots$
10	1

x	10^x
0	1
1	10
2	100
3	10^3
4	10^4
$\vdots$	$\vdots$
10	10^{10}

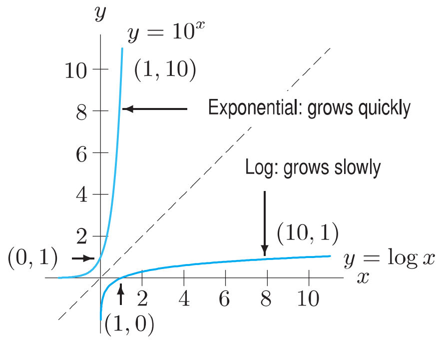


Figure 1.40: Graphs of $\log x$ and 10^x

The graph of $y = \ln x$ in Figure 1.41 has roughly the same shape as the graph of $y = \log x$. The x-intercept is $x = 1$, since $\ln 1 = 0$.The graph of $y = \ln x$ also climbs very slowly as x increases. Both graphs, $y = \log x$ and $y = \ln x$, have *vertical asymptotes* at $x = 0$.

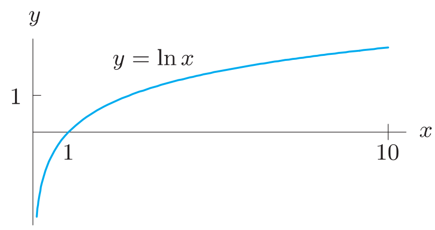


Figure 1.41: Graph of the natural logarithm

The following properties of logarithms may be deduced from the properties of exponents:

Properties of Logarithms

Note that $\log x$ and $\ln x$ are not defined when x is negative or 0.

1. $\log(AB) = \log A + \log B$	1. $\ln(AB) = \ln A + \ln B$
2. $\log\left(\frac{A}{B}\right) = \log A - \log B$	2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\log(A^p) = p \log A$	3. $\ln(A^p) = p \ln A$
4. $\log(10^x) = x$	4. $\ln e^x = x$
5. $10^{\log x} = x$	5. $e^{\ln x} = x$

In addition, $\log 1 = 0$ because $10^0 = 1$, and $\ln 1 = 0$ because $e^0 = 1$.

Solving Equations Using Logarithms

Logs are frequently useful when we have to solve for unknown exponents, as in the next examples.

Example 1 Find t such that $2^t = 7$.

Solution First, notice that we expect t to be between 2 and 3 (because $2^2 = 4$ and $2^3 = 8$). To calculate t, we take logs to base 10. (Natural logs could also be used.)

$$\log(2^t) = \log 7.$$

Then use the third property of logs, which says $\log(2^t) = t \log 2$, and get:

$$t \log 2 = \log 7.$$

Using a calculator to find the logs gives

$$t = \frac{\log 7}{\log 2} \approx 2.81.$$

Example 2 Find when the population of Nevada reaches 10 million by solving $10 = 2.020(1.036)^t$.

Solution Dividing both sides of the equation by 2.020, we get

$$\frac{10}{2.020} = (1.036)^t.$$

Now take logs of both sides:

$$\log\left(\frac{10}{2.020}\right) = \log(1.036^t).$$

Using the fact that $\log(A^t) = t \log A$, we get

$$\log\left(\frac{10}{2.020}\right) = t \log(1.036).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\log(10/2.020)}{\log(1.036)} = 45.23 \text{ years}$$

which is between $t = 45$ and $t = 46$. This value of t corresponds to the year 2045.

Example 3 The release of chlorofluorocarbons used in air conditioners and in household sprays (hair spray, shaving cream, etc.) destroys the ozone in the upper atmosphere. Currently, the amount of ozone, Q, is decaying exponentially at a continuous rate of 0.25% per year. What is the half-life of ozone?

Solution We want to find how long it takes for half the ozone to disappear. If Q_0 is the initial quantity of ozone, then

$$Q = Q_0 e^{-0.0025t}.$$

We want to find T, the value of t making $Q = Q_0/2$, that is,

$$Q_0 e^{-0.0025T} = \frac{Q_0}{2}.$$

Dividing by Q_0 and then taking natural logs yields

$$\ln\left(e^{-0.0025T}\right) = -0.0025T = \ln\left(\frac{1}{2}\right) \approx -0.6931,$$

so

$$T \approx 277 \text{ years}.$$

The half-life of ozone is about 277 years.

In Example 3 the decay rate was given. However, in many situations where we expect to find exponential growth or decay, the rate is not given. To find it, we must know the quantity at two different times and then solve for the growth or decay rate, as in the next example.

Example 4 The population of Mexico was 99.9 million in 2000 and 106.2 million in 2005. Assuming it increases exponentially, find a formula for the population of Mexico as a function of time.

Solution If we measure the population, P, in millions and time, t, in years since 2000, we can say

$$P = P_0 e^{kt} = 99.9e^{kt},$$

where $P_0 = 99.9$ is the initial value of P. We find k by using the fact that $P = 106.2$ when $t = 5$, so

$$106.2 = 99.9e^{k\cdot 5}.$$

To find k, we divide both sides by 99.9, giving

$$\frac{106.2}{99.9} = 1.063 = e^{5k}.$$

Now take natural logs of both sides:

$$\ln(1.063) = \ln(e^{5k}).$$

Using a calculator and the fact that $\ln(e^{5k}) = 5k$, this becomes

$$0.061 = 5k.$$

So

$$k \approx 0.012,$$

and therefore

$$P = 99.9e^{0.012t}.$$

Since $k = 0.012 = 1.2\%$, the population of Mexico was growing at a continuous rate of 1.2% per year.

In Example 4 we chose to use e for the base of the exponential function representing Mexico's population, making clear that the continuous growth rate was 1.2%. If we had wanted to emphasize the annual growth rate, we could have expressed the exponential function in the form $P = P_0 a^t$.

Example 5 Give a formula for the inverse of the following function (that is, solve for t in terms of P):

$$P = f(t) = 2.020(1.036)^t.$$

Solution We want a formula expressing t as a function of P. Take logs:

$$\log P = \log(2.020(1.036)^t).$$

Since $\log(AB) = \log A + \log B$, we have

$$\log P = \log 2.020 + \log((1.036)^t).$$

Now use $\log(A^t) = t \log A$:

$$\log P = \log 2.020 + t \log 1.036.$$

Solve for t in two steps, using a calculator at the final stage:

$$t\log 1.036 = \log P - \log 2.020$$
$$t = \frac{\log P}{\log 1.036} - \frac{\log 2.020}{\log 1.036} = 65.11\log P - 19.88.$$

Thus,

$$f^{-1}(P) = 65.11\log P - 19.88.$$

Note that

$$f^{-1}(10) = 65.11(\log 10) - 19.88 = 65.11(1) - 19.88 = 45.23,$$

which agrees with the result of Example 2.

Exercises and Problems for Section 1.4

Exercises

Simplify the expressions in Exercises 1–6 completely.

1. $e^{\ln(1/2)}$

2. $10^{\log(AB)}$

3. $5e^{\ln(A^2)}$

4. $\ln(e^{2AB})$

5. $\ln(1/e) + \ln(AB)$

6. $2\ln\left(e^A\right) + 3\ln B^e$

For Exercises 7–18, solve for x using logs.

7. $3^x = 11$

8. $17^x = 2$

9. $20 = 50(1.04)^x$

10. $4\cdot 3^x = 7\cdot 5^x$

11. $7 = 5e^{0.2x}$

12. $2^x = e^{x+1}$

13. $50 = 600e^{-0.4x}$

14. $2e^{3x} = 4e^{5x}$

15. $7^{x+2} = e^{17x}$

16. $10^{x+3} = 5e^{7-x}$

17. $2x - 1 = e^{\ln x^2}$

18. $4e^{2x-3} - 5 = e$

For Exercises 19–24, solve for t. Assume a and b are positive constants and k is nonzero.

19. $a = b^t$

20. $P = P_0\,a^t$

21. $Q = Q_0\,a^{nt}$

22. $P_0\,a^t = Q_0\,b^t$

23. $a = be^t$

24. $P = P_0\,e^{kt}$

In Exercises 25–28, put the functions in the form $P = P_0e^{kt}$.

25. $P = 15(1.5)^t$

26. $P = 10(1.7)^t$

27. $P = 174(0.9)^t$

28. $P = 4(0.55)^t$

Find the inverse function in Exercises 29–31.

29. $p(t) = (1.04)^t$

30. $f(t) = 50e^{0.1t}$

31. $f(t) = 1 + \ln t$

Problems

32. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs in Figure 1.42.

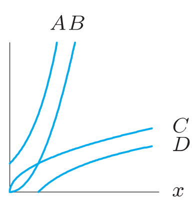


Figure 1.42

33. Find the equation of the line l in Figure 1.43.

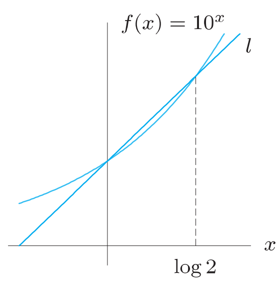


Figure 1.43

34. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 2$ and $y(1) = 1$. Find the constants C and α. What is $y(2)$?

35. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on

(a) The y-intercept? **(b)** The x-intercept?

36. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on

(a) The y-intercept? **(b)** The x-intercept?

37. If $f(x) = a\ln(x + 2)$, what is the effect of increasing a on the vertical asymptote?

38. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on the vertical asymptote?

39. Is there a difference between $\ln[\ln(x)]$ and $\ln^2(x)$? [Note: $\ln^2(x)$ is another way of writing $(\ln x)^2$.]

40. The population of a region is growing exponentially. There were 40,000,000 people in 1990 ($t = 0$) and 56,000,000 in 2000. Find an expression for the population at any time t, in years. What population would you predict for the year 2010? What is the doubling time?

41. What is the doubling time of prices which are increasing by 5% a year?

42. The size of an exponentially growing bacteria colony doubles in 5 hours. How long will it take for the number of bacteria to triple?

43. One hundred kilograms of a radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?

44. Find the half-life of a radioactive substance that is reduced by 30% in 20 hours.

45. The sales at Borders bookstores went from $2108 million in 2000 to $3880 million in 2005. Find an exponential function to model the sales as a function of years since 2000. What is the continuous percent growth rate, per year, of sales?

46. Owing to an innovative rural public health program, infant mortality in Senegal, West Africa, is being reduced at a rate of 10% per year. How long will it take for infant mortality to be reduced by 50%?

47. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.82)^t$.

(a) What was the initial amount taken?
(b) What percent of the drug leaves the body each hour?
(c) How much of the drug is left in the body 6 hours after the dose is administered?
(d) How long is it until only 1 mg of the drug remains in the body?

48. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
(b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
(c) Use logarithms to find the half-life of caffeine.

49. In 2000, there were about 213 million vehicles (cars and trucks) and about 281 million people in the US. The number of vehicles has been growing at 4% a year, while the population has been growing at 1% a year. If the growth rates remain constant, when is there, on average, one vehicle per person?

50. The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing according to the function $P = P_0e^{-kt}$, where t is time in hours. If 10% of the pollution is removed in the first five hours:

(a) What percentage of the pollution is left after 10 hours?
(b) How long is it before the pollution is reduced by 50%?
(c) Plot a graph of pollution against time. Show the results of your calculations on the graph.
(d) Explain why the quantity of pollutant might decrease in this way.

51. Air pressure, P, decreases exponentially with the height, h, in meters above sea level:

$$P = P_0e^{-0.00012h}$$

where P_0 is the air pressure at sea level.

(a) At the top of Mount McKinley, height 6194 meters (about 20,320 feet), what is the air pressure, as a percent of the pressure at sea level?
(b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?

52. The half-life of radioactive strontium-90 is 29 years. In 1960, radioactive strontium-90 was released into the atmosphere during testing of nuclear weapons, and was absorbed into people's bones. How many years does it take until only 10% of the original amount absorbed remains?

53. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.

1.5 TRIGONOMETRIC FUNCTIONS

Trigonometry originated as part of the study of triangles. The name *tri-gon-o-metry* means the measurement of three-cornered figures, and the first definitions of the trigonometric functions were in terms of triangles. However, the trigonometric functions can also be defined using the unit circle, a definition that makes them periodic, or repeating. Many naturally occurring processes are also periodic. The water level in a tidal basin, the blood pressure in a heart, an alternating current, and the position of the air molecules transmitting a musical note all fluctuate regularly. Such phenomena can be represented by trigonometric functions.

We use the three trigonometric functions found on a calculator: the sine, the cosine, and the tangent.

Radians

There are two commonly used ways to represent the input of the trigonometric functions: radians and degrees. The formulas of calculus, as you will see, are neater in radians than in degrees.

> An angle of 1 **radian** is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. (See Figure 1.44(a).) A unit circle has radius 1.

An angle of 2 radians cuts off an arc of length 2 on a unit circle. A negative angle, such as $-1/2$ radians, cuts off an arc of length $1/2$, but measured clockwise. (See Figure 1.44(b).)

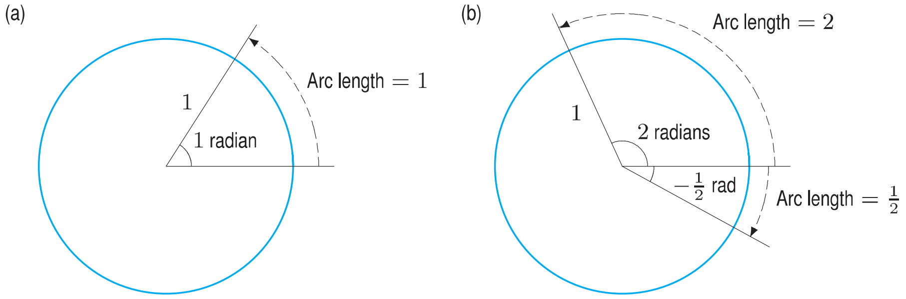


Figure 1.44: Radians defined using unit circle

It is useful to think of angles as rotations, since then we can make sense of angles larger than $360°$; for example, an angle of $720°$ represents two complete rotations counterclockwise. Since one full rotation of $360°$ cuts off an arc of length 2π, the circumference of the unit circle, it follows that

$$360° = 2\pi \text{ radians}, \quad \text{so} \quad 180° = \pi \text{ radians}.$$

In other words, 1 radian $= 180°/\pi$, so one radian is about $60°$. The word radians is often dropped, so if an angle or rotation is referred to without units, it is understood to be in radians.

Radians are useful for computing the length of an arc in any circle. If the circle has radius r and the arc cuts off an angle θ, as in Figure 1.45, then we have the following relation:

$$\boxed{\text{Arc length} = s = r\theta.}$$

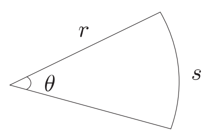


Figure 1.45: Arc length of a sector of a circle

The Sine and Cosine Functions

The two basic trigonometric functions—the sine and cosine—are defined using a unit circle. In Figure 1.46, an angle of t radians is measured counterclockwise around the circle from the point $(1, 0)$. If P has coordinates (x, y), we define

$$\boxed{\cos t = x \quad \text{and} \quad \sin t = y.}$$

We assume that the angles are *always* in radians unless specified otherwise.

Since the equation of the unit circle is $x^2 + y^2 = 1$, we have the following fundamental identity

$$\boxed{\cos^2 t + \sin^2 t = 1.}$$

As t increases and P moves around the circle, the values of $\sin t$ and $\cos t$ oscillate between 1 and -1, and eventually repeat as P moves through points where it has been before. If t is negative, the angle is measured clockwise around the circle.

Amplitude, Period, and Phase

The graphs of sine and cosine are shown in Figure 1.47. Notice that sine is an odd function, and cosine is even. The maximum and minimum values of sine and cosine are $+1$ and -1, because those are the maximum and minimum values of y and x on the unit circle. After the point P has moved around the complete circle once, the values of $\cos t$ and $\sin t$ start to repeat; we say the functions are *periodic*.

> For any periodic function of time, the
> - **Amplitude** is half the distance between the maximum and minimum values (if it exists).
> - **Period** is the smallest time needed for the function to execute one complete cycle.

The amplitude of $\cos t$ and $\sin t$ is 1, and the period is 2π. Why 2π? Because that's the value of t when the point P has gone exactly once around the circle. (Remember that $360° = 2\pi$ radians.)

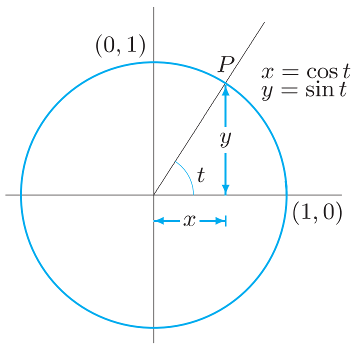


Figure 1.46: The definitions of $\sin t$ and $\cos t$

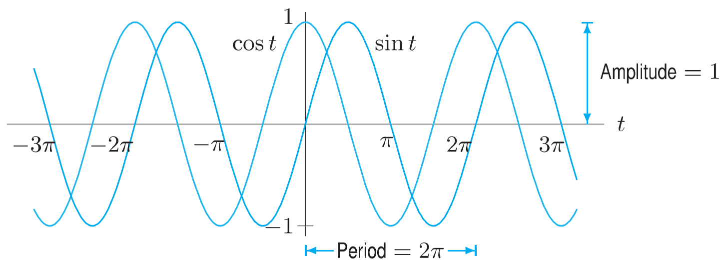


Figure 1.47: Graphs of $\cos t$ and $\sin t$

In Figure 1.47, we see that the sine and cosine graphs are exactly the same shape, only shifted horizontally. Since the cosine graph is the sine graph shifted $\pi/2$ to the left,

$$\cos t = \sin(t + \pi/2).$$

Equivalently, the sine graph is the cosine graph shifted $\pi/2$ to the right, so

$$\sin t = \cos(t - \pi/2).$$

We say that the *phase difference* or *phase shift*[8] between $\sin t$ and $\cos t$ is $\pi/2$.

Functions whose graphs are the shape of a sine or cosine curve are called *sinusoidal* functions.

> To describe arbitrary amplitudes and periods of sinusoidal functions, we use functions of the form
> $$f(t) = A\sin(Bt) \qquad \text{and} \qquad g(t) = A\cos(Bt),$$
> where $|A|$ is the amplitude and $2\pi/|B|$ is the period.
> The graph of a sinusoidal function is shifted horizontally by a distance $|h|$ when t is replaced by $t-h$ or $t+h$.
> Functions of the form $f(t) = A\sin(Bt) + C$ and $g(t) = A\cos(Bt) + C$ have graphs which are shifted vertically and oscillate about the value C.

Example 1 Find and show on a graph the amplitude and period of the functions

(a) $y = 5\sin(2t)$ (b) $y = -5\sin\left(\frac{t}{2}\right)$ (c) $y = 1 + 2\sin t$

Solution

(a) From Figure 1.48, you can see that the amplitude of $y = 5\sin(2t)$ is 5 because the factor of 5 stretches the oscillations up to 5 and down to -5. The period of $y = \sin(2t)$ is π, because when t changes from 0 to π, the quantity $2t$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(b) Figure 1.49 shows that the amplitude of $y = -5\sin(t/2)$ is again 5, because the negative sign reflects the oscillations in the t-axis, but does not change how far up or down they go. The period of $y = -5\sin(t/2)$ is 4π because when t changes from 0 to 4π, the quantity $t/2$ changes from 0 to 2π, so the sine function goes through one complete oscillation.

(c) The 1 shifts the graph $y = 2\sin t$ up by 1. Since $y = 2\sin t$ has an amplitude of 2 and a period of 2π, the graph of $y = 1 + 2\sin t$ goes up to 3 and down to -1, and has a period of 2π. (See Figure 1.50.) Thus, $y = 1 + 2\sin t$ also has amplitude 2.

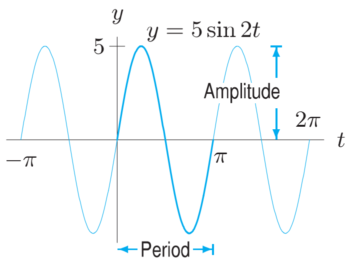


Figure 1.48: Amplitude $= 5$, period $= \pi$

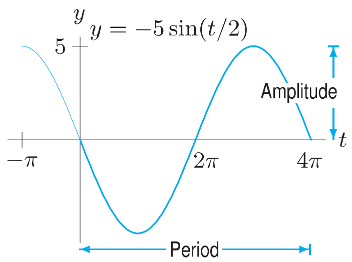


Figure 1.49: Amplitude $= 5$, period $= 4\pi$

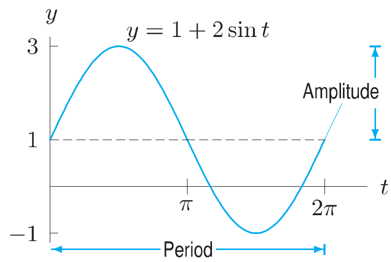
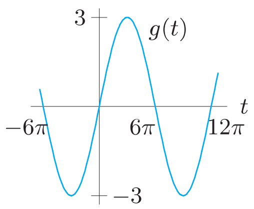


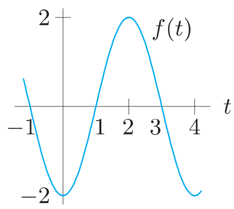
Figure 1.50: Amplitude $= 2$, period $= 2\pi$

Example 2 Find possible formulas for the following sinusoidal functions.

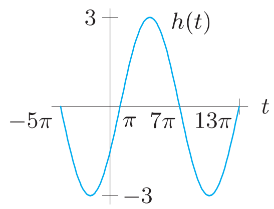
(a)



(b)



(c)



[8] Phase shift is defined in Section 11.10 on page 617.

Solution

(a) This function looks like a sine function of amplitude 3, so $g(t) = 3\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π, the quantity Bt changes by 2π. This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3\sin(t/6)$ has the graph shown.

(b) This function looks like an upside down cosine function with amplitude 2, so $f(t) = -2\cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π, so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2\cos(\pi t/2)$ has the graph shown.

(c) This function looks like the function $g(t)$ in part (a), but shifted a distance of π to the right. Since $g(t) = 3\sin(t/6)$, we replace t by $(t - \pi)$ to obtain $h(t) = 3\sin[(t - \pi)/6]$.

Example 3 On July 1, 2007, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; later, at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level in Boston as a function of time.

Solution Let y be the water level in feet, and let t be the time measured in hours from midnight. The oscillations have amplitude 4.9 feet $(= (9.9 - 0.1)/2)$ and period 12, so $12B = 2\pi$ and $B = \pi/6$. Since the water is highest at midnight, when $t = 0$, the oscillations are best represented by a cosine function. (See Figure 1.51.) We can say

$$\text{Height above average} = 4.9\cos\left(\frac{\pi}{6}t\right).$$

Since the average water level was 5 feet $(= (9.9 + 0.1)/2)$, we shift the cosine up by adding 5:

$$y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right).$$

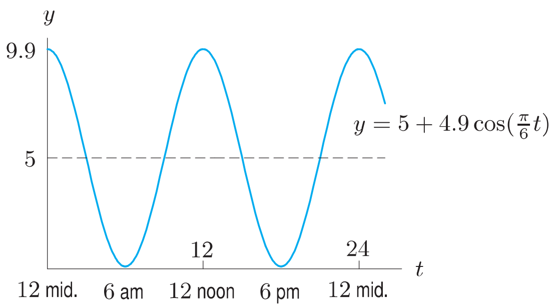


Figure 1.51: Function approximating the tide in Boston on July 1, 2007

Example 4 Of course, there's something wrong with the assumption in Example 3 that the next high tide is at noon. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as in fact it does. The interval between successive high tides actually averages about 12 hours 24 minutes. Using this, give a more accurate formula for the height of the water as a function of time.

Solution The period is 12 hours 24 minutes = 12.4 hours, so $B = 2\pi/12.4$, giving

$$y = 5 + 4.9\cos\left(\frac{2\pi}{12.4}t\right) = 5 + 4.9\cos(0.507t).$$

Example 5 Use the information from Example 4 to write a formula for the water level in Boston on a day when the high tide is at 2 pm.

Solution When the high tide is at midnight

$$y = 5 + 4.9\cos(0.507t).$$

Since 2 pm is 14 hours after midnight, we replace t by $(t-14)$. Therefore, on a day when the high tide is at 2 pm,

$$y = 5 + 4.9\cos(0.507(t-14)).$$

The Tangent Function

If t is any number with $\cos t \neq 0$, we define the tangent function as follows

$$\boxed{\tan t = \frac{\sin t}{\cos t}.}$$

Figure 1.46 on page 31 shows the geometrical meaning of the tangent function: $\tan t$ is the slope of the line through the origin $(0,0)$ and the point $P = (\cos t, \sin t)$ on the unit circle.

The tangent function is undefined wherever $\cos t = 0$, namely, at $t = \pm\pi/2, \pm 3\pi/2, \ldots$, and it has a vertical asymptote at each of these points. The function $\tan t$ is positive where $\sin t$ and $\cos t$ have the same sign. The graph of the tangent is shown in Figure 1.52.

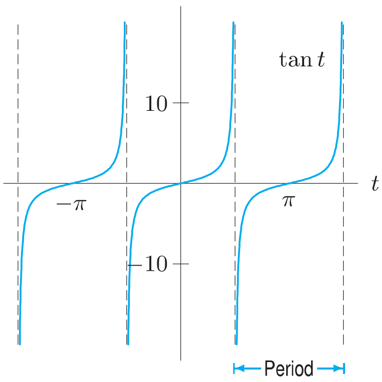


Figure 1.52: The tangent function

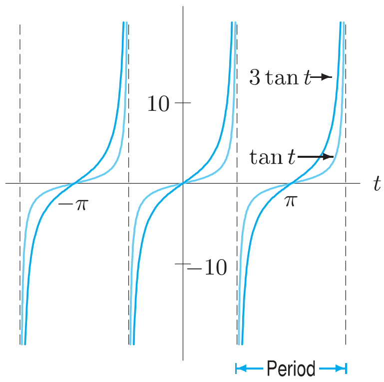


Figure 1.53: Multiple of tangent

The tangent function has period π, because it repeats every π units. Does it make sense to talk about the amplitude of the tangent function? Not if we're thinking of the amplitude as a measure of the size of the oscillation, because the tangent becomes infinitely large near each vertical asymptote. We can still multiply the tangent by a constant, but that constant no longer represents an amplitude. (See Figure 1.53.)

The Inverse Trigonometric Functions

On occasion, you may need to find a number with a given sine. For example, you might want to find x such that

$$\sin x = 0$$

or such that

$$\sin x = 0.3.$$

The first of these equations has solutions $x = 0, \pm\pi, \pm 2\pi, \ldots$. The second equation also has infinitely many solutions. Using a calculator and a graph, we get

$$x \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \ldots.$$

For each equation, we pick out the solution between $-\pi/2$ and $\pi/2$ as the preferred solution. For example, the preferred solution to $\sin x = 0$ is $x = 0$, and the preferred solution to $\sin x = 0.3$ is $x = 0.305$. We define the inverse sine, written "arcsin" or "$\sin^{-1}$," as the function which gives the preferred solution.

For $-1 \leq y \leq 1$,

$$\arcsin y = x$$

means

$$\sin x = y \quad \text{with} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Thus the arcsine is the inverse function to the piece of the sine function having domain $[-\pi/2, \pi/2]$. (See Table 1.14 and Figure 1.54.) On a calculator, the arcsine function[9] is usually denoted by $\boxed{\sin^{-1}}$.

Table 1.14 *Values of* $\sin x$ *and* $\sin^{-1} x$

x	$\sin x$
$-\frac{\pi}{2}$	-1.000
-1.0	-0.841
-0.5	-0.479
0.0	0.000
0.5	0.479
1.0	0.841
$\frac{\pi}{2}$	1.000

x	$\sin^{-1} x$
-1.000	$-\frac{\pi}{2}$
-0.841	-1.0
-0.479	-0.5
0.000	0.0
0.479	0.5
0.841	1.0
1.000	$\frac{\pi}{2}$

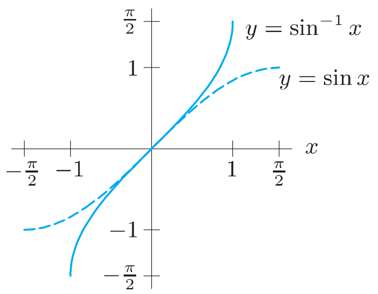


Figure 1.54: The arcsine function

The inverse tangent, written "arctan" or "$\tan^{-1}$," is the inverse function for the piece of the tangent function having the domain $-\pi/2 < x < \pi/2$. On a calculator, the inverse tangent is usually denoted by $\boxed{\tan^{-1}}$. The graph of the arctangent is shown in Figure 1.56.

For any y,

$$\arctan y = x$$

means

$$\tan x = y \quad \text{with} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse cosine function, written "arccos" or "$\cos^{-1}$," is discussed in Problem 52. The range of the arccosine function is $0 \leq x \leq \pi$.

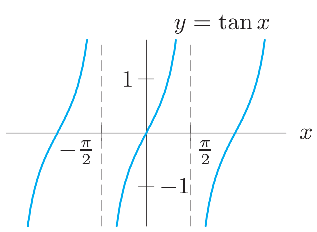


Figure 1.55: The tangent function

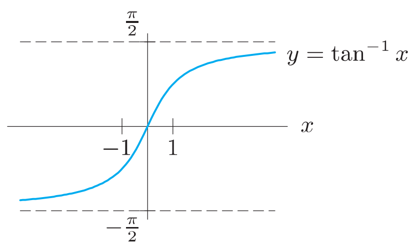


Figure 1.56: The arctangent function

[9]Note that $\sin^{-1} x = \arcsin x$ is not the same as $(\sin x)^{-1} = 1/\sin x$.

Exercises and Problems for Section 1.5

Exercises

For Exercises 1–9, draw the angle using a ray through the origin, and determine whether the sine, cosine, and tangent of that angle are positive, negative, zero, or undefined.

1. $\frac{3\pi}{2}$ **2.** 2π **3.** $\frac{\pi}{4}$

4. 3π **5.** $\frac{\pi}{6}$ **6.** $\frac{4\pi}{3}$

7. $\frac{-4\pi}{3}$ **8.** 4 **9.** -1

Given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$, compute (without using the trigonometric functions on your calculator) the quantities in Exercises 10–12. You may want to draw a picture showing the angles involved, and then check your answers on a calculator.

10. $\cos\left(-\frac{\pi}{5}\right)$ **11.** $\sin\frac{\pi}{5}$ **12.** $\cos\frac{\pi}{12}$

13. Consider the function $y = 5 + \cos(3x)$.

- **(a)** What is its amplitude?
- **(b)** What is its period?
- **(c)** Sketch its graph.

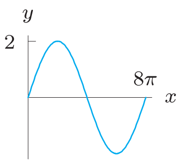
Find the period and amplitude in Exercises 14–17.

14. $y = 7\sin(3t)$ **15.** $z = 3\cos(u/4) + 5$

16. $w = 8 - 4\sin(2x + \pi)$ **17.** $r = 0.1\sin(\pi t) + 2$

For Exercises 18–27, find a possible formula for each graph.

18.



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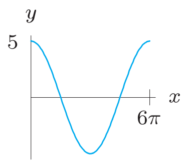

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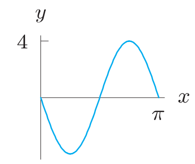

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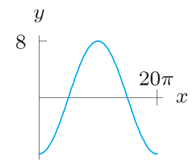

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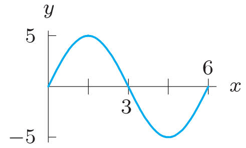

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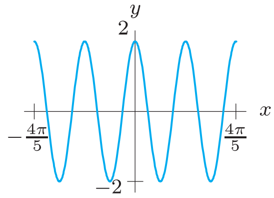

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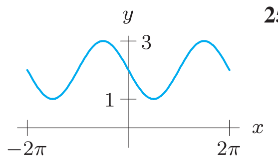

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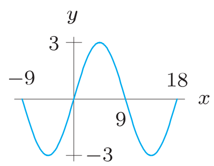

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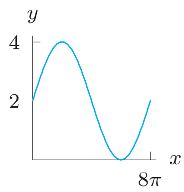

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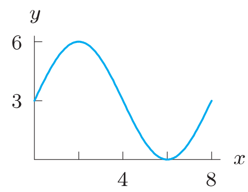


In Exercises 28–32, find a solution to the equation if possible. Give the answer in exact form and in decimal form.

28. $2 = 5\sin(3x)$ **29.** $1 = 8\cos(2x + 1) - 3$

30. $8 = 4\tan(5x)$ **31.** $1 = 8\tan(2x + 1) - 3$

32. $8 = 4\sin(5x)$

Problems

33. What is the difference between $\sin x^2$, $\sin^2 x$, and $\sin(\sin x)$? Express each of the three as a composition. (Note: $\sin^2 x$ is another way of writing $(\sin x)^2$.)

34. Without a calculator or computer, match the formulas with the graphs in Figure 1.57.

(a) $y = 2\cos(t - \pi/2)$ **(b)** $y = 2\cos t$

(c) $y = 2\cos(t + \pi/2)$

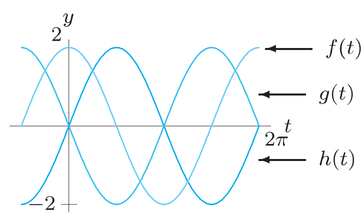


Figure 1.57

35. **(a)** Match the functions $\omega = f(t)$, $\omega = g(t)$, $\omega = h(t)$, $\omega = k(t)$, whose values are in the table, with the functions with formulas:

(i) $\omega = 1.5 + \sin t$ (ii) $\omega = 0.5 + \sin t$
(iii) $\omega = -0.5 + \sin t$ (iv) $\omega = -1.5 + \sin t$

(b) Based on the table, what is the relationship between the values of $g(t)$ and $k(t)$? Explain this relationship using the formulas you chose for g and k.

(c) Using the formulas you chose for g and h, explain why all the values of g are positive, whereas all the values of h are negative.

t	$f(t)$	t	$g(t)$	t	$h(t)$	t	$k(t)$
6.0	−0.78	3.0	1.64	5.0	−2.46	3.0	0.64
6.5	−0.28	3.5	1.15	5.1	−2.43	3.5	0.15
7.0	0.16	4.0	0.74	5.2	−2.38	4.0	−0.26
7.5	0.44	4.5	0.52	5.3	−2.33	4.5	−0.48
8.0	0.49	5.0	0.54	5.4	−2.27	5.0	−0.46

36. A compact disc spins at a rate of 200 to 500 revolutions per minute. What are the equivalent rates measured in radians per second?

37. When a car's engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?

38. What is the period of the earth's revolution around the sun?

39. What is the period of the motion of the minute hand of a clock?

40. What is the approximate period of the moon's revolution around the earth?

41. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos\left(B(t - C)\right).$$

(a) What is the physical meaning of D?
(b) What is the value of A?
(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of C?

42. In an electrical outlet, the voltage, V, in volts, is given as a function of time, t, in seconds, by the formula

$$V = V_0 \sin(120\pi t).$$

(a) What does V_0 represent in terms of voltage?
(b) What is the period of this function?
(c) How many oscillations are completed in 1 second?

43. In a US household, the voltage in volts in an electric outlet is given by

$$V = 156\sin(120\pi t),$$

where t is in seconds. However, in a European house, the voltage is given (in the same units) by

$$V = 339\sin(100\pi t).$$

Compare the voltages in the two regions, considering the maximum voltage and number of cycles (oscillations) per second.

44. A baseball hit at an angle of θ to the horizontal with initial velocity v_0 has horizontal range, R, given by

$$R = \frac{v_0^2}{g}\sin(2\theta).$$

Here g is the acceleration due to gravity. Sketch R as a function of θ for $0 \le \theta \le \pi/2$. What angle gives the maximum range? What is the maximum range?

45. A population of animals oscillates sinusoidally between a low of 700 on January 1 and a high of 900 on July 1.

(a) Graph the population against time.
(b) Find a formula for the population as a function of time, t, in months since the start of the year.

46. The desert temperature, H, oscillates daily between 40°F at 5 am and 80°F at 5 pm. Write a possible formula for H in terms of t, measured in hours from 5 am.

47. The visitors' guide to St. Petersburg, Florida, contains the chart shown in Figure 1.58 to advertise their good weather. Fit a trigonometric function approximately to the data, where H is temperature in degrees Fahrenheit, and the independent variable is time in months. In order to do this, you will need to estimate the amplitude and period of the data, and when the maximum occurs. (There are many possible answers to this problem, depending on how you read the graph.)

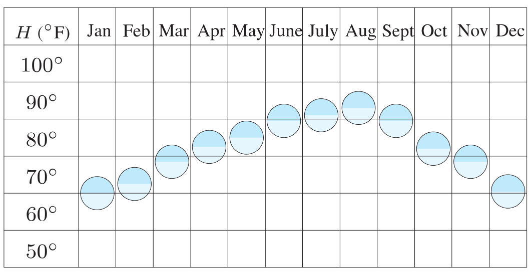


Figure 1.58: "St. Petersburg...where we're famous for our wonderful weather and year-round sunshine." (Reprinted with permission)

48. The point P is rotating around a circle of radius 5 shown in Figure 1.59. The angle θ, in radians, is given as a function of time, t, by the graph in Figure 1.60.

(a) Estimate the coordinates of P when $t = 1.5$.
(b) Describe in words the motion of the point P on the circle.

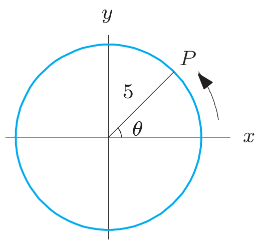


Figure 1.59

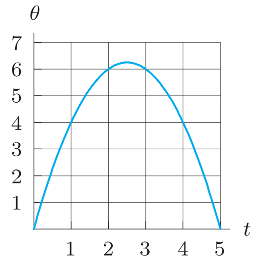


Figure 1.60

49. Find the area of the trapezoidal cross-section of the irrigation canal shown in Figure 1.61.

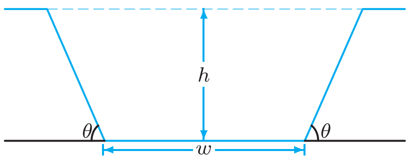


Figure 1.61

50. For a boat to float in a tidal bay, the water must be at least 2.5 meters deep. The depth of water around the boat, $d(t)$, in meters, where t is measured in hours since midnight, is

$$d(t) = 5 + 4.6\sin(0.5t).$$

(a) What is the period of the tides in hours?
(b) If the boat leaves the bay at midday, what is the latest time it can return before the water becomes too shallow?

51. Graph $y = \sin x$, $y = 0.4$, and $y = -0.4$.

(a) From the graph, estimate to one decimal place all the solutions of $\sin x = 0.4$ with $-\pi \le x \le \pi$.
(b) Use a calculator to find $\arcsin(0.4)$. What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (a)?
(c) Estimate all the solutions to $\sin x = -0.4$ with $-\pi \le x \le \pi$ (again, to one decimal place).
(d) What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (c)?

52. This problem introduces the arccosine function, or inverse cosine, denoted by $\boxed{\cos^{-1}}$ on most calculators.

(a) Using a calculator set in radians, make a table of values, to two decimal places, of $g(x) = \arccos x$, for $x = -1, -0.8, -0.6, \ldots, 0, \ldots, 0.6, 0.8, 1$.
(b) Sketch the graph of $g(x) = \arccos x$.
(c) Why is the domain of the arccosine the same as the domain of the arcsine?
(d) Why is the range of the arccosine *not* the same as the range of the arcsine? To answer this, look at how the domain of the original sine function was restricted to construct the arcsine. Why can't the domain of the cosine be restricted in exactly the same way to construct the arccosine?

1.6 POWERS, POLYNOMIALS, AND RATIONAL FUNCTIONS

Power Functions

A *power function* is a function in which the dependent variable is proportional to a power of the independent variable:

> A **power function** has the form
>
> $f(x) = kx^p$, where k and p are constant.

For example, the volume, V, of a sphere of radius r is given by

$$V = g(r) = \frac{4}{3}\pi r^3.$$

As another example, the gravitational force, F, on a unit mass at a distance r from the center of the earth is given by Newton's Law of Gravitation, which says that, for some positive constant k,

$$F = \frac{k}{r^2} \quad \text{or} \quad F = kr^{-2}.$$

We consider the graphs of the power functions x^n, with n a positive integer. Figures 1.62 and 1.63 show that the graphs fall into two groups: odd and even powers. For n greater than 1, the odd powers have a "seat" at the origin and are increasing everywhere else. The even powers are first decreasing and then increasing. For large x, the higher the power of x, the faster the function climbs.

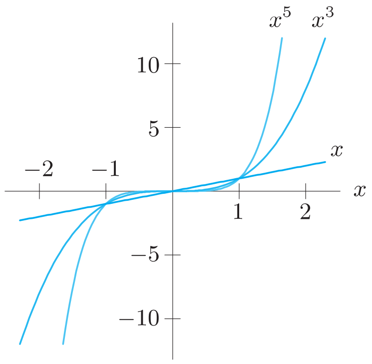


Figure 1.62: Odd powers of x: "Seat" shaped for $n > 1$

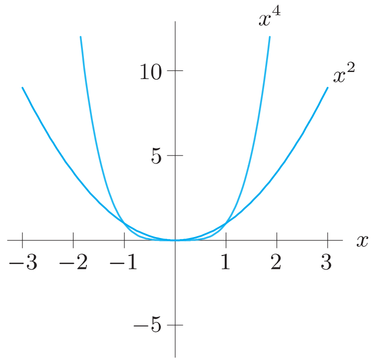


Figure 1.63: Even powers of x: $\bigcup$-shaped

Exponentials and Power Functions: Which Dominate?

In everyday language, the word exponential is often used to imply very fast growth. But do exponential functions always grow faster than power functions? To determine what happens "in the long run," we often want to know which functions *dominate* as x gets arbitrarily large.

Let's consider $y = 2^x$ and $y = x^3$. The close-up view in Figure 1.64(a) shows that between $x = 2$ and $x = 4$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. The far-away view in Figure 1.64(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. Figure 1.64(c), which gives a very far-away view, shows that, for large x, the value of x^3 is insignificant compared to 2^x. Indeed, 2^x is growing so much faster than x^3 that the graph of 2^x appears almost vertical in comparison to the more leisurely climb of x^3.

We say that Figure 1.64(a) gives a *local* view of the functions' behavior, whereas Figure 1.64(c) gives a *global* view.

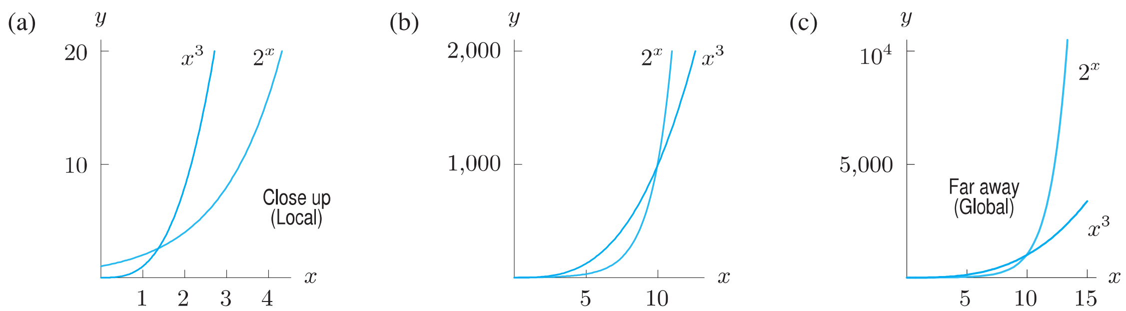


Figure 1.64: Comparison of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x, if we look at large enough x-values, a^x (with $a > 1$) will eventually dominate x^n, no matter what n is.

Polynomials

Polynomials are the sums of power functions with nonnegative integer exponents:

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Here n is a nonnegative integer called the *degree* of the polynomial, and $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants, with leading coefficient $a_n \neq 0$. An example of a polynomial of degree $n = 3$ is

$$y = p(x) = 2x^3 - x^2 - 5x - 7.$$

In this case $a_3 = 2, a_2 = -1, a_1 = -5$, and $a_0 = -7$. The shape of the graph of a polynomial depends on its degree; typical graphs are shown in Figure 1.65. These graphs correspond to a positive coefficient for x^n; a negative leading coefficient turns the graph upside down. Notice that the quadratic "turns around" once, the cubic "turns around" twice, and the quartic (fourth degree) "turns around" three times. An n^{th} degree polynomial "turns around" at most $n - 1$ times (where n is a positive integer), but there may be fewer turns.

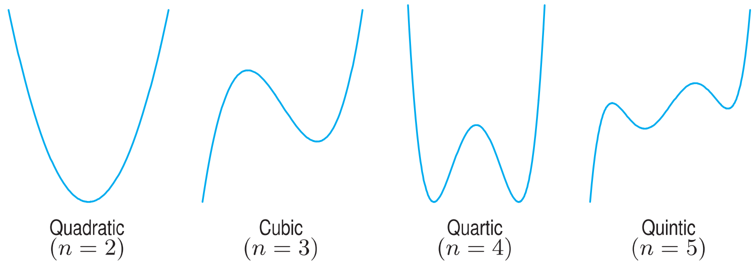


Figure 1.65: Graphs of typical polynomials of degree n

Example 1 Find possible formulas for the polynomials whose graphs are in Figure 1.66.

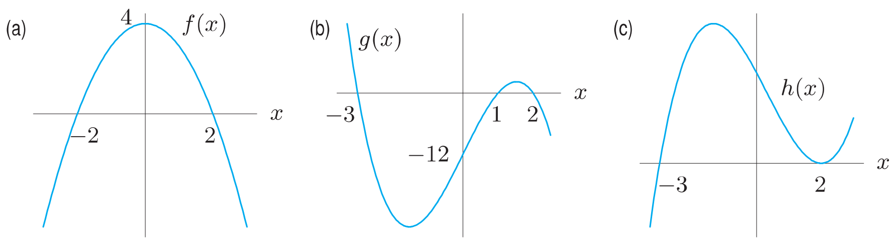


Figure 1.66: Graphs of polynomials

Solution (a) This graph appears to be a parabola, turned upside down, and moved up by 4, so

$$f(x) = -x^2 + 4.$$

The negative sign turns the parabola upside down and the $+4$ moves it up by 4. Notice that this formula does give the correct x-intercepts since $0 = -x^2 + 4$ has solutions $x = \pm 2$. These values of x are called *zeros* of f.

We can also solve this problem by looking at the x-intercepts first, which tell us that $f(x)$ has factors of $(x + 2)$ and $(x - 2)$. So

$$f(x) = k(x + 2)(x - 2).$$

To find k, use the fact that the graph has a y-intercept of 4, so $f(0) = 4$, giving

$$4 = k(0 + 2)(0 - 2),$$

or $k = -1$. Therefore, $f(x) = -(x + 2)(x - 2)$, which multiplies out to $-x^2 + 4$.

Note that $f(x) = 4 - x^4/4$ also has the same basic shape, but is flatter near $x = 0$. There are many possible answers to these questions.

(b) This looks like a cubic with factors $(x + 3)$, $(x - 1)$, and $(x - 2)$, one for each intercept:

$$g(x) = k(x + 3)(x - 1)(x - 2).$$

Since the y-intercept is -12, we have

$$-12 = k(0 + 3)(0 - 1)(0 - 2).$$

So $k = -2$, and we get the cubic polynomial

$$g(x) = -2(x + 3)(x - 1)(x - 2).$$

(c) This also looks like a cubic with zeros at $x = 2$ and $x = -3$. Notice that at $x = 2$ the graph of $h(x)$ touches the x-axis but does not cross it, whereas at $x = -3$ the graph crosses the x-axis. We say that $x = 2$ is a *double zero*, but that $x = -3$ is a single zero.

To find a formula for $h(x)$, imagine the graph of $h(x)$ to be slightly lower down, so that the graph has one x-intercept near $x = -3$ and two near $x = 2$, say at $x = 1.9$ and $x = 2.1$. Then a formula would be

$$h(x) \approx k(x + 3)(x - 1.9)(x - 2.1).$$

Now move the graph back to its original position. The zeros at $x = 1.9$ and $x = 2.1$ move toward $x = 2$, giving

$$h(x) = k(x + 3)(x - 2)(x - 2) = k(x + 3)(x - 2)^2.$$

The double zero leads to a repeated factor, $(x-2)^2$. Notice that when $x > 2$, the factor $(x-2)^2$ is positive, and when $x < 2$, the factor $(x - 2)^2$ is still positive. This reflects the fact that $h(x)$ does not change sign near $x = 2$. Compare this with the behavior near the single zero at $x = -3$, where h does change sign.

We cannot find k, as no coordinates are given for points off of the x-axis. Any positive value of k stretches the graph vertically but does not change the zeros, so any positive k works.

Example 2 Using a calculator or computer, graph $y = x^4$ and $y = x^4 - 15x^2 - 15x$ for $-4 \le x \le 4$ and for $-20 \le x \le 20$. Set the y range to $-100 \le y \le 100$ for the first domain, and to $-100 \le y \le 200{,}000$ for the second. What do you observe?

Solution From the graphs in Figure 1.67 we see that close up ($-4 \le x \le 4$) the graphs look different; from far away, however, they are almost indistinguishable. The reason is that the leading terms (those with the highest power of x) are the same, namely x^4, and for large values of x, the leading term dominates the other terms.

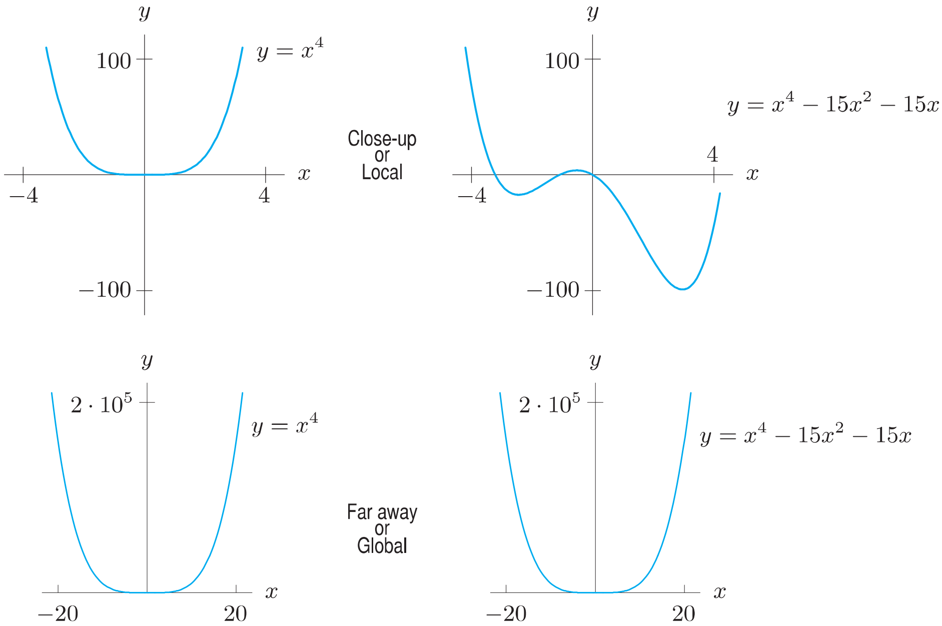


Figure 1.67: Local and global views of $y = x^4$ and $y = x^4 - 15x^2 - 15x$

Rational Functions

Rational functions are ratios of polynomials, p and q:

$$f(x) = \frac{p(x)}{q(x)}.$$

Example 3 Look at a graph and explain the behavior of $y = \dfrac{1}{x^2 + 4}$.

Solution The function is even, so the graph is symmetric about the y-axis. As x gets larger, the denominator gets larger, making the value of the function closer to 0. Thus the graph gets arbitrarily close to the x-axis as x increases without bound. See Figure 1.68.

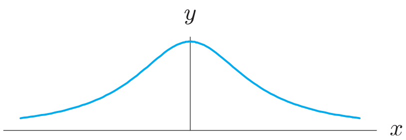


Figure 1.68: Graph of $y = \frac{1}{x^2+4}$

In the previous example, we say that $y = 0$ (i.e. the x-axis) is a *horizontal asymptote*. Writing "$\rightarrow$" to mean "tends to," we have $y \rightarrow 0$ as $x \rightarrow \infty$ and $y \rightarrow 0$ as $x \rightarrow -\infty$.

If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is called a **horizontal asymptote**.[10] This occurs when

$$f(x) \to L \quad \text{as} \quad x \to \infty \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to -\infty.$$

If the graph of $y = f(x)$ approaches the vertical line $x = K$ as $x \to K$ from one side or the other, that is, if

$$y \to \infty \quad \text{or} \quad y \to -\infty \quad \text{when} \quad x \to K,$$

then the line $x = K$ is called a **vertical asymptote**.

The graphs of rational functions may have vertical asymptotes where the denominator is zero. For example, the function in Example 3 has no vertical asymptotes as the denominator is never zero. The function in Example 4 has two vertical asymptotes corresponding to the two zeros in the denominator.

Rational functions have horizontal asymptotes if $f(x)$ approaches a finite number as $x \to \infty$ or $x \to -\infty$. We call the behavior of a function as $x \to \pm\infty$ its *end behavior*.

Example 4 Look at a graph and explain the behavior of $y = \dfrac{3x^2 - 12}{x^2 - 1}$, including end behavior.

Solution Factoring gives

$$y = \frac{3x^2 - 12}{x^2 - 1} = \frac{3(x+2)(x-2)}{(x+1)(x-1)}$$

so $x = \pm 1$ are vertical asymptotes. If $y = 0$, then $3(x+2)(x-2) = 0$ or $x = \pm 2$; these are the x-intercepts. Note that zeros of the denominator give rise to the vertical asymptotes, whereas zeros of the numerator give rise to x-intercepts. Substituting $x = 0$ gives $y = 12$; this is the y-intercept. The function is even, so the graph is symmetric about the y-axis.

To see what happens as $x \to \pm\infty$, look at the y-values in Table 1.15. Clearly y is getting closer to 3 as x gets large positively or negatively. Alternatively, realize that as $x \to \pm\infty$, only the highest powers of x matter. For large x, the 12 and the 1 are insignificant compared to x^2, so

$$y = \frac{3x^2 - 12}{x^2 - 1} \approx \frac{3x^2}{x^2} = 3 \quad \text{for large } x.$$

So $y \to 3$ as $x \to \pm\infty$, and therefore the horizontal asymptote is $y = 3$. See Figure 1.69. Since, for $x > 1$, the value of $(3x^2 - 12)/(x^2 - 1)$ is less than 3, the graph lies *below* its asymptote. (Why doesn't the graph lie below $y = 3$ when $-1 < x < 1$?)

Table 1.15 *Values of* $y = \frac{3x^2-12}{x^2-1}$

x	$y = \frac{3x^2-12}{x^2-1}$
± 10	2.909091
± 100	2.999100
± 1000	2.999991

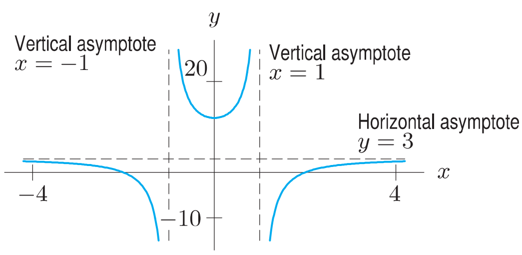


Figure 1.69: Graph of the function $y = \frac{3x^2-12}{x^2-1}$

[10] We are assuming that $f(x)$ gets arbitrarily close to L as $x \to \infty$.

Exercises and Problems for Section 1.6

Exercises

For Problems 1–2, what happens to the value of the function as $x \to \infty$ and as $x \to -\infty$?

1. $y = 0.25x^3 + 3$

2. $y = 2 \cdot 10^{4x}$

3. Determine the end behavior of each function as $x \to +\infty$ and as $x \to -\infty$.

(a) $f(x) = x^7$

(b) $f(x) = 3x+7x^3-12x^4$

(c) $f(x) = x^{-4}$

(d) $f(x) = \dfrac{6x^3 - 5x^2 + 2}{x^3 - 8}$

In Exercises 4–5, which function dominates as $x \to \infty$?

4. $10 \cdot 2^x$ or $72{,}000x^{12}$

5. $0.25\sqrt{x}$ or $25{,}000x^{-3}$

6. Each of the graphs in Figure 1.70 is of a polynomial. The windows are large enough to show end behavior.

(a) What is the minimum possible degree of the polynomial?

(b) Is the leading coefficient of the polynomial positive or negative?

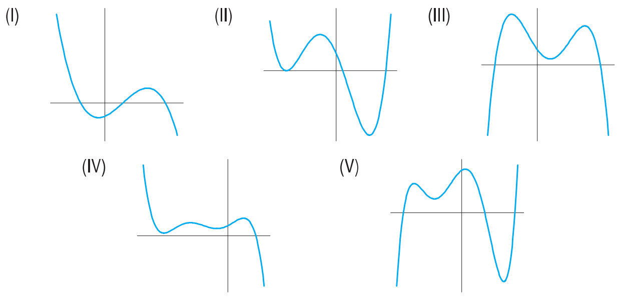
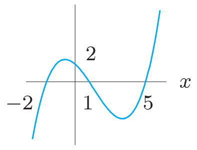


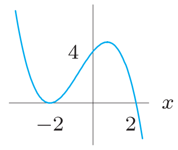
Figure 1.70

Find cubic polynomials for the graphs in Exercises 7–8.

7.

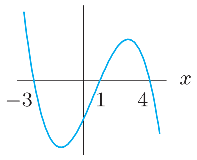


8.



Find possible formulas for the graphs in Exercises 9–12.

9.



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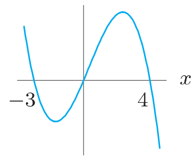

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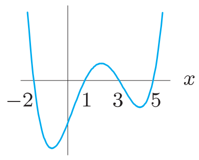

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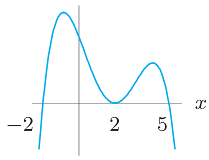


Problems

13. Which of the functions I–III meet each of the following descriptions? There may be more than one function for each description, or none at all.

(a) Horizontal asymptote of $y = 1$.
(b) The x-axis is a horizontal asymptote.
(c) Symmetric about the y-axis.
(d) An odd function.
(e) Vertical asymptotes at $x = \pm 1$.

I. $y = \dfrac{x-1}{x^2+1}$ II. $y = \dfrac{x^2-1}{x^2+1}$ III. $y = \dfrac{x^2+1}{x^2-1}$

14. The DuBois formula relates a person's surface area, s, in m^2, to weight w, in kg, and height h, in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

(a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?
(b) What is the weight of a person whose height is 180 cm and who has a surface area of $1.5\ \text{m}^2$?
(c) For people of fixed weight 70 kg, solve for h as a function of s. Simplify your answer.

15. A box of fixed volume V has a square base with side length x. Write a formula for the height, h, of the box in terms of x and V. Sketch a graph of h versus x.

16. According to *Car and Driver*, an Alfa Romeo going at 70 mph requires 177 feet to stop. Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph (its top speed).

17. Water is flowing down a cylindrical pipe of radius r.

(a) Write a formula for the volume, V, of water that emerges from the end of the pipe in one second if the water is flowing at a rate of

(i) 3 cm/sec (ii) k cm/sec

(b) Graph your answer to part (a)(ii) as a function of

(i) r, assuming k is constant

(ii) k, assuming r is constant

18. Poiseuille's Law gives the rate of flow, R, of a gas through a cylindrical pipe in terms of the radius of the pipe, r, for a fixed drop in pressure between the two ends of the pipe.

(a) Find a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.

(b) If $R = 400$ cm^3/sec in a pipe of radius 3 cm for a certain gas, find a formula for the rate of flow of that gas through a pipe of radius r cm.

(c) What is the rate of flow of the same gas through a pipe with a 5 cm radius?

19. The height of an object above the ground at time t is given by

$$s = v_0 t - \frac{g}{2}t^2,$$

where v_0 is the initial velocity and g is the acceleration due to gravity.

(a) At what height is the object initially?

(b) How long is the object in the air before it hits the ground?

(c) When will the object reach its maximum height?

(d) What is that maximum height?

20. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time t seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?

21. **(a)** If $f(x) = ax^2 + bx + c$, what can you say about the values of a, b, and c if:

(i) $(1, 1)$ is on the graph of $f(x)$?

(ii) $(1, 1)$ is the vertex of the graph of $f(x)$? [Hint: The axis of symmetry is $x = -b/(2a)$.]

(iii) The y intercept of the graph is $(0, 6)$?

(b) Find a quadratic function satisfying all three conditions.

22. Values of three functions are given in Table 1.16, rounded to two decimal places. One function is of the form $y = ab^t$, one is of the form $y = ct^2$, and one is of the form $y = kt^3$. Which function is which?

Table 1.16

t	$f(t)$	t	$g(t)$	t	$h(t)$
2.0	4.40	1.0	3.00	0.0	2.04
2.2	5.32	1.2	5.18	1.0	3.06
2.4	6.34	1.4	8.23	2.0	4.59
2.6	7.44	1.6	12.29	3.0	6.89
2.8	8.62	1.8	17.50	4.0	10.33
3.0	9.90	2.0	24.00	5.0	15.49

23. Values of three functions are given in Table 1.17, rounded to two decimal places. Two are power functions and one is an exponential. One of the power functions is a quadratic and one a cubic. Which one is exponential? Which one is quadratic? Which one is cubic?

Table 1.17

x	$f(x)$	x	$g(x)$	x	$k(x)$
8.4	5.93	5.0	3.12	0.6	3.24
9.0	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

24. A cubic polynomial with positive leading coefficient is shown in Figure 1.71 for $-10 \le x \le 10$ and $-10 \le y \le 10$. What can be concluded about the total number of zeros of this function? What can you say about the location of each of the zeros? Explain.

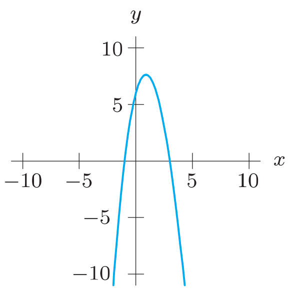


Figure 1.71

25. After running 3 miles at a speed of x mph, a man walked the next 6 miles at a speed that was 2 mph slower. Express the total time spent on the trip as a function of x. What horizontal and vertical asymptotes does the graph of this function have?

26. Match the following functions with the graphs in Figure 1.72. Assume $0 < b < a$.

(a) $y = \dfrac{a}{x} - x$ (b) $y = \dfrac{(x-a)(x+a)}{x}$

(c) $y = \dfrac{(x-a)(x^2+a)}{x^2}$ (d) $y = \dfrac{(x-a)(x+a)}{(x-b)(x+b)}$

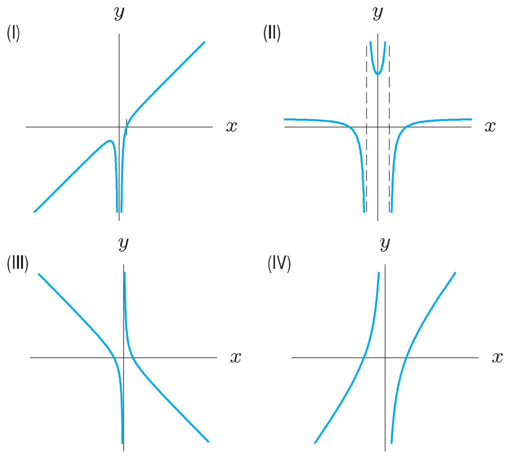


Figure 1.72

27. Consider the point P at the intersection of the circle $x^2 + y^2 = 2a^2$ and the parabola $y = x^2/a$ in Figure 1.73. If a is increased, the point P traces out a curve. For $a > 0$, find the equation of this curve.

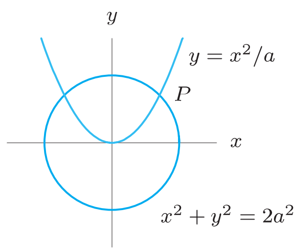


Figure 1.73

28. Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and ranges that give each of the graphs in Figure 1.74.

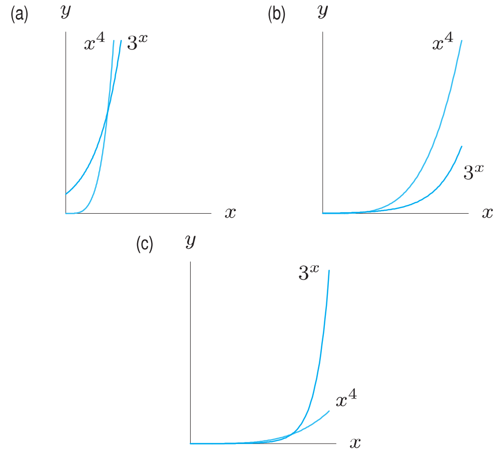


Figure 1.74

29. The rate, R, at which a population in a confined space increases is proportional to the product of the current population, P, and the difference between the carrying capacity, L, and the current population. (The carrying capacity is the maximum population the environment can sustain.)

(a) Write R as a function of P.
(b) Sketch R as a function of P.

30. When an object of mass m moves with a velocity v that is small compared to the velocity of light, c, its energy is given approximately by

$$E \approx \frac{1}{2}mv^2.$$

If v is comparable in size to c, then the energy must be computed by the exact formula

$$E = mc^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right).$$

(a) Plot a graph of both functions for E against v for $0 \le v \le 5 \cdot 10^8$ and $0 \le E \le 5 \cdot 10^{17}$. Take $m = 1$ kg and $c = 3 \cdot 10^8$ m/sec. Explain how you can predict from the exact formula the position of the vertical asymptote.
(b) What do the graphs tell you about the approximation? For what values of v does the first formula give a good approximation to E?

1.7 INTRODUCTION TO CONTINUITY

This section introduces the idea of *continuity* on an interval and at a point. This leads to the concept of limit, which is investigated in the next section.

Continuity of a Function on an Interval: Graphical Viewpoint

Roughly speaking, a function is said to be *continuous* on an interval if its graph has no breaks, jumps, or holes in that interval. Continuity is important because, as we shall see, continuous functions have many desirable properties.

For example, to locate the zeros of a function, we often look for intervals where the function changes sign. In the case of the function $f(x) = 3x^3 - x^2 + 2x - 1$, for instance, we expect[11] to find a zero between 0 and 1 because $f(0) = -1$ and $f(1) = 3$. (See Figure 1.75.) To be sure that $f(x)$ has a zero there, we need to know that the graph of the function has no breaks or jumps in it. Otherwise the graph could jump across the x-axis, changing sign but not creating a zero. For example, $f(x) = 1/x$ has opposite signs at $x = -1$ and $x = 1$, but no zeros for $-1 \leq x \leq 1$ because of the break at $x = 0$. (See Figure 1.76.) To be certain that a function has a zero in an interval on which it changes sign, we need to know that the function is defined and continuous in that interval.

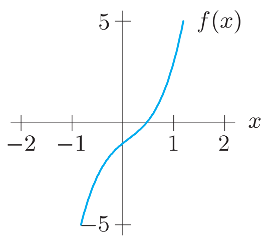


Figure 1.75: The graph of $f(x) = 3x^3 - x^2 + 2x - 1$

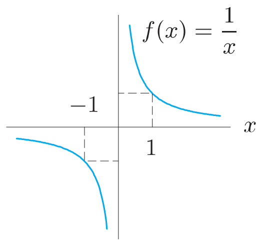


Figure 1.76: No zero although $f(-1)$ and $f(1)$ have opposite signs

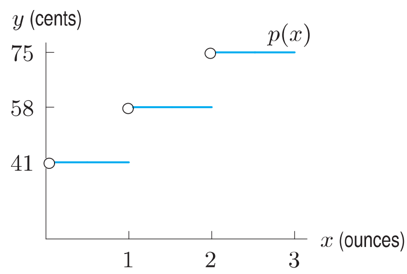


Figure 1.77: Cost of mailing a letter

A continuous function has a graph which can be drawn without lifting the pencil from the paper.

Example: The function $f(x) = 3x^3 - x^2 + 2x - 1$ is continuous on any interval. (See Figure 1.75.)

Example: The function $f(x) = 1/x$ is not defined at $x = 0$. It is continuous on any interval not containing the origin. (See Figure 1.76.)

Example: Suppose $p(x)$ is the price of mailing a first-class letter weighing x ounces. It costs 41¢ for one ounce or less, 58¢ between the first and second ounces, and so on. So the graph (in Figure 1.77) is a series of steps. This function is not continuous on any open interval containing a positive integer because the graph jumps at these points.

Which Functions are Continuous?

Requiring a function to be continuous on an interval is not asking very much, as any function whose graph is an unbroken curve over the interval is continuous. For example, exponential functions, polynomials, and the sine and cosine are continuous on every interval. Rational functions are continuous on any interval in which their denominators are not zero. Functions created by adding, multiplying, or composing continuous functions are also continuous.

[11]This is due to the Intermediate Value Theorem, which is discussed on page 48.

The Intermediate Value Theorem

Continuity tells us about the values taken by a function. In particular, a continuous function cannot skip values. For example, the function in the next example must have a zero because its graph cannot skip over the x-axis.

Example 1 What do the values in Table 1.18 tell you about the zeros of $f(x) = \cos x - 2x^2$?

Table 1.18

x	$f(x)$
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	−0.58
1.0	−1.46

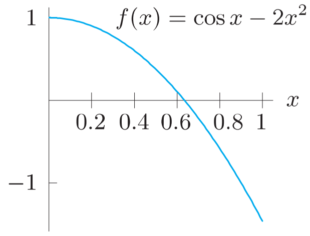


Figure 1.78: Zeros occur where the graph of a continuous function crosses the horizontal axis

Solution Since $f(x)$ is the difference of two continuous functions, it is continuous. We conclude that $f(x)$ has at least one zero in the interval $0.6 < x < 0.8$, since $f(x)$ changes from positive to negative on that interval. The graph of $f(x)$ in Figure 1.78 suggests that there is only one zero in the interval $0 \le x \le 1$, but we cannot be sure of this from the graph or the table of values.

In the previous example, we concluded that $f(x) = \cos x - 2x^2$ has a zero between $x = 0$ and $x = 1$ because $f(x)$ is positive at $x = 0$ and negative at $x = 1$. More generally, an intuitive notion of continuity tells us that, as we follow the graph of a continuous function f from some point $(a, f(a))$ to another point $(b, f(b))$, then f takes on all intermediate values between $f(a)$ and $f(b)$. (See Figure 1.79.) This is:

Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

The Intermediate Value Theorem depends on the formal definition of continuity given in Section 1.8. See also www.wiley.com/college/hugheshallett. The key idea is to find successively smaller subintervals of $[a, b]$ on which f changes from less than k to more than k. These subintervals converge on the number c.

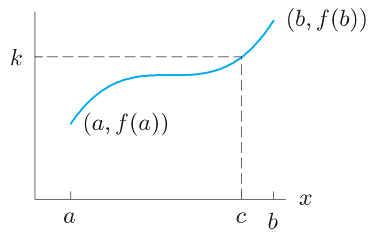


Figure 1.79: The Intermediate Value Theorem

Continuity of a Function at a Point: Numerical Viewpoint

A function is continuous if nearby values of the independent variable give nearby values of the function. In practical work, continuity is important because it means that small errors in the independent variable lead to small errors in the value of the function.

Example: Suppose that $f(x) = x^2$ and that we want to compute $f(\pi)$. Knowing f is continuous tells us that taking $x = 3.14$ should give a good approximation to $f(\pi)$, and that we can get as accurate an approximation to $f(\pi)$ as we want by using enough decimals of π.

Example: If $p(x)$ is the cost of mailing a letter weighing x ounces, then $p(0.99) = p(1) = 41$¢, whereas $p(1.01) = 58$¢, because as soon as we get over 1 ounce, the price jumps up to 58¢. So a small difference in the weight of a letter can lead to a significant difference in its mailing cost. Hence p is not continuous at $x = 1$.

In other words, if $f(x)$ is continuous at $x = c$, the values of $f(x)$ approach $f(c)$ as x approaches c. In Section 1.8, we discuss the concept of a *limit*, which allows us to define more precisely what it means for the values of $f(x)$ to approach $f(c)$ as x approaches c.

Example 2 Investigate the continuity of $f(x) = x^2$ at $x = 2$.

Solution From Table 1.19, it appears that the values of $f(x) = x^2$ approach $f(2) = 4$ as x approaches 2. Thus f appears to be continuous at $x = 2$. Continuity at a point describes behavior of a function *near* a point, as well as *at* the point.

Table 1.19 *Values of x^2 near $x = 2$*

x	1.9	1.99	1.999	2.001	2.01	2.1
x^2	3.61	3.96	3.996	4.004	4.04	4.41

Exercises and Problems for Section 1.7

Exercises

Are the functions in Exercises 1–10 continuous on the given intervals?

1. $2x + x^{2/3}$ on $[-1, 1]$

2. $2x + x^{-1}$ on $[-1, 1]$

3. $\dfrac{1}{x-2}$ on $[-1, 1]$

4. $\dfrac{1}{x-2}$ on $[0, 3]$

5. $\dfrac{1}{\sqrt{2x-5}}$ on $[3, 4]$

6. $\dfrac{x}{x^2+2}$ on $[-2, 2]$

7. $\dfrac{1}{\cos x}$ on $[0, \pi]$

8. $\dfrac{1}{\sin x}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

9. $\dfrac{e^x}{e^x - 1}$ on $[-1, 1]$

10. $\dfrac{e^{\sin\theta}}{\cos\theta}$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

In Exercises 11–14, show that there is a number c, with $0 \le c \le 1$, such that $f(c) = 0$.

11. $f(x) = x^3 + x^2 - 1$

12. $f(x) = e^x - 3x$

13. $f(x) = x - \cos x$

14. $f(x) = 2^x - 1/x$

15. Are the following functions continuous? Explain.

(a) $f(x) = \begin{cases} x & x \le 1 \\ x^2 & 1 < x \end{cases}$

(b) $g(x) = \begin{cases} x & x \le 3 \\ x^2 & 3 < x \end{cases}$

Problems

16. Which of the following are continuous functions of time?

(a) The quantity of gas in the tank of a car on a journey between New York and Boston.
(b) The number of students enrolled in a class during a semester.
(c) The age of the oldest person alive.

17. An electrical circuit switches instantaneously from a 6 volt battery to a 12 volt battery 7 seconds after being turned on. Graph the battery voltage against time. Give formulas for the function represented by your graph. What can you say about the continuity of this function?

18. A car is coasting down a hill at a constant speed. A truck collides with the rear of the car, causing it to lurch ahead. Graph the car's speed from a time shortly before impact to a time shortly after impact. Graph the distance from the top of the hill for this time period. What can you say about the continuity of each of these functions?

19. Find k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} kx & x \le 3 \\ 5 & 3 < x \end{cases}$$

20. Find k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ 3x^2 & 2 \le x \end{cases}$$

21. Is the following function continuous on $[-1, 1]$?

$$f(x) = \begin{cases} \dfrac{x}{|x|} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

22. If possible, choose k so that the following function is continuous on any interval:

$$f(x) = \begin{cases} \dfrac{5x^3 - 10x^2}{x - 2} & x \ne 2 \\ k & x = 2 \end{cases}$$

23. Find k so that the following function is continuous on any interval:

$$j(x) = \begin{cases} k\cos x & x \le 0 \\ e^x - k & x > 0 \end{cases}$$

24. Discuss the continuity of the function g graphed in Figure 1.80 and defined as follows:

$$g(\theta) = \begin{cases} \dfrac{\sin\theta}{\theta} & \text{for } \theta \ne 0 \\ 1/2 & \text{for } \theta = 0. \end{cases}$$

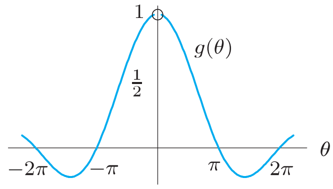


Figure 1.80

25. **(a)** What does a graph of $y = e^x$ and $y = 4 - x^2$ tell you about the solutions to the equation $e^x = 4 - x^2$?
(b) Evaluate $f(x) = e^x + x^2 - 4$ at $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$. In which intervals do the solutions to $e^x = 4 - x^2$ lie?

26. Let $p(x)$ be a cubic polynomial with $p(5) < 0$, $p(10) > 0$, and $p(12) < 0$. What can you say about the number and location of zeros of $p(x)$?

27. Sketch the graphs of three different functions that are continuous on $0 \le x \le 1$ and that have the values given in the table. The first function is to have exactly one zero in $[0, 1]$, the second is to have at least two zeros in the interval $[0.6, 0.8]$, and the third is to have at least two zeros in the interval $[0, 0.6]$.

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	0.90	0.60	0.11	−0.58	−1.46

28. **(a)** Sketch the graph of a continuous function f with *all* of the following properties:
(i) $f(0) = 2$
(ii) $f(x)$ is decreasing for $0 \le x \le 3$
(iii) $f(x)$ is increasing for $3 < x \le 5$
(iv) $f(x)$ is decreasing for $x > 5$
(v) $f(x) \to 9$ as $x \to \infty$
(b) Is it possible that the graph of f is concave down for all $x > 6$? Explain.

29. A 0.6 ml dose of a drug is injected into a patient steadily for half a second. At the end of this time, the quantity, Q, of the drug in the body starts to decay exponentially at a continuous rate of 0.2% per second. Using formulas, express Q as a continuous function of time, t in seconds.

30. Use a computer or calculator to sketch the functions

$$y(x) = \sin x \quad \text{and} \quad z_k(x) = ke^{-x}$$

for $k = 1, 2, 4, 6, 8, 10$. In each case find the smallest positive solution of the equation $y(x) = z_k(x)$. Now define a new function f by

$$f(k) = \{\text{Smallest positive solution of } y(x) = z_k(x)\}.$$

Explain why the function $f(k)$ is not continuous on the interval $0 \le k \le 10$.

1.8 LIMITS

The concept of *limit* is the underpinning of calculus. In Section 1.7, we said that a function f is continuous at $x = c$ if the values of $f(x)$ approach $f(c)$ as x approaches c. In this section, we define a limit, which makes precise what we mean by approaching.

The Idea of a Limit

We first introduce some notation:

> We write $\lim_{x \to c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c.

How should we find L, or even know whether such a number exists? We will look for trends in the values of $f(x)$ as x gets closer to c, but $x \neq c$. A graph from a calculator or computer often helps.

Example 1 Use a graph to estimate $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right)$. (Use radians.)

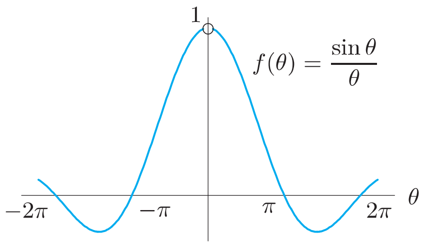


Figure 1.81: Find the limit as $\theta \to 0$

Solution Figure 1.81 shows that as θ approaches 0 from either side, the value of $\sin \theta / \theta$ appears to approach 1, suggesting that $\lim_{\theta \to 0} (\sin \theta / \theta) = 1$. Zooming in on the graph near $\theta = 0$ provides further support for this conclusion. Notice that $\sin \theta / \theta$ is undefined at $\theta = 0$.

Figure 1.81 strongly suggests that $\lim_{\theta \to 0} (\sin \theta / \theta) = 1$, but to be sure we need to be more precise about words like "approach" and "close."

Definition of Limit

By the beginning of the 19th century, calculus had proved its worth, and there was no doubt about the correctness of its answers. However, it was not until the work of the French mathematician Augustin Cauchy (1789–1857) that calculus was put on a rigorous footing. Cauchy gave a formal definition of the limit, similar to the following:

> A function f is defined on an interval around c, except perhaps at the point $x = c$. We define the **limit** of the function $f(x)$ as x approaches c, written $\lim_{x \to c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write
> $$\lim_{x \to c} f(x) = L.$$

Shortly, we see how "as close as we want" and "sufficiently close" are expressed using inequalities. First, we look at $\lim_{\theta \to 0} (\sin \theta / \theta)$ more closely (see Example 1).

Example 2 By graphing $y = (\sin\theta)/\theta$ in an appropriate window, find how close θ should be to 0 in order to make $(\sin\theta)/\theta$ within 0.01 of 1.

Solution Since we want $(\sin\theta)/\theta$ to be within 0.01 of 1, we set the y-range on the graphing window to go from 0.99 to 1.01. Our first attempt with $-0.5 \le \theta \le 0.5$ yields the graph in Figure 1.82. Since we want the y-values to stay within the range $0.99 < y < 1.01$, we do not want the graph to leave the window through the top or bottom. By trial and error, we find that changing the θ-range to $-0.2 \le \theta \le 0.2$ gives the graph in Figure 1.83. Thus, the graph suggests that $(\sin\theta)/\theta$ is within 0.01 of 1 whenever θ is within 0.2 of 0. Proving this requires an analytical argument, not just graphs from a calculator.

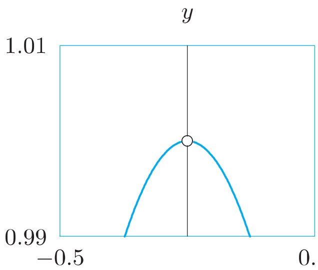


Figure 1.82: $(\sin\theta)/\theta$ with $-0.5 \le \theta \le 0.5$

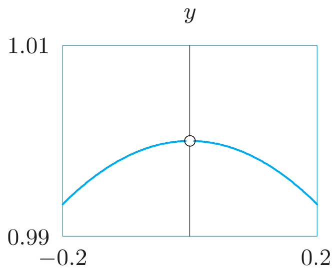


Figure 1.83: $(\sin\theta)/\theta$ with $-0.2 \le \theta \le 0.2$

When we say "$f(x)$ is close to L," we measure closeness by the distance between $f(x)$ and L, expressed using absolute values:

$$|f(x) - L| = \text{Distance between } f(x) \text{ and } L.$$

When we say "as close to L as we want," we use ϵ (the Greek letter epsilon) to specify how close. We write

$$|f(x) - L| < \epsilon$$

to indicate that we want the distance between $f(x)$ and L to be less than ϵ. In Example 2 we used $\epsilon = 0.01$. Similarly, we interpret "x is sufficiently close to c" as specifying a distance between x and c:

$$|x - c| < \delta,$$

where δ (the Greek letter delta) tells us how close x should be to c. In Example 2 we found $\delta = 0.2$.

If $\lim_{x\to c} f(x) = L$, we know that no matter how narrow the horizontal band determined by ϵ in Figure 1.84, there is always a δ which makes the graph stay within that band, for $c - \delta < x < c + \delta$.

Thus we restate the definition of a limit, using symbols:

Definition of Limit

We define $\lim_{x\to c} f(x)$ to be the number L (if one exists) such that for every $\epsilon > 0$ (as small as we want), there is a $\delta > 0$ (sufficiently small) such that if $|x - c| < \delta$ and $x \ne c$, then $|f(x) - L| < \epsilon$.

We have arrived at a formal definition of limit. Let's see if it agrees with our intuition.

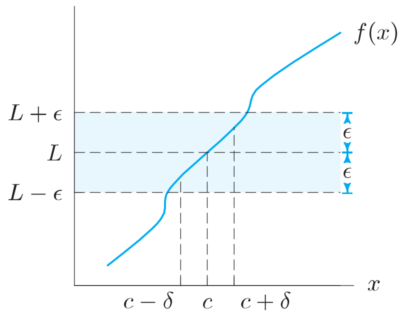


Figure 1.84: What the definition of the limit means graphically

Example 3 Use the definition of limit to show that $\lim_{x\to 3} 2x = 6$.

Solution We must show how, given any $\epsilon > 0$, we can find a $\delta > 0$ such that

$$\text{If } |x-3| < \delta \text{ and } x \neq 3, \text{ then } |2x-6| < \epsilon.$$

Since $|2x-6| = 2|x-3|$, to get $|2x-6| < \epsilon$ we require that $|x-3| < \epsilon/2$. Thus we take $\delta = \epsilon/2$.

It is important to understand that the ϵ, δ definition does not make it easier to calculate limits; rather the ϵ, δ definition makes it possible to put calculus on a rigorous foundation. From this foundation, we can prove the following properties. See Problems 58–60.

Theorem 1.2: Properties of Limits

Assuming all the limits on the right hand side exist:

1. If b is a constant, then $\lim_{x\to c}(bf(x)) = b\left(\lim_{x\to c} f(x)\right)$.
2. $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
3. $\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$.
4. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$.
5. For any constant k, $\lim_{x\to c} k = k$.
6. $\lim_{x\to c} x = c$.

These properties underlie many limit calculations, though we may not acknowledge them explicitly.

Example 4 Explain how the limit properties are used in the following calculation:

$$\lim_{x\to 3}\frac{x^2+5x}{x+9} = \frac{3^2+5\cdot 3}{3+9} = 2.$$

Solution We calculate this limit in stages, using the limit properties to justify each step:

$$\lim_{x\to 3}\frac{x^2+5x}{x+9} = \frac{\lim\limits_{x\to 3}(x^2+5x)}{\lim\limits_{x\to 3}(x+9)} \quad \text{(Property 4, since } \lim_{x\to 3}(x+9)\neq 0\text{)}$$

$$= \frac{\lim\limits_{x\to 3}(x^2)+\lim\limits_{x\to 3}(5x)}{\lim\limits_{x\to 3}x+\lim\limits_{x\to 3}9} \quad \text{(Property 2)}$$

$$= \frac{\left(\lim\limits_{x\to 3}x\right)^2+5\left(\lim\limits_{x\to 3}x\right)}{\lim\limits_{x\to 3}x+\lim\limits_{x\to 3}9} \quad \text{(Properties 1 and 3)}$$

$$= \frac{3^2+5\cdot 3}{3+9} = 2. \quad \text{(Properties 5 and 6)}$$

One- and Two-Sided Limits

When we write

$$\lim_{x\to 2} f(x),$$

we mean the number that $f(x)$ approaches as x approaches 2 *from both sides*. We examine values of $f(x)$ as x approaches 2 through values greater than 2 (such as 2.1, 2.01, 2.003) and values less than 2 (such as 1.9, 1.99, 1.994). If we want x to approach 2 only through values greater than 2, we write

$$\lim_{x\to 2^+} f(x)$$

for the number that $f(x)$ approaches (assuming such a number exists). Similarly,

$$\lim_{x\to 2^-} f(x)$$

denotes the number (if it exists) obtained by letting x approach 2 through values less than 2. We call $\lim\limits_{x\to 2^+} f(x)$ a *right-hand limit* and $\lim\limits_{x\to 2^-} f(x)$ a *left-hand limit*. Problems 22 and 23 ask for formal definitions of left and right-hand limits.

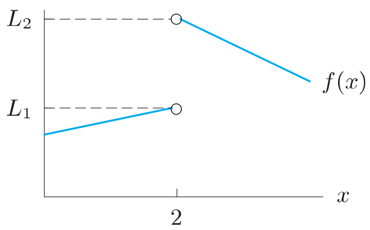


Figure 1.85: Left- and right-hand limits at $x=2$

For the function graphed in Figure 1.85, we have

$$\lim_{x\to 2^-} f(x) = L_1 \qquad \lim_{x\to 2^+} f(x) = L_2.$$

If the left- and right-hand limits were equal, that is, if $L_1 = L_2$, then it can be proved that $\lim\limits_{x\to 2} f(x)$ exists and $\lim\limits_{x\to 2} f(x) = L_1 = L_2$. Since $L_1 \neq L_2$ in Figure 1.85, we see that $\lim\limits_{x\to 2} f(x)$ does not exist in this case.

When Limits Do Not Exist

Whenever there is no number L such that $\lim\limits_{x\to c} f(x) = L$, we say $\lim\limits_{x\to c} f(x)$ does not exist. Here are three examples in which limits fail to exist.

Example 5 Explain why $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist.

Solution Figure 1.86 shows the problem: The right-hand limit and the left-hand limit are different. For $x > 2$, we have $|x-2| = x-2$, so as x approaches 2 from the right,

$$\lim_{x\to 2^+}\frac{|x-2|}{x-2} = \lim_{x\to 2^+}\frac{x-2}{x-2} = \lim_{x\to 2^+} 1 = 1.$$

Similarly, if $x < 2$, then $|x-2| = 2-x$ so

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2} = \lim_{x\to 2^-}\frac{2-x}{x-2} = \lim_{x\to 2^-}(-1) = -1.$$

So if $\lim_{x\to 2}\dfrac{|x-2|}{x-2} = L$ then L would have to be both 1 and -1. Since L cannot have two different values, the limit does not exist.

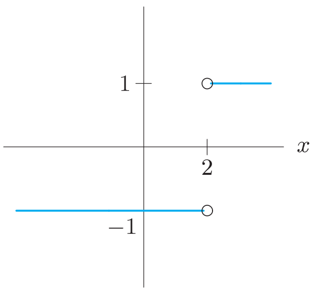


Figure 1.86: Graph of $|x-2|/(x-2)$

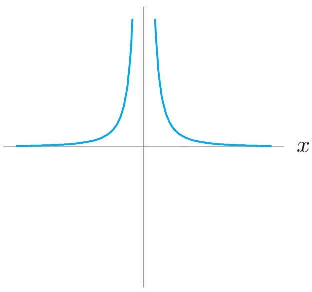


Figure 1.87: Graph of $1/x^2$

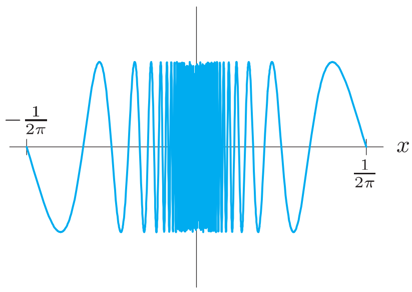


Figure 1.88: Graph of $\sin(1/x)$

Example 6 Explain why $\lim_{x\to 0}\dfrac{1}{x^2}$ does not exist.

Solution As x approaches zero, $1/x^2$ becomes arbitrarily large, so it cannot approach any finite number L. See Figure 1.87. Therefore we say $1/x^2$ has no limit as $x \to 0$.

If $\lim_{x\to a} f(x)$ does not exist because $f(x)$ gets arbitrarily large on both sides of a, we also say $\lim_{x\to a} f(x) = \infty$. So in Example 6 we could say $\lim_{x\to 0} 1/x^2 = \infty$. This behavior may also be described as "diverging to infinity."

Example 7 Explain why $\lim_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist.

Solution The sine function has values between -1 and 1. The graph in Figure 1.88 oscillates more and more rapidly as $x \to 0$. There are x-values approaching 0 where $\sin(1/x) = -1$. There are also x-values approaching 0 where $\sin(1/x) = 1$. So if the limit existed, it would have to be both -1 and 1. Thus, the limit does not exist.

Limits at Infinity

Sometimes we want to know what happens to $f(x)$ as x gets large, that is, the end behavior of f.

If $f(x)$ gets as close to a number L as we please when x gets sufficiently large, then we write

$$\lim_{x\to\infty} f(x) = L.$$

Similarly, if $f(x)$ approaches L when x is negative and has a sufficiently large absolute value, then we write

$$\lim_{x\to-\infty} f(x) = L.$$

The symbol ∞ does not represent a number. Writing $x \to \infty$ means that we consider arbitrarily large values of x. If the limit of $f(x)$ as $x \to \infty$ or $x \to -\infty$ is L, we say that the graph of f has $y = L$ as a *horizontal asymptote*. Problem 24 asks for a formal definition of $\lim_{x\to\infty} f(x)$.

Example 8 Investigate $\displaystyle\lim_{x\to\infty}\frac{1}{x}$ and $\displaystyle\lim_{x\to-\infty}\frac{1}{x}$.

Solution A graph of $f(x) = 1/x$ in a large window shows $1/x$ approaching zero as x increases in either the positive or the negative direction (See Figure 1.89). This is as we would expect, since dividing 1 by larger and larger numbers yields answers which are smaller and smaller. This suggests that

$$\lim_{x\to\infty}\frac{1}{x} = \lim_{x\to-\infty}\frac{1}{x} = 0,$$

and that $f(x) = 1/x$ has $y = 0$ as a horizontal asymptote as $x \to \pm\infty$.

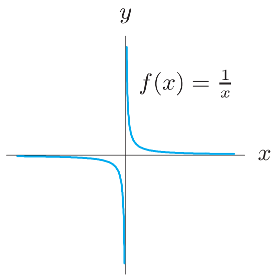


Figure 1.89: The end behavior of $f(x) = 1/x$

Definition of Continuity

We can now give a precise definition of continuity using limits.

The function f is **continuous** at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x\to c} f(x) = f(c).$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c. The function is **continuous on an interval** $[a, b]$ if it is continuous at every point in the interval.[12]

Constant functions and $f(x) = x$ are continuous for all x. Using the continuity of sums and products, we can show that any polynomial is continuous. Proving that $\sin x$, $\cos x$, and e^x are

[12]If c is an endpoint of the interval, we define continuity at $x = c$ using one-sided limits at c.

continuous is more difficult. The following theorem, based on the properties of limits on page 53, makes it easier to decide whether certain combinations of functions are continuous.

Theorem 1.3: Continuity of Sums, Products, and Quotients of Functions

Suppose that f and g are continuous on an interval and that b is a constant. Then, on that same interval,

1. $bf(x)$ is continuous.
2. $f(x) + g(x)$ is continuous.
3. $f(x)g(x)$ is continuous.
4. $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

We prove the third of these properties.

Proof Let c be any point in the interval. We must show that $\lim_{x \to c}(f(x)g(x)) = f(c)g(c)$. Since $f(x)$ and $g(x)$ are continuous, we know that $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. So, by the first property of limits in Theorem 1.2,

$$\lim_{x \to c}(f(x)g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right) = f(c)g(c).$$

Since c was chosen arbitrarily, we have shown that $f(x)g(x)$ is continuous at every point in the interval.

Theorem 1.4: Continuity of Composite Functions

If f and g are continuous, and if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.

Assuming the continuity of $\sin x$ and e^x, Theorem 1.4 shows us, for example, that $\sin(e^x)$ and $e^{\sin x}$ are both continuous.

Although we now have a formal definition of continuity, some properties of continuous functions, such as the Intermediate Value Theorem, can be difficult to prove. For a further treatment of limits and continuity, see www.wiley.com/college/hugheshallett.

Exercises and Problems for Section 1.8

Exercises

1. Use Figure 1.90 to give approximate values for the following limits (if they exist).

(a) $\lim_{x \to -2} f(x)$ **(b)** $\lim_{x \to 0} f(x)$

(c) $\lim_{x \to 2} f(x)$ **(d)** $\lim_{x \to 4} f(x)$

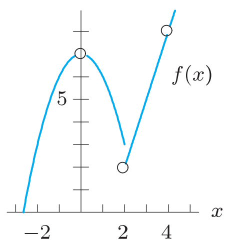


Figure 1.90

2. Use Figure 1.91 to estimate the limits if they exist:

(a) $\lim_{x\to 0} f(x)$ **(b)** $\lim_{x\to 1} f(x)$

(c) $\lim_{x\to 2} f(x)$ **(d)** $\lim_{x\to 3^-} f(x)$

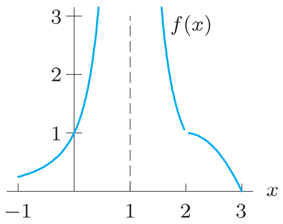


Figure 1.91

3. Using Figures 1.92 and 1.93, estimate

(a) $\lim_{x\to 1^-} (f(x)+g(x))$ **(b)** $\lim_{x\to 1^+} (f(x)+2g(x))$

(c) $\lim_{x\to 1^-} f(x)g(x)$ **(d)** $\lim_{x\to 1^+} \dfrac{f(x)}{g(x)}$

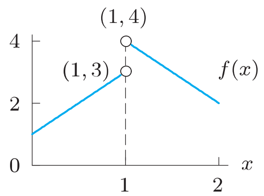


Figure 1.92

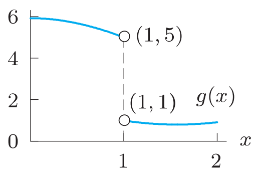


Figure 1.93

Estimate the limits in Exercises 4–5 graphically.

4. $\lim_{x\to 0} \dfrac{|x|}{x}$

5. $\lim_{x\to 0} x \ln |x|$

6. Does $f(x) = \dfrac{|x|}{x}$ have right or left limits at 0? Is $f(x)$ continuous?

Use a graph to estimate each of the limits in Exercises 7–10. Use radians unless degrees are indicated by θ°.

7. $\lim_{\theta\to 0} \dfrac{\sin(2\theta)}{\theta}$

8. $\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta}$

9. $\lim_{\theta\to 0} \dfrac{\sin\theta^\circ}{\theta^\circ}$

10. $\lim_{\theta\to 0} \dfrac{\theta}{\tan(3\theta)}$

For the functions in Exercises 11–13, use algebra to evaluate the limits $\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$, and $\lim_{x\to a} f(x)$ if they exist. Sketch a graph to confirm your answers.

11. $a = 4, \quad f(x) = \dfrac{|x-4|}{x-4}$

12. $a = 2, \quad f(x) = \dfrac{|x-2|}{x}$

13. $a = 3, \quad f(x) = \begin{cases} x^2 - 2, & 0 < x < 3 \\ 2, & x = 3 \\ 2x+1, & 3 < x \end{cases}$

14. Estimate how close θ should be to 0 to make $(\sin\theta)/\theta$ stay within 0.001 of 1.

15. Write the definition of the following statement both in words and in symbols:

$$\lim_{h\to a} g(h) = K.$$

Problems

In Problems 16–19, is the function continuous for all x? If not, say where it is not continuous and explain in what way the definition of continuity is not satisfied.

16. $f(x) = 1/x$

17. $f(x) = \begin{cases} |x|/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

18. $f(x) = \begin{cases} x/x & x \neq 0 \\ 1 & x = 0 \end{cases}$

19. $f(x) = \begin{cases} 2x/x & x \neq 0 \\ 3 & x = 0 \end{cases}$

20. By graphing $y = (1+x)^{1/x}$, estimate $\lim_{x\to 0}(1+x)^{1/x}$. You should recognize the answer you get. What does the limit appear to be?

21. What does a calculator suggest about $\lim_{x\to 0^+} xe^{1/x}$? Does the limit appear to exist? Explain.

In Problems 22–24, modify the definition of limit on page 52 to give a definition of each of the following.

22. A right-hand limit

23. A left-hand limit

24. $\lim_{x\to\infty} f(x) = L$

25. If $p(x)$ is the function on page 47 giving the price of mailing a first-class letter, explain why $\lim_{x\to 1} p(x)$ does not exist.

26. Investigate $\lim_{h\to 0}(1+h)^{1/h}$ numerically.

27. The notation $\lim_{x\to 0^+}$ means that we only consider values of x greater than 0. Estimate the limit

$$\lim_{x\to 0^+} x^x,$$

either by evaluating x^x for smaller and smaller positive values of x (say $x = 0.1, 0.01, 0.001, \ldots$) or by zooming in on the graph of $y = x^x$ near $x = 0$.

For the functions in Exercises 28–35, do the following:

(a) Make a table of values of $f(x)$ for $x = 0.1, 0.01, 0.001, 0.0001, -0.1, -0.01, -0.001$, and -0.0001.

(b) Make a conjecture about the value of $\lim_{x\to 0} f(x)$.

(c) Graph the function to see if it is consistent with your answers to parts (a) and (b).

(d) Find an interval for x near 0 such that the difference between your conjectured limit and the value of the function is less than 0.01. (In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom of the window.)

28. $f(x) = 3x + 1$

29. $f(x) = x^2 - 1$

30. $f(x) = \sin 2x$

31. $f(x) = \sin 3x$

32. $f(x) = \dfrac{\sin 2x}{x}$

33. $f(x) = \dfrac{\sin 3x}{x}$

34. $f(x) = \dfrac{e^x - 1}{x}$

35. $f(x) = \dfrac{e^{2x} - 1}{x}$

Assuming that limits as $x \to \infty$ have the properties listed for limits as $x \to c$ on page 53, use algebraic manipulations to evaluate $\lim_{x\to\infty}$ for the functions in Problems 36–45.

36. $f(x) = \dfrac{x+3}{2-x}$

37. $f(x) = \dfrac{\pi + 3x}{\pi x - 3}$

38. $f(x) = \dfrac{x-5}{5+2x^2}$

39. $f(x) = \dfrac{x^2+2x-1}{3+3x^2}$

40. $f(x) = \dfrac{x^2+4}{x+3}$

41. $f(x) = \dfrac{2x^3 - 16x^2}{4x^2 + 3x^3}$

42. $f(x) = \dfrac{x^4+3x}{x^4+2x^5}$

43. $f(x) = \dfrac{3e^x + 2}{2e^x + 3}$

44. $f(x) = \dfrac{2^{-x}+5}{3^{-x}+7}$

45. $f(x) = \dfrac{2e^{-x}+3}{3e^{-x}+2}$

In Problems 46–53, find a value of the constant k such that the limit exists.

46. $\displaystyle\lim_{x\to 4} \frac{x^2 - k^2}{x - 4}$

47. $\displaystyle\lim_{x\to 1} \frac{x^2 - kx + 4}{x - 1}$

48. $\displaystyle\lim_{x\to -2} \frac{x^2 + 4x + k}{x + 2}$

49. $\displaystyle\lim_{x\to\infty} \frac{x^2 + 3x + 5}{4x + 1 + x^k}$

50. $\displaystyle\lim_{x\to -\infty} \frac{e^{2x} - 5}{e^{kx} + 3}$

51. $\displaystyle\lim_{x\to\infty} \frac{x^3 - 6}{x^k + 3}$

52. $\displaystyle\lim_{x\to\infty} \frac{3^{kx} + 6}{3^{2x} + 4}$

53. $\displaystyle\lim_{x\to -\infty} \frac{3^{kx} + 6}{3^{2x} + 4}$

For each value of ϵ in Problems 54–55, find a positive value of δ such that the graph of the function leaves the window $a - \delta < x < a + \delta$, $b - \epsilon < y < b + \epsilon$ by the sides and not through the top or bottom.

54. $f(x) = -2x + 3$; $a = 0$; $b = 3$; $\epsilon = 0.2, 0.1, 0.02, 0.01, 0.002, 0.001$.

55. $g(x) = -x^3 + 2$; $a = 0$; $b = 2$; $\epsilon = 0.1, 0.01, 0.001$.

56. Show that $\lim_{x\to 0}(-2x + 3) = 3$. [Hint: Use Problem 54.]

57. Consider the function $f(x) = \sin(1/x)$.

(a) Find a sequence of x-values that approach 0 such that $\sin(1/x) = 0$.
[Hint: Use the fact that $\sin(\pi) = \sin(2\pi) = \sin(3\pi) = \ldots = \sin(n\pi) = 0$.]

(b) Find a sequence of x-values that approach 0 such that $\sin(1/x) = 1$.
[Hint: Use the fact that $\sin(n\pi/2) = 1$ if $n = 1, 5, 9, \ldots$.]

(c) Find a sequence of x-values that approach 0 such that $\sin(1/x) = -1$.

(d) Explain why your answers to any two of parts (a)–(c) show that $\lim_{x\to 0} \sin(1/x)$ does not exist.

58. This problem suggests a proof of the first property of limits on page 53: $\lim_{x\to c} bf(x) = b \lim_{x\to c} f(x)$.

(a) First, prove the property in the case $b = 0$.

(b) Now suppose that $b \neq 0$. Let $\epsilon > 0$. Show that if $|f(x) - L| < \epsilon/|b|$, then $|bf(x) - bL| < \epsilon$.

(c) Finally, prove that if $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} bf(x) = bL$. [Hint: Choose δ so that if $|x - c| < \delta$, then $|f(x) - L| < \epsilon/|b|$.]

59. Prove the second property of limits: $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$. Assume that the limits on the right exist.

60. This problem suggests a proof of the third property of limits (assuming the limits on the right exist):

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$$

Let $L_1 = \lim_{x\to c} f(x)$ and $L_2 = \lim_{x\to c} g(x)$.

(a) First, show that if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$, then $\lim_{x\to c}(f(x)g(x)) = 0$.

(b) Show algebraically that $f(x)g(x) = (f(x) - L_1)(g(x) - L_2) + L_1 g(x) + L_2 f(x) - L_1 L_2$.

(c) Use the second limit property (see Problem 59) to explain why $\lim_{x\to c}(f(x) - L_1) = \lim_{x\to c}(g(x) - L_2) = 0$.

(d) Use parts (a) and (c) to explain why $\lim_{x\to c}(f(x) - L_1)(g(x) - L_2) = 0$.

(e) Finally, use parts (b) and (d) and the first and second limit properties to show that
$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$

61. Show $f(x) = x$ is continuous everywhere.

62. Use Problem 61 to show that for any positive integer n, the function x^n is continuous everywhere.

63. Use Theorem 1.2 on page 53 to explain why if f and g are continuous on an interval, then so are $f + g$, fg, and f/g (assuming $g(x) \neq 0$ on the interval).

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Function terminology**
  Domain/range, increasing/decreasing, concavity, zeros (roots), even/odd, end behavior, asymptotes.
- **Linear functions**
  Slope, vertical intercept. Grow by equal amounts in equal times.
- **Exponential functions**
  Exponential growth and decay, with base e, growth rate, continuous growth rate, doubling time, half life. Grow by equal percentages in equal times.
- **Logarithmic functions**
  Log base 10, natural logarithm.
- **Trigonometric functions**
  Sine and cosine, tangent, amplitude, period, arcsine, arctangent.
- **Power functions**
- **Polynomials and rational functions**
- **New functions from old**
  Inverse functions, composition of functions, shifting, stretching, shrinking.
- **Working with functions**
  Find a formula for a linear, exponential, power, logarithmic, or trigonometric function, given graph, table of values, or verbal description. Find vertical and horizontal asymptotes. End behavior. Proportional relationships.
- **Comparisons between functions**
  Exponential functions dominate power and linear functions.
- **Continuity**
  Interpret graphically and numerically. Intermediate Value Theorem.
- **Limits**
  Graphical interpretation, ϵ-δ definition, properties, one-sided limits, limits to infinity.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER ONE

Exercises

Find formulas for the functions described in Exercises 1–8.

1. A line with slope 2 and x-intercept 5.

2. A parabola opening downward with its vertex at $(2, 5)$.

3. A parabola with x-intercepts ± 1 and y-intercept 3.

4. The bottom half of a circle centered at the origin and with radius $\sqrt{2}$.

5. The top half of a circle with center $(-1, 2)$ and radius 3.

6. A cubic polynomial having x-intercepts at 1, 5, 7.

7. A rational function of the form $y = ax/(x + b)$ with a vertical asymptote at $x = 2$ and a horizontal asymptote of $y = -5$.

8. A cosine curve with a maximum at $(0, 5)$, a minimum at $(\pi, -5)$, and no maxima or minima in between.

9. The pollutant PCB (polychlorinated biphenyl) affects the thickness of pelican eggs. Thinking of the thickness, T, of the eggs, in mm, as a function of the concentration, P, of PCBs in ppm (parts per million), we have $T = f(P)$. Explain the meaning of $f(200)$ in terms of thickness of pelican eggs and concentration of PCBs.

10. In a California town, the monthly charge for a waste collection is \$8 for 32 gallons of waste and \$12.32 for 68 gallons of waste.

(a) Find a linear formula for the cost, C, of waste collection as a function of the number of gallons of waste, w.

(b) What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.

(c) What is the vertical intercept of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.

11. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a $950 refrigerator depreciates completely in seven years, find a formula for its value as a function of time.

12. Table 1.20 shows some values of a linear function f and an exponential function g. Find exact values (not decimal approximations) for each of the missing entries.

Table 1.20

x	0	1	2	3	4
$f(x)$	10	?	20	?	?
$g(x)$	10	?	20	?	?

13. **(a)** Write an equation for a graph obtained by vertically stretching the graph of $y = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
(b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
(c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

14. How many distinct roots can a polynomial of degree 5 have? (List all possibilities.) Sketch a possible graph for each case.

15. A rational function $y = f(x)$ is graphed in Figure 1.94. If $f(x) = g(x)/h(x)$ with $g(x)$ and $h(x)$ both quadratic functions, give possible formulas for $g(x)$ and $h(x)$.

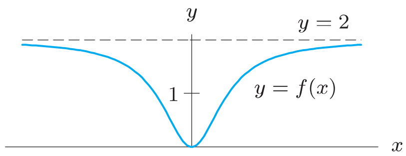


Figure 1.94

16. Find a calculator window in which the graphs of $f(x) = x^3 + 1000x^2 + 1000$ and $g(x) = x^3 - 1000x^2 - 1000$ appear indistinguishable.

17. The table gives the average temperature in Wallingford, Connecticut, for the first 10 days in March.
(a) Over which intervals was the average temperature increasing? Decreasing?
(b) Find a pair of consecutive intervals over which the average temperature was increasing at a decreasing rate. Find another pair of consecutive intervals over which the average temperature was increasing at an increasing rate.

Day	1	2	3	4	5	6	7	8	9	10
°F	42°	42°	34°	25°	22°	34°	38°	40°	49°	49°

18. The entire graph of $f(x)$ is shown in Figure 1.95.
(a) What is the domain of $f(x)$?
(b) What is the range of $f(x)$?
(c) List all zeros of $f(x)$.
(d) List all intervals on which $f(x)$ is decreasing.
(e) Is $f(x)$ concave up or concave down at $x = 6$?
(f) What is $f(4)$?
(g) Is this function invertible? Explain.

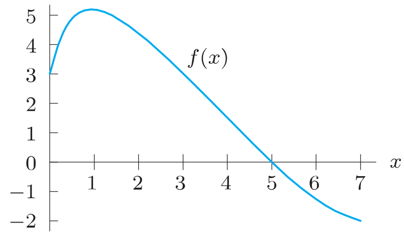


Figure 1.95

19. Use Figure 1.96 to estimate the following limits, if they exist.
(a) $\lim_{x\to 1^-} f(x)$ **(b)** $\lim_{x\to 1^+} f(x)$ **(c)** $\lim_{x\to 1} f(x)$
(d) $\lim_{x\to 2^-} f(x)$ **(e)** $\lim_{x\to 2^+} f(x)$ **(f)** $\lim_{x\to 2} f(x)$

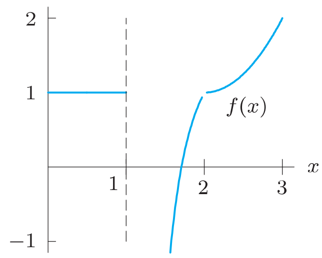


Figure 1.96

20. Use Figure 1.97 to graph each of the following. Label any intercepts or asymptotes that can be determined.
(a) $y = f(x) + 3$ **(b)** $y = 2f(x)$
(c) $y = f(x + 4)$ **(d)** $y = 4 - f(x)$

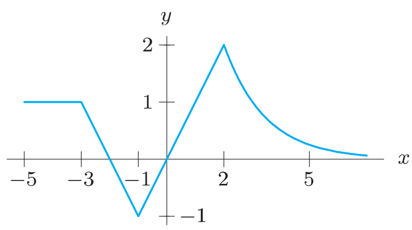


Figure 1.97

21. For each function, fill in the blanks in the statements:
$f(x) \to$ ________ as $x \to -\infty$,
$f(x) \to$ ________ as $x \to +\infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$

(b) $f(x) = \dfrac{3x^2 - 5x + 2}{2x^2 - 8}$

(c) $f(x) = e^x$

For Exercises 22–25, solve for t using logs.

22. $5^t = 7$ **23.** $2 = (1.02)^t$ **24.** $7 \cdot 3^t = 5 \cdot 2^t$

25. $5.02(1.04)^t = 12.01(1.03)^t$

In Exercises 26–27, put the functions in the form $P = P_0 e^{kt}$.

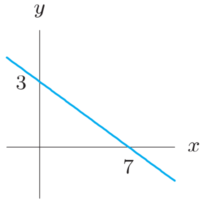
26. $P = P_0 2^t$ **27.** $P = 5.23(0.2)^t$

Find the amplitudes and periods in Exercises 28–29.

28. $y = 5\sin(x/3)$ **29.** $y = 4 - 2\cos(5x)$

Find possible formulas for the graphs in Exercises 30–43.

30.



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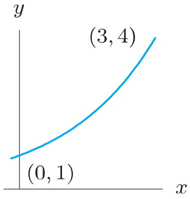

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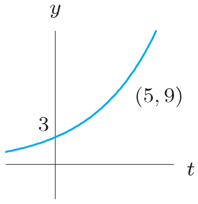

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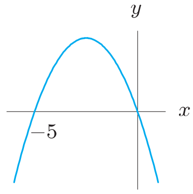

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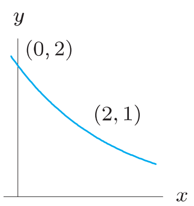

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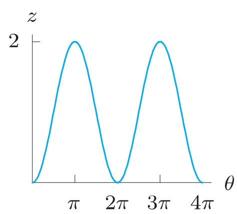

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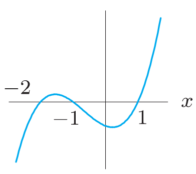

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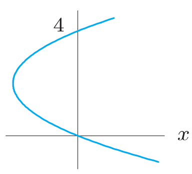

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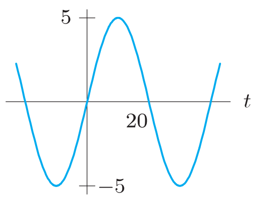

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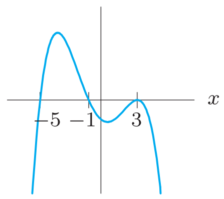

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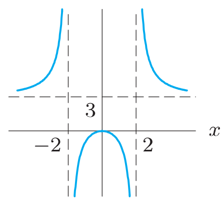

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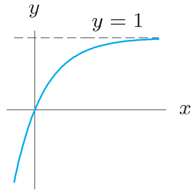

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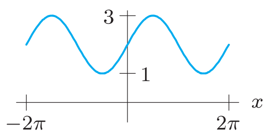

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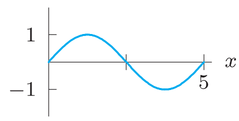

For Exercises 44–45, find functions f and g such that $h(x) = f(g(x))$. [Note: Do not choose $f(x) = x$ or $g(x) = x$.]

44. $h(x) = \ln(x^3)$ **45.** $h(x) = (\ln x)^3$

Are the functions in Exercises 46–47 continuous on $[-1, 1]$?

46. $g(x) = \dfrac{1}{x^2 + 1}$ **47.** $h(x) = \dfrac{1}{1 - x^2}$

For the functions in Exercises 48–49, use algebra to evaluate the limits $\lim\limits_{x\to a+} f(x)$, $\lim\limits_{x\to a-} f(x)$, and $\lim\limits_{x\to a} f(x)$ if they exist. Sketch a graph to confirm your answers.

48. $a = 3, \quad f(x) = \dfrac{x^3|2x - 6|}{x - 3}$

49. $a = 0, \quad f(x) = \begin{cases} e^x & -1 < x < 0 \\ 1 & x = 0 \\ \cos x & 0 < x < 1 \end{cases}$

Problems

50. Match the functions $h(s)$, $f(s)$, and $g(s)$, whose values are in Table 1.21, with the formulas

$$y = a(1.1)^s, \quad y = b(1.05)^s, \quad y = c(1.03)^s,$$

assuming a, b, and c are constants. Note that the function values have been rounded to two decimal places.

Table 1.21

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

51. Complete the following table with values for the functions f, g, and h, given that:

(a) f is an even function.
(b) g is an odd function.
(c) h is the composition $h(x) = g(f(x))$.

x	$f(x)$	$g(x)$	$h(x)$
-3	0	0	
-2	2	2	
-1	2	2	
0	0	0	
1			
2			
3			

52. If $f(x) = a\ln(x+2)$, how does increasing a affect

(a) The y-intercept? **(b)** The x-intercept?

53. **(a)** How does the parameter A affect the graph of $y = A\sin(Bx)$? (Plot for $A = 1, 2, 3$ with $B = 1$.)
(b) How does the parameter B affect the graph of $y = A\sin(Bx)$? (Plot for $B = 1, 2, 3$ with $A = 1$.)

54. Find k so that the following function is continuous on any interval:

$$g(t) = \begin{cases} t + k & t \le 5 \\ kt & 5 < t \end{cases}$$

55. Find k so that the following function is continuous on any interval:

$$h(x) = \begin{cases} k\cos x & 0 \le x \le \pi \\ 12 - x & \pi < x \end{cases}$$

56. The yield, Y, of an apple orchard (in bushels) as a function of the amount, a, of fertilizer (in pounds) used on the orchard is shown in Figure 1.98.

(a) Describe the effect of the amount of fertilizer on the yield of the orchard.
(b) What is the vertical intercept? Explain what it means in terms of apples and fertilizer.
(c) What is the horizontal intercept? Explain what it means in terms of apples and fertilizer.
(d) What is the range of this function for $0 \le a \le 80$?
(e) Is the function increasing or decreasing at $a = 60$?
(f) Is the graph concave up or down near $a = 40$?

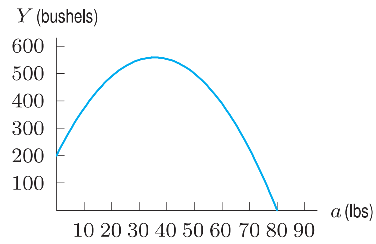

Figure 1.98

57. The table gives the average weight, w, in pounds, of American men in their sixties for height, h, in inches.[13]

(a) How do you know that the data in this table could represent a linear function?
(b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?
(c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

h (inches)	68	69	70	71	72	73	74	75
w (pounds)	166	171	176	181	186	191	196	201

58. A rock is dropped from a window and falls to the ground below. The height, s (in meters), of the rock above ground is a function of the time, t (in seconds), since the rock was dropped, so $s = f(t)$.

(a) Sketch a possible graph of s as a function of t.
(b) Explain what the statement $f(7) = 12$ tells us about the rock's fall.
(c) The graph drawn as the answer for part (a) should have a horizontal and vertical intercept. Interpret each intercept in terms of the rock's fall.

[13] Adapted from "Average Weight of Americans by Height and Age," *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 956.

59. A culture of 100 bacteria doubles after 2 hours. How long will it take for the number of bacteria to reach 3,200?

60. Different isotopes (versions) of the same element can have very different half-lives. With t in years, the decay of plutonium-240 is described by the formula

$$Q = Q_0 e^{-0.00011t},$$

whereas the decay of plutonium-242 is described by

$$Q = Q_0 e^{-0.0000018t}.$$

Find the half-lives of plutonium-240 and plutonium-242.

61. A culture of bacteria originally numbers 500. After 2 hours there are 1500 bacteria in the culture. Assuming exponential growth, how many are there after 6 hours?

62. The population of the US was 248.7 million in 1990 and 281.4 million in 2000. Assuming exponential growth,

(a) In what year is the population expected to go over 300 million?
(b) What population is predicted for the 2010 census?

63. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a continuous rate of approximately 2.47% per year. After the Chernobyl disaster, it was suggested that it would be about 100 years before the region would again be safe for human habitation. What percent of the original strontium-90 would still remain then?

64. In the early 1920s, Germany had tremendously high inflation, called hyperinflation. Photographs of the time show people going to the store with wheelbarrows full of money. If a loaf of bread cost $1/4$ RM in 1919 and 2,400,000 RM in 1922, what was the average yearly inflation rate between 1919 and 1922?

65. An airplane uses a fixed amount of fuel for takeoff, a (different) fixed amount for landing, and a third fixed amount per mile when it is in the air. How does the total quantity of fuel required depend on the length of the trip? Write a formula for the function involved. Explain the meaning of the constants in your formula.

66. A closed cylindrical can of fixed volume V has radius r.

(a) Find the surface area, S, as a function of r.
(b) What happens to the value of S as $r \to \infty$?
(c) Sketch a graph of S against r, if $V = 10$ cm^3.

67. On the graph of $y = \sin x$, points P and Q are at consecutive lowest and highest points. Find the slope of the line through P and Q.

68. The depth of water in a tank oscillates sinusoidally once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

69. The voltage, V, of an electrical outlet in a home as a function of time, t (in seconds), is $V = V_0 \cos(120\pi t)$.

(a) What is the period of the oscillation?
(b) What does V_0 represent?
(c) Sketch the graph of V against t. Label the axes.

70. Figure 1.99 shows $f(t)$, the number (in millions) of motor vehicles registered[14] in the world in the year t.

(a) Is f invertible? Explain.
(b) What is the meaning of $f^{-1}(400)$ in practical terms? Evaluate $f^{-1}(400)$.
(c) Sketch the graph of f^{-1}.

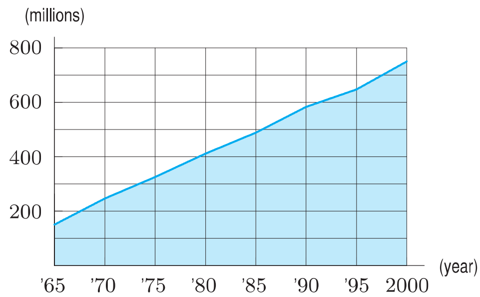

Figure 1.99

71. Each of the functions in the table is increasing over its domain, but each increases in a different way. Match the functions f, g, h to the graphs in Figure 1.100.

x	$f(x)$
1	1
2	2
4	3
7	4
11	5
16	6
22	7
29	8
37	9
47	10

x	$g(x)$
3.0	1
3.2	2
3.4	3
3.6	4
3.8	5
4.0	6
4.2	7
4.4	8
4.6	9
4.8	10

x	$h(x)$
10	1
20	2
28	3
34	4
39	5
43	6
46.5	7
49	8
51	9
52	10

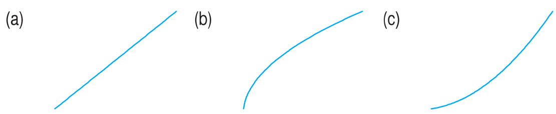

Figure 1.100

[14] www.earth-policy.org, accessed May 18, 2007. In 2000, about 30% of the registered vehicles were in the US.

72. **(a)** Use a graphing calculator or computer to estimate the period of $2\sin\theta + 3\cos(2\theta)$.
(b) Explain your answer, given that the period of $\sin\theta$ is 2π and the period of $\cos(2\theta)$ is π.

For the functions in Exercises 73–74, do the following:

(a) Make a table of values of $f(x)$ for $x = a+0.1, a+0.01, a+0.001, a+0.0001, a-0.1, a-0.01, a-0.001$, and $a-0.0001$.
(b) Make a conjecture about the value of $\lim_{x\to a} f(x)$.
(c) Graph the function to see if it is consistent with your answers to parts (a) and (b).
(d) Find an interval for x containing a such that the difference between your conjectured limit and the value of the function is less than 0.01 on that interval. (In other words, find a window of height 0.02 such that the graph exits the sides of the window and not the top or bottom of the window.)

73. $f(x) = \dfrac{\cos 2x - 1 + 2x^2}{x^3}, \quad a = 0$

74. $f(x) = \dfrac{\cos 3x - 1 + 4.5x^2}{x^3}, \quad a = 0$

For each value of ϵ in Problems 75–76, find a positive value of δ such that the graph of the function leaves the window $a-\delta < x < a+\delta$, $b-\epsilon < y < b+\epsilon$ by the sides and not through the top or bottom.

75. $h(x) = \sin x$, $a = b = 0$, $\epsilon = 0.1, 0.05, 0.0007$.

76. $k(x) = \cos x$, $a = 0$, $b = 1$, $\epsilon = 0.1, 0.001, 0.00001$.

CAS Challenge Problems

77. **(a)** Factor $f(x) = x^4 + bx^3 - cx^3 - a^2x^2 - bcx^2 - a^2bx + a^2cx + a^2bc$ using a computer algebra system.
(b) Assuming a, b, c are constants with $0 < a < b < c$, use your answer to part (a) to make a hand sketch of the graph of f. Explain how you know its shape.

78. **(a)** Using a computer algebra system, factor $f(x) = -x^5 + 11x^4 - 46x^3 + 90x^2 - 81x + 27$.
(b) Use your answer to part (a) to make a hand sketch of the graph of f. Explain how you know its shape.

79. Let $f(x) = e^{6x} + e^{5x} - 2e^{4x} - 10e^{3x} - 8e^{2x} + 16e^{x} + 16$.
(a) What happens to the value of $f(x)$ as $x \to \infty$? As $x \to -\infty$? Explain your answer.
(b) Using a computer algebra system, factor $f(x)$ and predict the number of zeros of the function $f(x)$.
(c) What are the exact values of the zeros? What is the relationship between successive zeros?

80. Let $f(x) = x^2 - x$.
(a) Find the polynomials $f(f(x))$ and $f(f(f(x)))$ in expanded form.
(b) What do you expect to be the degree of the polynomial $f(f(f(f(f(f(x))))))$? Explain.

81. **(a)** Use a computer algebra system to rewrite the rational function
$$f(x) = \frac{x^3 - 30}{x - 3}$$
in the form
$$f(x) = p(x) + \frac{r(x)}{q(x)},$$
where $p(x), q(x), r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $q(x)$.
(b) What is the vertical asymptote of f? Use your answer to part (a) to write the formula for a function whose graph looks like the graph of f for x near the vertical asymptote.
(c) Use your answer to part (a) to write the formula for a function whose graph looks like the graph of f for $x \to \infty$ and $x \to -\infty$.
(d) Using graphs, confirm the asymptote you found in part (b) and the formula you found in part (c).

We say that a function can be written as a polynomial in $\sin x$ (or $\cos x$) if it is of the form $p(\sin x)$ (or $p(\cos x)$) for some polynomial $p(x)$. For example, $\cos 2x$ can be written as a polynomial in $\sin x$ because $\cos(2x) = 1 - 2\sin^2 x = p(\sin x)$, where $p(x) = 1 - 2x^2$.

82. Use the trigonometric capabilities of your computer algebra system to express $\sin(5x)$ as a polynomial in $\sin x$.

83. Use the trigonometric capabilities of your computer algebra system to express $\cos(4x)$ as a polynomial in
(a) $\sin x$
(b) $\cos x$.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–34 true or false? Give an explanation for your answer.

1. For any two points in the plane, there is a linear function whose graph passes through them.
2. The graph of $f(x) = 100(10^x)$ is a horizontal shift of the graph of $g(x) = 10^x$.
3. The graph of $f(x) = \ln x$ is concave down.
4. The graph of $g(x) = \log(x - 1)$ crosses the x-axis at $x = 1$.

5. Every polynomial of odd degree has at least one zero.

6. The function $y = 2 + 3e^{-t}$ has a y-intercept of $y = 3$.

7. The function $y = 5 - 3e^{-4t}$ has a horizontal asymptote of $y = 5$.

8. If $y = f(x)$ is a linear function, then increasing x by 1 unit changes the corresponding y by m units, where m is the slope.

9. If $y = f(x)$ is an exponential function and if increasing x by 1 increases y by a factor of 5, then increasing x by 2 increases y by a factor of 10.

10. If $y = Ab^x$ and increasing x by 1 increases y by a factor of 3, then increasing x by 2 increases y by a factor of 9.

11. The function $f(\theta) = \cos\theta - \sin\theta$ is increasing on $0 \le \theta \le \pi/2$.

12. The function $f(t) = \sin(0.05\pi t)$ has period 0.05.

13. If t is in seconds, $g(t) = \cos(200\pi t)$ executes 100 cycles in one second.

14. The function $f(\theta) = \tan(\theta - \pi/2)$ is not defined at $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$.

15. $\sin|x| = \sin x$ for $-2\pi < x < 2\pi$

16. $\sin|x| = |\sin x|$ for $-2\pi < x < 2\pi$

17. $\cos|x| = |\cos x|$ for $-2\pi < x < 2\pi$

18. $\cos|x| = \cos x$ for $-2\pi < x < 2\pi$

19. The function $f(x) = \sin(x^2)$ is periodic, with period 2π.

20. The function $g(\theta) = e^{\sin\theta}$ is periodic.

21. If $f(x)$ is a periodic function with period k, then $f(g(x))$ is periodic with period k for every function $g(x)$.

22. If $g(x)$ is a periodic function, then $f(g(x))$ is periodic for every function $f(x)$.

23. The function $f(x) = e^{-x^2}$ is decreasing for all x.

24. The inverse function of $y = \log x$ is $y = 1/\log x$.

25. If f is an increasing function, then f^{-1} is an increasing function.

26. The function $f(x) = |\sin x|$ is even.

27. If a function is even, then it does not have an inverse.

28. If a function is odd, then it does not have an inverse function.

29. If a and b are positive constants, $b \ne 1$, then $y = a + ab^x$ has a horizontal asymptote.

30. If a and b are positive constants, then $y = \ln(ax + b)$ has no vertical asymptote.

31. The function $y = 20/(1 + 2e^{-kt})$ with $k > 0$, has a horizontal asymptote at $y = 20$.

32. If $g(x)$ is an even function then $f(g(x))$ is even for every function $f(x)$.

33. If $f(x)$ is an even function then $f(g(x))$ is even for every function $g(x)$.

34. If $\lim_{h\to 0} f(h) = L$, then $f(0.0001)$ is closer to L than is $f(0.01)$.

In Problems 35–40, give an example of a function with the specified properties. Express your answer using formulas.

35. Continuous on $[0, 1]$ but not continuous on $[1, 3]$.

36. Increasing but not continuous on $[0, 10]$.

37. Has a vertical asymptote at $x = -7\pi$.

38. Has exactly 17 vertical asymptotes.

39. Has a vertical asymptote which is crossed by a horizontal asymptote.

40. Two functions $f(x)$ and $g(x)$ such that moving the graph of f to the left 2 units gives the graph of g and moving the graph of f up 3 also gives the graph of g.

Suppose f is an increasing function and g is a decreasing function. In Problems 41–44, give an example for f and g for which the statement is true, or say why such an example is impossible.

41. $f(x) + g(x)$ is decreasing for all x.

42. $f(x) - g(x)$ is decreasing for all x.

43. $f(x)g(x)$ is decreasing for all x.

44. $f(g(x))$ is increasing for all x.

Are the statements in Problems 45–54 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

45. Every rational function that is not a polynomial has a vertical asymptote.

46. If a function is increasing on an interval, then it is concave up on that interval.

47. If y is a linear function of x, then the ratio y/x is constant for all points on the graph at which $x \ne 0$.

48. An exponential function can be decreasing.

49. If $y = f(x)$ is a linear function, then increasing x by 2 units adds $m + 2$ units to the corresponding y, where m is the slope.

50. If a function is not continuous at a point, then it is not defined at that point.

51. If f is continuous on the interval $[0, 10]$ and $f(0) = 0$ and $f(10) = 100$, then $f(c)$ cannot be negative for c in $[0, 10]$.

52. If $f(x)$ is not continuous on the interval $[a, b]$, then $f(x)$ must omit at least one value between $f(a)$ and $f(b)$.

53. There is a function which is both even and odd.

54. If $\lim_{x\to c^+}(f(x) + g(x))$ exists, then $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^+} g(x)$ must exist.

Suppose that $\lim_{x\to 3} f(x) = 7$. Are the statements in Problems 55–61 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

55. $\lim_{x\to 3}(xf(x)) = 21$.

56. If $g(3) = 4$, then $\lim_{x\to 3}(f(x)g(x)) = 28$.

57. If $\lim_{x\to 3} g(x) = 5$, then $\lim_{x\to 3}(f(x) + g(x)) = 12$.

58. If $\lim_{x\to 3}(f(x) + g(x)) = 12$, then $\lim_{x\to 3} g(x) = 5$.

59. $f(2.99)$ is closer to 7 than $f(2.9)$ is.

60. If $f(3.1) > 0$, then $f(3.01) > 0$.

61. If $\lim_{x\to 3} g(x)$ does not exist, then $\lim_{x\to 3}(f(x)g(x))$ does not exist.

Which of the statements in Problems 62–66 are true about every function $f(x)$ such that $\lim\limits_{x\to c} f(x) = L$? Give a reason for your answer.

62. If $f(x)$ is within 10^{-3} of L, then x is within 10^{-3} of c.

63. There is a positive ϵ such that, provided x is within 10^{-3} of c, and $x \neq c$, we can be sure $f(x)$ is within ϵ of L.

64. For any positive ϵ, we can find a positive δ such that, provided x is within δ of c, and $x \neq c$, we can be sure that $f(x)$ is within ϵ of L.

65. For each $\epsilon > 0$, there is a $\delta > 0$ such that if x is not within δ of c, then $f(x)$ is not within ϵ of L.

66. For each $\epsilon > 0$, there is some $\delta > 0$ such that if $f(x)$ is within ϵ of L, then we can be sure that x is within δ of c.

67. Which of the following statements is a direct consequence of the statement: "If f and g are continuous at $x = a$ and $g(a) \neq 0$ then f/g is continuous at $x = a$?"

(a) If f and g are continuous at $x = a$ and $f(a) \neq 0$ then g/f is continuous at $x = a$.
(b) If f and g are continuous at $x = a$ and $g(a) = 0$, then f/g is not continuous at $x = a$.
(c) If f, g, are continuous at $x = a$, but f/g is not continuous at $x = a$, then $g(a) = 0$.
(d) If f and f/g are continuous at $x = a$ and $g(a) \neq 0$, then g is continuous at $x = a$.

PROJECTS FOR CHAPTER ONE

1. Matching Functions to Data

From the data in Table 1.22, determine a possible formula for each function.[15] Write an explanation of your reasoning.

Table 1.22

x	$f(x)$	$g(x)$	$h(x)$	$F(x)$	$G(x)$	$H(x)$
−5	−10	20	25	0.958924	0.544021	2.958924
−4.5	− 9	19	20.25	0.97753	−0.412118	2.97753
−4	− 8	18	16	0.756802	−0.989358	2.756802
−3.5	− 7	17	12.25	0.350783	−0.656987	2.350783
−3	− 6	16	9	−0.14112	0.279415	1.85888
−2.5	− 5	15	6.25	−0.598472	0.958924	1.401528
−2	− 4	14	4	−0.909297	0.756802	1.090703
−1.5	− 3	13	2.25	−0.997495	−0.14112	1.002505
−1	− 2	12	1	−0.841471	−0.909297	1.158529
−0.5	− 1	11	0.25	−0.479426	−0.841471	1.520574
0	0	10	0	0	0	2
0.5	1	9	0.25	0.479426	0.841471	2.479426
1	2	8	1	0.841471	0.909297	2.841471
1.5	3	7	2.25	0.997495	0.14112	2.997495
2	4	6	4	0.909297	−0.756802	2.909297
2.5	5	5	6.25	0.598472	−0.958924	2.598472
3	6	4	9	0.14112	−0.279415	2.14112
3.5	7	3	12.25	−0.350783	0.656987	1.649217
4	8	2	16	−0.756802	0.989358	1.243198
4.5	9	1	20.25	−0.97753	0.412118	1.02247
5	10	0	25	−0.958924	−0.544021	1.041076

[15] Based on a problem by Lee Zia

2. Which Way is the Wind Blowing?

Mathematicians name a wind by giving the angle *toward* which it is blowing measured counterclockwise from east. Meteorologists give the angle *from* which it is blowing measured clockwise from north. Both use values from 0° to 360°. Figure 1.101 shows the two angles for a wind blowing from the northeast.

(a) Graph the mathematicians' angle θ_{math} as a function of the meteorologists' angle θ_{met}.
(b) Find a piecewise formula that gives θ_{math} in terms of θ_{met}.

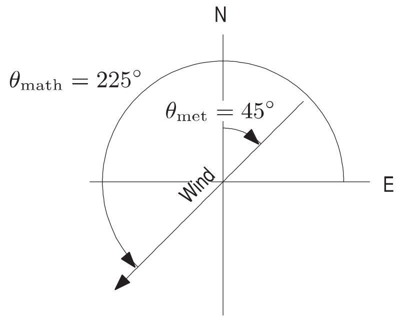

Figure 1.101