Chapter Two

KEY CONCEPT: THE DERIVATIVE

Contents

2.1 HOW DO WE MEASURE SPEED?

The speed of an object at an instant in time is surprisingly difficult to define precisely. Consider the statement: "At the instant it crossed the finish line, the horse was traveling at 42 mph." How can such a claim be substantiated? A photograph taken at that instant will show the horse motionless—it is no help at all. There is some paradox in trying to study the horse's motion at a particular instant in time, since by focusing on a single instant we stop the motion!

Problems of motion were of central concern to Zeno and other philosophers as early as the fifth century B.C. The modern approach, made famous by Newton's calculus, is to stop looking for a simple notion of speed at an instant, and instead to look at speed over small time intervals containing the instant. This method sidesteps the philosophical problems mentioned earlier but introduces new ones of its own.

We illustrate the ideas discussed above by an idealized example, called a thought experiment. It is idealized in the sense that we assume that we can make measurements of distance and time as accurately as we wish.

A Thought Experiment: Average and Instantaneous Velocity

We look at the speed of a small object (say, a grapefruit) that is thrown straight upward into the air at $t = 0$ seconds. The grapefruit leaves the thrower's hand at high speed, slows down until it reaches its maximum height, and then speeds up in the downward direction and finally, "Splat!" (See Figure 2.1.)

Suppose that we want to determine the speed, say, at $t = 1$ second. Table 2.1 gives the height, y, of the grapefruit above the ground as a function of time. During the first second the grapefruit travels $90 - 6 = 84$ feet, and during the second second it travels only $142 - 90 = 52$ feet. Hence the grapefruit traveled faster over the first interval, $0 \leq t \leq 1$, than the second interval, $1 \leq t \leq 2$.
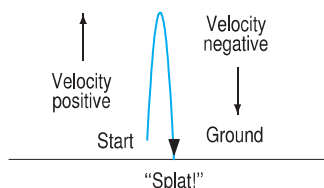

Figure 2.1: The grapefruit's path is straight up and down

Table 2.1   *Height of the grapefruit above the ground*

t (sec)	0	1	2	3	4	5	6
y (feet)	6	90	142	162	150	106	30

Velocity versus Speed

From now on, we will distinguish between velocity and speed. Suppose an object moves along a line. We pick one direction to be positive and say that the *velocity* is positive if it is in the same direction, and negative if it is in the opposite direction. For the grapefruit, upward is positive and downward is negative. (See Figure 2.1.) *Speed* is the magnitude of the velocity and so is always positive or zero.

If $s(t)$ is the position of an object at time t, then the **average velocity** of the object over the interval $a \leq t \leq b$ is

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b) - s(a)}{b - a}.$$

In words, the **average velocity** of an object over an interval is the net change in position during the interval divided by the change in time.

Example 1   Compute the average velocity of the grapefruit over the interval $4 \leq t \leq 5$. What is the significance of the sign of your answer?

Solution   During this interval, the grapefruit moves $(106 - 150) = -44$ feet. Therefore the average velocity is -44 ft/sec. The negative sign means the height is decreasing and the grapefruit is moving downward.

Example 2   Compute the average velocity of the grapefruit over the interval $1 \leq t \leq 3$.

Solution   Average velocity $= (162 - 90)/(3 - 1) = 72/2 = 36$ ft/sec.

The average velocity is a useful concept since it gives a rough idea of the behavior of the grapefruit: If two grapefruits are hurled into the air, and one has an average velocity of 10 ft/sec over the interval $0 \leq t \leq 1$ while the second has an average velocity of 100 ft/sec over the same interval, the second one is moving faster.

But average velocity over an interval does not solve the problem of measuring the velocity of the grapefruit at *exactly* $t = 1$ second. To get closer to an answer to that question, we have to look at what happens near $t = 1$ in more detail. The data[1] in Figure 2.2 shows the average velocity over small intervals on either side of $t = 1$.

Notice that the average velocity before $t = 1$ is slightly more than the average velocity after $t = 1$. We expect to define the velocity *at* $t = 1$ to be between these two average velocities. As the size of the interval shrinks, the values of the velocity before $t = 1$ and the velocity after $t = 1$ get closer together. In the smallest interval in Figure 2.2, both velocities are 68.0 ft/sec (to one decimal place), so we define the velocity at $t = 1$ to be 68.0 ft/sec (to one decimal place).
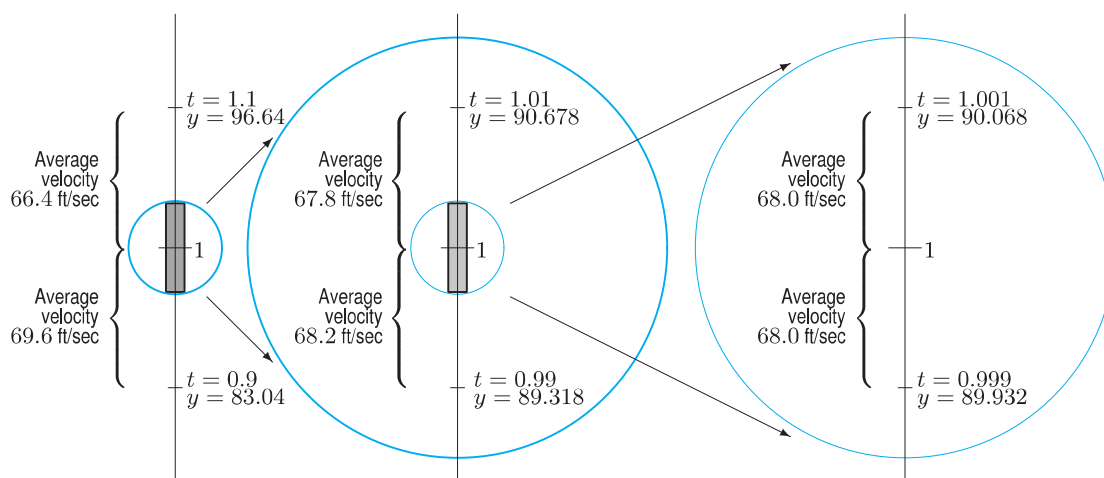
Figure 2.2: Average velocities over intervals on either side of $t = 1$: showing successively smaller intervals

Of course, if we calculate to more decimal places, the average velocities before and after $t = 1$ would no longer agree. To calculate the velocity at $t = 1$ to more decimal places of accuracy, we take smaller and smaller intervals on either side of $t = 1$ until the average velocities agree to the number of decimal places we want. In this way, we can estimate the velocity at $t = 1$ to any accuracy.

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[1]The data is in fact calculated from the formula $y = 6 + 100t - 16t^2$.

Defining Instantaneous Velocity Using Limit Notation

When we take smaller and smaller intervals, it turns out that the average velocities get closer and closer to 68 ft/sec. It seems natural, then, to define *instantaneous velocity* at the instant $t = 1$ to be 68 ft/sec. Its definition depends on our being convinced that smaller and smaller intervals give average speeds that come arbitrarily close to 68; that is, the average speeds approach 68 as a limit.

Notice how we have replaced the original difficulty of computing velocity at a point by a search for an argument to convince ourselves that the average velocities approach a limit as the time intervals shrink in size. Showing that the limit is exactly 68 requires the precise definition of limit given in Section 1.8.

To define instantaneous velocity at an arbitrary point $t = a$, we use the same method as for $t = 1$. On small intervals of size h around $t = a$, we calculate

$$\text{Average velocity} = \frac{s(a + h) - s(a)}{h}.$$

The instantaneous velocity is the number that the average velocities approach as the intervals decrease in size, that is, as h becomes smaller. So we make the following definition:

Let $s(t)$ be the position at time t. Then the **instantaneous velocity** at $t = a$ is defined as

$$\begin{array}{rcl} \text{Instantaneous velocity} \\ \text{at } t = a \end{array} \quad = \quad \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}.$$

In words, the **instantaneous velocity** of an object at time $t = a$ is given by the limit of the average velocity over an interval, as the interval shrinks around a.

This limit refers to the number that the average velocities approach as the intervals shrink. To estimate the limit, we look at intervals of smaller and smaller, but never zero, length.

Visualizing Velocity: Slope of Curve

Now we visualize velocity using a graph of height. The cornerstone of the idea is the fact that, on a very small scale, most functions look almost like straight lines. Imagine taking the graph of a function near a point and "zooming in" to get a close-up view. (See Figure 2.3.) The more we zoom in, the more the curve appears to be a straight line. We call the slope of this line the *slope of the curve* at the point.

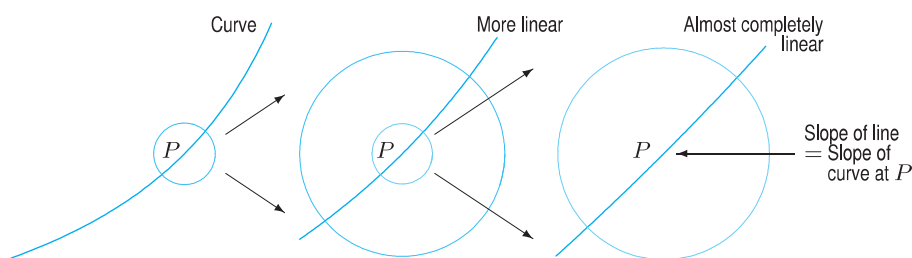**Figure 2.3:** Estimating the slope of the curve at the point by "zooming in"

To visualize the instantaneous velocity, we think about how we calculated it. We took average velocities over small intervals containing at $t = 1$. Two such velocities are represented by the slopes of the lines in Figure 2.4. As the length of the interval shrinks, the slope of the line gets closer to the slope of the curve at $t = 1$.
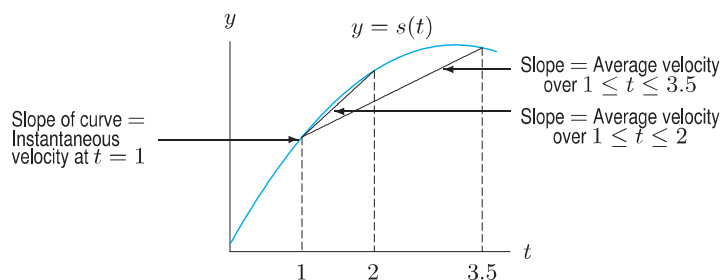
Figure 2.4: Average velocities over small intervals

The **instantaneous velocity** is the slope of the curve at a point.

Let's go back to the grapefruit. Figure 2.5 shows the height of the grapefruit plotted against time. (Note that this is not a picture of the grapefruit's path, which is straight up and down.)

How can we visualize the average velocity on this graph? Suppose $y = s(t)$. We consider the interval $1 \leq t \leq 2$ and the expression

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(2) - s(1)}{2 - 1} = \frac{142 - 90}{1} = 52 \text{ ft/sec.}$$

Now $s(2) - s(1)$ is the change in position over the interval, and it is marked vertically in Figure 2.5. The 1 in the denominator is the time elapsed and is marked horizontally in Figure 2.5. Therefore,

$$\text{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \text{Slope of line joining } B \text{ and } C.$$

(See Figure 2.5.) A similar argument shows the following:

The **average velocity** over any time interval $a \leq t \leq b$ is the slope of the line joining the points on the graph of $s(t)$ corresponding to $t = a$ and $t = b$.

Figure 2.5 shows how the grapefruit's velocity varies during its journey. At points A and B the curve has a large positive slope, indicating that the grapefruit is traveling up rapidly. Point D is almost at the top: the grapefruit is slowing down. At the peak, the slope of the curve is zero: the fruit has slowed to zero velocity for an instant in preparation for its return to earth. At point E the curve has a small negative slope, indicating a slow velocity of descent. Finally, the slope of the curve at point G is large and negative, indicating a large downward velocity that is responsible for the "Splat."
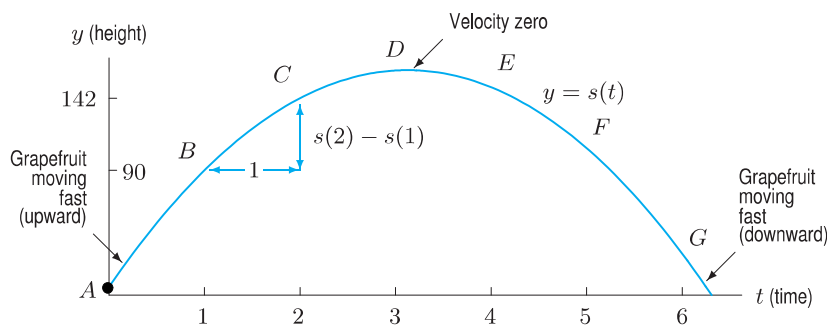
Figure 2.5: The height, y, of the grapefruit at time t

Using Limits to Compute the Instantaneous Velocity

Suppose we want to calculate the instantaneous velocity for $s(t) = t^2$ at $t = 3$. We must find:

$$\lim_{h \to 0} \frac{s(3+h) - s(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 9}{h}.$$

We show two possible approaches.

Example 3   Estimate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$ numerically.

Solution   The limit is the value approached by this expression as h approaches 0. The values in Table 2.2 seem to be converging to 6 as $h \to 0$. So it is a reasonable guess that

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = 6.$$

However, we cannot be sure that the limit is *exactly* 6 by looking at the table. To calculate the limit exactly requires algebra.

Table 2.2   *Values of $\left((3+h)^2 - 9\right)/h$ near $h = 0$*

h	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\left((3+h)^2 - 9\right)/h$	5.9	5.99	5.999	6.001	6.01	6.1

Example 4   Use algebra to find $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

Solution   Expanding the numerator gives

$$\frac{(3+h)^2 - 9}{h} = \frac{9 + 6h + h^2 - 9}{h} = \frac{6h + h^2}{h}.$$

Since taking the limit as $h \to 0$ means looking at values of h near, but not equal, to 0, we can cancel h, giving

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0}(6 + h).$$

As h approaches 0, the values of $(6 + h)$ approach 6, so

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0}(6 + h) = 6.$$

Exercises and Problems for Section 2.1

Exercises

1. The distance, s, a car has traveled on a trip is shown in the table as a function of the time, t, since the trip started. Find the average velocity between $t = 2$ and $t = 5$.

t (hours)	0	1	2	3	4	5
s (km)	0	45	135	220	300	400

2. At time t in seconds, a particle's distance $s(t)$, in cm, from a point is given in the table. What is the average velocity of the particle from $t = 3$ to $t = 10$?

t	0	3	6	10	13
$s(t)$	0	72	92	144	180

3. The table gives the position of a particle moving along the x-axis as a function of time in seconds, where x is in angstroms. What is the average velocity of the particle from $t = 2$ to $t = 8$?

t	0	2	4	6	8
$x(t)$	0	14	-6	-18	-4

4. Figure 2.6 shows a particle's distance from a point. What is the particle's average velocity from $t = 0$ to $t = 3$?
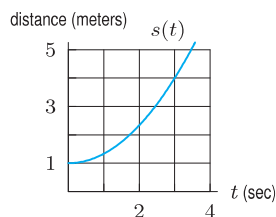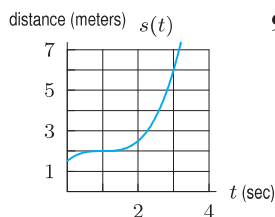

Figure 2.6

Figure 2.7

5. Figure 2.7 shows a particle's distance from a point. What is the particle's average velocity from $t = 1$ to $t = 3$?

6. At time t in seconds, a particle's distance $s(t)$, in centimeters, from a point is given by $s(t) = 4 + 3\sin t$. What is the average velocity of the particle from $t = \pi/3$ to $t = 7\pi/3$?

7. At time t in seconds, a particle's distance $s(t)$, in micrometers (μm), from a point is given by $s(t) = e^t - 1$. What is the average velocity of the particle from $t = 2$ to $t = 4$?

8. In a time of t seconds, a particle moves a distance of s meters from its starting point, where $s = 3t^2$.

   (a) Find the average velocity between $t = 1$ and $t = 1 + h$ if:

      (i) $h = 0.1$,   **(ii)** $h = 0.01$, **(iii)** $h = 0.001$.

   (b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

9. In a time of t seconds, a particle moves a distance of s meters from its starting point, where $s = 4t^2 + 3$.

   (a) Find the average velocity between $t = 1$ and $t = 1 + h$ if:

      (i) $h = 0.1$,   **(ii)** $h = 0.01$, **(iii)** $h = 0.001$.

   (b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

10. In a time of t seconds, a particle moves a distance of s meters from its starting point, where $s = \sin(2t)$.

   (a) Find the average velocity between $t = 1$ and $t = 1 + h$ if:

      (i) $h = 0.1$,   **(ii)** $h = 0.01$, **(iii)** $h = 0.001$.

   (b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $t = 1$.

Problems

Estimate the limits in Problems 11–14 by substituting smaller and smaller values of h. For trigonometric functions, use radians. Give answers to one decimal place.

11. $\displaystyle\lim_{h \to 0} \frac{(3 + h)^3 - 27}{h}$ **12.** $\displaystyle\lim_{h \to 0} \frac{\cos h - 1}{h}$

13. $\displaystyle\lim_{h \to 0} \frac{7^h - 1}{h}$ **14.** $\displaystyle\lim_{h \to 0} \frac{e^{1+h} - e}{h}$

15. Match the points labeled on the curve in Figure 2.8 with the given slopes.

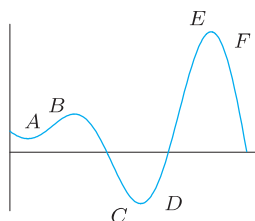| Slope	Point
-3	
-1	
0	
$1/2$	
1	
2	

Figure 2.8

16. For the function shown in Figure 2.9, at what labeled points is the slope of the graph positive? Negative? At which labeled point does the graph have the greatest (i.e., most positive) slope? The least slope (i.e., negative and with the largest magnitude)?
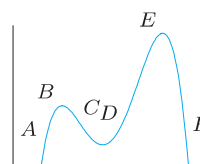

Figure 2.9

17. A car is driven at a constant speed. Sketch a graph of the distance the car has traveled as a function of time.

18. A car is driven at an increasing speed. Sketch a graph of the distance the car has traveled as a function of time.

19. A car starts at a high speed, and its speed then decreases slowly. Sketch a graph of the distance the car has traveled as a function of time.

20. For the graph $y = f(x)$ in Figure 2.10, arrange the following numbers from smallest to largest:

- The slope of the graph at A.
- The slope of the graph at B.
- The slope of the graph at C.
- The slope of the line AB.
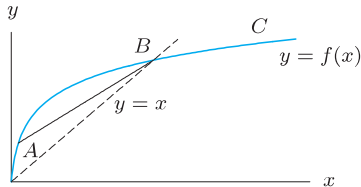- The number 0.
- The number 1.

Figure 2.10

21. The graph of $f(t)$ in Figure 2.11 gives the position of a particle at time t. List the following quantities in order, smallest to largest.

- A, average velocity between $t = 1$ and $t = 3$,
- B, average velocity between $t = 5$ and $t = 6$,
- C, instantaneous velocity at $t = 1$,
- D, instantaneous velocity at $t = 3$,
- E, instantaneous velocity at $t = 5$,
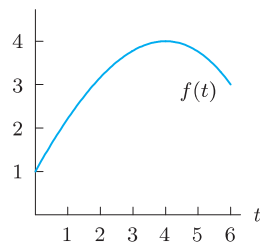- F, instantaneous velocity at $t = 6$.

Figure 2.11

22. A particle moves at varying velocity along a line and $s = f(t)$ represents the particle's distance from a point as a function of time, t. Sketch a possible graph for f if the average velocity of the particle between $t = 2$ and $t = 6$ is the same as the instantaneous velocity at $t = 5$.

23. Find the average velocity over the interval $0 \leq t \leq 0.2$, and estimate the velocity at $t = 0.2$ of a car whose position, s, is given by the following table.

t (sec)	0	0.2	0.4	0.6	0.8	1.0
s (ft)	0	0.5	1.8	3.8	6.5	9.6

24. A ball is tossed into the air from a bridge, and its height, y (in feet), above the ground t seconds after it is thrown is given by

$$y = f(t) = -16t^2 + 50t + 36.$$

(a) How high above the ground is the bridge?
(b) What is the average velocity of the ball for the first second?
(c) Approximate the velocity of the ball at $t = 1$ second.
(d) Graph f, and determine the maximum height the ball reaches. What is the velocity at the time the ball is at the peak?
(e) Use the graph to decide at what time, t, the ball reaches its maximum height.

Use algebra to evaluate the limits in Problems 25–28.

25. $\lim\limits_{h \to 0} \dfrac{(2 + h)^2 - 4}{h}$ **26.** $\lim\limits_{h \to 0} \dfrac{(1 + h)^3 - 1}{h}$

27. $\lim\limits_{h \to 0} \dfrac{3(2 + h)^2 - 12}{h}$

28. $\lim\limits_{h \to 0} \dfrac{(3 + h)^2 - (3 - h)^2}{2h}$

2.2   THE DERIVATIVE AT A POINT

Average Rate of Change

In Section 2.1, we looked at the change in height divided by the change in time, which tells us

$$\text{Average rate of change of height} \atop \text{with respect to time} = \frac{s(a + h) - s(a)}{h}.$$

This ratio is called the *difference quotient*. Now we apply the same analysis to any function f, not necessarily a function of time. We say:

$$\boxed{\text{Average rate of change of } f \atop \text{over the interval from } a \text{ to } a + h} = \frac{f(a + h) - f(a)}{h}.$$

The numerator, $f(a + h) - f(a)$, measures the change in the value of f over the interval from a to $a + h$. The difference quotient is the change in f divided by the change in x. See Figure 2.12.
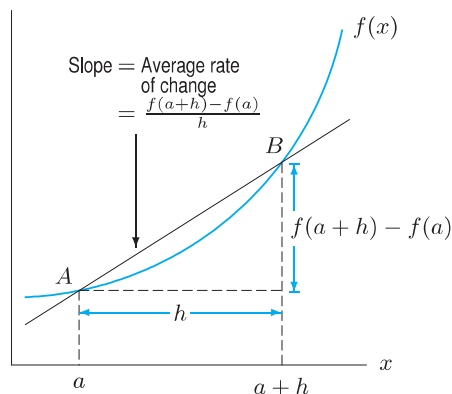

Figure 2.12: Visualizing the average rate of change of f

Although the interval is no longer necessarily a time interval, we still talk about the *average rate of change* of f over the interval. If we want to emphasize the independent variable, we talk about the average rate of change of f *with respect to x*.

Average Rate of Change versus Absolute Change

The average rate of change of a function over an interval is not the same as the absolute change. Absolute change is just the difference in the values of f at the ends of the interval:

$$f(a + h) - f(a).$$

The average rate of change is the absolute change divided by the size of the interval:

$$\frac{f(a + h) - f(a)}{h}.$$

The average rate of change tells how quickly (or slowly) the function changes from one end of the interval to the other, relative to the size of the interval. It is often more useful to know the rate of change than the absolute change. For example, if someone offers you a $100 salary, you will want to know how long to work to make that money. Just knowing the absolute change in your money, $100, is not enough, but knowing the rate of change (i.e., $100 divided by the time it takes to make it) helps you decide whether or not to accept the salary.

Blowing Up a Balloon

Consider the function which gives the radius of a sphere in terms of its volume. For example, think of blowing air into a balloon. You've probably noticed that a balloon seems to blow up faster at the start and then slows down as you blow more air into it. What you're seeing is variation in the rate of change of the radius with respect to volume.

Example 1   The volume, V, of a sphere of radius r is given by $V = 4\pi r^3/3$. Solving for r in terms of V gives

$$r = f(V) = \left(\frac{3V}{4\pi}\right)^{1/3}.$$

Calculate the average rate of change of r with respect to V over the intervals $0.5 \leq V \leq 1$ and $1 \leq V \leq 1.5$.

Solution   Using the formula for the average rate of change gives

$$\text{Average rate of change of radius for } 0.5 \le V \le 1 = \frac{f(1) - f(0.5)}{0.5} = 2\left(\left(\frac{3}{4\pi}\right)^{1/3} - \left(\frac{1.5}{4\pi}\right)^{1/3}\right) \approx 0.26.$$

$$\text{Average rate of change of radius for } 1 \le V \le 1.5 = \frac{f(1.5) - f(1)}{0.5} = 2\left(\left(\frac{4.5}{4\pi}\right)^{1/3} - \left(\frac{3}{4\pi}\right)^{1/3}\right) \approx 0.18.$$

So we see that the rate decreases as the volume increases.

Instantaneous Rate of Change: The Derivative

We define the *instantaneous rate of change* of a function at a point in the same way that we defined instantaneous velocity: we look at the average rate of change over smaller and smaller intervals. This instantaneous rate of change is called the *derivative of f at a*, denoted by $f'(a)$.

> The **derivative of f at a**, written $f'(a)$, is defined as
>
> $$\text{Rate of change of } f \text{ at } a = f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$
>
> If the limit exists, then f is said to be **differentiable at a**.

To emphasize that $f'(a)$ is the rate of change of $f(x)$ as the variable x changes, we call $f'(a)$ the derivative of f *with respect to* x at $x = a$. When the function $y = s(t)$ represents the position of an object, the derivative $s'(t)$ is the velocity.

Example 2   By choosing small values for h, estimate the instantaneous rate of change of the radius, r, of a sphere with respect to change in volume at $V = 1$.

Solution   The formula for $r = f(V)$ is given in Example 1. With $h = 0.01$ and $h = -0.01$, we have the difference quotients

$$\frac{f(1.01) - f(1)}{0.01} \approx 0.2061 \qquad \text{and} \qquad \frac{f(0.99) - f(1)}{-0.01} \approx 0.2075.$$

With $h = 0.001$ and $h = -0.001$,

$$\frac{f(1.001) - f(1)}{0.001} \approx 0.2067 \qquad \text{and} \qquad \frac{f(0.999) - f(1)}{-0.001} \approx 0.2069.$$

The values of these difference quotients suggest that the limit is between 0.2067 and 0.2069. We conclude that the value is about 0.207; taking smaller h values confirms this. So we say

$$f'(1) = \text{Instantaneous rate of change of radius with respect to volume at } V = 1 \approx 0.207.$$

In this example we found an approximation to the instantaneous rate of change, or derivative, by substituting in smaller and smaller values of h. Now we see how to visualize the derivative.

Visualizing the Derivative: Slope of Curve and Slope of Tangent

As with velocity, we can visualize the derivative $f'(a)$ as the slope of the graph of f at $x = a$. In addition, there is another way to think of $f'(a)$. Consider the difference quotient $(f(a + h) - f(a))/h$. The numerator, $f(a + h) - f(a)$, is the vertical distance marked in Figure 2.13 and h is the horizontal distance, so

$$\text{Average rate of change of } f = \frac{f(a + h) - f(a)}{h} = \text{Slope of line } AB.$$

As h becomes smaller, the line AB approaches the tangent line to the curve at A. (See Figure 2.14.) We say

$$\begin{array}{l} \text{Instantaneous rate of change} \\ \text{of } f \text{ at } a \end{array} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \text{Slope of tangent at } A.$$
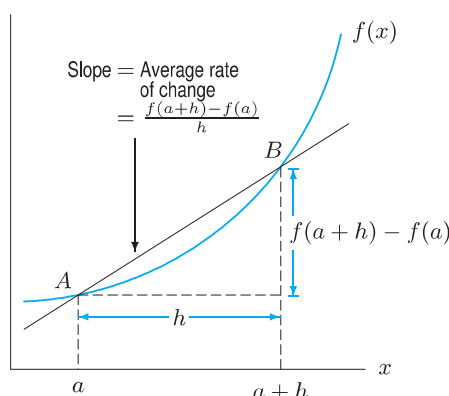

Figure 2.13: Visualizing the average rate of change of f

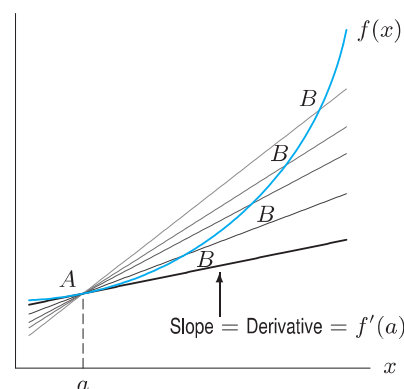**Figure 2.14:** Visualizing the instantaneous rate of change of f

The derivative at point A can be interpreted as:
- The slope of the curve at A.
- The slope of the tangent line to the curve at A.

The slope interpretation is often useful in gaining rough information about the derivative, as the following examples show.

Example 3   Is the derivative of $\sin x$ at $x = \pi$ positive or negative?

Solution   Looking at a graph of $\sin x$ in Figure 2.15 (remember, x is in radians), we see that a tangent line drawn at $x = \pi$ has negative slope. So the derivative at this point is negative.
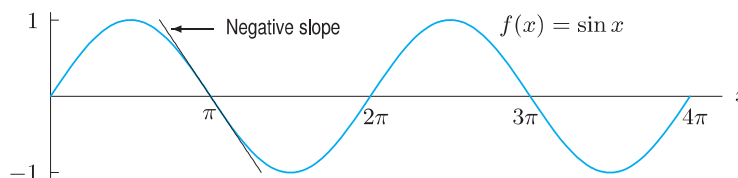

Figure 2.15: Tangent line to $\sin x$ at $x = \pi$

Recall that if we zoom in on the graph of a function $y = f(x)$ at the point $x = a$, we usually find that the graph looks like a straight line with slope $f'(a)$.

Example 4   By zooming in on the point $(0,0)$ on the graph of the sine function, estimate the value of the derivative of $\sin x$ at $x = 0$, with x in radians.

Solution   Figure 2.16 shows graphs of $\sin x$ with smaller and smaller scales. On the interval $-0.1 \leq x \leq 0.1$, the graph looks like a straight line of slope 1. Thus, the derivative of $\sin x$ at $x = 0$ is about 1.
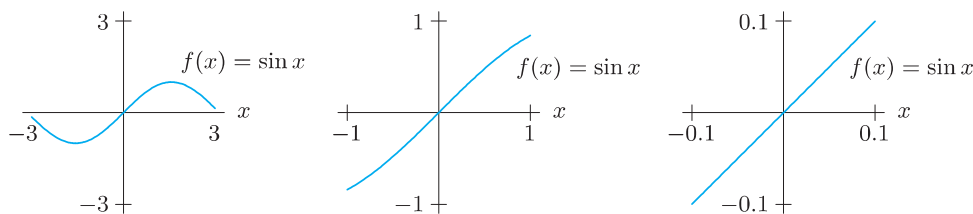
Figure 2.16: Zooming in on the graph of $\sin x$ near $x = 0$ shows the derivative is about 1 at $x = 0$

Later we will show that the derivative of $\sin x$ at $x = 0$ is exactly 1. (See page 143 in Section 3.5.) From now on we will assume that this is so.

Example 5   Use the tangent line at $x = 0$ to estimate values of $\sin x$ near $x = 0$.

Solution   In the previous example we see that near $x = 0$, the graph of $y = \sin x$ looks like the straight line $y = x$; we can use this line to estimate values of $\sin x$ when x is close to 0. For example, the point on the straight line $y = x$ with x coordinate 0.32 is $(0.32, 0.32)$. Since the line is close to the graph of $y = \sin x$, we estimate that $\sin 0.32 \approx 0.32$. (See Figure 2.17.) Checking on a calculator, we find that $\sin 0.32 \approx 0.3146$, so our estimate is quite close. Notice that the graph suggests that the real value of $\sin 0.32$ is slightly less than 0.32.
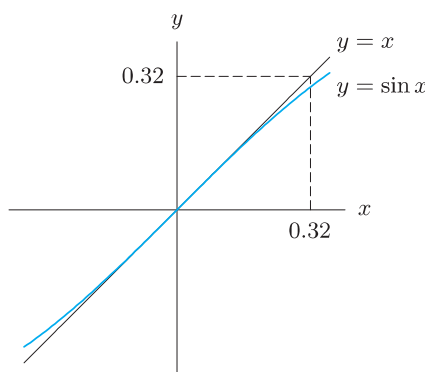
Figure 2.17: Approximating $y = \sin x$ by the line $y = x$

Why Do We Use Radians and Not Degrees?

After Example 4 we stated that the derivative of $\sin x$ at $x = 0$ is 1, when x is in radians. This is the reason we choose to use radians. If we had done Example 4 in degrees, the derivative of $\sin x$ would have turned out to be a much messier number. (See Problem 23, page 84.)

Estimating the Derivative of an Exponential Function

Example 6   Estimate the value of the derivative of $f(x) = 2^x$ at $x = 0$ graphically and numerically.

Solution   Graphically: Figure 2.18 indicates that the graph is concave up. Assuming this, the slope at A is between the slope of BA and the slope of AC. Since

$$\text{Slope of line } BA = \frac{(2^0 - 2^{-1})}{(0 - (-1))} = \frac{1}{2} \quad \text{and} \quad \text{Slope of line } AC = \frac{(2^1 - 2^0)}{(1 - 0)} = 1,$$

we know that at $x = 0$ the derivative of 2^x is between $1/2$ and 1.

Numerically: To estimate the derivative at $x = 0$, we look at values of the difference quotient

$$\frac{f(0 + h) - f(0)}{h} = \frac{2^h - 2^0}{h} = \frac{2^h - 1}{h}$$

for small h. Table 2.3 shows some values of 2^h together with values of the difference quotients. (See Problem 31 on page 85 for what happens for very small values of h.)
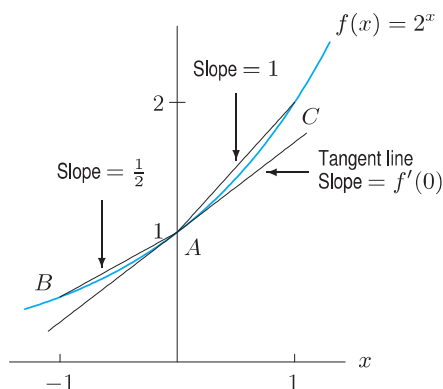

Figure 2.18: Graph of $y = 2^x$ showing the derivative at $x = 0$

Table 2.3 *Numerical values for difference quotient of 2^x at $x = 0$*

h	2^h	Difference quotient: $\frac{2^h - 1}{h}$
-0.0003	0.999792078	0.693075
-0.0002	0.999861380	0.693099
-0.0001	0.999930688	0.693123
0	1	
0.0001	1.00006932	0.693171
0.0002	1.00013864	0.693195
0.0003	1.00020797	0.693219

The concavity of the curve tells us that difference quotients calculated with negative h's are smaller than the derivative, and those calculated with positive h's are larger. From Table 2.3 we see that the derivative is between 0.693123 and 0.693171. To three decimal places, $f'(0) = 0.693$.

Example 7   Find an approximate equation for the tangent line to $f(x) = 2^x$ at $x = 0$.

Solution   From the previous example, we know the slope of the tangent line is about 0.693. Since the tangent line has y-intercept 1, its equation is

$$y = 0.693x + 1.$$

Computing the Derivative of $1/x$ at $x = 2$

The graph of $f(x) = 1/x$ in Figure 2.19 leads us to expect that $f'(2)$ is negative. To compute $f'(2)$ exactly, we use algebra.
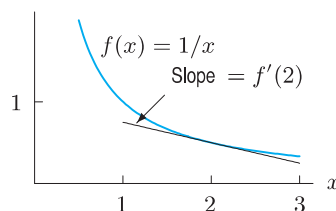

Figure 2.19: Tangent line to $f(x) = 1/x$ at $x = 2$

Example 8   Find the derivative of $f(x) = 1/x$ at the point $x = 2$.

Solution   The derivative is the limit of the difference quotient, so we look at

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}.$$

Using the formula for f and simplifying gives

$$f'(2) = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{2+h} - \frac{1}{2}\right) = \lim_{h \to 0}\left(\frac{2 - (2+h)}{2h(2+h)}\right) = \lim_{h \to 0} \frac{-h}{2h(2+h)}.$$

Since the limit only examines values of h close to, but not equal to, zero, we can cancel h. We get

$$f'(2) = \lim_{h \to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}.$$

Thus, $f'(2) = -1/4$. The slope of the tangent line in Figure 2.19 is $-1/4$.

Exercises and Problems for Section 2.2

Exercises

1. The table shows values of $f(x) = x^3$ near $x = 2$ (to three decimal places). Use it to estimate $f'(2)$.

x	1.998	1.999	2.000	2.001	2.002
x^3	7.976	7.988	8.000	8.012	8.024

2. (a) Make a table of values rounded to two decimal places for the function $f(x) = e^x$ for $x = 1, 1.5, 2, 2.5,$ and 3. Then use the table to answer parts (b) and (c).
   (b) Find the average rate of change of $f(x)$ between $x = 1$ and $x = 3$.
   (c) Use average rates of change to approximate the instantaneous rate of change of $f(x)$ at $x = 2$.

3. (a) Make a table of values, rounded to two decimal places, for $f(x) = \log x$ (that is, log base 10) with $x = 1, 1.5, 2, 2.5, 3$. Then use this table to answer parts (b) and (c).
   (b) Find the average rate of change of $f(x)$ between $x = 1$ and $x = 3$.
   (c) Use average rates of change to approximate the instantaneous rate of change of $f(x)$ at $x = 2$.

4. (a) Let $f(x) = x^2$. Explain what Table 2.4 tells us about $f'(1)$.
   (b) Find $f'(1)$ exactly.
   (c) If x changes by 0.1 near $x = 1$, what does $f'(1)$ tell us about how $f(x)$ changes? Illustrate your answer with a sketch.

Table 2.4

x	x^2	Difference in successive x^2 values
0.998	0.996004	
		0.001997
0.999	0.998001	
		0.001999
1.000	1.000000	
		0.002001
1.001	1.002001	
		0.002003
1.002	1.004004	

5. If $f(x) = x^3 + 4x$, estimate $f'(3)$ using a table with values of x near 3, spaced by 0.001.

6. Graph $f(x) = \sin x$, and use the graph to decide whether the derivative of $f(x)$ at $x = 3\pi$ is positive or negative.

7. For the function $f(x) = \log x$, estimate $f'(1)$. From the graph of $f(x)$, would you expect your estimate to be greater than or less than $f'(1)$?

8. Estimate $f'(2)$ for $f(x) = 3^x$. Explain your reasoning.

9. Figure 2.20 shows the graph of f. Match the derivatives in the table with the points a, b, c, d, e.
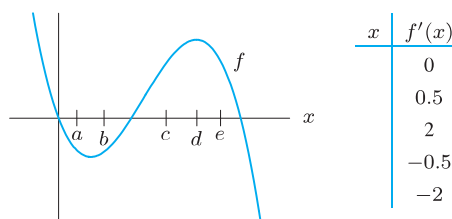

x	$f'(x)$
	0
	0.5
	2
	-0.5
	-2

Figure 2.20

10. Label points $A, B, C, D, E,$ and F on the graph of $y = f(x)$ in Figure 2.21.

    (a) Point A is a point on the curve where the derivative is negative.
    (b) Point B is a point on the curve where the value of the function is negative.
    (c) Point C is a point on the curve where the derivative is largest.
    (d) Point D is a point on the curve where the derivative is zero.
    (e) Points E and F are different points on the curve where the derivative is about the same.
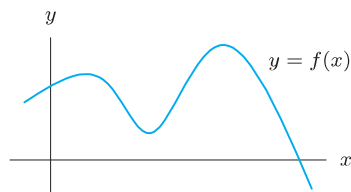

Figure 2.21

11. The graph of $y = f(x)$ is shown in Figure 2.22. Which is larger in each of the following pairs?
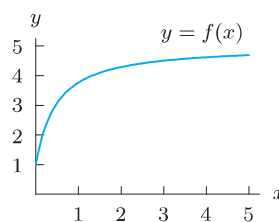

Figure 2.22

    (a) Average rate of change: Between $x = 1$ and $x = 3$? Or between $x = 3$ and $x = 5$?
    (b) $f(2)$ or $f(5)$?
    (c) $f'(1)$ or $f'(4)$?

Problems

12. Suppose that $f(x)$ is a function with $f(100) = 35$ and $f'(100) = 3$. Estimate $f(102)$.

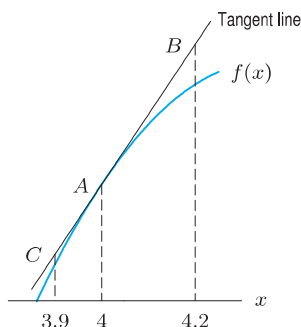13. The function in Figure 2.23 has $f(4) = 25$ and $f'(4) = 1.5$. Find the coordinates of the points A, B, C.

Figure 2.23

14. Use Figure 2.24 to fill in the blanks in the following statements about the function g at point B.

    (a) $g(\underline{\quad}) = \underline{\quad}$ **(b)** $g'(\underline{\quad}) = \underline{\quad}$

Figure 2.24

15. Show how to represent the following on Figure 2.25.

    (a) $f(4)$ **(b)** $f(4) - f(2)$
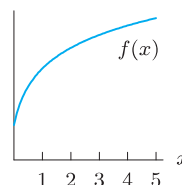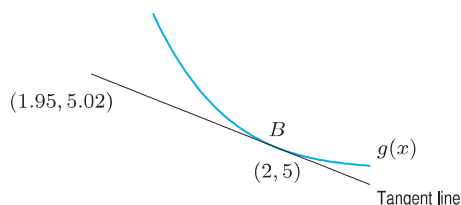   **(c)** $\dfrac{f(5) - f(2)}{5 - 2}$ **(d)** $f'(3)$

Figure 2.25

16. For each of the following pairs of numbers, use Figure 2.25 to decide which is larger. Explain your answer.

    (a) $f(3)$ or $f(4)$?
    (b) $f(3) - f(2)$ or $f(2) - f(1)$?
    (c) $\dfrac{f(2) - f(1)}{2 - 1}$ or $\dfrac{f(3) - f(1)}{3 - 1}$?
    (d) $f'(1)$ or $f'(4)$?

17. With the function f given by Figure 2.25, arrange the following quantities in ascending order:

$$0, \quad f'(2), \quad f'(3), \quad f(3) - f(2)$$

18. On a copy of Figure 2.26, mark lengths that represent the quantities in parts (a) – (d). (Pick any positive x and h.)

(a) $f(x)$ (b) $f(x + h)$

(c) $f(x + h) - f(x)$ (d) h

(e) Using your answers to parts (a)–(d), show how the quantity $\dfrac{f(x + h) - f(x)}{h}$ can be represented as the slope of a line in Figure 2.26.
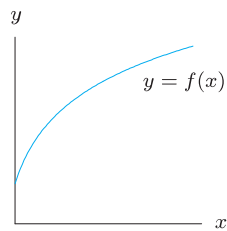

Figure 2.26

19. On a copy of Figure 2.27, mark lengths that represent the quantities in parts (a) – (d). (Pick any convenient x, and assume $h > 0$.)

(a) $f(x)$ (b) $f(x+h)$ (c) $f(x+h) - f(x)$ (d) h

(e) Using your answers to parts (a)–(d), show how the quantity $\dfrac{f(x + h) - f(x)}{h}$ can be represented as the slope of a line on the graph.
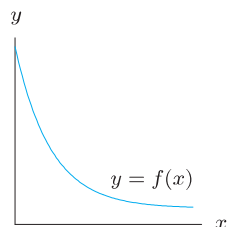

Figure 2.27

20. Consider the function shown in Figure 2.28.

(a) Write an expression involving f for the slope of the line joining A and B.

(b) Draw the tangent line at C. Compare its slope to the slope of the line in part (a).

(c) Are there any other points on the curve at which the slope of the tangent line is the same as the slope of the tangent line at C? If so, mark them on the graph. If not, why not?
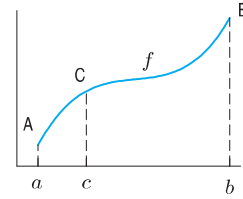

Figure 2.28

21. (a) If f is even and $f'(10) = 6$, what is $f'(-10)$?

(b) If f is any even function and $f'(0)$ exists, what is $f'(0)$?

22. If g is an odd function and $g'(4) = 5$, what is $g'(-4)$?

23. (a) Estimate $f'(0)$ if $f(x) = \sin x$, with x in degrees.

(b) In Example 4 on page 80, we found that the derivative of $\sin x$ at $x = 0$ was 1. Why do we get a different result here? (This problem shows why radians are almost always used in calculus.)

24. Estimate the instantaneous rate of change of the function $f(x) = x \ln x$ at $x = 1$ and at $x = 2$. What do these values suggest about the concavity of the graph between 1 and 2?

25. Estimate the derivative of $f(x) = x^x$ at $x = 2$.

26. For $y = f(x) = 3x^{3/2} - x$, use your calculator to construct a graph of $y = f(x)$, for $0 \le x \le 2$. From your graph, estimate $f'(0)$ and $f'(1)$.

27. Let $f(x) = \ln(\cos x)$. Use your calculator to approximate the instantaneous rate of change of f at the point $x = 1$. Do the same thing for $x = \pi/4$. (Note: Be sure that your calculator is set in radians.)

28. The population, $P(t)$, of China,[2] in billions, can be approximated by

$$P(t) = 1.267(1.007)^t,$$

where t is the number of years since the start of 2000. According to this model, how fast was the population growing at the start of 2000 and at the start of 2007? Give your answers in millions of people per year.

29. On October 17, 2006, in an article called "US Population Reaches 300 Million," the BBC reported that the US gains 1 person every 11 seconds. If $f(t)$ is the US population in millions t years after October 17, 2006, find $f(0)$ and $f'(0)$.

30. (a) Graph $f(x) = \frac{1}{2}x^2$ and $g(x) = f(x) + 3$ on the same set of axes. What can you say about the slopes of the tangent lines to the two graphs at the point $x = 0$? $x = 2$? Any point $x = x_0$?

(b) Explain why adding a constant value, C, to any function does not change the value of the slope of its graph at any point. [Hint: Let $g(x) = f(x) + C$, and calculate the difference quotients for f and g.]

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[2]www.unescap.org/stat/data/apif/index.asp, accessed May 1, 2007.

31. Suppose Table 2.3 on page 81 is continued with smaller values of h. A particular calculator gives the results in Table 2.5. (Your calculator may give slightly different results.) Comment on the values of the difference quotient in Table 2.5. In particular, why is the last value of $(2^h - 1)/h$ zero? What do you expect the calculated value of $(2^h - 1)/h$ to be when $h = 10^{-20}$?

Table 2.5 *Questionable values of difference quotients of 2^x near $x = 0$*

h	Difference quotient: $(2^h - 1)/h$
10^{-4}	0.6931712
10^{-6}	0.693147
10^{-8}	0.6931
10^{-10}	0.69
10^{-12}	0

Use algebra to evaluate the limits in Problems 32–37.

32. $\lim\limits_{h \to 0} \dfrac{(-3 + h)^2 - 9}{h}$

33. $\lim\limits_{h \to 0} \dfrac{(2 - h)^3 - 8}{h}$

34. $\lim\limits_{h \to 0} \dfrac{1/(1 + h) - 1}{h}$

35. $\lim\limits_{h \to 0} \dfrac{1/(1 + h)^2 - 1}{h}$

36. $\lim\limits_{h \to 0} \dfrac{\sqrt{4 + h} - 2}{h}$ [Hint: Multiply by $\sqrt{4 + h} + 2$ in numerator and denominator.]

37. $\lim\limits_{h \to 0} \dfrac{1/\sqrt{4 + h} - 1/2}{h}$

Find the derivatives in Problems 38–43 algebraically.

38. $f(x) = 5x^2$ at $x = 10$

39. $f(x) = x^3$ at $x = -2$

40. $g(t) = t^2 + t$ at $t = -1$

41. $f(x) = x^3 + 5$ at $x = 1$

42. $g(x) = 1/x$ at $x = 2$

43. $g(z) = z^{-2}$, find $g'(2)$

For Problems 44–47, find the equation of the line tangent to the function at the given point.

44. $f(x) = 5x^2$ at $x = 10$

45. $f(x) = x^3$ at $x = -2$

46. $f(x) = x$ at $x = 20$

47. $f(x) = 1/x^2$ at $(1, 1)$

2.3 THE DERIVATIVE FUNCTION

In the previous section we looked at the derivative of a function at a fixed point. Now we consider what happens at a variety of points. The derivative generally takes on different values at different points and is itself a function.

First, remember that the derivative of a function at a point tells us the rate at which the value of the function is changing at that point. Geometrically, we can think of the derivative as the slope of the curve or of the tangent line at the point.

Example 1   Estimate the derivative of the function $f(x)$ graphed in Figure 2.29 at $x = -2, -1, 0, 1, 2, 3, 4, 5$.
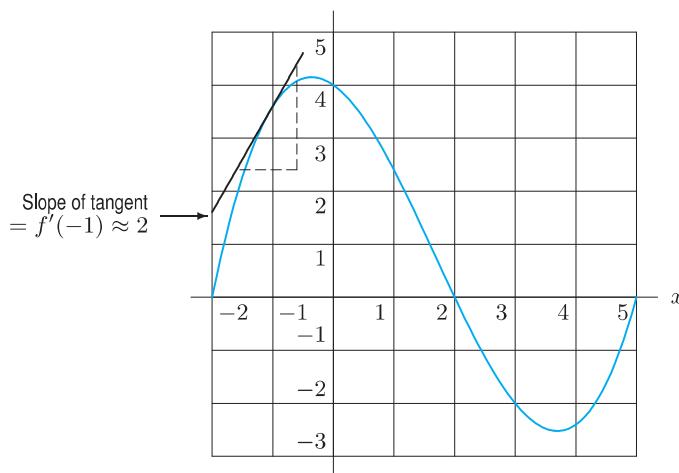

Figure 2.29: Estimating the derivative graphically as the slope of the tangent line

Solution   From the graph we estimate the derivative at any point by placing a straightedge so that it forms the tangent line at that point, and then using the grid squares to estimate the slope of the straightedge. For example, the tangent at $x = -1$ is drawn in Figure 2.29, and has a slope of about 2, so $f'(-1) \approx 2$. Notice that the slope at $x = -2$ is positive and fairly large; the slope at $x = -1$ is positive but smaller. At $x = 0$, the slope is negative, by $x = 1$ it has become more negative, and so on. Some estimates of the derivative are listed in Table 2.6. You should check these values. Are they reasonable? Is the derivative positive where you expect? Negative?

Table 2.6   *Estimated values of derivative of function in Figure 2.29*

x	-2	-1	0	1	2	3	4	5
$f'(x)$	6	2	-1	-2	-2	-1	1	4

Notice that for every x-value, there's a corresponding value of the derivative. Therefore, the derivative is itself a function of x.

For any function f, we define the **derivative function**, f', by

$$f'(x) = \text{ Rate of change of } f \text{ at } x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For every x-value for which this limit exists, we say f is *differentiable at* that x-value. If the limit exists for all x in the domain of f, we say f is *differentiable everywhere*. Most functions we meet are differentiable at every point in their domain, except perhaps for a few isolated points.

The Derivative Function: Graphically

Example 2   Sketch the graph of the derivative of the function shown in Figure 2.29.

Solution   We plot the values of this derivative given in Table 2.6. We obtain Figure 2.30, which shows a graph of the derivative (the black curve), along with the original function (color).

You should check that this graph of f' makes sense. Where the values of f' are positive, f is increasing ($x < -0.3$ or $x > 3.8$) and where f' is negative, f is decreasing. Notice that at the points
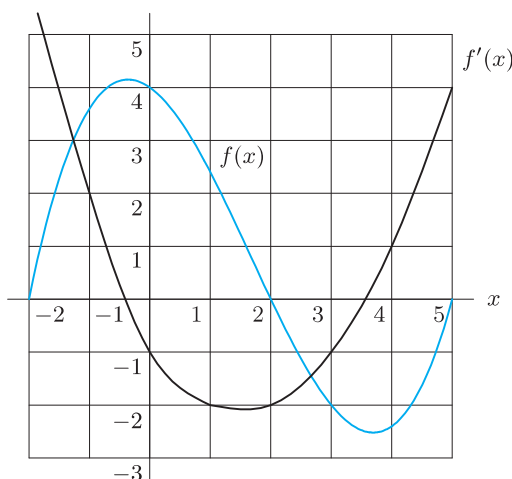

Figure 2.30: Function (colored) and derivative (black) from Example 1

where f has large positive slope, such as $x = -2$, the graph of the derivative is far above the x-axis, as it should be, since the value of the derivative is large there. At points where the slope is gentler, such as $x = -1$, the graph of f' is closer to the x-axis, since the derivative is smaller.

What Does the Derivative Tell Us Graphically?

Where f' is positive, the tangent line to f is sloping up; where f' is negative, the tangent line to f is sloping down. If $f' = 0$ everywhere, then the tangent line to f is horizontal everywhere, and f is constant. We see that the sign of f' tells us whether f is increasing or decreasing.

> If $f' > 0$ on an interval, then f is *increasing* over that interval.
> If $f' < 0$ on an interval, then f is *decreasing* over that interval.

Moreover, the magnitude of the derivative gives us the magnitude of the rate of change; so if f' is large (positive or negative), then the graph of f is steep (up or down), whereas if f' is small the graph of f slopes gently. With this in mind, we can learn about the behavior of a function from the behavior of its derivative.

The Derivative Function: Numerically

If we are given values of a function instead of its graph, we can estimate values of the derivative.

Example 3   Table 2.7 gives values of $c(t)$, the concentration (μg/cm^3) of a drug in the bloodstream at time t (min). Construct a table of estimated values for $c'(t)$, the rate of change of $c(t)$ with respect to time.

Table 2.7   *Concentration as a function of time*

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c(t)$ (μg/cm^3)	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

Solution   We estimate values of c' using the values in the table. To do this, we have to assume that the data points are close enough together that the concentration does not change wildly between them. From the table, we see that the concentration is increasing between $t = 0$ and $t = 0.4$, so we expect a positive derivative there. However, the increase is quite slow, so we expect the derivative to be small. The concentration does not change between 0.4 and 0.5, so we expect the derivative to be roughly 0 there. From $t = 0.5$ to $t = 1.0$, the concentration starts to decrease, and the rate of decrease gets larger and larger, so we expect the derivative to be negative and of greater and greater magnitude.

Using the data in the table, we estimate the derivative using the difference quotient:

$$c'(t) \approx \frac{c(t + h) - c(t)}{h}.$$

Since the data points are 0.1 apart, we use $h = 0.1$, giving

$$c'(0) \approx \frac{c(0.1) - c(0)}{0.1} = \frac{0.89 - 0.84}{0.1} = 0.5 \ \mu\text{g/cm}^3/\text{min}$$

$$c'(0.1) \approx \frac{c(0.2) - c(0.1)}{0.1} = \frac{0.94 - 0.89}{0.1} = 0.5 \ \mu\text{g/cm}^3/\text{min}$$

$$c'(0.2) \approx \frac{c(0.3) - c(0.2)}{0.1} = \frac{0.98 - 0.94}{0.1} = 0.4 \ \mu\text{g/cm}^3/\text{min}$$

$$c'(0.3) \approx \frac{c(0.4) - c(0.3)}{0.1} = \frac{1.00 - 0.98}{0.1} = 0.2 \ \mu\text{g/cm}^3/\text{min}$$

$$c'(0.4) \approx \frac{c(0.5) - c(0.4)}{0.1} = \frac{1.00 - 1.00}{0.1} = 0.0 \ \mu\text{g/cm}^3/\text{min}$$

and so on.

These values are tabulated in Table 2.8. Notice that the derivative has small positive values up until $t = 0.4$, where it is roughly 0, and then it gets more and more negative, as we expected. The slopes are shown on the graph of $c(t)$ in Figure 2.31.

Table 2.8 *Estimated derivative of concentration*

t	$c'(t)$
0	0.5
0.1	0.5
0.2	0.4
0.3	0.2
0.4	0.0
0.5	−0.3
0.6	−0.7
0.7	−1.1
0.8	−1.6
0.9	−2.2
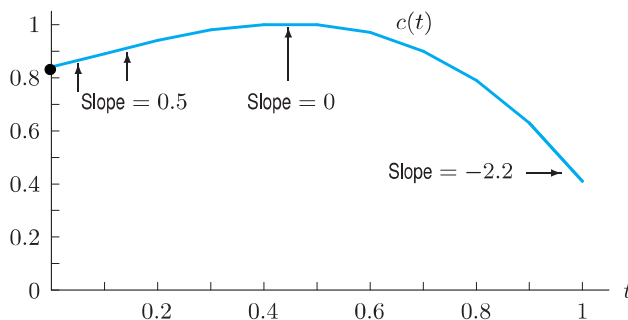
Figure 2.31: Graph of concentration as a function of time

Improving Numerical Estimates for the Derivative

In the previous example, the estimate for the derivative at 0.2 used the interval to the right; we found the average rate of change between $t = 0.2$ and $t = 0.3$. However, we could equally well have gone to the left and used the rate of change between $t = 0.1$ and $t = 0.2$ to approximate the derivative at 0.2. For a more accurate result, we could average these slopes and say

$$c'(0.2) \approx \frac{1}{2}\left(\begin{array}{c}\text{Slope to left}\\ \text{of 0.2}\end{array} + \begin{array}{c}\text{Slope to right}\\ \text{of 0.2}\end{array}\right) = \frac{0.5 + 0.4}{2} = 0.45.$$

In general, averaging the slopes leads to a more accurate answer.

Derivative Function: From a Formula

If we are given a formula for f, can we come up with a formula for f'? We often can, as shown in the next example. Indeed, much of the power of calculus depends on our ability to find formulas for the derivatives of all the functions we described earlier. This is done systematically in Chapter 3.

Derivative of a Constant Function

The graph of a constant function $f(x) = k$ is a horizontal line, with a slope of 0 everywhere. Therefore, its derivative is 0 everywhere. (See Figure 2.32.)

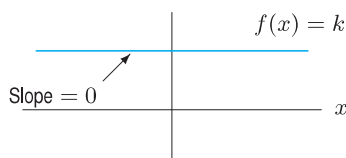$$\boxed{\text{If } f(x) = k, \text{ then } f'(x) = 0.}$$

Figure 2.32: A constant function

Derivative of a Linear Function

We already know that the slope of a straight line is constant. This tells us that the derivative of a linear function is constant.

$$\text{If } f(x) = b + mx, \text{ then } f'(x) = \text{Slope} = m.$$

Derivative of a Power Function

Example 4   Find a formula for the derivative of $f(x) = x^2$.

Solution   Before computing the formula for $f'(x)$ algebraically, let's try to guess the formula by looking for a pattern in the values of $f'(x)$. Table 2.9 contains values of $f(x) = x^2$ (rounded to three decimals), which we can use to estimate the values of $f'(1)$, $f'(2)$, and $f'(3)$.

Table 2.9   *Values of $f(x) = x^2$ near $x = 1$, $x = 2$, $x = 3$ (rounded to three decimals)*

x	x^2	x	x^2	x	x^2
0.999	0.998	1.999	3.996	2.999	8.994
1.000	1.000	2.000	4.000	3.000	9.000
1.001	1.002	2.001	4.004	3.001	9.006
1.002	1.004	2.002	4.008	3.002	9.012

Near $x = 1$, the value of x^2 increases by about 0.002 each time x increases by 0.001, so

$$f'(1) \approx \frac{0.002}{0.001} = 2.$$

Similarly, near $x = 2$ and $x = 3$, the value of x^2 increases by about 0.004 and 0.006, respectively, when x increases by 0.001. So

$$f'(2) \approx \frac{0.004}{0.001} = 4 \qquad \text{and} \qquad f'(3) \approx \frac{0.006}{0.001} = 6.$$

Knowing the value of f' at specific points can never tell us the formula for f', but it certainly can be suggestive: Knowing $f'(1) \approx 2$, $f'(2) \approx 4$, $f'(3) \approx 6$ suggests that $f'(x) = 2x$.

The derivative is calculated by forming the difference quotient and taking the limit as h goes to zero. The difference quotient is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h}.$$

Since h never actually reaches zero, we can cancel it in the last expression to get $2x + h$. The limit of this as h goes to zero is $2x$, so

$$f'(x) = \lim_{h \to 0} (2x + h) = 2x.$$

Example 5   Calculate $f'(x)$ if $f(x) = x^3$.

Solution   We look at the difference quotient

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}.$$

Multiplying out gives $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, so

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}.$$

Since in taking the limit as $h \to 0$, we consider values of h near, but not equal to, zero, we can cancel h giving

$$f'(x) = \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2).$$

As $h \to 0$, the value of $(3xh + h^2) \to 0$ so

$$f'(x) = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2.$$

The previous two examples show how to compute the derivatives of power functions of the form $f(x) = x^n$, when n is 2 or 3. We can use the Binomial Theorem to show the *power rule* for a positive integer n:

> If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

This result is in fact valid for any real value of n.

Exercises and Problems for Section 2.3

Exercises

1. **(a)** Estimate $f'(2)$ using the values of f in the table.
   (b) For what values of x does $f'(x)$ appear to be positive? Negative?

x	0	2	4	6	8	10	12
$f(x)$	10	18	24	21	20	18	15

2. Find approximate values for $f'(x)$ at each of the x-values given in the following table.

x	0	5	10	15	20
$f(x)$	100	70	55	46	40

For Exercises 3–12, graph the derivative of the given functions.
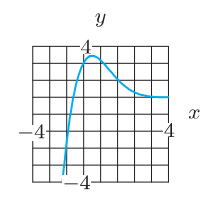
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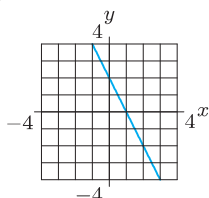
4.
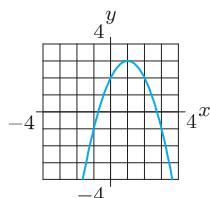
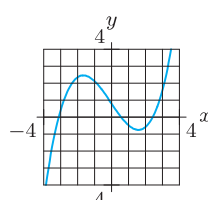**5.**
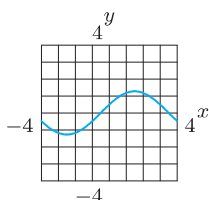
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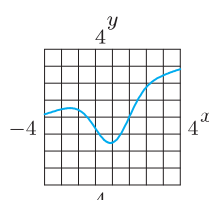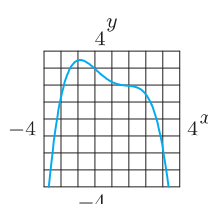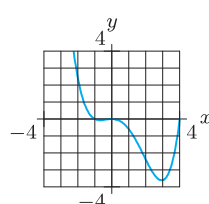

9.

10.

11.

12.
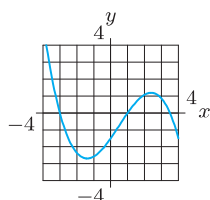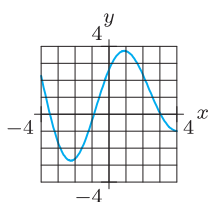
In Exercises 13–14, find a formula for the derivative using the power rule. Confirm it using difference quotients.

13. $k(x) = 1/x$ **14.** $l(x) = 1/x^2$

Find a formula for the derivatives of the functions in Exercises 15–16 using difference quotients.

15. $g(x) = 2x^2 - 3$ **16.** $m(x) = 1/(x+1)$

For Exercises 17–22, sketch the graph of $f(x)$, and use this graph to sketch the graph of $f'(x)$.

17. $f(x) = 5x$ **18.** $f(x) = x^2$

19. $f(x) = e^x$ **20.** $f(x) = x(x - 1)$

21. $f(x) = \cos x$ **22.** $f(x) = \log x$

Problems

23. In each case, graph a smooth curve whose slope meets the condition.

  (a) Everywhere positive and increasing gradually.
  (b) Everywhere positive and decreasing gradually.
  (c) Everywhere negative and increasing gradually (becoming less negative).
  (d) Everywhere negative and decreasing gradually (becoming more negative).

24. For $f(x) = \ln x$, construct tables, rounded to four decimals, near $x = 1$, $x = 2$, $x = 5$, and $x = 10$. Use the tables to estimate $f'(1)$, $f'(2)$, $f'(5)$, and $f'(10)$. Then guess a general formula for $f'(x)$.

25. Given the numerical values shown, find approximate values for the derivative of $f(x)$ at each of the x-values given. Where is the rate of change of $f(x)$ positive? Where is it negative? Where does the rate of change of $f(x)$ seem to be greatest?
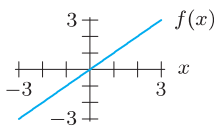
x	0	1	2	3	4	5	6	7	8
$f(x)$	18	13	10	9	9	11	15	21	30

26. Values of x and $g(x)$ are given in the table. For what value of x is $g'(x)$ closest to 3?

x	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2
$g(x)$	3.4	4.4	5.0	5.4	6.0	7.4	9.0	11.0

For Problems 27–36, sketch the graph of $f'(x)$.

27.

28.
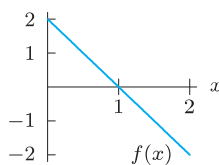
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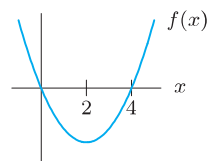
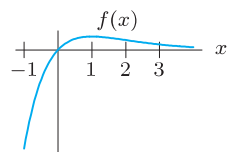**30.**
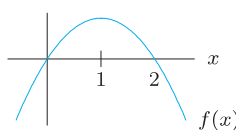
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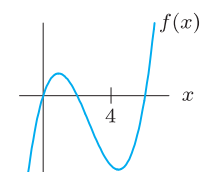
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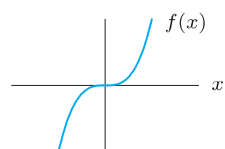
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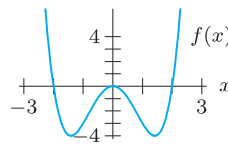
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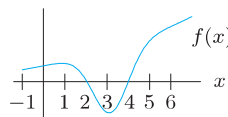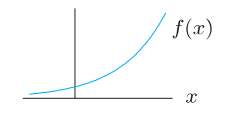**36.**

37. A vehicle moving along a straight road has distance $f(t)$ from its starting point at time t. Which of the graphs in Figure 2.33 could be $f'(t)$ for the following scenarios? (Assume the scales on the vertical axes are all the same.)

  (a) A bus on a popular route, with no traffic
  (b) A car with no traffic and all green lights
  (c) A car in heavy traffic conditions
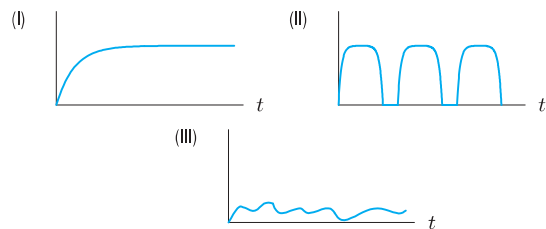
Figure 2.33

38. A child inflates a balloon, admires it for a while and then lets the air out at a constant rate. If $V(t)$ gives the volume of the balloon at time t, then Figure 2.34 shows $V'(t)$ as a function of t. At what time does the child:

  (a) Begin to inflate the balloon?
  (b) Finish inflating the balloon?
  (c) Begin to let the air out?
  (d) What would the graph of $V'(t)$ look like if the child had alternated between pinching and releasing the open end of the balloon, instead of letting the air out at a constant rate?

Figure 2.34

39. Figure 2.35 shows a graph of voltage across an electrical capacitor as a function of time. The current is proportional to the derivative of the voltage; the constant of proportionality is positive. Sketch a graph of the current as a function of time.
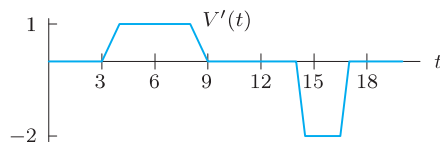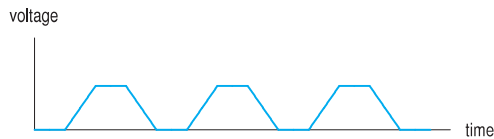

Figure 2.35

40. In the graph of f in Figure 2.36, at which of the labeled x-values is

(a) $f(x)$ greatest?   (b) $f(x)$ least?
(c) $f'(x)$ greatest?   (d) $f'(x)$ least?
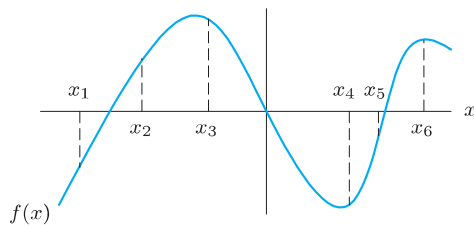

Figure 2.36

41. Figure 2.37 is the graph of f', the derivative of a function f. On what interval(s) is the function f

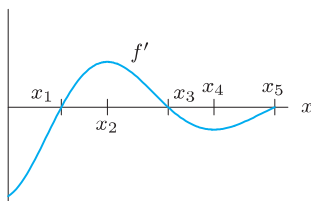(a) Increasing?   (b) Decreasing?

Figure 2.37: Graph of f', not f

42. The derivative of f is the spike function in Figure 2.38. What can you say about the graph of f?
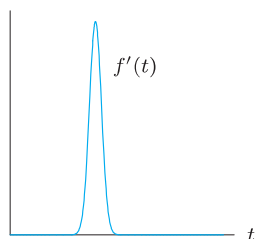

Figure 2.38

43. The population of a herd of deer is modeled by

$$P(t) = 4000 + 500 \sin\left(2\pi t - \frac{\pi}{2}\right)$$

where t is measured in years from January 1.

(a) How does this population vary with time? Sketch a graph of $P(t)$ for one year.
(b) Use the graph to decide when in the year the population is a maximum. What is that maximum? Is there a minimum? If so, when?
(c) Use the graph to decide when the population is growing fastest. When is it decreasing fastest?
(d) Estimate roughly how fast the population is changing on the first of July.

44. The graph in Figure 2.39 shows the accumulated federal debt since 1970. Sketch the derivative of this function. What does it represent?
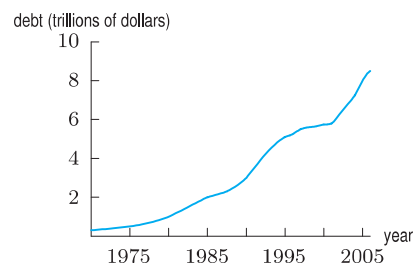
Figure 2.39

45. Draw the graph of a continuous function $y = f(x)$ that satisfies the following three conditions.

- $f'(x) > 0$ for $x < -2$,
- $f'(x) < 0$ for $-2 < x < 2$,
- $f'(x) = 0$ for $x > 2$.

46. Draw the graph of a continuous function $y = f(x)$ that satisfies the following three conditions:

- $f'(x) > 0$ for $1 < x < 3$
- $f'(x) < 0$ for $x < 1$ and $x > 3$
- $f'(x) = 0$ at $x = 1$ and $x = 3$

47. If $\lim_{x \to \infty} f(x) = 50$ and $f'(x)$ is positive for all x, what is $\lim_{x \to \infty} f'(x)$? (Assume this limit exists.) Explain your answer with a picture.

48. Using a graph, explain why if $f(x)$ is an even function, then $f'(x)$ is odd.

49. Using a graph, explain why if $g(x)$ is an odd function, then $g'(x)$ is even.

2.4   INTERPRETATIONS OF THE DERIVATIVE

We have seen the derivative interpreted as a slope and as a rate of change. In this section, we see other interpretations. The purpose of these examples is not to make a catalog of interpretations but to illustrate the process of obtaining them.

An Alternative Notation for the Derivative

So far we have used the notation f' to stand for the derivative of the function f. An alternative notation for derivatives was introduced by the German mathematician Wilhelm Gottfried Leibniz (1646–1716). If the variable y depends on the variable x, that is, if

$$y = f(x),$$

then he wrote dy/dx for the derivative, so

$$\frac{dy}{dx} = f'(x).$$

Leibniz's notation is quite suggestive if we think of the letter d in dy/dx as standing for "small difference in . . . ." The notation dy/dx reminds us that the derivative is a limit of ratios of the form

$$\frac{\text{Difference in } y\text{-values}}{\text{Difference in } x\text{-values}}.$$

The notation dy/dx suggests the units for the derivative: the units for y divided by the units for x. The separate entities dy and dx officially have no independent meaning: they are all part of one notation. In fact, a good way to view the notation dy/dx is to think of d/dx as a single symbol meaning "the derivative with respect to x of . . . ." So dy/dx can be viewed as

$$\frac{d}{dx}(y), \quad \text{meaning "the derivative with respect to } x \text{ of } y\text{."}$$

On the other hand, many scientists and mathematicians think of dy and dx as separate entities representing "infinitesimally" small differences in y and x, even though it is difficult to say exactly how small "infinitesimal" is. Although not formally correct, it can be helpful to think of dy/dx as a small change in y divided by a small change in x.

For example, recall that if $s = f(t)$ is the position of a moving object at time t, then $v = f'(t)$ is the velocity of the object at time t. Writing

$$v = \frac{ds}{dt}$$

reminds us that v is a velocity, since the notation suggests a distance, ds, over a time, dt, and we know that distance over time is velocity. Similarly, we recognize

$$\frac{dy}{dx} = f'(x)$$

as the slope of the graph of $y = f(x)$ since slope is vertical rise, dy, over horizontal run, dx.

The disadvantage of Leibniz's notation is that it is awkward to specify the x-value at which we are evaluating the derivative. To specify $f'(2)$, for example, we have to write

$$\left.\frac{dy}{dx}\right|_{x=2}.$$

Using Units to Interpret the Derivative

The following examples illustrate how useful units can be in suggesting interpretations of the derivative. We use the fact that the units of the instantaneous and the average rate of change are the same.

For example, suppose $s = f(t)$ gives the distance, in meters, of a body from a fixed point as a function of time, t, in seconds. Then knowing that

$$\frac{ds}{dt}\bigg|_{t=2} = f'(2) = 10 \text{ meters/sec}$$

tells us that when $t = 2$ seconds, the body is moving at an instantaneous velocity of 10 meters/sec. This means that if the body continued to move at this speed for a whole second, it would move 10 meters. In practice, however, the velocity of the body may not remain 10 meters/sec for long. Notice that the units of instantaneous velocity and of average velocity are the same.

Example 1
The cost C (in dollars) of building a house A square feet in area is given by the function $C = f(A)$. What is the practical interpretation of the function $f'(A)$?

Solution
In the alternative notation,

$$f'(A) = \frac{dC}{dA}.$$

This is a cost divided by an area, so it is measured in dollars per square foot. You can think of dC as the extra cost of building an extra dA square feet of house. Then you can think of dC/dA as the additional cost per square foot. So if you are planning to build a house roughly A square feet in area, $f'(A)$ is the cost per square foot of the *extra* area involved in building a slightly larger house, and is called the *marginal cost*. The marginal cost is probably smaller than the average cost per square foot for the entire house, since once you are already set up to build a large house, the cost of adding a few square feet is likely to be small.

Example 2
The cost of extracting T tons of ore from a copper mine is $C = f(T)$ dollars. What does it mean to say that $f'(2000) = 100$?

Solution
In the alternative notation,

$$f'(2000) = \frac{dC}{dT}\bigg|_{T=2000} = 100.$$

Since C is measured in dollars and T is measured in tons, dC/dT must be measured in dollars per ton. So the statement $f'(2000) = 100$ says that when 2000 tons of ore have already been extracted from the mine, the cost of extracting the next ton is approximately \$100.

Example 3
If $q = f(p)$ gives the number of pounds of sugar produced when the price per pound is p dollars, then what are the units and the meaning of the statement $f'(3) = 50$?

Solution
Since $f'(3)$ is the limit as $h \to 0$ of the difference quotient

$$\frac{f(3 + h) - f(3)}{h},$$

the units of $f'(3)$ and the difference quotient are the same. Since $f(3 + h) - f(3)$ is in pounds and h is in dollars, the units of the difference quotient and $f'(3)$ are pounds/dollar. The statement

$$f'(3) = 50 \text{ pounds/dollar}$$

tells us that the instantaneous rate of change of q with respect to p is 50 when $p = 3$. In other words, when the price is \$3, the quantity produced is increasing at 50 pounds/dollar. Thus, if the price increased by a dollar, the quantity produced would increase by approximately 50 pounds.

Example 4   You are told that water is flowing through a pipe at a constant rate of 10 cubic feet per second. Interpret this rate as the derivative of some function.

Solution   You might think at first that the statement has something to do with the velocity of the water, but in fact a flow rate of 10 cubic feet per second could be achieved either with very slowly moving water through a large pipe, or with very rapidly moving water through a narrow pipe. If we look at the units—cubic feet per second—we realize that we are being given the rate of change of a quantity measured in cubic feet. But a cubic foot is a measure of volume, so we are being told the rate of change of a volume. One way to visualize this is to imagine all the water that is flowing through the pipe ending up in a tank somewhere. Let $V(t)$ be the volume of water in the tank at time t. Then we are being told that the rate of change of $V(t)$ is 10, or

$$V'(t) = \frac{dV}{dt} = 10.$$

Example 5   Suppose $P = f(t)$ is the population of Mexico in millions, where t is the number of years since 1980. Explain the meaning of the statements:

(a) $f'(6) = 2$ (b) $f^{-1}(95.5) = 16$ (c) $(f^{-1})'(95.5) = 0.46$

Solution   (a)   The units of P are millions of people, the units of t are years, so the units of $f'(t)$ are millions of people per year. Therefore the statement $f'(6) = 2$ tells us that at $t = 6$ (that is, in 1986), the population of Mexico was increasing at 2 million people per year.
(b)   The statement $f^{-1}(95.5) = 16$ tells us that the year when the population was 95.5 million was $t = 16$ (that is, in 1996).
(c)   The units of $(f^{-1})'(P)$ are years per million of population. The statement $(f^{-1})'(95.5) = 0.46$ tells us that when the population was 95.5 million, it took about 0.46 years for the population to increase by 1 million.

Exercises and Problems for Section 2.4

Exercises

1. The cost, C (in dollars), to produce q quarts of ice cream is $C = f(q)$. In each of the following statements, what are the units of the two numbers? In words, what does each statement tell us?

(a) $f(200) = 600$ (b) $f'(200) = 2$

2. The temperature, H, in degrees Celsius, of a cup of coffee placed on the kitchen counter is given by $H = f(t)$, where t is in minutes since the coffee was put on the counter.

(a)   Is $f'(t)$ positive or negative? Give a reason for your answer.
(b)   What are the units of $f'(20)$? What is its practical meaning in terms of the temperature of the coffee?

3. The temperature, T, in degrees Fahrenheit, of a cold yam placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the yam was put in the oven.

(a)   What is the sign of $f'(t)$? Why?
(b)   What are the units of $f'(20)$? What is the practical meaning of the statement $f'(20) = 2$?

4. The cost, C (in dollars) to produce g gallons of a chemical can be expressed as $C = f(g)$. Using units, explain the meaning of the following statements in terms of the chemical.

(a) $f(200) = 1300$ (b) $f'(200) = 6$

5. The time for a chemical reaction, T (in minutes), is a function of the amount of catalyst present, a (in milliliters), so $T = f(a)$.

(a)   If $f(5) = 18$, what are the units of 5? What are the units of 18? What does this statement tell us about the reaction?
(b)   If $f'(5) = -3$, what are the units of 5? What are the units of -3? What does this statement tell us?

6. After investing \$1000 at an annual interest rate of 7% compounded continuously for t years, your balance is \$$B$, where $B = f(t)$. What are the units of dB/dt? What is the financial interpretation of dB/dt?

7. Suppose $C(r)$ is the total cost of paying off a car loan borrowed at an annual interest rate of $r\%$. What are the units of $C'(r)$? What is the practical meaning of $C'(r)$? What is its sign?

8. Suppose $P(t)$ is the monthly payment, in dollars, on a mortgage which will take t years to pay off. What are the units of $P'(t)$? What is the practical meaning of $P'(t)$? What is its sign?

9. Investing \$1000 at an annual interest rate of $r\%$, compounded continuously, for 10 years gives you a balance of \$$B$, where $B = g(r)$. Give a financial interpretation of the statements:

    (a) $g(5) \approx 1649$.
    (b) $g'(5) \approx 165$. What are the units of $g'(5)$?

10. Let $f(x)$ be the elevation in feet of the Mississippi river x miles from its source. What are the units of $f'(x)$? What can you say about the sign of $f'(x)$?

11. An economist is interested in how the price of a certain item affects its sales. At a price of \$$p$, a quantity, q, of the item is sold. If $q = f(p)$, explain the meaning of each of the following statements:

    (a) $f(150) = 2000$ (b) $f'(150) = -25$

12. Meteorologists define the temperature lapse rate to be $-dT/dz$ where T is the air temperature in Celsius at altitude z kilometers above the ground.

    (a) What are the units of the lapse rate?
    (b) What is the practical meaning of a lapse rate of 6.5?

Problems

13. A laboratory study investigating the relationship between diet and weight in adult humans found that the weight of a subject, W, in pounds, was a function, $W = f(c)$, of the average number of Calories per day, c, consumed by the subject.

    (a) Interpret the statements $f(1800) = 155$, $f'(2000) = 0$, and $f^{-1}(162) = 2200$ in terms of diet and weight.
    (b) What are the units of $f'(c) = dW/dc$?

14. A city grew in population throughout the 1980s. The population was at its largest in 1990, and then shrank throughout the 1990s. Let $P = f(t)$ represent the population of the city t years since 1980. Sketch graphs of $f(t)$ and $f'(t)$, labeling the units on the axes.

15. If t is the number of years since 2003, the population, P, of China, in billions, can be approximated by the function

    $$P = f(t) = 1.291(1.006)^t.$$

    Estimate $f(6)$ and $f'(6)$, giving units. What do these two numbers tell you about the population of China?

16. For some painkillers, the size of the dose, D, given depends on the weight of the patient, W. Thus, $D = f(W)$, where D is in milligrams and W is in pounds.

    (a) Interpret the statements $f(140) = 120$ and $f'(140) = 3$ in terms of this painkiller.
    (b) Use the information in the statements in part (a) to estimate $f(145)$.

17. On May 9, 2007, CBS Evening News had a 4.3 point rating. (Ratings measure the number of viewers.) News executives estimated that a 0.1 drop in the ratings for the CBS Evening News corresponds to a \$5.5 million drop in revenue.[3] Express this information as a derivative. Specify the function, the variables, the units, and the point at which the derivative is evaluated.

18. Let $f(t)$ be the number of centimeters of rainfall that has fallen since midnight, where t is the time in hours. Interpret the following in practical terms, giving units.

    (a) $f(10) = 3.1$ (b) $f^{-1}(5) = 16$
    (c) $f'(10) = 0.4$ (d) $(f^{-1})'(5) = 2$

19. Water is flowing into a tank; the depth, in feet, of the water at time t in hours is $h(t)$. Interpret, with units, the following statements.

    (a) $h(5) = 3$ (b) $h'(5) = 0.7$
    (c) $h^{-1}(5) = 7$ (d) $(h^{-1})'(5) = 1.2$

20. Let $p(h)$ be the pressure in dynes per cm^2 on a diver at a depth of h meters below the surface of the ocean. What do each of the following quantities mean to the diver? Give units for the quantities.

    (a) $p(100)$ (b) h such that $p(h) = 1.2 \cdot 10^6$
    (c) $p(h) + 20$ (d) $p(h + 20)$
    (e) $p'(100)$ (f) h such that $p'(h) = 20$

21. If $g(v)$ is the fuel efficiency, in miles per gallon, of a car going at v miles per hour, what are the units of $g'(90)$? What is the practical meaning of the statement $g'(55) = -0.54$?

22. Let P be the total petroleum reservoir on earth in the year t. (In other words, P represents the total quantity of petroleum, including what's not yet discovered, on earth at time t.) Assume that no new petroleum is being made and that P is measured in barrels. What are the units of dP/dt? What is the meaning of dP/dt? What is its sign? How would you set about estimating this derivative in practice? What would you need to know to make such an estimate?

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[3]OC Register, May 9, 2007; *The New York Times*, May 14, 2007.

23. (a) If you jump out of an airplane without a parachute, you fall faster and faster until air resistance causes you to approach a steady velocity, called the *terminal* velocity. Sketch a graph of your velocity against time.

(b) Explain the concavity of your graph.

(c) Assuming air resistance to be negligible at $t = 0$, what natural phenomenon is represented by the slope of the graph at $t = 0$?

24. Let W be the amount of water, in gallons, in a bathtub at time t, in minutes.

(a) What are the meaning and units of dW/dt?

(b) Suppose the bathtub is full of water at time t_0, so that $W(t_0) > 0$. Subsequently, at time $t_p > t_0$, the plug is pulled. Is dW/dt positive, negative, or zero:

   (i) For $t_0 < t < t_p$?

   (ii) After the plug is pulled, but before the tub is empty?

   (iii) When all the water has drained from the tub?

25. A company's revenue from car sales, C (in thousands of dollars), is a function of advertising expenditure, a, in thousands of dollars, so $C = f(a)$.

(a) What does the company hope is true about the sign of f'?

(b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

(c) Suppose the company plans to spend about $100,000 on advertising. If $f'(100) = 2$, should the company spend more or less than $100,000 on advertising? What if $f'(100) = 0.5$?

26. Let $P(x)$ be the number of people of height $\leq x$ inches in the US. What is the meaning of $P'(66)$? What are its units? Estimate $P'(66)$ (using common sense). Is $P'(x)$ ever negative? [Hint: You may want to approximate $P'(66)$ by a difference quotient, using $h = 1$. Also, you may assume the US population is about 300 million, and note that 66 inches = 5 feet 6 inches.]

27. In May 2007 in the US, there was one birth every 8 seconds, one death every 13 seconds, one new international migrant every 27 seconds.[4]

(a) Let $f(t)$ be the population of the US, where t is time in seconds measured from the start of May 2007. Find $f'(0)$. Give units.

(b) To the nearest second, how long did it take for the US population to add one person in May 2007?

28. When you breathe, a muscle (called the diaphragm) reduces the pressure around your lungs and they expand to fill with air. The table shows the volume of a lung as a function of the reduction in pressure from the diaphragm.

Pulmonologists (lung doctors) define the *compliance* of the lung as the derivative of this function.[5]

(a) What are the units of compliance?

(b) Estimate the maximum compliance of the lung.

(c) Explain why the compliance gets small when the lung is nearly full (around 1 liter).

Pressure reduction (cm of water)	Volume (liters)
0	0.20
5	0.29
10	0.49
15	0.70
20	0.86
25	0.95
30	1.00

29. The compressibility index, γ, of cold matter (in a neutron star or black hole) is given by

$$\gamma = \frac{\delta + (p/c^2)}{p} \frac{dp}{d\delta},$$

where p is the pressure (in dynes/cm^2), δ is the density (in g/cm^3), and $c \approx 3 \cdot 10^{10}$ is the speed of light (in cm/sec). Figure 2.40 shows the relationship between δ, γ, and p. Values of $\log p$ are marked along the graph.[6]

(a) Estimate $dp/d\delta$ for cold iron, which has a density of about 10 g/cm^3. What does the magnitude of your answer tell you about cold iron?

(b) Estimate $dp/d\delta$ for the matter inside a white dwarf star, which has a density of about 10^6 g/cm^3. What does your answer tell you about matter inside a white dwarf?
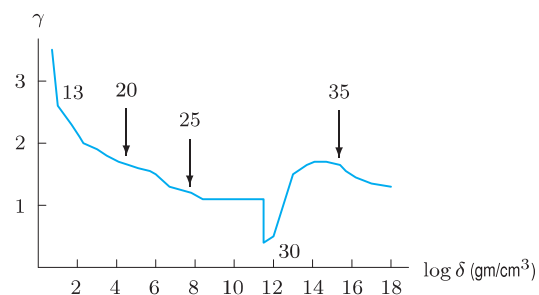

Figure 2.40:

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[4] www.census.gov, accessed May 14, 2007.

[5] Adapted from John B. West, *Respiratory Physiology* 4th Ed. (New York: Williams and Wilkins, 1990).

[6] From C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (San Francisco: W. H. Freeman and Company, 1973).

2.5 THE SECOND DERIVATIVE

Since the derivative is itself a function, we can consider its derivative. For a function f, the derivative of its derivative is called the *second derivative*, and written f'' (read "f double-prime"). If $y = f(x)$, the second derivative can also be written as $\dfrac{d^2 y}{dx^2}$, which means $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$, the derivative of $\dfrac{dy}{dx}$.

What Do Derivatives Tell Us?

Recall that the derivative of a function tells you whether a function is increasing or decreasing:

- If $f' > 0$ on an interval, then f is *increasing* over that interval. Thus, f is monotonic over that interval.

- If $f' < 0$ on an interval, then f is *decreasing* over that interval. Thus, f is monotonic over that interval.

Since f'' is the derivative of f',

- If $f'' > 0$ on an interval, then f' is *increasing* over that interval.

- If $f'' < 0$ on an interval, then f' is *decreasing* over that interval.

What does it mean for f' to be increasing or decreasing? An example in which f' is increasing is shown in Figure 2.41, where the curve is bending upward, or is *concave up*. In the example shown in Figure 2.42, in which f' is decreasing, the graph is bending downward, or is *concave down*. These figures suggest the following result:

> If $f'' > 0$ on an interval, then f' is increasing, so the graph of f is concave up there.
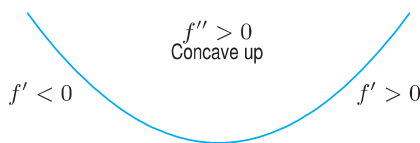> If $f'' < 0$ on an interval, then f' is decreasing, so the graph of f is concave down there.

Figure 2.41: Meaning of f'': The slope increases from left to right, f'' is positive, and f is concave up
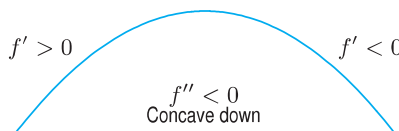

Figure 2.42: Meaning of f'': The slope decreases from left to right, f'' is negative, and f is concave down

Warning! The graph of a function f can be concave up everywhere and yet have $f'' = 0$ at some point. For instance, the graph of $f(x) = x^4$ in Figure 2.43 is concave up, but it can be shown that $f''(0) = 0$. If we are told that the graph of a function f is concave up, we can be sure that f'' is not negative, that is $f'' \geq 0$, but not that f'' is positive, $f'' > 0$.
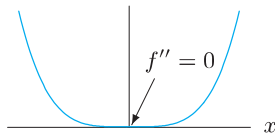

Figure 2.43: Graph of $f(x) = x^4$

> If the graph of f is concave up on an interval, then $f'' \geq 0$ there.
> If the graph of f is concave down on an interval, then $f'' \leq 0$ there.

Example 1   For the functions graphed in Figure 2.44, what can be said about the sign of the second derivative?.
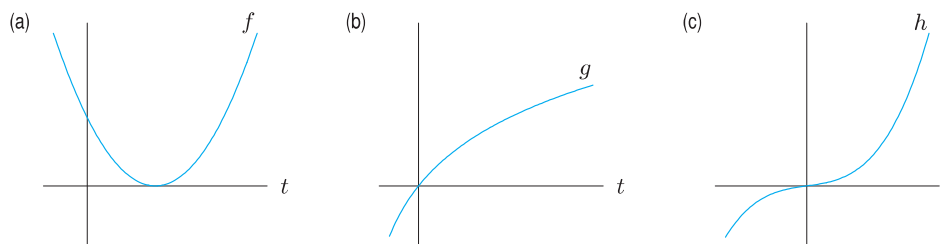

Figure 2.44: What signs do the second derivatives have?

Solution   (a) The graph of f is concave up everywhere, so $f'' \geq 0$ everywhere.
(b) The graph of g is concave down everywhere, so $g'' \leq 0$ everywhere.
(c) For $t < 0$, the graph of h is concave down, so $h'' \leq 0$ there. For $t > 0$, the graph of h is concave up, so $h'' \geq 0$ there.

Example 2   Sketch the second derivative f'' for the function f of Example 1 on page 85, graphed with its derivative, f', in Figure 2.45. Relate the resulting graph of f'' to the graphs of f and f'.
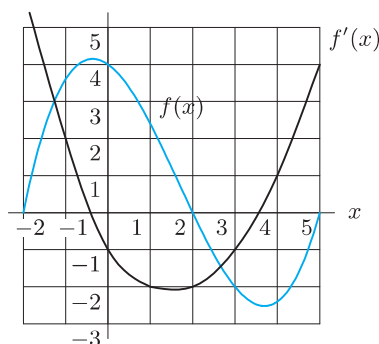

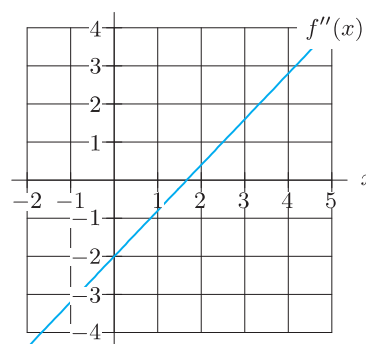**Figure 2.45**: Function, f in color; derivative, f', in black

Figure 2.46: Graph of f''

Solution   We want to sketch the derivative of f'. We do this by estimating the slopes of f' and plotting them, obtaining Figure 2.46.

We observe that where $f'' > 0$, the graph of f is concave up and f' is increasing, and that where $f'' < 0$, the graph of f is concave down and f' is decreasing. Where $f''(x) = 0$, the graph of f changes from concave up to concave down, and f' changes from decreasing to increasing.

Interpretation of the Second Derivative as a Rate of Change

If we think of the derivative as a rate of change, then the second derivative is a rate of change of a rate of change. If the second derivative is positive, the rate of change of f is increasing; if the second derivative is negative, the rate of change of f is decreasing.

The second derivative can be a matter of practical concern. In 1985 a newspaper headline reported the Secretary of Defense as saying that Congress had cut the defense budget. As his opponents pointed out, however, Congress had merely cut the rate at which the defense budget was increasing.[7] In other words, the derivative of the defense budget was still positive (the budget was increasing), but the second derivative was negative (the budget's rate of increase had slowed).

[7]In the *Boston Globe*, March 13, 1985, Representative William Gray (D–Pa.) was reported as saying: "It's confusing to the American people to imply that Congress threatens national security with reductions when you're really talking about a reduction in the increase."

Example 3   A population, P, growing in a confined environment often follows a *logistic* growth curve, like that shown in Figure 2.47; Relate the sign of d^2P/dt^2 to how the rate of growth, dP/dt, changes over time. What are practical interpretations of t_0 and L?
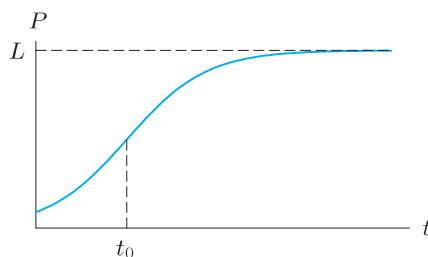

Figure 2.47: Logistic growth curve

Solution   For $t < t_0$, the rate of growth, dP/dt, is increasing and $d^2P/dt^2 \geq 0$. At t_0, the rate dP/dt is a maximum. In other words, at time t_0 the population is growing fastest. For $t > t_0$, the rate of growth, dP/dt, is decreasing and $dP^2/dt^2 \leq 0$. At t_0, the curve changes from concave up to concave down, and $d^2P/dt^2 = 0$ there.

The quantity L represents the limiting value of the population as $t \to \infty$. Biologists call L the *carrying capacity* of the environment.

Example 4   Tests on the C5 Chevy Corvette sports car gave the results[8] in Table 2.10.

(a) Estimate dv/dt for the time intervals shown.

(b) What can you say about the sign of d^2v/dt^2 over the period shown?

Table 2.10   *Velocity of C5 Chevy Corvette*

Time, t (sec)	0	3	6	9	12
Velocity, v (meters/sec)	0	20	33	43	51

Solution   (a) For each time interval we can calculate the average rate of change of velocity. For example, from $t = 0$ to $t = 3$ we have

$$\frac{dv}{dt} \approx \text{Average rate of change of velocity} = \frac{20 - 0}{3 - 0} = 6.67 \ \frac{\text{m/sec}}{\text{sec}}.$$

Estimated values of dv/dt are in Table 2.11.

(b) Since the values of dv/dt are decreasing between the points shown, we expect $d^2v/dt^2 \leq 0$. The graph of v against t in Figure 2.48 supports this; it is concave down. The fact that $dv/dt > 0$ tells us that the car is speeding up; the fact that $d^2v/dt^2 \leq 0$ tells us that the rate of increase decreased (actually, did not increase) over this time period.

Figure 2.48: Velocity of C5 Chevy Corvette

Table 2.11   *Estimates for dv/dt (meters/sec/sec)*

Time interval (sec)	$0 - 3$	$3 - 6$	$6 - 9$	$9 - 12$
Average rate of change (dv/dt)	6.67	4.33	3.33	2.67

[8]Adapted from the report in *Car and Driver*, February 1997.

Velocity and Acceleration

When a car is speeding up, we say that it is accelerating. We define *acceleration* as the rate of change of velocity with respect to time. If $v(t)$ is the velocity of an object at time t, we have

$$\begin{matrix} \text{Average acceleration} \\ \text{from } t \text{ to } t + h \end{matrix} = \frac{v(t+h) - v(t)}{h},$$

$$\text{Instantaneous acceleration} = v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$

If the term velocity or acceleration is used alone, it is assumed to be instantaneous. Since velocity is the derivative of position, acceleration is the second derivative of position. Summarizing:

If $y = s(t)$ is the position of an object at time t, then

- Velocity: $v(t) = \dfrac{dy}{dt} = s'(t)$.

- Acceleration: $a(t) = \dfrac{d^2y}{dt^2} = s''(t) = v'(t)$.

Example 5 A particle is moving along a straight line; its acceleration is zero only once. Its distance, s, to the right of a fixed point is given by Figure 2.49. Estimate:

(a) When the particle is moving to the right and when it is moving to the left.
(b) When the acceleration of the particle is zero, when it is negative, and when it is positive.
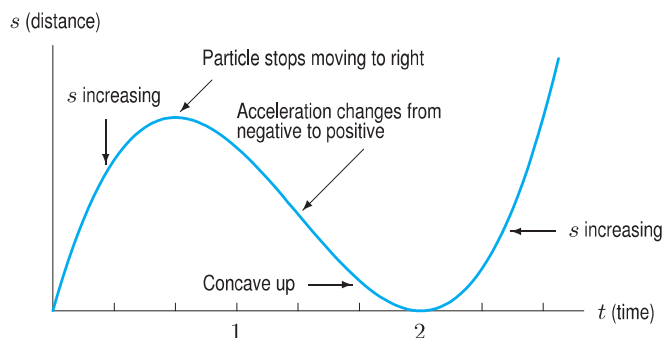

Figure 2.49: Distance of particle to right of a fixed point

Solution

(a) The particle is moving to the right whenever s is increasing. From the graph, this appears to be for $0 < t < \frac{2}{3}$ and for $t > 2$. For $\frac{2}{3} < t < 2$, the value of s is decreasing, so the particle is moving to the left.

(b) Since the acceleration is zero only once, this must be when the curve changes concavity, at about $t = \frac{4}{3}$. Then the acceleration is negative for $t < \frac{4}{3}$, since the graph is concave down there, and the acceleration is positive for $t > \frac{4}{3}$, since the graph is concave up there.

Exercises and Problems for Section 2.5

Exercises

1. For the function graphed in Figure 2.50, are the following nonzero quantities positive or negative?
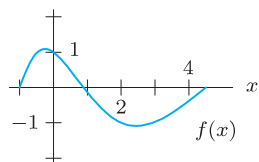
   (a) $f(2)$ **(b)** $f'(2)$ **(c)** $f''(2)$

Figure 2.50

2. At one of the labeled points on the graph in Figure 2.51 both dy/dx and d^2y/dx^2 are positive. Which is it?
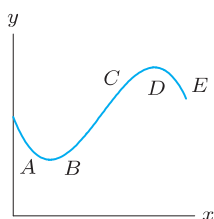

Figure 2.51

3. At exactly two of the labeled points in Figure 2.52, the derivative f' is 0; the second derivative f'' is not zero at any of the labeled points. On a copy of the table, give the signs of f, f', f'' at each marked point.
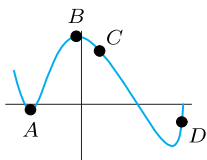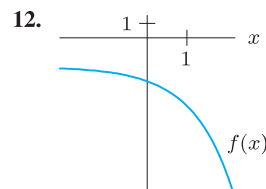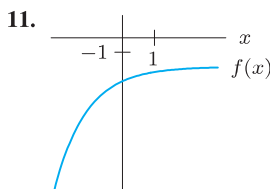

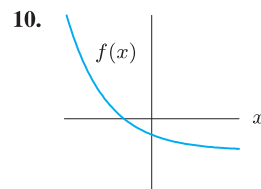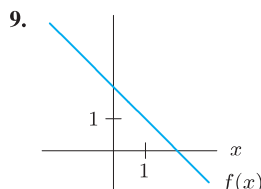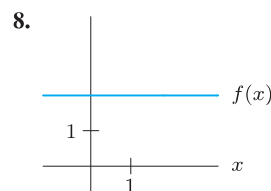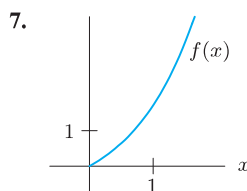| Point	f	f'	f''
A			
B			
C			
D			

Figure 2.52

4. Graph the functions described in parts (a)–(d).

   (a) First and second derivatives everywhere positive.
   (b) Second derivative everywhere negative; first derivative everywhere positive.
   (c) Second derivative everywhere positive; first derivative everywhere negative.
   (d) First and second derivatives everywhere negative.

5. Sketch the graph of a function whose first derivative is everywhere negative and whose second derivative is positive for some x-values and negative for other x-values.

6. Sketch the graph of the height of a particle against time if velocity is positive and acceleration is negative.

For Exercises 7–12, give the signs of the first and second derivatives for the following functions. Each derivative is either positive everywhere, zero everywhere, or negative everywhere.

7.

8.

9.

10.

11.

12.

13. The position of a particle moving along the x-axis is given by $s(t) = 5t^2 + 3$. Use difference quotients to find the velocity $v(t)$ and acceleration $a(t)$.

Problems

14. The table gives the number of passenger cars, $C = f(t)$, in millions,[9] in the US in the year t.

   (a) Do $f'(t)$ and $f''(t)$ appear to be positive or negative during the period 1940–1980?

   (b) Estimate $f'(1975)$. Using units, interpret your answer in terms of passenger cars.

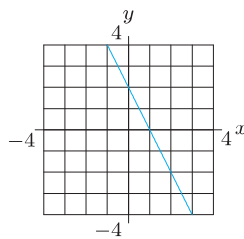| t	1940	1950	1960	1970	1980	1990	2000
C	27.5	40.3	61.7	89.2	121.6	133.7	133.6

[9]www.bts.gov/publications/national_transportation_statistics/html/table_01_11.html. Accessed June 22, 2008.

15. An accelerating sports car goes from 0 mph to 60 mph in five seconds. Its velocity is given in the following table, converted from miles per hour to feet per second, so that all time measurements are in seconds. (Note: 1 mph is 22/15 ft/sec.) Find the average acceleration of the car over each of the first two seconds.
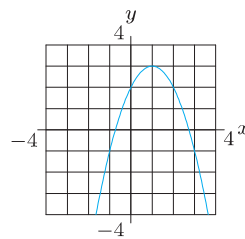
Time, t (sec)	0	1	2	3	4	5
Velocity, $v(t)$ (ft/sec)	0	30	52	68	80	88

In Problems 16–21, graph the second derivative of the function.
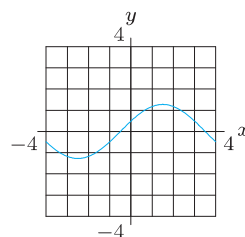
16.

17.

18.

19.

20.

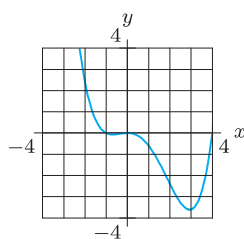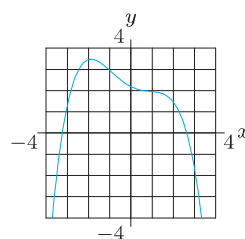**21.**

22. Let $P(t)$ represent the price of a share of stock of a corporation at time t. What does each of the following statements tell us about the signs of the first and second derivatives of $P(t)$?

(a) "The price of the stock is rising faster and faster."

(b) "The price of the stock is close to bottoming out."

23. In economics, *total utility* refers to the total satisfaction from consuming some commodity. According to the economist Samuelson:[10]

> As you consume more of the same good, the total (psychological) utility increases. However, ...with successive new units of the good, your total utility will grow at a slower and slower rate because of a fundamental tendency for your psychological ability to appreciate more of the good to become less keen.

(a) Sketch the total utility as a function of the number of units consumed.

(b) In terms of derivatives, what is Samuelson saying?

24. "Winning the war on poverty" has been described cynically as slowing the rate at which people are slipping below the poverty line. Assuming that this is happening:

(a) Graph the total number of people in poverty against time.

(b) If N is the number of people below the poverty line at time t, what are the signs of dN/dt and d^2N/dt^2? Explain.

25. In April 1991, the *Economist* carried an article[11] which said:

> Suddenly, everywhere, it is not the rate of change of things that matters, it is the rate of change of rates of change. Nobody cares much about inflation; only whether it is going up or down. Or rather, whether it is going up fast or down fast. "Inflation drops by disappointing two points," cries the billboard. Which roughly translated means that prices are still rising, but less fast than they were, though not quite as much less fast as everybody had hoped.

In the last sentence, there are three statements about prices. Rewrite these as statements about derivatives.

26. An industry is being charged by the Environmental Protection Agency (EPA) with dumping unacceptable levels of toxic pollutants in a lake. Over a period of several months, an engineering firm makes daily measurements of the rate at which pollutants are being discharged into the lake. The engineers produce a graph similar to either Figure 2.53(a) or Figure 2.53(b). For each case, give an idea of what argument the EPA might make in court against the industry and of the industry's defense.
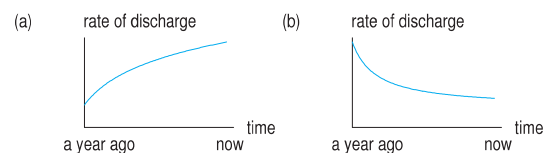

Figure 2.53

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[10]From Paul A. Samuelson, *Economics*, 11th edition (New York: McGraw-Hill, 1981).

[11]From "The Tyranny of Differential Calculus: $d^2P/dt^2 > 0 = $ misery." *The Economist* (London: April 6, 1991).

27. At which of the marked x-values in Figure 2.54 can the following statements be true?

(a) $f(x) < 0$
(b) $f'(x) < 0$
(c) $f(x)$ is decreasing
(d) $f'(x)$ is decreasing
(e) Slope of $f(x)$ is positive
(f) Slope of $f(x)$ is increasing

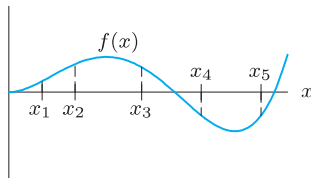**Figure 2.54**

28. Figure 2.55 gives the position, $f(t)$, of a particle at time t. At which of the marked values of t can the following statements be true?

(a) The position is positive
(b) The velocity is positive
(c) The acceleration is positive
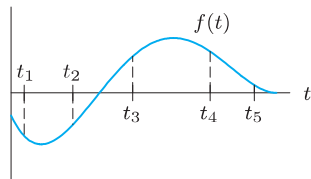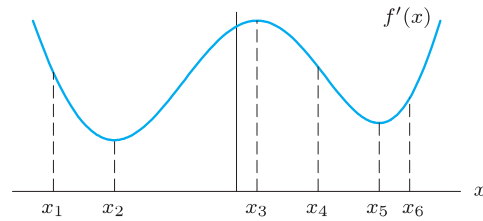(d) The position is decreasing
(e) The velocity is decreasing

Figure 2.55

29. The graph of f' (not f) is given in Figure 2.56. At which of the marked values of x is

(a) $f(x)$ greatest?   (b) $f(x)$ least?
(c) $f'(x)$ greatest?   (d) $f'(x)$ least?
(e) $f''(x)$ greatest?   (f) $f''(x)$ least?

Figure 2.56: Graph of f', not f

30. A function f has $f(5) = 20$, $f'(5) = 2$, and $f''(x) < 0$, for $x \geq 5$. Which of the following are possible values for $f(7)$ and which are impossible?

(a)   26   (b)   24   (c)   22

31. A continuous function defined for all x has the following properties:

• f is increasing   • f is concave down
• $f(5) = 2$ • $f'(5) = \frac{1}{2}$

(a) Sketch a possible graph for f.
(b) How many zeros does f have?
(c) What can you say about the location of the zeros?
(d) What is $\lim\limits_{x \to -\infty} f(x)$?
(e) Is it possible that $f'(1) = 1$?
(f) Is it possible that $f'(1) = \frac{1}{4}$?

2.6 DIFFERENTIABILITY

What Does It Mean for a Function to be Differentiable?

A function is differentiable at a point if it has a derivative there. In other words:

> The function f is **differentiable** at x if
>
> $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{exists.}$$
>
> Thus, the graph of f has a nonvertical tangent line at x. The value of the limit and the slope of the tangent line are the derivative of f at x.

Occasionally we meet a function which fails to have a derivative at a few points. A function fails to be differentiable at a point if:

• The function is not continuous at the point.

• The graph has a sharp corner at that point.

• The graph has a vertical tangent line.

Figure 2.57 shows a function which appears to be differentiable at all points except $x = a$ and $x = b$. There is no tangent at A because the graph has a corner there. As x approaches a from the left, the slope of the line joining P to A converges to some positive number. As x approaches a from the right, the slope of the line joining P to A converges to some negative number. Thus the slopes approach different numbers as we approach $x = a$ from different sides. Therefore the function is not differentiable at $x = a$. At B, the graph has a vertical tangent. As x approaches b, the slope of the line joining B to Q does not approach a limit; it just keeps growing larger and larger. Again, the limit defining the derivative does not exist and the function is not differentiable at $x = b$.
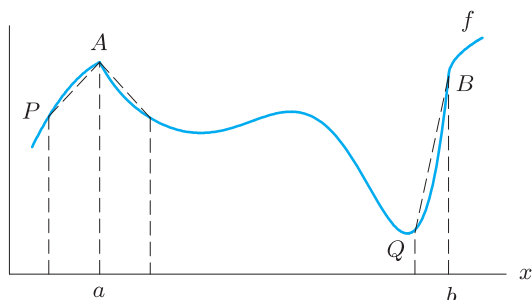

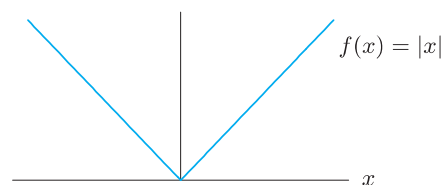**Figure 2.57:** A function which is not differentiable at A or B

Figure 2.58: Graph of absolute value function, showing point of non-differentiability at $x = 0$

Examples of Nondifferentiable Functions

An example of a function whose graph has a corner is the *absolute value* function defined as follows:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

This function is called *piecewise linear* because each part of it is linear. Its graph is in Figure 2.58. Near $x = 0$, even close-up views of the graph of $f(x)$ look the same, so this is a corner which can't be straightened out by zooming in.

Example 1   Try to compute the derivative of the function $f(x) = |x|$ at $x = 0$. Is f differentiable there?

Solution   To find the derivative at $x = 0$, we want to look at

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

As h approaches 0 from the right, h is positive, so $|h| = h$, and the ratio is always 1. As h approaches 0 from the left, h is negative, so $|h| = -h$, and the ratio is -1. Since the limits are different from each side, the limit of the difference quotient does not exist. Thus, the absolute value function is not differentiable at $x = 0$. The limits of 1 and -1 correspond to the fact that the slope of the right-hand part of the graph is 1, and the slope of the left-hand part is -1.

Example 2   Investigate the differentiability of $f(x) = x^{1/3}$ at $x = 0$.

Solution   This function is smooth at $x = 0$ (no sharp corners) but appears to have a vertical tangent there. (See Figure 2.59.) Looking at the difference quotient at $x = 0$, we see

$$\lim_{h \to 0} \frac{(0 + h)^{1/3} - 0^{1/3}}{h} = \lim_{h \to 0} \frac{h^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}.$$
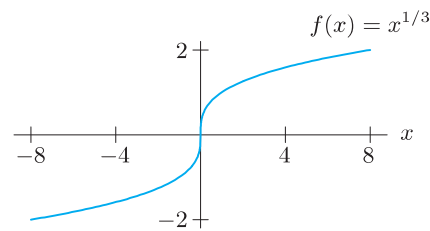
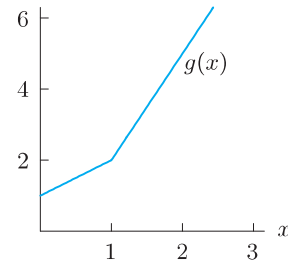**Figure 2.59:** Continuous function not differentiable at $x = 0$: Vertical tangent

Figure 2.60: Continuous function not differentiable at $x = 1$

As $h \to 0$ the denominator becomes small, so the fraction grows without bound. Hence, the function fails to have a derivative at $x = 0$.

Example 3 Consider the function given by the formulas

$$g(x) = \begin{cases} x + 1 & \text{if} \quad x \le 1 \\ 3x - 1 & \text{if} \quad x > 1. \end{cases}$$

Draw the graph of g. Is g continuous? Is g differentiable at $x = 1$?

Solution The graph in Figure 2.60 has no breaks in it, so the function is continuous. However, the graph has a corner at $x = 1$ which no amount of magnification will remove. To the left of $x = 1$, the slope is 1; to the right of $x = 1$, the slope is 3. Thus, the difference quotient at $x = 1$ has no limit, so the function g is not differentiable at $x = 1$.

The Relationship Between Differentiability and Continuity

The fact that a function which is differentiable at a point has a tangent line suggests that the function is continuous there, as the next theorem shows.

Theorem 2.1: A Differentiable Function Is Continuous

If $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is continuous at $x = a$.

Proof We assume f is differentiable at $x = a$. Then we know that $f'(a)$ exists where

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

To show that f is continuous at $x = a$, we want to show that $\lim_{x \to a} f(x) = f(a)$. We calculate $\lim_{x \to a}(f(x) - f(a))$, hoping to get 0. By algebra, we know that for $x \ne a$,

$$f(x) - f(a) = (x - a) \cdot \frac{f(x) - f(a)}{x - a}.$$

Taking the limits, we have

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}\left((x-a)\frac{f(x) - f(a)}{x - a}\right)$$

$$= \left(\lim_{x \to a}(x-a)\right) \cdot \left(\lim_{x \to a}\frac{f(x) - f(a)}{x - a}\right) \qquad \text{(By Theorem 1.2, Property 3)}$$

$$= 0 \cdot f'(a) = 0. \qquad \text{(Since } f'(a) \text{ exists)}$$

Thus we know that $\lim_{x \to a} f(x) = f(a)$, which means $f(x)$ is continuous at $x = a$.
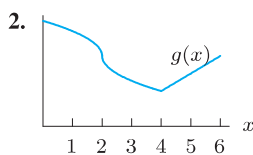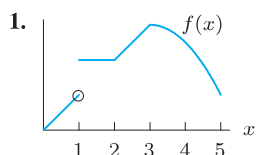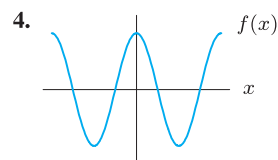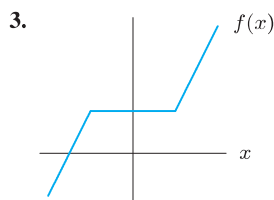
Exercises and Problems for Section 2.6

Exercises

For the graphs in Exercises 1–2, list the x-values for which the function appears to be

(a) Not continuous.   (b) Not differentiable.

1.

2.

In Exercises 3–4, does the function appear to be differentiable on the interval of x-values shown?

3.

4.

Problems

Decide if the functions in Problems 5–7 are differentiable at $x = 0$. Try zooming in on a graphing calculator, or calculating the derivative $f'(0)$ from the definition.

5. $f(x) = (x + |x|)^2 + 1$

6. $f(x) = \begin{cases} x\sin(1/x) + x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

7. $f(x) = \begin{cases} x^2\sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

8. In each of the following cases, sketch the graph of a continuous function $f(x)$ with the given properties.

(a) $f''(x) > 0$ for $x < 2$ and for $x > 2$ and $f'(2)$ is undefined.

(b) $f''(x) > 0$ for $x < 2$ and $f''(x) < 0$ for $x > 2$ and $f'(2)$ is undefined.

9. Look at the graph of $f(x) = (x^2 + 0.0001)^{1/2}$ shown in Figure 2.61. The graph of f appears to have a sharp corner at $x = 0$. Do you think f has a derivative at $x = 0$?
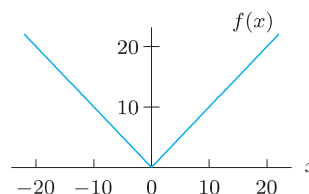

Figure 2.61

10. The acceleration due to gravity, g, varies with height above the surface of the earth, in a certain way. If you go down below the surface of the earth, g varies in a different way. It can be shown that g is given by

$$g = \begin{cases} \dfrac{GMr}{R^3} & \text{for } r < R \\[2mm] \dfrac{GM}{r^2} & \text{for } r \geq R \end{cases}$$

where R is the radius of the earth, M is the mass of the earth, G is the gravitational constant, and r is the distance to the center of the earth.

   (a) Sketch a graph of g against r.
   (b) Is g a continuous function of r? Explain your answer.
   (c) Is g a differentiable function of r? Explain your answer.

11. An electric charge, Q, in a circuit is given as a function of time, t, by

$$Q = \begin{cases} C & \text{for } t \leq 0 \\ Ce^{-t/RC} & \text{for } t > 0, \end{cases}$$

where C and R are positive constants. The electric current, I, is the rate of change of charge, so

$$I = \frac{dQ}{dt}.$$

   (a) Is the charge, Q, a continuous function of time?
   (b) Do you think the current, I, is defined for all times, t? [Hint: To graph this function, take, for example, $C = 1$ and $R = 1$.]

12. A magnetic field, B, is given as a function of the distance, r, from the center of a wire as follows:

$$B = \begin{cases} \dfrac{r}{r_0}B_0 & \text{for } r \leq r_0 \\[2ex] \dfrac{r_0}{r}B_0 & \text{for } r > r_0. \end{cases}$$

   (a) Sketch a graph of B against r. What is the meaning of the constant B_0?
   (b) Is B continuous at $r = r_0$? Give reasons.
   (c) Is B differentiable at $r = r_0$? Give reasons.

13. A cable is made of an insulating material in the shape of a long, thin cylinder of radius r_0. It has electric charge distributed evenly throughout it. The electric field, E, at a distance r from the center of the cable is given by

$$E = \begin{cases} kr & \text{for } r \leq r_0 \\ k\dfrac{r_0^2}{r} & \text{for } r > r_0. \end{cases}$$

   (a) Is E continuous at $r = r_0$?
   (b) Is E differentiable at $r = r_0$?
   (c) Sketch a graph of E as a function of r.

14. Graph the function defined by

$$g(r) = \begin{cases} 1 + \cos(\pi r/2) & \text{for } -2 \leq r \leq 2 \\ 0 & \text{for } r < -2 \text{ or } r > 2. \end{cases}$$

   (a) Is g continuous at $r = 2$? Explain your answer.
   (b) Do you think g is differentiable at $r = 2$? Explain your answer.

15. The potential, ϕ, of a charge distribution at a point on the y-axis is given by

$$\phi = \begin{cases} 2\pi\sigma\left(\sqrt{y^2 + a^2} - y\right) & \text{for } y \geq 0 \\[1ex] 2\pi\sigma\left(\sqrt{y^2 + a^2} + y\right) & \text{for } y < 0 \end{cases}$$

where σ and a are positive constants. [Hint: To graph this function, take, for example, $2\pi\sigma = 1$ and $a = 1$.]

   (a) Is ϕ continuous at $y = 0$?
   (b) Do you think ϕ is differentiable at $y = 0$?

16. Sometimes, odd behavior can be hidden beneath the surface of a rather normal-looking function. Consider the following function:

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

   (a) Sketch a graph of this function. Does it have any vertical segments or corners? Is it differentiable everywhere? If so, sketch the derivative f' of this function. [Hint: You may want to use the result of Example 4 on page 89.]
   (b) Is the derivative function, $f'(x)$, differentiable everywhere? If not, at what point(s) is it not differentiable? Draw the second derivative of $f(x)$ wherever it exists. Is the second derivative function, $f''(x)$, differentiable? Continuous?

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Rate of change**
  Average, instantaneous.
- **Definition of derivative**
  Difference quotient, limit.
- **Estimating and computing derivatives**
  Estimate derivatives from a graph, table of values, or formula. Use definition to find derivatives of simple functions algebraically. Know derivatives of constant, linear, and power functions.

- **Interpretation of derivatives**
  Rate of change, instantaneous velocity, slope, using units.
- **Second derivative**
  Concavity, acceleration.
- **Working with derivatives**
  Understand relation between sign of f' and whether f is increasing or decreasing. Sketch graph of f' from graph of f.
- **Differentiability**

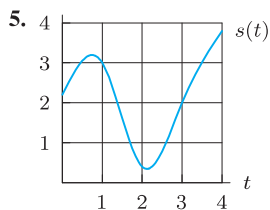# REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TWO

Exercises

In Exercises 1–6, find the average velocity for the position function $s(t)$, in mm, over the interval $1 \leq t \leq 3$, where t is in seconds.
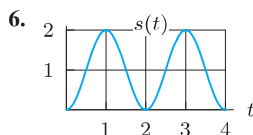
1. $s(t) = 12t - t^2$

2. $s(t) = \ln(t)$

3.

t	0	1	2	3
$s(t)$	7	3	7	11

4.

t	0	1	2	3
$s(t)$	8	4	2	4

5.

6.

Sketch the graphs of the derivatives of the functions shown in Exercises 7–12. Be sure your sketches are consistent with the important features of the graphs of the original functions.
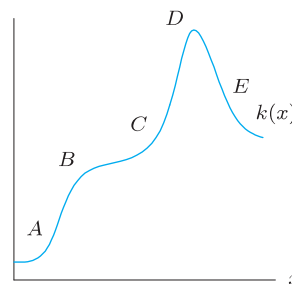
7.

8.

9.

10.

11.

12.

In Exercises 13–14, graph the second derivative of each of the given functions.

13.

14.

Find a formula for the derivatives of the functions in Exercises 15–16 algebraically.

15. $f(x) = 5x^2 + x$

16. $n(x) = (1/x) + 1$

17. Find the derivative of $f(x) = x^2 + 1$ at $x = 3$ algebraically. Find the equation of the tangent line to f at $x = 3$.

18. **(a)** Between which pair of consecutive points in Figure 2.62 is the average rate of change of k

  (a) Greatest? **(b)** Closest to zero?

  (b) Between which two pairs of consecutive points are the average rates of change of k closest?

Figure 2.62

Use algebra to evaluate the limits in Exercises 19–23. Assume $a > 0$.

19. $\displaystyle\lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}$

20. $\displaystyle\lim_{h \to 0} \frac{1/(a+h) - 1/a}{h}$

21. $\displaystyle\lim_{h \to 0} \frac{1/(a+h)^2 - 1/a^2}{h}$

22. $\displaystyle\lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}$ [Hint: Multiply by $\sqrt{a+h} + \sqrt{a}$ in numerator and denominator.]

23. $\displaystyle\lim_{h \to 0} \frac{1/\sqrt{a+h} - 1/\sqrt{a}}{h}$

Problems

24. Sketch the graph of a function whose first and second derivatives are everywhere positive.

25. Figure 2.63 gives the position, $y = s(t)$, of a particle at time t. Arrange the following numbers from smallest to largest:

- The instantaneous velocity at A.
- The instantaneous velocity at B.
- The instantaneous velocity at C.
- The average velocity between A and B.
- The number 0.
- The number 1.

Figure 2.63

26. The table[12] gives the number of passenger cars, $C = f(t)$, in millions, in the US in the year t.

(a) During the period 2002–2006, when is $f'(t)$ positive? Negative?

(b) Estimate $f'(2003)$. Using units, interpret your answer in terms of passenger cars.

t (year)	2002	2003	2004	2005	2006
C (cars, in millions)	135.9	135.7	136.4	136.6	135.4

27. Let $f(t)$ be the depth, in centimeters, of water in a tank at time t, in minutes.

(a) What does the sign of $f'(t)$ tell us?

(b) Explain the meaning of $f'(30) = 20$. Include units.

(c) Using the information in part (b), at time $t = 30$ minutes, find the rate of change of depth, in meters, with respect to time in hours. Give units.
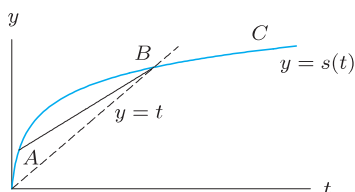
28. The revenue, in thousands of dollars, earned by a gas station when the price of gas is $\$p$ per gallon is $R(p)$.

(a) What are the units of $R'(3)$? Interpret this quantity.

(b) What are the units of $(R^{-1})'(5)$? Interpret this quantity.

29. (a) Give an example of a function with $\lim_{x \to 2} f(x) = \infty$.

(b) Give an example of a function with $\lim_{x \to 2} f(x) = -\infty$.

30. Suppose $f(2) = 3$ and $f'(2) = 1$. Find $f(-2)$ and $f'(-2)$, assuming that $f(x)$ is

(a) Even   (b) Odd.

31. Do the values for the function $y = k(x)$ in the table suggest that the graph of $k(x)$ is concave up or concave down for $1 \leq x \leq 3.3$? Write a sentence in support of your conclusion.

x	1.0	1.2	1.5	1.9	2.5	3.3
$k(x)$	4.0	3.8	3.6	3.4	3.2	3.0

32. Suppose that $f(x)$ is a function with $f(20) = 345$ and $f'(20) = 6$. Estimate $f(22)$.

33. Students were asked to evaluate $f'(4)$ from the following table which shows values of the function f:

x	1	2	3	4	5	6
$f(x)$	4.2	4.1	4.2	4.5	5.0	5.7

- Student A estimated the derivative as $f'(4) \approx \dfrac{f(5) - f(4)}{5 - 4} = 0.5$.
- Student B estimated the derivative as $f'(4) \approx \dfrac{f(4) - f(3)}{4 - 3} = 0.3$.
- Student C suggested that they should split the difference and estimate the average of these two results, that is, $f'(4) \approx \frac{1}{2}(0.5 + 0.3) = 0.4$.

(a) Sketch the graph of f, and indicate how the three estimates are represented on the graph.

(b) Explain which answer is likely to be best.

(c) Use Student C's method to find an algebraic formula to approximate $f'(x)$ using increments of size h.

34. Use Figure 2.64 to fill in the blanks in the following statements about the function f at point A.

(a) $f(\_\_) = \_\_$ (b) $f'(\_\_) = \_\_$

Figure 2.64

35. Use Figure 2.65. At point A, we are told that $x = 1$. In addition, $f(1) = 3$, $f'(1) = 2$, and $h = 0.1$. What are the values of $x_1, x_2, x_3, y_1, y_2, y_3$?
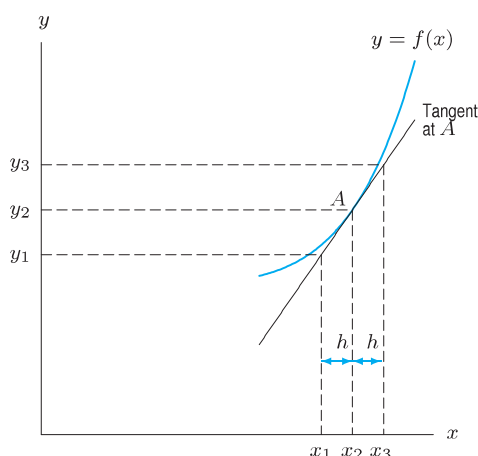

Figure 2.65

36. Given all of the following information about a function f, sketch its graph.

- $f(x) = 0$ at $x = -5, x = 0$, and $x = 5$
- $\lim\limits_{x \to -\infty} f(x) = \infty$
- $\lim\limits_{x \to \infty} f(x) = -3$
- $f'(x) = 0$ at $x = -3$, $x = 2.5$, and $x = 7$

37. A yam has just been taken out of the oven and is cooling off before being eaten. The temperature, T, of the yam (measured in degrees Fahrenheit) is a function of how long it has been out of the oven, t (measured in minutes). Thus, we have $T = f(t)$.

- **(a)** Is $f'(t)$ positive or negative? Why?
- **(b)** What are the units for $f'(t)$?

38. An economist is interested in how the price of a certain commodity affects its sales. Suppose that at a price of $\$p$, a quantity q of the commodity is sold. If $q = f(p)$, explain in economic terms the meaning of the statements $f(10) = 240,000$ and $f'(10) = -29,000$.

39. At time, t, in years, the US population is growing at 0.8% per year times its size, $P(t)$, at that moment. Using the derivative, write an equation representing this statement.

40. (a) Using the table, estimate $f'(0.6)$ and $f'(0.5)$.
- **(b)** Estimate $f''(0.6)$.
- **(c)** Where do you think the maximum and minimum values of f occur in the interval $0 \leq x \leq 1$?

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	3.7	3.5	3.5	3.9	4.0	3.9

41. Let $g(x) = \sqrt{x}$ and $f(x) = kx^2$, where k is a constant.

- **(a)** Find the slope of the tangent line to the graph of g at the point $(4, 2)$.
- **(b)** Find the equation of this tangent line.
- **(c)** If the graph of f contains the point $(4, 2)$, find k.
- **(d)** Where does the graph of f intersect the tangent line found in part (b)?

42. A circle with center at the origin and radius of length $\sqrt{19}$ has equation $x^2 + y^2 = 19$. Graph the circle.

- **(a)** Just from looking at the graph, what can you say about the slope of the line tangent to the circle at the point $(0, \sqrt{19})$? What about the slope of the tangent at $(\sqrt{19}, 0)$?
- **(b)** Estimate the slope of the tangent to the circle at the point $(2, -\sqrt{15})$ by graphing the tangent carefully at that point.
- **(c)** Use the result of part (b) and the symmetry of the circle to find slopes of the tangents drawn to the circle at $(-2, \sqrt{15})$, $(-2, -\sqrt{15})$, and $(2, \sqrt{15})$.

43. Each of the graphs in Figure 2.66 shows the position of a particle moving along the x-axis as a function of time, $0 \leq t \leq 5$. The vertical scales of the graphs are the same. During this time interval, which particle has

- **(a)** Constant velocity?
- **(b)** The greatest initial velocity?
- **(c)** The greatest average velocity?
- **(d)** Zero average velocity?
- **(e)** Zero acceleration?
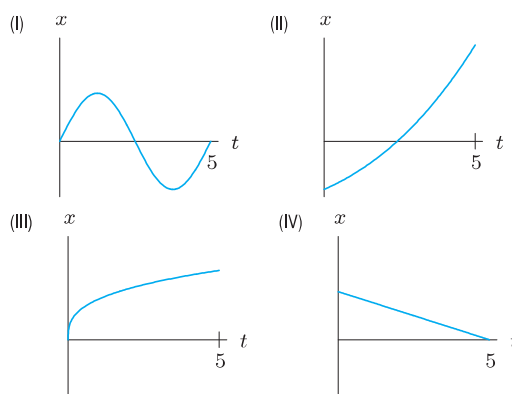- **(f)** Positive acceleration throughout?

Figure 2.66

44. The population of a herd of deer is modeled by

$$P(t) = 4000 + 400 \sin\left(\frac{\pi}{6}t\right) + 180 \sin\left(\frac{\pi}{3}t\right)$$

where t is measured in months from the first of April.

- **(a)** Use a calculator or computer to sketch a graph showing how this population varies with time.

Use the graph to answer the following questions.

(b) When is the herd largest? How many deer are in it at that time?

(c) When is the herd smallest? How many deer are in it then?

(d) When is the herd growing the fastest? When is it shrinking the fastest?

(e) How fast is the herd growing on April 1?

45. The number of hours, H, of daylight in Madrid is a function of t, the number of days since the start of the year. Figure 2.67 shows a one-month portion of the graph of H.
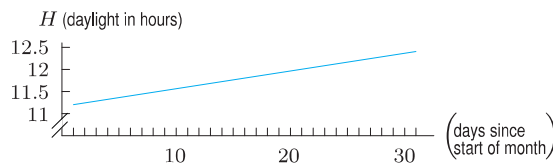

Figure 2.67

(a) Comment on the shape of the graph. Why does it look like a straight line?

(b) What month does this graph show? How do you know?

(c) What is the approximate slope of this line? What does the slope represent in practical terms?

46. Suppose you put a yam in a hot oven, maintained at a constant temperature of $200°$C. As the yam picks up heat from the oven, its temperature rises.[13]

(a) Draw a possible graph of the temperature T of the yam against time t (minutes) since it is put into the oven. Explain any interesting features of the graph, and in particular explain its concavity.

(b) Suppose that, at $t = 30$, the temperature T of the yam is $120°$ and increasing at the (instantaneous) rate of $2°$/min. Using this information, plus what you know about the shape of the T graph, estimate the temperature at time $t = 40$.

(c) Suppose in addition you are told that at $t = 60$, the temperature of the yam is $165°$. Can you improve your estimate of the temperature at $t = 40$?

(d) Assuming all the data given so far, estimate the time at which the temperature of the yam is $150°$.

47. You are given the following values for the error function, $\mathrm{erf}(x)$.

$$\mathrm{erf}(0) = 0 \quad \mathrm{erf}(1) = 0.84270079$$
$$\mathrm{erf}(0.1) = 0.11246292 \quad \mathrm{erf}(0.01) = 0.01128342.$$

(a) Use all this information to determine your best estimate for $\mathrm{erf}'(0)$. (Give only those digits of which you feel reasonably certain.)

(b) Suppose you find that $\mathrm{erf}(0.001) = 0.00112838$. How does this extra information change your answer to part (a)?

48. (a) Use your calculator to approximate the derivative of the hyperbolic sine function (written $\sinh x$) at the points 0, 0.3, 0.7, and 1.

(b) Can you find a relation between the values of this derivative and the values of the hyperbolic cosine (written $\cosh x$)?

CAS Challenge Problems

49. Use a computer algebra system to find the derivative of $f(x) = \sin^2 x + \cos^2 x$ and simplify your answer. Explain your result.

50. (a) Use a computer algebra system to find the derivative of $f(x) = 2 \sin x \cos x$.

(b) Simplify $f(x)$ and $f'(x)$ using double angle formulas. Write down the derivative formula that you get after doing this simplification.

51. (a) Use a computer algebra system to find the second derivative of $g(x) = e^{-ax^2}$ with respect to x.

(b) Graph $g(x)$ and $g''(x)$ on the same axes for $a = 1, 2, 3$ and describe the relation between the two graphs.

(c) Explain your answer to part (b) in terms of concavity.

52. (a) Use a computer algebra system to find the derivative of $f(x) = \ln(x)$, $g(x) = \ln(2x)$, and $h(x) = \ln(3x)$. What is the relationship between the answers?

(b) Use the properties of logarithms to explain what you see in part (a).

53. (a) Use a computer algebra system to find the derivative of $(x^2 + 1)^2$, $(x^2 + 1)^3$, and $(x^2 + 1)^4$.

(b) Conjecture a formula for the derivative of $(x^2 + 1)^n$ that works for any integer n. Check your formula using the computer algebra system.

54. (a) Use a computer algebra system to find the derivatives of $\sin x$, $\cos x$ and $\sin x \cos x$.

(b) Is the derivative of a product of two functions always equal to the product of their derivatives?

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[13]From Peter D. Taylor, *Calculus: The Analysis of Functions* (Toronto: Wall & Emerson, Inc., 1992).

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–22 true or false? Give an explanation for your answer.

1. If a car is going 50 miles per hour at 2 pm and 60 miles per hour at 3 pm then it travels between 50 and 60 miles during the hour between 2 pm and 3 pm.

2. If a car travels 80 miles between 2 and 4 pm, then its velocity is close to 40 mph at 2 pm.

3. If the time interval is short enough, then the average velocity of a car over the time interval and the instantaneous velocity at a time in the interval can be expected to be close.

4. If an object moves with the same average velocity over every time interval, then its average velocity equals its instantaneous velocity at any time.

5. The formula Distance traveled $=$ Average velocity $\times$ Time is valid for every moving object for every time interval.

6. By definition, the instantaneous velocity of an object equals a difference quotient.

7. If $f(x)$ is concave up, then $f'(a) < (f(b) - f(a))/(b - a)$ for $a < b$.

8. You cannot be sure of the exact value of a derivative of a function at a point using only the information in a table of values of the function. The best you can do is find an approximation.

9. If $f'(x)$ is increasing, then $f(x)$ is also increasing.

10. If $f(a) \neq g(a)$, then $f'(a) \neq g'(a)$.

11. The derivative of a linear function is constant.

12. If $g(x)$ is a vertical shift of $f(x)$, then $f'(x) = g'(x)$.

13. If $f(x)$ is defined for all x but $f'(0)$ is not defined, then the graph of $f(x)$ must have a corner at the point where $x = 0$.

14. If $y = f(x)$, then $\left.\dfrac{dy}{dx}\right|_{x=a} = f'(a)$.

15. If you zoom in (with your calculator) on the graph of $y = f(x)$ in a small interval around $x = 10$ and see a straight line, then the slope of that line equals the derivative $f'(10)$.

16. If $f''(x) > 0$ then $f'(x)$ is increasing.

17. The instantaneous acceleration of a moving particle at time t is the limit of difference quotients.

18. If $f(t)$ is the quantity in grams of a chemical produced after t minutes and $g(t)$ is the same quantity in kilograms, then $f'(t) = 1000g'(t)$.

19. If $f(t)$ is the quantity in kilograms of a chemical produced after t minutes and $g(t)$ is the quantity in kilograms produced after t seconds, then $f'(t) = 60g'(t)$.

20. A function which is monotonic on an interval is either increasing or decreasing on the interval.

21. The function $f(x) = x^3$ is monotonic on any interval.

22. The function $f(x) = x^2$ is monotonic on any interval.

Are the statements in Problems 23–27 true or false? If a statement is true, give an example illustrating it. If a statement is false, give a counterexample.

23. There is a function which is continuous on $[1, 5]$ but not differentiable at $x = 3$.

24. If a function is differentiable, then it is continuous.

25. If a function is continuous, then it is differentiable.

26. If a function is not continuous, then it is not differentiable.

27. If a function is not differentiable, then it is not continuous.

28. Which of the following would be a counterexample to the statement: "If f is differentiable at $x = a$ then f is continuous at $x = a$"?

    (a) A function which is not differentiable at $x = a$ but is continuous at $x = a$.
    (b) A function which is not continuous at $x = a$ but is differentiable at $x = a$.
    (c) A function which is both continuous and differentiable at $x = a$.
    (d) A function which is neither continuous nor differentiable at $x = a$.

PROJECTS FOR CHAPTER TWO

1. **Hours of Daylight as a Function of Latitude**

   Let $S(x)$ be the number of sunlight hours on a cloudless June 21, as a function of latitude, x, measured in degrees.

   (a) What is $S(0)$?

   (b) Let x_0 be the latitude of the Arctic Circle ($x_0 \approx 66°30'$). In the northern hemisphere, $S(x)$ is given, for some constants a and b, by the formula:

   $$S(x) = \begin{cases} a + b \arcsin \left(\dfrac{\tan x}{\tan x_0} \right) & \text{for } 0 \le x < x_0 \\ 24 & \text{for } x_0 \le x \le 90. \end{cases}$$

   Find a and b so that $S(x)$ is continuous.

   (c) Calculate $S(x)$ for Tucson, Arizona ($x = 32°13'$) and Walla Walla, Washington ($46°4'$).

   (d) Graph $S(x)$, for $0 \le x \le 90$.

   (e) Does $S(x)$ appear to be differentiable?

2. **US Population**

   Census figures for the US population (in millions) are listed in Table 2.12. Let f be the function such that $P = f(t)$ is the population (in millions) in year t.

   Table 2.12 *US population (in millions), 1790–2000*

Year	Population	Year	Population	Year	Population	Year	Population
1790	3.9	1850	23.1	1910	92.0	1970	205.0
1800	5.3	1860	31.4	1920	105.7	1980	226.5
1810	7.2	1870	38.6	1930	122.8	1990	248.7
1820	9.6	1880	50.2	1940	131.7	2000	281.4
1830	12.9	1890	62.9	1950	150.7		
1840	17.1	1900	76.0	1960	179.0		

   (a) (i) Estimate the rate of change of the population for the years 1900, 1945, and 2000.

   (ii) When, approximately, was the rate of change of the population greatest?

   (iii) Estimate the US population in 1956.

   (iv) Based on the data from the table, what would you predict for the census in the year 2010?

   (b) Assume that f is increasing (as the values in the table suggest). Then f is invertible.

   (i) What is the meaning of $f^{-1}(100)$?

   (ii) What does the derivative of $f^{-1}(P)$ at $P = 100$ represent? What are its units?

   (iii) Estimate $f^{-1}(100)$.

   (iv) Estimate the derivative of $f^{-1}(P)$ at $P = 100$.

   (c) (i) Usually we think the US population $P = f(t)$ as a smooth function of time. To what extent is this justified? What happens if we zoom in at a point of the graph? What about events such as the Louisiana Purchase? Or the moment of your birth?

   (ii) What do we in fact mean by the rate of change of the population at a particular time t?

   (iii) Give another example of a real-world function which is not smooth but is usually treated as such.