

## Chapter Three

# SHORT-CUTS TO DIFFERENTIATION

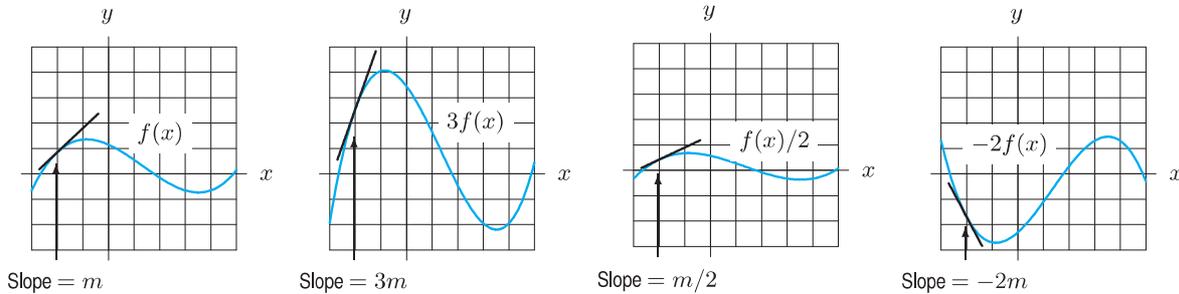
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### 3.1 POWERS AND POLYNOMIALS

#### Derivative of a Constant Times a Function

Figure 3.1 shows the graph of  $y = f(x)$  and of three multiples:  $y = 3f(x)$ ,  $y = \frac{1}{2}f(x)$ , and  $y = -2f(x)$ . What is the relationship between the derivatives of these functions? In other words, for a particular  $x$ -value, how are the slopes of these graphs related?



**Figure 3.1:** A function and its multiples: Derivative of multiple is multiple of derivative

Multiplying the value of a function by a constant stretches or shrinks the original graph (and reflects it in the  $x$ -axis if the constant is negative). This changes the slope of the curve at each point. If the graph has been stretched, the “rises” have all been increased by the same factor, whereas the “runs” remain the same. Thus, the slopes are all steeper by the same factor. If the graph has been shrunk, the slopes are all smaller by the same factor. If the graph has been reflected in the  $x$ -axis, the slopes will all have their signs reversed. In other words, if a function is multiplied by a constant,  $c$ , so is its derivative:

**Theorem 3.1: Derivative of a Constant Multiple**

If  $f$  is differentiable and  $c$  is a constant, then

$$\frac{d}{dx} [cf(x)] = cf'(x).$$

**Proof** Although the graphical argument makes the theorem plausible, to prove it we must use the definition of the derivative:

$$\begin{aligned} \frac{d}{dx} [cf(x)] &= \lim_{h \rightarrow 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \rightarrow 0} c \frac{f(x+h) - f(x)}{h} \\ &= c \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = cf'(x). \end{aligned}$$

We can take  $c$  across the limit sign by the properties of limits (part 1 of Theorem 1.2 on page 53).

#### Derivatives of Sums and Differences

Suppose we have two functions,  $f(x)$  and  $g(x)$ , with the values listed in Table 3.1. Values of the sum  $f(x) + g(x)$  are given in the same table.

**Table 3.1** Sum of Functions

$x$	$f(x)$	$g(x)$	$f(x) + g(x)$
0	100	0	100
1	110	0.2	110.2
2	130	0.4	130.4
3	160	0.6	160.6
4	200	0.8	200.8

We see that adding the increments of  $f(x)$  and the increments of  $g(x)$  gives the increments of  $f(x) + g(x)$ . For example, as  $x$  increases from 0 to 1,  $f(x)$  increases by 10 and  $g(x)$  increases by 0.2, while  $f(x) + g(x)$  increases by  $110.2 - 100 = 10.2$ . Similarly, as  $x$  increases from 3 to 4,  $f(x)$  increases by 40 and  $g(x)$  by 0.2, while  $f(x) + g(x)$  increases by  $200.8 - 160.6 = 40.2$ .

From this example, we see that the rate at which  $f(x) + g(x)$  is increasing is the sum of the rates at which  $f(x)$  and  $g(x)$  are increasing. Similar reasoning applies to the difference,  $f(x) - g(x)$ . In terms of derivatives:

### Theorem 3.2: Derivative of Sum and Difference

If  $f$  and  $g$  are differentiable, then

$$\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x) \quad \text{and} \quad \frac{d}{dx} [f(x) - g(x)] = f'(x) - g'(x).$$

**Proof** Using the definition of the derivative:

$$\begin{aligned} \frac{d}{dx} [f(x) + g(x)] &= \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \left[ \underbrace{\frac{f(x+h) - f(x)}{h}}_{\text{Limit of this is } f'(x)} + \underbrace{\frac{g(x+h) - g(x)}{h}}_{\text{Limit of this is } g'(x)} \right] \\ &= f'(x) + g'(x). \end{aligned}$$

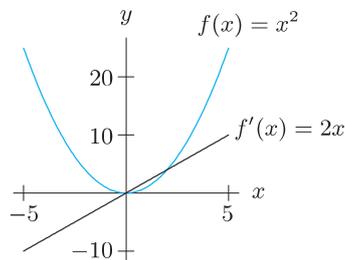
We have used the fact that the limit of a sum is the sum of the limits, part 2 of Theorem 1.2 on page 53. The proof for  $f(x) - g(x)$  is similar.

## Powers of $x$

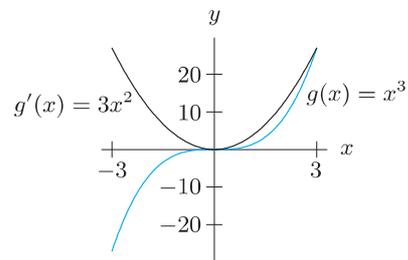
In Chapter 2 we showed that

$$f'(x) = \frac{d}{dx}(x^2) = 2x \quad \text{and} \quad g'(x) = \frac{d}{dx}(x^3) = 3x^2.$$

The graphs of  $f(x) = x^2$  and  $g(x) = x^3$  and their derivatives are shown in Figures 3.2 and 3.3. Notice  $f'(x) = 2x$  has the behavior we expect. It is negative for  $x < 0$  (when  $f$  is decreasing), zero for  $x = 0$ , and positive for  $x > 0$  (when  $f$  is increasing). Similarly,  $g'(x) = 3x^2$  is zero when  $x = 0$ , but positive everywhere else, as  $g$  is increasing everywhere else.



**Figure 3.2:** Graphs of  $f(x) = x^2$  and its derivative  $f'(x) = 2x$



**Figure 3.3:** Graphs of  $g(x) = x^3$  and its derivative  $g'(x) = 3x^2$

These examples are special cases of the power rule which we justify for any positive integer  $n$  on page 119:

### The Power Rule

For any constant real number  $n$ ,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Problem 73 asks you to show that this rule holds for negative integral powers; such powers can also be differentiated using the quotient rule (Section 3.3). In Section 3.6 we indicate how to justify the power rule for powers of the form  $1/n$ .

**Example 1** Use the power rule to differentiate (a)  $\frac{1}{x^3}$ , (b)  $x^{1/2}$ , (c)  $\frac{1}{\sqrt[3]{x}}$ .

**Solution**

(a) For  $n = -3$ :  $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$ .

(b) For  $n = 1/2$ :  $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ .

(c) For  $n = -1/3$ :  $\frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{d}{dx}(x^{-1/3}) = -\frac{1}{3}x^{(-1/3)-1} = -\frac{1}{3}x^{-4/3} = -\frac{1}{3x^{4/3}}$ .

**Example 2** Use the definition of the derivative to justify the power rule for  $n = -2$ : Show  $\frac{d}{dx}(x^{-2}) = -2x^{-3}$ .

**Solution** Provided  $x \neq 0$ , we have

$$\begin{aligned} \frac{d}{dx}(x^{-2}) &= \frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{h \rightarrow 0} \left( \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \right) = \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{x^2 - (x+h)^2}{(x+h)^2 x^2} \right] && \text{(Combining fractions over a common denominator)} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{x^2 - (x^2 + 2xh + h^2)}{(x+h)^2 x^2} \right] && \text{(Multiplying out)} \\ &= \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h(x+h)^2 x^2} && \text{(Simplifying numerator)} \\ &= \lim_{h \rightarrow 0} \frac{-2x - h}{(x+h)^2 x^2} && \text{(Dividing numerator and denominator by } h) \\ &= \frac{-2x}{x^4} && \text{(Letting } h \rightarrow 0) \\ &= -2x^{-3}. \end{aligned}$$

The graphs of  $x^{-2}$  and its derivative,  $-2x^{-3}$ , are shown in Figure 3.4. Does the graph of the derivative have the features you expect?

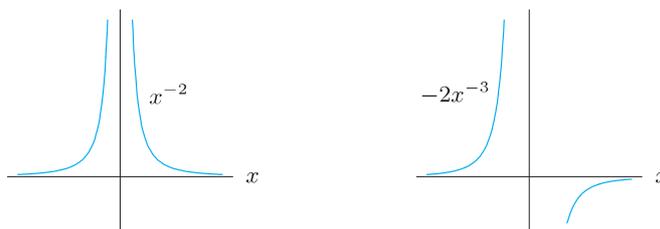


Figure 3.4: Graphs of  $x^{-2}$  and its derivative,  $-2x^{-3}$

### Justification of $\frac{d}{dx}(x^n) = nx^{n-1}$ , for $n$ a Positive Integer

To calculate the derivatives of  $x^2$  and  $x^3$ , we had to expand  $(x+h)^2$  and  $(x+h)^3$ . To calculate the derivative of  $x^n$ , we must expand  $(x+h)^n$ . Let's look back at the previous expansions:

$$(x+h)^2 = x^2 + 2xh + h^2, \quad (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$

and multiply out a few more examples:

$$\begin{aligned} (x+h)^4 &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4, \\ (x+h)^5 &= x^5 + 5x^4h + \underbrace{10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}_{\text{Terms involving } h^2 \text{ and higher powers of } h}. \end{aligned}$$

In general, we can say

$$(x+h)^n = x^n + nx^{n-1}h + \underbrace{\dots + h^n}_{\text{Terms involving } h^2 \text{ and higher powers of } h}.$$

We have just seen this is true for  $n = 2, 3, 4, 5$ . It can be proved in general using the Binomial Theorem (see [www.wiley.com/college/hugheshallett](http://www.wiley.com/college/hugheshallett)). Now to find the derivative,

$$\begin{aligned} \frac{d}{dx}(x^n) &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^n + nx^{n-1}h + \dots + h^n) - x^n}{h} \\ &\quad \text{Terms involving } h^2 \text{ and higher powers of } h \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \dots + h^n}{h}. \end{aligned}$$

When we factor out  $h$  from terms involving  $h^2$  and higher powers of  $h$ , each term will still have an  $h$  in it. Factoring and dividing, we get:

$$\frac{d}{dx}(x^n) = \lim_{h \rightarrow 0} \frac{h(nx^{n-1} + \dots + h^{n-1})}{h} = \lim_{h \rightarrow 0} (nx^{n-1} + \dots + h^{n-1}).$$

Terms involving  $h$  and higher powers of  $h$

But as  $h \rightarrow 0$ , all terms involving an  $h$  will go to 0, so

$$\frac{d}{dx}(x^n) = \lim_{h \rightarrow 0} (nx^{n-1} + \underbrace{\dots + h^{n-1}}_{\text{These terms go to 0}}) = nx^{n-1}.$$

## Derivatives of Polynomials

Now that we know how to differentiate powers, constant multiples, and sums, we can differentiate any polynomial.

**Example 3** Find the derivatives of (a)  $5x^2 + 3x + 2$ , (b)  $\sqrt{3}x^7 - \frac{x^5}{5} + \pi$ .

**Solution**

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(5x^2 + 3x + 2) &= 5 \frac{d}{dx}(x^2) + 3 \frac{d}{dx}(x) + \frac{d}{dx}(2) \\ &= 5 \cdot 2x + 3 \cdot 1 + 0 \quad \text{(Since the derivative of a constant, } \frac{d}{dx}(2), \text{ is zero.)} \\ &= 10x + 3. \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \frac{d}{dx} \left( \sqrt{3}x^7 - \frac{x^5}{5} + \pi \right) &= \sqrt{3} \frac{d}{dx}(x^7) - \frac{1}{5} \frac{d}{dx}(x^5) + \frac{d}{dx}(\pi) \\
 &= \sqrt{3} \cdot 7x^6 - \frac{1}{5} \cdot 5x^4 + 0 \quad (\text{Since } \pi \text{ is a constant, } d\pi/dx = 0.) \\
 &= 7\sqrt{3}x^6 - x^4.
 \end{aligned}$$

We can also use the rules we have seen so far to differentiate expressions which are not polynomials.

**Example 4** Differentiate (a)  $5\sqrt{x} - \frac{10}{x^2} + \frac{1}{2\sqrt{x}}$  (b)  $0.1x^3 + 2x\sqrt{2}$

**Solution**

$$\begin{aligned}
 \text{(a)} \quad \frac{d}{dx} \left( 5\sqrt{x} - \frac{10}{x^2} + \frac{1}{2\sqrt{x}} \right) &= \frac{d}{dx} \left( 5x^{1/2} - 10x^{-2} + \frac{1}{2}x^{-1/2} \right) \\
 &= 5 \cdot \frac{1}{2}x^{-1/2} - 10(-2)x^{-3} + \frac{1}{2} \left( -\frac{1}{2} \right) x^{-3/2} \\
 &= \frac{5}{2\sqrt{x}} + \frac{20}{x^3} - \frac{1}{4x^{3/2}}. \\
 \text{(b)} \quad \frac{d}{dx}(0.1x^3 + 2x\sqrt{2}) &= 0.1 \frac{d}{dx}(x^3) + 2 \frac{d}{dx}(x\sqrt{2}) = 0.3x^2 + 2\sqrt{2}x^{\sqrt{2}-1}.
 \end{aligned}$$

**Example 5** Find the second derivative and interpret its sign for

(a)  $f(x) = x^2$ , (b)  $g(x) = x^3$ , (c)  $k(x) = x^{1/2}$ .

**Solution**

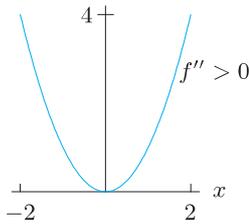
(a) If  $f(x) = x^2$ , then  $f'(x) = 2x$ , so  $f''(x) = \frac{d}{dx}(2x) = 2$ . Since  $f''$  is always positive,  $f$  is concave up, as expected for a parabola opening upward. (See Figure 3.5.)

(b) If  $g(x) = x^3$ , then  $g'(x) = 3x^2$ , so  $g''(x) = \frac{d}{dx}(3x^2) = 3 \frac{d}{dx}(x^2) = 3 \cdot 2x = 6x$ . This is positive for  $x > 0$  and negative for  $x < 0$ , which means  $x^3$  is concave up for  $x > 0$  and concave down for  $x < 0$ . (See Figure 3.6.)

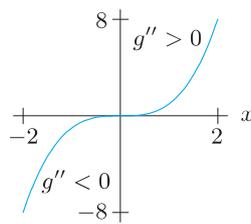
(c) If  $k(x) = x^{1/2}$ , then  $k'(x) = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2}$ , so

$$k''(x) = \frac{d}{dx} \left( \frac{1}{2}x^{-1/2} \right) = \frac{1}{2} \cdot \left( -\frac{1}{2} \right) x^{-(1/2)-1} = -\frac{1}{4}x^{-3/2}.$$

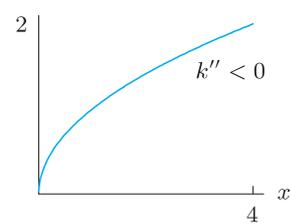
Now  $k'$  and  $k''$  are only defined on the domain of  $k$ , that is,  $x \geq 0$ . When  $x > 0$ , we see that  $k''(x)$  is negative, so  $k$  is concave down. (See Figure 3.7.)



**Figure 3.5:**  $f(x) = x^2$  and  $f''(x) = 2$



**Figure 3.6:**  $g(x) = x^3$  and  $g''(x) = 6x$



**Figure 3.7:**  $k(x) = x^{1/2}$  and  $k''(x) = -\frac{1}{4}x^{-3/2}$

**Example 6** If the position of a body, in meters, is given as a function of time  $t$ , in seconds, by

$$s = -4.9t^2 + 5t + 6,$$

find the velocity and acceleration of the body at time  $t$ .

**Solution** The velocity,  $v$ , is the derivative of the position:

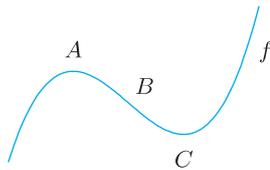
$$v = \frac{ds}{dt} = \frac{d}{dt}(-4.9t^2 + 5t + 6) = -9.8t + 5,$$

and the acceleration,  $a$ , is the derivative of the velocity:

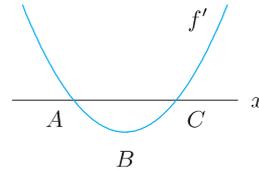
$$a = \frac{dv}{dt} = \frac{d}{dt}(-9.8t + 5) = -9.8.$$

Note that  $v$  is in meters/second and  $a$  is in meters/second<sup>2</sup>.

**Example 7** Figure 3.8 shows the graph of a cubic polynomial. Both graphically and algebraically, describe the behavior of the derivative of this cubic.



**Figure 3.8:** The cubic of Example 7



**Figure 3.9:** Derivative of the cubic of Example 7

**Solution** **Graphical approach:** Suppose we move along the curve from left to right. To the left of  $A$ , the slope is positive; it starts very positive and decreases until the curve reaches  $A$ , where the slope is 0. Between  $A$  and  $C$  the slope is negative. Between  $A$  and  $B$  the slope is decreasing (getting more negative); it is most negative at  $B$ . Between  $B$  and  $C$  the slope is negative but increasing; at  $C$  the slope is zero. From  $C$  to the right, the slope is positive and increasing. The graph of the derivative function is shown in Figure 3.9.

**Algebraic approach:**  $f$  is a cubic that goes to  $+\infty$  as  $x \rightarrow +\infty$ , so

$$f(x) = ax^3 + bx^2 + cx + d$$

with  $a > 0$ . Hence,

$$f'(x) = 3ax^2 + 2bx + c,$$

whose graph is a parabola opening upward, as in Figure 3.9.

## Exercises and Problems for Section 3.1

### Exercises

- Let  $f(x) = 7$ . Using the definition of the derivative, show that  $f'(x) = 0$  for all values of  $x$ .
- Let  $f(x) = 17x + 11$ . Use the definition of the derivative to calculate  $f'(x)$ .

For Exercises 3–5, determine if the derivative rules from this section apply. If they do, find the derivative. If they don't apply, indicate why.

- $y = x^3$
- $y = 3^x$
- $y = x^\pi$

For Exercises 6–47, find the derivatives of the given functions. Assume that  $a$ ,  $b$ ,  $c$ , and  $k$  are constants.

- $y = x^{11}$
- $y = x^{-12}$
- $y = x^{3.2}$
- $y = x^{3/4}$
- $y = x^{12}$
- $y = -x^{-11}$
- $y = x^{4/3}$
- $y = x^{-3/4}$

14.  $f(x) = \frac{1}{x^4}$       15.  $g(t) = \frac{1}{t^5}$       34.  $h(t) = \frac{3}{t} + \frac{4}{t^2}$       35.  $y = \sqrt{\theta} \left( \sqrt{\theta} + \frac{1}{\sqrt{\theta}} \right)$
16.  $f(z) = -\frac{1}{z^{6.1}}$       17.  $y = \frac{1}{r^{7/2}}$       36.  $y = \frac{x^2 + 1}{x}$       37.  $f(z) = \frac{z^2 + 1}{3z}$
18.  $y = \sqrt{x}$       19.  $f(x) = \sqrt[4]{x}$       38.  $f(t) = \frac{t^2 + t^3 - 1}{t^4}$       39.  $y = \frac{\theta - 1}{\sqrt{\theta}}$
20.  $h(\theta) = \frac{1}{\sqrt[3]{\theta}}$       21.  $f(x) = \sqrt{\frac{1}{x^3}}$       40.  $j(x) = \frac{x^3}{a} + \frac{a}{b}x^2 - cx$       41.  $f(x) = \frac{ax + b}{x}$
22.  $f(x) = x^e$       23.  $y = 4x^{3/2} - 5x^{1/2}$       42.  $h(x) = \frac{ax + b}{c}$       43.  $g(t) = \frac{\sqrt{t}(1 + t)}{t^2}$
24.  $f(t) = 3t^2 - 4t + 1$       25.  $y = 17x + 24x^{1/2}$       44.  $\frac{dV}{dr}$  if  $V = \frac{4}{3}\pi r^2 b$       45.  $\frac{dw}{dq}$  if  $w = 3ab^2q$
26.  $y = z^2 + \frac{1}{2z}$       27.  $f(x) = 5x^4 + \frac{1}{x^2}$       46.  $\frac{dy}{dx}$  if  $y = ax^2 + bx + c$       47.  $\frac{dP}{dt}$  if  $P = a + b\sqrt{t}$
28.  $h(w) = -2w^{-3} + 3\sqrt{w}$       29.  $y = 6x^3 + 4x^2 - 2x$
30.  $y = 3t^5 - 5\sqrt{t} + \frac{7}{t}$       31.  $y = 3t^2 + \frac{12}{\sqrt{t}} - \frac{1}{t^2}$
32.  $y = \sqrt{x}(x + 1)$       33.  $y = t^{3/2}(2 + \sqrt{t})$

### Problems

For Problems 48–53, determine if the derivative rules from this section apply. If they do, find the derivative. If they don't apply, indicate why.

48.  $y = (x + 3)^{1/2}$       49.  $y = \pi^x$
50.  $g(x) = x^\pi - x^{-\pi}$       51.  $y = 3x^2 + 4$
52.  $y = \frac{1}{3x^2 + 4}$       53.  $y = \frac{1}{3z^2} + \frac{1}{4}$
54. The graph of  $y = x^3 - 9x^2 - 16x + 1$  has a slope of 5 at two points. Find the coordinates of the points.
55. Find the equation of the line tangent to the graph of  $f$  at  $(1, 1)$ , where  $f$  is given by  $f(x) = 2x^3 - 2x^2 + 1$ .
56. (a) Find the equation of the tangent line to  $f(x) = x^3$  at the point where  $x = 2$ .  
 (b) Graph the tangent line and the function on the same axes. If the tangent line is used to estimate values of the function, will the estimates be overestimates or underestimates?
57. Using a graph to help you, find the equations of all lines through the origin tangent to the parabola
- $$y = x^2 - 2x + 4.$$
- Sketch the lines on the graph.
58. On what intervals is the function  $f(x) = x^4 - 4x^3$  both decreasing and concave up?
59. For what values of  $x$  is the graph of  $y = x^5 - 5x$  both increasing and concave up?
60. If  $f(x) = x^3 - 6x^2 - 15x + 20$ , find analytically all values of  $x$  for which  $f'(x) = 0$ . Show your answers on a graph of  $f$ .
61. (a) Find the *eighth* derivative of  $f(x) = x^7 + 5x^5 - 4x^3 + 6x - 7$ . Think ahead!  
 (The  $n^{\text{th}}$  derivative,  $f^{(n)}(x)$ , is the result of differentiating  $f(x)$   $n$  times.)  
 (b) Find the seventh derivative of  $f(x)$ .
62. Given  $p(x) = x^n - x$ , find the intervals over which  $p$  is a decreasing function when:
- (a)  $n = 2$       (b)  $n = \frac{1}{2}$       (c)  $n = -1$
63. The height of a sand dune (in centimeters) is represented by  $f(t) = 700 - 3t^2$ , where  $t$  is measured in years since 2005. Find  $f(5)$  and  $f'(5)$ . Using units, explain what each means in terms of the sand dune.
64. A ball is dropped from the top of the Empire State building to the ground below. The height,  $y$ , of the ball above the ground (in feet) is given as a function of time,  $t$ , (in seconds) by
- $$y = 1250 - 16t^2.$$
- (a) Find the velocity of the ball at time  $t$ . What is the sign of the velocity? Why is this to be expected?  
 (b) Show that the acceleration of the ball is a constant. What are the value and sign of this constant?  
 (c) When does the ball hit the ground, and how fast is it going at that time? Give your answer in feet per second and in miles per hour (1 ft/sec = 15/22 mph).

65. At a time  $t$  seconds after it is thrown up in the air, a tomato is at a height of  $f(t) = -4.9t^2 + 25t + 3$  meters.
- What is the average velocity of the tomato during the first 2 seconds? Give units.
  - Find (exactly) the instantaneous velocity of the tomato at  $t = 2$ . Give units.
  - What is the acceleration at  $t = 2$ ?
  - How high does the tomato go?
  - How long is the tomato in the air?
66. The gravitational attraction,  $F$ , between the earth and a satellite of mass  $m$  at a distance  $r$  from the center of the earth is given by
- $$F = \frac{GMm}{r^2},$$
- where  $M$  is the mass of the earth, and  $G$  is a constant. Find the rate of change of force with respect to distance.
67. The period,  $T$ , of a pendulum is given in terms of its length,  $l$ , by
- $$T = 2\pi\sqrt{\frac{l}{g}},$$
- where  $g$  is the acceleration due to gravity (a constant).
- Find  $dT/dl$ .
  - What is the sign of  $dT/dl$ ? What does this tell you about the period of pendulums?
68. (a) Use the formula for the area of a circle of radius  $r$ ,  $A = \pi r^2$ , to find  $dA/dr$ .  
 (b) The result from part (a) should look familiar. What does  $dA/dr$  represent geometrically?  
 (c) Use the difference quotient to explain the observation you made in part (b).
69. What is the formula for  $V$ , the volume of a sphere of radius  $r$ ? Find  $dV/dr$ . What is the geometrical meaning of  $dV/dr$ ?
70. Show that for any power function  $f(x) = x^n$ , we have  $f'(1) = n$ .
71. Given a power function of the form  $f(x) = ax^n$ , with  $f'(2) = 3$  and  $f'(4) = 24$ , find  $n$  and  $a$ .
72. Is there a value of  $n$  which makes  $y = x^n$  a solution to the equation  $13x(dy/dx) = y$ ? If so, what value?
73. Using the definition of derivative, justify the formula  $d(x^n)/dx = nx^{n-1}$ .
- For  $n = -1$ ; for  $n = -3$ .
  - For any negative integer  $n$ .

## 3.2 THE EXPONENTIAL FUNCTION

What do we expect the graph of the derivative of the exponential function  $f(x) = a^x$  to look like? The exponential function in Figure 3.10 increases slowly for  $x < 0$  and more rapidly for  $x > 0$ , so the values of  $f'$  are small for  $x < 0$  and larger for  $x > 0$ . Since the function is increasing for all values of  $x$ , the graph of the derivative must lie above the  $x$ -axis. It appears that the graph of  $f'$  may resemble the graph of  $f$  itself.

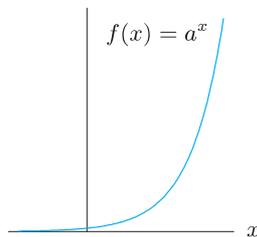


Figure 3.10:  $f(x) = a^x$ , with  $a > 1$

In this section we see that  $f'(x) = k \cdot a^x$ , so in fact  $f'(x)$  is proportional to  $f(x)$ . This property of exponential functions makes them particularly useful in modeling because many quantities have rates of change which are proportional to themselves. For example, the simplest model of population growth has this property.

### Derivatives of Exponential Functions and the Number $e$

We start by calculating the derivative of  $g(x) = 2^x$ , which is given by

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \left( \frac{2^{x+h} - 2^x}{h} \right) = \lim_{h \rightarrow 0} \left( \frac{2^x 2^h - 2^x}{h} \right) = \lim_{h \rightarrow 0} 2^x \left( \frac{2^h - 1}{h} \right) \\ &= \lim_{h \rightarrow 0} \left( \frac{2^h - 1}{h} \right) \cdot 2^x. \quad (\text{Since } x \text{ and } 2^x \text{ are fixed during this calculation). \end{aligned}$$

To find  $\lim_{h \rightarrow 0} (2^h - 1)/h$ , see Table 3.2 where we have substituted values of  $h$  near 0. The table suggests that the limit exists and has value 0.693. Let us call the limit  $k$ , so  $k = 0.693$ . Then

$$\frac{d}{dx}(2^x) = k \cdot 2^x = 0.693 \cdot 2^x.$$

So the derivative of  $2^x$  is proportional to  $2^x$  with constant of proportionality 0.693. A similar calculation shows that the derivative of  $f(x) = a^x$  is

$$f'(x) = \lim_{h \rightarrow 0} \left( \frac{a^{x+h} - a^x}{h} \right) = \lim_{h \rightarrow 0} \left( \frac{a^h - 1}{h} \right) \cdot a^x.$$

Table 3.2

$h$	$(2^h - 1)/h$
-0.1	0.6697
-0.01	0.6908
-0.001	0.6929
0.001	0.6934
0.01	0.6956
0.1	0.7177

Table 3.3

$a$	$k = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$
2	0.693
3	1.099
4	1.386
5	1.609
6	1.792
7	1.946

Table 3.4

$h$	$(1 + h)^{1/h}$
-0.001	2.7196422
-0.0001	2.7184178
-0.00001	2.7182954
0.00001	2.7182682
0.0001	2.7181459
0.001	2.7169239

The quantity  $\lim_{h \rightarrow 0} (a^h - 1)/h$  is also a constant, although the value of the constant depends on  $a$ . Writing  $k = \lim_{h \rightarrow 0} (a^h - 1)/h$ , we see that the derivative of  $f(x) = a^x$  is proportional to  $a^x$ :

$$\frac{d}{dx}(a^x) = k \cdot a^x.$$

For particular values of  $a$ , we can estimate  $k$  by substituting values of  $h$  near 0 into the expression  $(a^h - 1)/h$ . Table 3.3 shows the results. Notice that for  $a = 2$ , the value of  $k$  is less than 1, while for  $a = 3, 4, 5, \dots$ , the values of  $k$  are greater than 1. The values of  $k$  appear to be increasing, so we guess that there is a value of  $a$  between 2 and 3 for which  $k = 1$ . If so, we have found a value of  $a$  with the remarkable property that the function  $a^x$  is equal to its own derivative.

So let's look for such an  $a$ . This means we want to find  $a$  such that

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1, \quad \text{or, for small } h, \quad \frac{a^h - 1}{h} \approx 1.$$

Solving for  $a$ , we can estimate  $a$  as follows:

$$a^h - 1 \approx h, \quad \text{or} \quad a^h \approx 1 + h, \quad \text{so} \quad a \approx (1 + h)^{1/h}.$$

Taking small values of  $h$ , as in Table 3.4, we see  $a \approx 2.718 \dots$ . This is the number  $e$  introduced in Chapter 1. In fact, it can be shown that if

$$e = \lim_{h \rightarrow 0} (1 + h)^{1/h} = 2.718 \dots \quad \text{then} \quad \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

This means that  $e^x$  is its own derivative:

$$\frac{d}{dx}(e^x) = e^x.$$

Figure 3.11 shows the graphs  $2^x$ ,  $3^x$ , and  $e^x$  together with their derivatives. Notice that the derivative of  $2^x$  is below the graph of  $2^x$ , since  $k < 1$  there, and the graph of the derivative of  $3^x$  is above the graph of  $3^x$ , since  $k > 1$  there. With  $e \approx 2.718$ , the function  $e^x$  and its derivative are identical.

### Note on Round-Off Error and Limits

If we try to evaluate  $(1+h)^{1/h}$  on a calculator by taking smaller and smaller values of  $h$ , the values of  $(1+h)^{1/h}$  at first get closer to  $2.718\dots$ . However, they will eventually move away again because of the *round-off error* (i.e., errors introduced by the fact that the calculator can only hold a certain number of digits).

As we try smaller and smaller values of  $h$ , how do we know when to stop? Unfortunately, there is no fixed rule. A calculator can only suggest the value of a limit, but can never confirm that this value is correct. In this case, it looks like the limit is  $2.718\dots$  because the values of  $(e^h - 1)/h$  approach this number for a while. To be sure this is correct, we have to find the limit analytically.

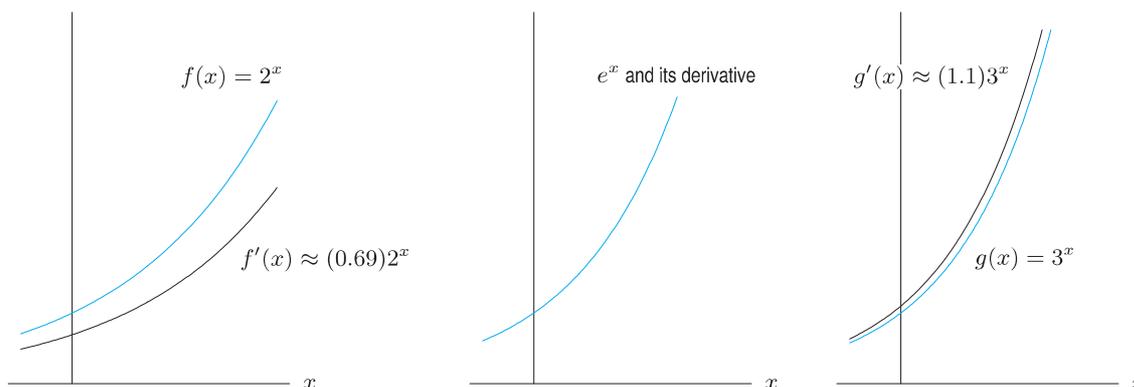


Figure 3.11: Graphs of the functions  $2^x$ ,  $e^x$ , and  $3^x$  and their derivatives

### A Formula for the Derivative of $a^x$

To get a formula for the derivative of  $a^x$ , we must calculate

$$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \underbrace{\left( \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right)}_k a^x.$$

However, without knowing the value of  $a$ , we can't use a calculator to estimate  $k$ . We take a different approach, rewriting  $a = e^{\ln a}$ , so

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \lim_{h \rightarrow 0} \frac{(e^{\ln a})^h - 1}{h} = \lim_{h \rightarrow 0} \frac{e^{(\ln a)h} - 1}{h}.$$

To evaluate this limit, we use a limit that we already know

$$\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1.$$

In order to use this limit, we substitute  $t = (\ln a)h$ . Since  $t$  approaches 0 as  $h$  approaches 0, we have

$$\lim_{h \rightarrow 0} \frac{e^{(\ln a)h} - 1}{h} = \lim_{t \rightarrow 0} \frac{e^t - 1}{(t/\ln a)} = \lim_{t \rightarrow 0} \left( \ln a \cdot \frac{e^t - 1}{t} \right) = \ln a \left( \lim_{t \rightarrow 0} \frac{e^t - 1}{t} \right) = (\ln a) \cdot 1 = \ln a.$$

Thus, we have

$$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \left( \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right) a^x = (\ln a) a^x.$$

In Section 3.6 we obtain the same result by another method. We conclude that:

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

Thus, for any  $a$ , the derivative of  $a^x$  is proportional to  $a^x$ . The constant of proportionality is  $\ln a$ . The derivative of  $a^x$  is equal to  $a^x$  if the constant of proportionality is 1, that is, if  $\ln a = 1$ , then  $a = e$ . The fact that the constant of proportionality is 1 when  $a = e$  makes  $e$  a particularly convenient base for exponential functions.

**Example 1** Differentiate  $2 \cdot 3^x + 5e^x$ .

Solution

$$\frac{d}{dx}(2 \cdot 3^x + 5e^x) = 2 \frac{d}{dx}(3^x) + 5 \frac{d}{dx}(e^x) = 2 \ln 3 \cdot 3^x + 5e^x \approx (2.1972)3^x + 5e^x.$$

## Exercises and Problems for Section 3.2

### Exercises

Find the derivatives of the functions in Exercises 1–26. Assume that  $a$ ,  $b$ ,  $c$ , and  $k$  are constants.

1.  $f(x) = 2e^x + x^2$
2.  $y = 5t^2 + 4e^t$
3.  $y = 5^x + 2$
4.  $f(x) = 12e^x + 11^x$
5.  $y = 5x^2 + 2^x + 3$
6.  $f(x) = 2^x + 2 \cdot 3^x$
7.  $y = 4 \cdot 10^x - x^3$
8.  $y = 3x - 2 \cdot 4^x$
9.  $y = 2^x + \frac{2}{x^3}$
10.  $y = \frac{3^x}{3} + \frac{33}{\sqrt{x}}$
11.  $z = (\ln 4)e^x$
12.  $z = (\ln 4)4^x$
13.  $f(t) = (\ln 3)^t$
14.  $y = 5 \cdot 5^t + 6 \cdot 6^t$
15.  $h(z) = (\ln 2)^z$
16.  $f(x) = e^2 + x^e$
17.  $f(x) = x^3 + 3^x$
18.  $y = \pi^2 + \pi^x$
19.  $f(x) = e^\pi + \pi^x$
20.  $f(x) = \pi^x + x^\pi$
21.  $f(x) = e^k + k^x$
22.  $f(x) = e^{1+x}$

$$23. f(t) = e^{t+2}$$

$$24. y = e^{\theta-1}$$

$$25. y(x) = a^x + x^a.$$

$$26. f(x) = x^{\pi^2} + (\pi^2)^x$$

Which of the functions in Exercises 27–35 can be differentiated using the rules we have developed so far? Differentiate if you can; otherwise, indicate why the rules discussed so far do not apply.

$$27. y = x^2 + 2^x$$

$$28. y = \sqrt{x} - \left(\frac{1}{2}\right)^x$$

$$29. y = x^2 \cdot 2^x$$

$$30. y = \frac{2^x}{x}$$

$$31. y = e^{x+5}$$

$$32. y = e^{5x}$$

$$33. y = 4^{(x^2)}$$

$$34. f(z) = (\sqrt{4})^z$$

$$35. f(\theta) = 4^{\sqrt{\theta}}$$

### Problems

For Problems 36–37, determine if the derivative rules from this section apply. If they do, find the derivative. If they don't apply, indicate why.

$$36. f(x) = 4^{(3^x)}$$

$$37. f(s) = 5^s e^s$$

38. With a yearly inflation rate of 5%, prices are given by

$$P = P_0(1.05)^t,$$

where  $P_0$  is the price in dollars when  $t = 0$  and  $t$  is time in years. Suppose  $P_0 = 1$ . How fast (in cents/year) are prices rising when  $t = 10$ ?

39. With  $t$  in years since January 1, 1980, the population  $P$  of Slim Chance has been given by

$$P = 35,000(0.98)^t.$$

At what rate was the population changing on January 1, 2003?

40. The population of the world in billions of people can be modeled by the function  $f(t) = 5.3(1.018)^t$ , where  $t$  is years since 1990. Find  $f(0)$  and  $f'(0)$ . Find  $f(30)$  and  $f'(30)$ . Using units, explain what each answer tells you about the population of the world.

41. During the 2000s, the population of Hungary<sup>1</sup> was approximated by

$$P = 10.186(0.997)^t,$$

where  $P$  is in millions and  $t$  is in years since 2000. Assume the trend continues.

- (a) What does this model predict for the population of Hungary in the year 2020?  
 (b) How fast (in people/year) does this model predict Hungary's population will be decreasing in 2020?
42. In 2005, the population of Mexico was 103 million and growing 1.0% annually, while the population of the US was 296 million and growing 0.9% annually.<sup>2</sup> If we measure growth rates in people/year, which population was growing faster in 2005?
43. The value of an automobile purchased in 2007 can be approximated by the function  $V(t) = 25(0.85)^t$ , where  $t$  is the time, in years, from the date of purchase, and  $V(t)$  is the value, in thousands of dollars.
- (a) Evaluate and interpret  $V(4)$ .  
 (b) Find an expression for  $V'(t)$ , including units.  
 (c) Evaluate and interpret  $V'(4)$ .  
 (d) Use  $V(t)$ ,  $V'(t)$ , and any other considerations you think are relevant to write a paragraph in support of or in opposition to the following statement: "From a monetary point of view, it is best to keep this vehicle as long as possible."
44. (a) Find the slope of the graph of  $f(x) = 1 - e^x$  at the point where it crosses the  $x$ -axis.  
 (b) Find the equation of the tangent line to the curve at this point.  
 (c) Find the equation of the line perpendicular to the tangent line at this point. (This is the *normal* line.)

45. Find the value of  $c$  in Figure 3.12, where the line  $l$  tangent to the graph of  $y = 2^x$  at  $(0, 1)$  intersects the  $x$ -axis.

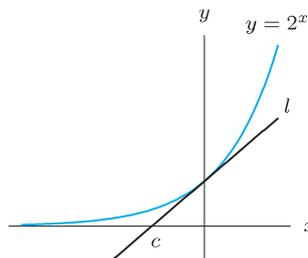


Figure 3.12

46. Find the quadratic polynomial  $g(x) = ax^2 + bx + c$  which best fits the function  $f(x) = e^x$  at  $x = 0$ , in the sense that

$$g(0) = f(0), \text{ and } g'(0) = f'(0), \text{ and } g''(0) = f''(0).$$

Using a computer or calculator, sketch graphs of  $f$  and  $g$  on the same axes. What do you notice?

47. Using the equation of the tangent line to the graph of  $e^x$  at  $x = 0$ , show that

$$e^x \geq 1 + x$$

for all values of  $x$ . A sketch may be helpful.

48. For what value(s) of  $a$  are  $y = a^x$  and  $y = 1 + x$  tangent at  $x = 0$ ? Explain.  
 49. Explain for which values of  $a$  the function  $a^x$  is increasing and for which values it is decreasing. Use the fact that, for  $a > 0$ ,

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

### 3.3 THE PRODUCT AND QUOTIENT RULES

We now know how to find derivatives of powers and exponentials, and of sums and constant multiples of functions. This section shows how to find the derivatives of products and quotients.

#### Using $\Delta$ Notation

To express the difference quotients of general functions, some additional notation is helpful. We write  $\Delta f$ , read "delta  $f$ ," for a small change in the value of  $f$  at the point  $x$ ,

$$\Delta f = f(x + h) - f(x).$$

In this notation, the derivative is the limit of the ratio  $\Delta f/h$ :

$$f'(x) = \lim_{h \rightarrow 0} \frac{\Delta f}{h}.$$

<sup>1</sup><http://sedac.ciesin.columbia.edu/gpw/country.jsp?iso=HUN> and <https://www.cia.gov/library/publications/the-world-factbook/print/hu.html>, accessed February 19, 2008.

<sup>2</sup>[en.wikipedia.org/wiki/Demographics\\_of\\_Mexico](http://en.wikipedia.org/wiki/Demographics_of_Mexico) and [www.census.gov/Press-Release/www/releases/archives/population/006142.html](http://www.census.gov/Press-Release/www/releases/archives/population/006142.html), accessed May 27, 2007.

## The Product Rule

Suppose we know the derivatives of  $f(x)$  and  $g(x)$  and want to calculate the derivative of the product,  $f(x)g(x)$ . The derivative of the product is calculated by taking the limit, namely,

$$\frac{d[f(x)g(x)]}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

To picture the quantity  $f(x+h)g(x+h) - f(x)g(x)$ , imagine the rectangle with sides  $f(x+h)$  and  $g(x+h)$  in Figure 3.13, where  $\Delta f = f(x+h) - f(x)$  and  $\Delta g = g(x+h) - g(x)$ . Then

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &= (\text{Area of whole rectangle}) - (\text{Unshaded area}) \\ &= \text{Area of the three shaded rectangles} \\ &= \Delta f \cdot g(x) + f(x) \cdot \Delta g + \Delta f \cdot \Delta g. \end{aligned}$$

Now divide by  $h$ :

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{\Delta f}{h} \cdot g(x) + f(x) \cdot \frac{\Delta g}{h} + \frac{\Delta f \cdot \Delta g}{h}.$$

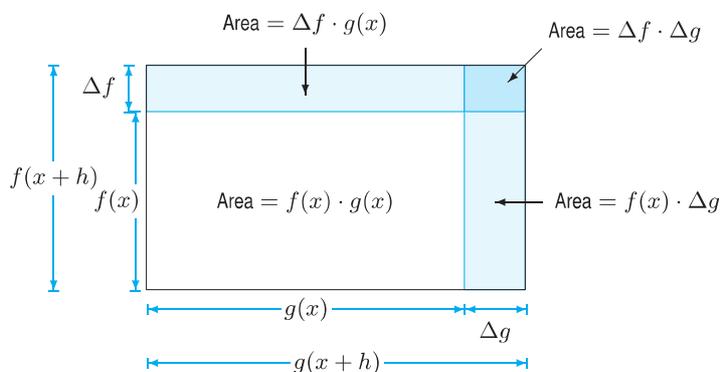


Figure 3.13: Illustration for the product rule (with  $\Delta f, \Delta g$  positive)

To evaluate the limit as  $h \rightarrow 0$ , we examine the three terms on the right separately. Notice that

$$\lim_{h \rightarrow 0} \frac{\Delta f}{h} \cdot g(x) = f'(x)g(x) \quad \text{and} \quad \lim_{h \rightarrow 0} f(x) \cdot \frac{\Delta g}{h} = f(x)g'(x).$$

In the third term we multiply the top and bottom by  $h$  to get  $\frac{\Delta f}{h} \cdot \frac{\Delta g}{h} \cdot h$ . Then,

$$\lim_{h \rightarrow 0} \frac{\Delta f \cdot \Delta g}{h} = \lim_{h \rightarrow 0} \frac{\Delta f}{h} \cdot \frac{\Delta g}{h} \cdot h = \lim_{h \rightarrow 0} \frac{\Delta f}{h} \cdot \lim_{h \rightarrow 0} \frac{\Delta g}{h} \cdot \lim_{h \rightarrow 0} h = f'(x) \cdot g'(x) \cdot 0 = 0.$$

Therefore, we conclude that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h \rightarrow 0} \left( \frac{\Delta f}{h} \cdot g(x) + f(x) \cdot \frac{\Delta g}{h} + \frac{\Delta f \cdot \Delta g}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{\Delta f}{h} \cdot g(x) + \lim_{h \rightarrow 0} f(x) \cdot \frac{\Delta g}{h} + \lim_{h \rightarrow 0} \frac{\Delta f \cdot \Delta g}{h} \\ &= f'(x)g(x) + f(x)g'(x). \end{aligned}$$

Thus we have proved the following rule:

**Theorem 3.3: The Product Rule**

If  $u = f(x)$  and  $v = g(x)$  are differentiable, then

$$(fg)' = f'g + fg'$$

The product rule can also be written

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

In words:

The derivative of a product is the derivative of the first times the second plus the first times the derivative of the second.

Another justification of the product rule is given in Problem 36 on page 163.

**Example 1** Differentiate (a)  $x^2e^x$ , (b)  $(3x^2 + 5x)e^x$ , (c)  $\frac{e^x}{x^2}$ .

Solution

$$(a) \quad \frac{d(x^2e^x)}{dx} = \left(\frac{d(x^2)}{dx}\right)e^x + x^2\frac{d(e^x)}{dx} = 2xe^x + x^2e^x = (2x + x^2)e^x.$$

$$(b) \quad \frac{d((3x^2 + 5x)e^x)}{dx} = \left(\frac{d(3x^2 + 5x)}{dx}\right)e^x + (3x^2 + 5x)\frac{d(e^x)}{dx} \\ = (6x + 5)e^x + (3x^2 + 5x)e^x = (3x^2 + 11x + 5)e^x.$$

(c) First we must write  $\frac{e^x}{x^2}$  as the product  $x^{-2}e^x$ :

$$\frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{d(x^{-2}e^x)}{dx} = \left(\frac{d(x^{-2})}{dx}\right)e^x + x^{-2}\frac{d(e^x)}{dx} \\ = -2x^{-3}e^x + x^{-2}e^x = (-2x^{-3} + x^{-2})e^x.$$

**The Quotient Rule**

Suppose we want to differentiate a function of the form  $Q(x) = f(x)/g(x)$ . (Of course, we have to avoid points where  $g(x) = 0$ .) We want a formula for  $Q'$  in terms of  $f'$  and  $g'$ .

Assuming that  $Q(x)$  is differentiable,<sup>3</sup> we can use the product rule on  $f(x) = Q(x)g(x)$ :

$$f'(x) = Q'(x)g(x) + Q(x)g'(x) \\ = Q'(x)g(x) + \frac{f(x)}{g(x)}g'(x).$$

Solving for  $Q'(x)$  gives

$$Q'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}.$$

Multiplying the top and bottom by  $g(x)$  to simplify gives

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

So we have the following rule:

<sup>3</sup>The method in Example 6 on page 137 can be used to explain why  $Q(x)$  must be differentiable.

**Theorem 3.4: The Quotient Rule**

If  $u = f(x)$  and  $v = g(x)$  are differentiable, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2},$$

or equivalently,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - u \cdot \frac{dv}{dx}}{v^2}.$$

In words:

The derivative of a quotient is the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all over the denominator squared.

**Example 2** Differentiate (a)  $\frac{5x^2}{x^3+1}$ , (b)  $\frac{1}{1+e^x}$ , (c)  $\frac{e^x}{x^2}$ .

**Solution**

(a)

$$\begin{aligned} \frac{d}{dx}\left(\frac{5x^2}{x^3+1}\right) &= \frac{\left(\frac{d}{dx}(5x^2)\right)(x^3+1) - 5x^2\frac{d}{dx}(x^3+1)}{(x^3+1)^2} = \frac{10x(x^3+1) - 5x^2(3x^2)}{(x^3+1)^2} \\ &= \frac{-5x^4 + 10x}{(x^3+1)^2}. \end{aligned}$$

(b)

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1+e^x}\right) &= \frac{\left(\frac{d}{dx}(1)\right)(1+e^x) - 1\frac{d}{dx}(1+e^x)}{(1+e^x)^2} = \frac{0(1+e^x) - 1(0+e^x)}{(1+e^x)^2} \\ &= \frac{-e^x}{(1+e^x)^2}. \end{aligned}$$

(c) This is the same as part (c) of Example 1, but this time we do it by the quotient rule.

$$\begin{aligned} \frac{d}{dx}\left(\frac{e^x}{x^2}\right) &= \frac{\left(\frac{d(e^x)}{dx}\right)x^2 - e^x\left(\frac{d(x^2)}{dx}\right)}{(x^2)^2} = \frac{e^xx^2 - e^x(2x)}{x^4} \\ &= e^x\left(\frac{x^2 - 2x}{x^4}\right) = e^x\left(\frac{x-2}{x^3}\right). \end{aligned}$$

This is, in fact, the same answer as before, although it looks different. Can you show that it is the same?

**Exercises and Problems for Section 3.3****Exercises**

- If  $f(x) = x^2(x^3 + 5)$ , find  $f'(x)$  two ways: by using the product rule and by multiplying out before taking the derivative. Do you get the same result? Should you?
- If  $f(x) = 2^x \cdot 3^x$ , find  $f'(x)$  two ways: by using the product rule and by using the fact that  $2^x \cdot 3^x = 6^x$ . Do you get the same result?
- For Exercises 3–30, find the derivative. It may be to your advantage to simplify first. Assume that  $a$ ,  $b$ ,  $c$ , and  $k$  are constants.
- $f(x) = xe^x$
- $y = \sqrt{x} \cdot 2^x$
- $y = x \cdot 2^x$
- $y = (t^2 + 3)e^t$

7.  $f(x) = (x^2 - \sqrt{x})3^x$       8.  $z = (s^2 - \sqrt{s})(s^2 + \sqrt{s})$   
 9.  $f(y) = 4^y(2 - y^2)$       10.  $y = (t^3 - 7t^2 + 1)e^t$       21.  $w = \frac{y^3 - 6y^2 + 7y}{y}$       22.  $y = \frac{\sqrt{t}}{t^2 + 1}$   
 11.  $f(x) = \frac{x}{e^x}$       12.  $g(x) = \frac{25x^2}{e^x}$       23.  $f(z) = \frac{z^2 + 1}{\sqrt{z}}$       24.  $w = \frac{5 - 3z}{5 + 3z}$   
 13.  $y = \frac{t + 1}{2^t}$       14.  $g(w) = \frac{w^{3.2}}{5w}$       25.  $h(r) = \frac{r^2}{2r + 1}$       26.  $f(z) = \frac{3z^2}{5z^2 + 7z}$   
 15.  $q(r) = \frac{3r}{5r + 2}$       16.  $g(t) = \frac{t - 4}{t + 4}$       27.  $w(x) = \frac{17e^x}{2^x}$       28.  $h(p) = \frac{1 + p^2}{3 + 2p^2}$   
 17.  $z = \frac{3t + 1}{5t + 2}$       18.  $z = \frac{t^2 + 5t + 2}{t + 3}$       29.  $f(x) = \frac{1 + x}{2 + 3x + 4x^2}$       30.  $f(x) = \frac{ax + b}{cx + k}$   
 19.  $z = \frac{t^2 + 3t + 1}{t + 1}$       20.  $f(x) = \frac{x^2 + 3}{x}$

**Problems**

In Problems 31–33, use Figure 3.14 to estimate the derivative, or state that it does not exist. The graph of  $f(x)$  has a sharp corner at  $x = 2$ .

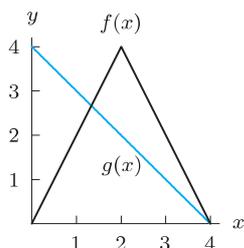


Figure 3.14

31. Let  $h(x) = f(x) \cdot g(x)$ . Find:  
 (a)  $h'(1)$       (b)  $h'(2)$       (c)  $h'(3)$   
 32. Let  $k(x) = (f(x))/g(x)$ . Find:  
 (a)  $k'(1)$       (b)  $k'(2)$       (c)  $k'(3)$   
 33. Let  $j(x) = (g(x))/f(x)$ . Find:  
 (a)  $j'(1)$       (b)  $j'(2)$       (c)  $j'(3)$

For Problems 34–39, let  $h(x) = f(x) \cdot g(x)$ , and  $k(x) = f(x)/g(x)$ , and  $l(x) = g(x)/f(x)$ . Use Figure 3.15 to estimate the derivatives.

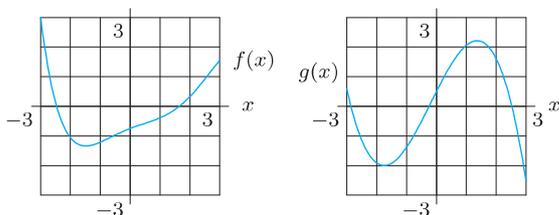


Figure 3.15

34.  $h'(1)$       35.  $k'(1)$       36.  $h'(2)$   
 37.  $k'(2)$       38.  $l'(1)$       39.  $l'(2)$

40. Differentiate  $f(t) = e^{-t}$  by writing it as  $f(t) = \frac{1}{e^t}$ .  
 41. Differentiate  $f(x) = e^{2x}$  by writing it as  $f(x) = e^x \cdot e^x$ .  
 42. Differentiate  $f(x) = e^{3x}$  by writing it as  $f(x) = e^x \cdot e^{2x}$  and using the result of Problem 41.  
 43. For what intervals is  $f(x) = xe^x$  concave up?  
 44. For what intervals is  $g(x) = \frac{1}{x^2 + 1}$  concave down?  
 45. Find the equation of the tangent line to the graph of  $f(x) = \frac{2x - 5}{x + 1}$  at the point at which  $x = 0$ .  
 46. Find the equation of the tangent line at  $x = 1$  to  $y = f(x)$  where  $f(x) = \frac{3x^2}{5x^2 + 7x}$ .  
 47. (a) Differentiate  $y = \frac{e^x}{x}$ ,  $y = \frac{e^x}{x^2}$ , and  $y = \frac{e^x}{x^3}$ .  
 (b) What do you anticipate the derivative of  $y = \frac{e^x}{x^n}$  will be? Confirm your guess.  
 48. Suppose  $f$  and  $g$  are differentiable functions with the values shown in the following table. For each of the following functions  $h$ , find  $h'(2)$ .

- (a)  $h(x) = f(x) + g(x)$       (b)  $h(x) = f(x)g(x)$   
 (c)  $h(x) = \frac{f(x)}{g(x)}$

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	5	-2

49. If  $H(3) = 1$ ,  $H'(3) = 3$ ,  $F(3) = 5$ ,  $F'(3) = 4$ , find:
- $G'(3)$  if  $G(z) = F(z) \cdot H(z)$
  - $G'(3)$  if  $G(w) = F(w)/H(w)$
50. Find a possible formula for a function  $y = f(x)$  such that  $f'(x) = 10x^9e^x + x^{10}e^x$ .
51. The quantity,  $q$ , of a certain skateboard sold depends on the selling price,  $p$ , in dollars, so we write  $q = f(p)$ . You are given that  $f(140) = 15,000$  and  $f'(140) = -100$ .
- What do  $f(140) = 15,000$  and  $f'(140) = -100$  tell you about the sales of skateboards?
  - The total revenue,  $R$ , earned by the sale of skateboards is given by  $R = pq$ . Find  $\left. \frac{dR}{dp} \right|_{p=140}$ .
  - What is the sign of  $\left. \frac{dR}{dp} \right|_{p=140}$ ? If the skateboards are currently selling for \$140, what happens to revenue if the price is increased to \$141?
52. When an electric current passes through two resistors with resistance  $r_1$  and  $r_2$ , connected in parallel, the combined resistance,  $R$ , can be calculated from the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}.$$

Find the rate at which the combined resistance changes with respect to changes in  $r_1$ . Assume that  $r_2$  is constant.

53. A museum has decided to sell one of its paintings and to invest the proceeds. If the picture is sold between the years 2000 and 2020 and the money from the sale is invested in a bank account earning 5% interest per year compounded annually, then  $B(t)$ , the balance in the year 2020, depends on the year,  $t$ , in which the painting is sold and the sale price  $P(t)$ . If  $t$  is measured from the year 2000 so that  $0 < t < 20$  then

$$B(t) = P(t)(1.05)^{20-t}.$$

- Explain why  $B(t)$  is given by this formula.
- Show that the formula for  $B(t)$  is equivalent to

$$B(t) = (1.05)^{20} \frac{P(t)}{(1.05)^t}.$$

- Find  $B'(10)$ , given that  $P(10) = 150,000$  and  $P'(10) = 5000$ .
54. Let  $f(v)$  be the gas consumption (in liters/km) of a car going at velocity  $v$  (in km/hr). In other words,  $f(v)$  tells you how many liters of gas the car uses to go one kilometer, if it is going at velocity  $v$ . You are told that

$$f(80) = 0.05 \text{ and } f'(80) = 0.0005.$$

- Let  $g(v)$  be the distance the same car goes on one liter of gas at velocity  $v$ . What is the relationship between  $f(v)$  and  $g(v)$ ? Find  $g(80)$  and  $g'(80)$ .

- Let  $h(v)$  be the gas consumption in liters per hour. In other words,  $h(v)$  tells you how many liters of gas the car uses in one hour if it is going at velocity  $v$ . What is the relationship between  $h(v)$  and  $f(v)$ ? Find  $h(80)$  and  $h'(80)$ .
- How would you explain the practical meaning of the values of these functions and their derivatives to a driver who knows no calculus?

55. The function  $f(x) = e^x$  has the properties

$$f'(x) = f(x) \text{ and } f(0) = 1.$$

Explain why  $f(x)$  is the only function with both these properties. [Hint: Assume  $g'(x) = g(x)$ , and  $g(0) = 1$ , for some function  $g(x)$ . Define  $h(x) = g(x)/e^x$ , and compute  $h'(x)$ . Then use the fact that a function with a derivative of 0 must be a constant function.]

56. Find  $f'(x)$  for the following functions with the product rule, rather than by multiplying out.

- $f(x) = (x-1)(x-2)$ .
- $f(x) = (x-1)(x-2)(x-3)$ .
- $f(x) = (x-1)(x-2)(x-3)(x-4)$ .

57. Use the answer from Problem 56 to guess  $f'(x)$  for the following function:

$$f(x) = (x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)$$

where  $r_1, r_2, \dots, r_n$  are any real numbers.

58. (a) Provide a three-dimensional analogue for the geometrical demonstration of the formula for the derivative of a product, given in Figure 3.13 on page 128. In other words, find a formula for the derivative of  $F(x) \cdot G(x) \cdot H(x)$  using Figure 3.16.
- Confirm your results by writing  $F(x) \cdot G(x) \cdot H(x)$  as  $[F(x) \cdot G(x)] \cdot H(x)$  and using the product rule twice.
  - Generalize your result to  $n$  functions: what is the derivative of

$$f_1(x) \cdot f_2(x) \cdot f_3(x) \cdots f_n(x)?$$

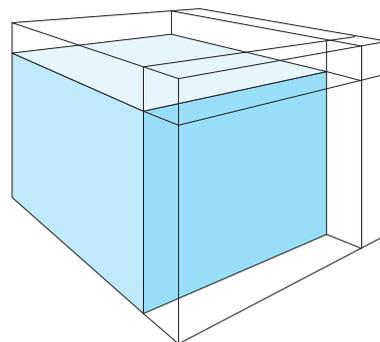


Figure 3.16: A graphical representation of the three-dimensional product rule

59. If  $P(x) = (x - a)^2 Q(x)$ , where  $Q(x)$  is a polynomial, we call  $x = a$  a double zero of the polynomial  $P(x)$ .
- (a) If  $x = a$  is a double zero of a polynomial  $P(x)$ , show that  $P(a) = P'(a) = 0$ .
- (b) If  $P(x)$  is a polynomial and  $P(a) = P'(a) = 0$ , show that  $x = a$  is a double zero of  $P(x)$ .
60. Find and simplify  $\frac{d^2}{dx^2} (f(x)g(x))$ .

## 3.4 THE CHAIN RULE

The chain rule enables us to differentiate composite functions such as  $\sin(3t)$  or  $e^{-x^2}$ . Before seeing a formula, let's think about the derivative of a composite function in a practical situation.

### Intuition Behind the Chain Rule

Imagine we are moving straight upward in a hot air balloon. Let  $y$  be our distance from the ground. The temperature,  $H$ , is changing as a function of altitude, so  $H = f(y)$ . How does our temperature change with time?

The rate of change of our temperature is affected both by how fast the temperature is changing with altitude (about  $16^\circ\text{F}$  per mile), and by how fast we are climbing (say 2 mph). Then our temperature changes by  $16^\circ$  for every mile we climb, and since we move 2 miles in an hour, our temperature changes by  $16 \cdot 2 = 32$  degrees in an hour.

Since temperature is a function of height,  $H = f(y)$ , and height is a function of time,  $y = g(t)$ , we can think of temperature as a composite function of time,  $H = f(g(t))$ , with  $f$  as the outside function and  $g$  as the inside function. The example suggests the following result, which turns out to be true:

$$\begin{array}{ccccc} \text{Rate of change} & = & \text{Rate of change} & \times & \text{Rate of change} \\ \text{of composite function} & & \text{of outside function} & & \text{of inside function} \end{array}$$

### The Derivative of a Composition of Functions

We now obtain a formula for the chain rule. Suppose  $f(g(x))$  is a composite function, with  $f$  being the outside function and  $g$  being the inside. Let us write

$$z = g(x) \quad \text{and} \quad y = f(z), \quad \text{so} \quad y = f(g(x)).$$

Then a small change in  $x$ , called  $\Delta x$ , generates a small change in  $z$ , called  $\Delta z$ . In turn,  $\Delta z$  generates a small change in  $y$  called  $\Delta y$ . Provided  $\Delta x$  and  $\Delta z$  are not zero, we can say:

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x}.$$

Since  $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ , this suggests that in the limit as  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$  get smaller and smaller, we have:

#### The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

In other words:

The rate of change of a composite function is the product of the rates of change of the outside and inside functions.

Since  $\frac{dy}{dz} = f'(z)$  and  $\frac{dz}{dx} = g'(x)$ , we can also write

$$\frac{d}{dx}f(g(x)) = f'(z) \cdot g'(x).$$

Substituting  $z = g(x)$ , we can rewrite this as follows:

### Theorem 3.5: The Chain Rule

If  $f$  and  $g$  are differentiable functions, then

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

In words:

The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

A justification of the chain rule is given in Problem 37 on page 164. The following example shows how units confirm that the rate of change of a composite function is the product of the rates of change of the outside and inside functions.

**Example 1** The length,  $L$ , in micrometers ( $\mu\text{m}$ ), of steel depends on the air temperature,  $H^\circ\text{C}$ , and the temperature  $H$  depends on time,  $t$ , measured in hours. If the length of a steel bridge increases by  $0.2 \mu\text{m}$  for every degree increase in temperature, and the temperature is increasing at  $3^\circ\text{C}$  per hour, how fast is the length of the bridge increasing? What are the units for your answer?

**Solution** We want to know how much the length of the bridge changes in one hour; this rate is in  $\mu\text{m}/\text{hr}$ . We are told that the length of the bridge changes by  $0.2 \mu\text{m}$  for each degree that the temperature changes, and that the temperature changes by  $3^\circ\text{C}$  each hour. Thus, in one hour, the length of the bridge changes by  $0.2 \cdot 3 = 0.6 \mu\text{m}$ .

Now we do the same calculation using derivative notation and the chain rule. We know that

$$\text{Rate length increasing with respect to temperature} = \frac{dL}{dH} = 0.2 \mu\text{m}/^\circ\text{C}$$

$$\text{Rate temperature increasing with respect to time} = \frac{dH}{dt} = 3^\circ\text{C}/\text{hr}.$$

We want to calculate the rate at which the length is increasing with respect to time, or  $dL/dt$ . We think of  $L$  as a function of  $H$ , and  $H$  as a function of  $t$ . The chain rule tells us that

$$\frac{dL}{dt} = \frac{dL}{dH} \cdot \frac{dH}{dt} = \left(0.2 \frac{\mu\text{m}}{^\circ\text{C}}\right) \cdot \left(3 \frac{^\circ\text{C}}{\text{hr}}\right) = 0.6 \mu\text{m}/\text{hr}.$$

Thus, the length is increasing at  $0.6 \mu\text{m}/\text{hr}$ . Notice that the units work out as we expect.

Example 1 shows us how to interpret the chain rule in practical terms. The next examples show how the chain rule is used to compute derivatives of functions given by formulas.

**Example 2** Find the derivatives of the following functions:

(a)  $(x^2 + 1)^{100}$       (b)  $\sqrt{3x^2 + 5x - 2}$       (c)  $\frac{1}{x^2 + x^4}$       (d)  $e^{3x}$       (e)  $e^{x^2}$

**Solution** (a) Here  $z = g(x) = x^2 + 1$  is the inside function;  $f(z) = z^{100}$  is the outside function. Now  $g'(x) = 2x$  and  $f'(z) = 100z^{99}$ , so

$$\frac{d}{dx}((x^2 + 1)^{100}) = 100z^{99} \cdot 2x = 100(x^2 + 1)^{99} \cdot 2x = 200x(x^2 + 1)^{99}.$$

(b) Here  $z = g(x) = 3x^2 + 5x - 2$  and  $f(z) = \sqrt{z}$ , so  $g'(x) = 6x + 5$  and  $f'(z) = \frac{1}{2\sqrt{z}}$ . Hence

$$\frac{d}{dx}(\sqrt{3x^2 + 5x - 2}) = \frac{1}{2\sqrt{z}} \cdot (6x + 5) = \frac{1}{2\sqrt{3x^2 + 5x - 2}} \cdot (6x + 5).$$

(c) Let  $z = g(x) = x^2 + x^4$  and  $f(z) = 1/z$ , so  $g'(x) = 2x + 4x^3$  and  $f'(z) = -z^{-2} = -\frac{1}{z^2}$ . Then

$$\frac{d}{dx} \left( \frac{1}{x^2 + x^4} \right) = -\frac{1}{z^2} (2x + 4x^3) = -\frac{2x + 4x^3}{(x^2 + x^4)^2}.$$

We could have done this problem using the quotient rule. Try it and see that you get the same answer!

(d) Let  $z = g(x) = 3x$  and  $f(z) = e^z$ . Then  $g'(x) = 3$  and  $f'(z) = e^z$ , so

$$\frac{d}{dx}(e^{3x}) = e^z \cdot 3 = 3e^{3x}.$$

(e) To figure out which is the inside function and which is the outside, notice that to evaluate  $e^{x^2}$  we first evaluate  $x^2$  and then take  $e$  to that power. This tells us that the inside function is  $z = g(x) = x^2$  and the outside function is  $f(z) = e^z$ . Therefore,  $g'(x) = 2x$ , and  $f'(z) = e^z$ , giving

$$\frac{d}{dx}(e^{x^2}) = e^z \cdot 2x = e^{x^2} \cdot 2x = 2xe^{x^2}.$$

To differentiate a complicated function, we may have to use the chain rule more than once, as in the following example.

**Example 3** Differentiate: (a)  $\sqrt{e^{-x/7} + 5}$       (b)  $(1 - e^{2\sqrt{t}})^{19}$

**Solution** (a) Let  $z = g(x) = e^{-x/7} + 5$  be the inside function; let  $f(z) = \sqrt{z}$  be the outside function. Now  $f'(z) = \frac{1}{2\sqrt{z}}$ , but we need the chain rule to find  $g'(x)$ .

We choose new inside and outside functions whose composition is  $g(x)$ . Let  $u = h(x) = -x/7$  and  $k(u) = e^u + 5$  so  $g(x) = k(h(x)) = e^{-x/7} + 5$ . Then  $h'(x) = -1/7$  and  $k'(u) = e^u$ , so

$$g'(x) = e^u \cdot \left(-\frac{1}{7}\right) = -\frac{1}{7}e^{-x/7}.$$

Using the chain rule to combine the derivatives of  $f(z)$  and  $g(x)$ , we have

$$\frac{d}{dx}(\sqrt{e^{-x/7} + 5}) = \frac{1}{2\sqrt{z}} \left(-\frac{1}{7}e^{-x/7}\right) = -\frac{e^{-x/7}}{14\sqrt{e^{-x/7} + 5}}.$$

(b) Let  $z = g(t) = 1 - e^{2\sqrt{t}}$  be the inside function and  $f(z) = z^{19}$  be the outside function. Then  $f'(z) = 19z^{18}$  but we need the chain rule to differentiate  $g(t)$ .

Now we choose  $u = h(t) = 2\sqrt{t}$  and  $k(u) = 1 - e^u$ , so  $g(t) = k(h(t))$ . Then  $h'(t) = 2 \cdot \frac{1}{2}t^{-1/2} = \frac{1}{\sqrt{t}}$  and  $k'(u) = -e^u$ , so

$$g'(t) = -e^u \cdot \frac{1}{\sqrt{t}} = -\frac{e^{2\sqrt{t}}}{\sqrt{t}}.$$

Using the chain rule to combine the derivatives of  $f(z)$  and  $g(t)$ , we have

$$\frac{d}{dx}(1 - e^{2\sqrt{t}})^{19} = 19z^{18} \left( -\frac{e^{2\sqrt{t}}}{\sqrt{t}} \right) = -19 \frac{e^{2\sqrt{t}}}{\sqrt{t}} (1 - e^{2\sqrt{t}})^{18}.$$

It is often faster to use the chain rule without introducing new variables, as in the following examples.

**Example 4** Differentiate  $\sqrt{1 + e^{\sqrt{3+x^2}}}$ .

**Solution** The chain rule is needed four times:

$$\begin{aligned} \frac{d}{dx} \left( \sqrt{1 + e^{\sqrt{3+x^2}}} \right) &= \frac{1}{2} \left( 1 + e^{\sqrt{3+x^2}} \right)^{-1/2} \cdot \frac{d}{dx} \left( 1 + e^{\sqrt{3+x^2}} \right) \\ &= \frac{1}{2} \left( 1 + e^{\sqrt{3+x^2}} \right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{d}{dx} \left( \sqrt{3+x^2} \right) \\ &= \frac{1}{2} \left( 1 + e^{\sqrt{3+x^2}} \right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{2} (3+x^2)^{-1/2} \cdot \frac{d}{dx} (3+x^2) \\ &= \frac{1}{2} \left( 1 + e^{\sqrt{3+x^2}} \right)^{-1/2} \cdot e^{\sqrt{3+x^2}} \cdot \frac{1}{2} (3+x^2)^{-1/2} \cdot 2x. \end{aligned}$$

**Example 5** Find the derivative of  $e^{2x}$  by the chain rule and by the product rule.

**Solution** Using the chain rule, we have

$$\frac{d}{dx}(e^{2x}) = e^{2x} \cdot \frac{d}{dx}(2x) = e^{2x} \cdot 2 = 2e^{2x}.$$

Using the product rule, we write  $e^{2x} = e^x \cdot e^x$ . Then

$$\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x e^x) = \left( \frac{d}{dx}(e^x) \right) e^x + e^x \left( \frac{d}{dx}(e^x) \right) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}.$$

## Using the Product and Chain Rules to Differentiate a Quotient

If you prefer, you can differentiate a quotient by the product and chain rules, instead of by the quotient rule. The resulting formulas may look different, but they will be equivalent.

**Example 6** Find  $k'(x)$  if  $k(x) = \frac{x}{x^2 + 1}$ .

**Solution** One way is to use the quotient rule:

$$\begin{aligned} k'(x) &= \frac{1 \cdot (x^2 + 1) - x \cdot (2x)}{(x^2 + 1)^2} \\ &= \frac{1 - x^2}{(x^2 + 1)^2}. \end{aligned}$$

Alternatively, we can write the original function as a product,

$$k(x) = x \frac{1}{x^2 + 1} = x \cdot (x^2 + 1)^{-1},$$

and use the product rule:

$$k'(x) = 1 \cdot (x^2 + 1)^{-1} + x \cdot \frac{d}{dx} [(x^2 + 1)^{-1}].$$

Now use the chain rule to differentiate  $(x^2 + 1)^{-1}$ , giving

$$\frac{d}{dx} [(x^2 + 1)^{-1}] = -(x^2 + 1)^{-2} \cdot 2x = \frac{-2x}{(x^2 + 1)^2}.$$

Therefore,

$$k'(x) = \frac{1}{x^2 + 1} + x \cdot \frac{-2x}{(x^2 + 1)^2} = \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}.$$

Putting these two fractions over a common denominator gives the same answer as the quotient rule.

## Exercises and Problems for Section 3.4

### Exercises

Find the derivatives of the functions in Exercises 1–50. Assume that  $a$ ,  $b$ ,  $c$ , and  $k$  are constants.

- |                               |                                |   |   |
|-------------------------------|--------------------------------|---|---|
| 1. $f(x) = (x + 1)^{99}$      | 2. $w = (t^3 + 1)^{100}$       | 21. $y = e^{-4t}$                             | 22. $y = \sqrt{s^3 + 1}$                  |
| 3. $(4x^2 + 1)^7$             | 4. $f(x) = \sqrt{1 - x^2}$     | 23. $y = te^{-t^2}$                           | 24. $f(z) = \sqrt{z}e^{-z}$               |
| 5. $\sqrt{e^x + 1}$           | 6. $w = (\sqrt{t} + 1)^{100}$  | 25. $z(x) = \sqrt[3]{2^x + 5}$                | 26. $z = 2^{5t-3}$                        |
| 7. $h(w) = (w^4 - 2w)^5$      | 8. $w(r) = \sqrt{r^4 + 1}$     | 27. $w = \sqrt{(x^2 \cdot 5^x)^3}$            | 28. $f(y) = \sqrt{10^{(5-y)}}$            |
| 9. $k(x) = (x^3 + e^x)^4$     | 10. $f(x) = e^{2x}(x^2 + 5^x)$ | 29. $f(z) = \frac{\sqrt{z}}{e^z}$             | 30. $y = \frac{\sqrt{z}}{2^z}$            |
| 11. $f(t) = e^{3t}$           | 12. $g(x) = e^{\pi x}$         | 31. $y = \left(\frac{x^2 + 2}{3}\right)^2$    | 32. $h(x) = \sqrt{\frac{x^2 + 9}{x + 3}}$ |
| 13. $f(\theta) = 2^{-\theta}$ | 14. $y = \pi^{(x+2)}$          | 33. $y = \frac{e^{2x}}{x^2 + 1}$              | 34. $y = \frac{1}{e^{3x} + x^2}$          |
| 15. $g(x) = 3^{(2x+7)}$       | 16. $f(t) = te^{5-2t}$         | 35. $h(z) = \left(\frac{b}{a + z^2}\right)^4$ | 36. $f(z) = \frac{1}{(e^z + 1)^2}$        |
| 17. $p(t) = e^{4t+2}$         | 18. $v(t) = t^2e^{-ct}$        |   |   |
| 19. $g(t) = e^{(1+3t)^2}$     | 20. $w = e^{\sqrt{s}}$         |   |   |

37.  $w = (t^2 + 3t)(1 - e^{-2t})$     38.  $h(x) = 2e^{3x}$     45.  $f(y) = e^{e^{(y^2)}}$     46.  $f(t) = 2e^{-2e^{2t}}$   
 39.  $f(x) = 6e^{5x} + e^{-x^2}$     40.  $f(x) = e^{-(x-1)^2}$     47.  $f(x) = (ax^2 + b)^3$     48.  $f(t) = ae^{bt}$   
 41.  $f(w) = (5w^2 + 3)e^{w^2}$     42.  $f(\theta) = (e^\theta + e^{-\theta})^{-1}$     49.  $f(x) = axe^{-bx}$     50.  $g(\alpha) = e^{\alpha e^{-2\alpha}}$   
 43.  $y = \sqrt{e^{-3t^2} + 5}$     44.  $z = (te^{3t} + e^{5t})^9$

### Problems

In Problems 51–54, use Figure 3.14 to estimate the derivative, or state it does not exist. The graph of  $f(x)$  has a sharp corner at  $x = 2$ .

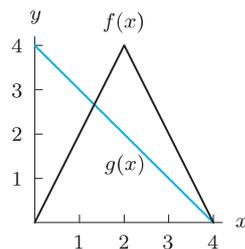


Figure 3.17

51. Let  $h(x) = f(g(x))$ . Find:  
 (a)  $h'(1)$     (b)  $h'(2)$     (c)  $h'(3)$   
 52. Let  $u(x) = g(f(x))$ . Find:  
 (a)  $u'(1)$     (b)  $u'(2)$     (c)  $u'(3)$   
 53. Let  $v(x) = f(f(x))$ . Find:  
 (a)  $v'(1)$     (b)  $v'(2)$     (c)  $v'(3)$   
 54. Let  $w(x) = g(g(x))$ . Find:  
 (a)  $w'(1)$     (b)  $w'(2)$     (c)  $w'(3)$

In Problems 55–58, use Figure 3.18 to evaluate the derivative.

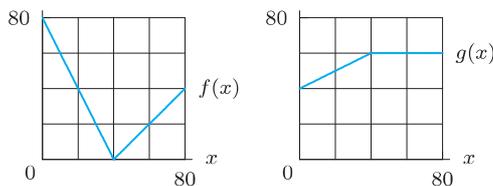


Figure 3.18

55.  $\frac{d}{dx}f(g(x))|_{x=30}$     56.  $\frac{d}{dx}f(g(x))|_{x=70}$   
 57.  $\frac{d}{dx}g(f(x))|_{x=30}$     58.  $\frac{d}{dx}g(f(x))|_{x=70}$

59. Find the equation of the tangent line to  $f(x) = (x - 1)^3$  at the point where  $x = 2$ .  
 60. Find the equation of the line tangent to  $y = f(x)$  at  $x = 1$ , where  $f(x)$  is the function in Problem 39.  
 61. For what values of  $x$  is the graph of  $y = e^{-x^2}$  concave down?  
 62. For what intervals is  $f(x) = xe^{-x}$  concave down?  
 63. Suppose  $f(x) = (2x + 1)^{10}(3x - 1)^7$ . Find a formula for  $f'(x)$ . Decide on a reasonable way to simplify your result, and find a formula for  $f''(x)$ .  
 64. A fish population is approximated by  $P(t) = 10e^{0.6t}$ , where  $t$  is in months. Calculate and use units to explain what each of the following tells us about the population:  
 (a)  $P(12)$     (b)  $P'(12)$   
 65. At any time,  $t$ , a population,  $P(t)$ , is growing at a rate proportional to the population at that moment.  
 (a) Using derivatives, write an equation representing the growth of the population. Let  $k$  be the constant of proportionality.  
 (b) Show that the function  $P(t) = Ae^{kt}$  satisfies the equation in part (a) for any constant  $A$ .  
 66. Find the mean and variance of the normal distribution of statistics using parts (a) and (b) with  $m(t) = e^{\mu t + \sigma^2 t^2 / 2}$ .  
 (a) Mean =  $m'(0)$   
 (b) Variance =  $m''(0) - (m'(0))^2$   
 67. If the derivative of  $y = k(x)$  equals 2 when  $x = 1$ , what is the derivative of  
 (a)  $k(2x)$  when  $x = \frac{1}{2}$ ?  
 (b)  $k(x + 1)$  when  $x = 0$ ?  
 (c)  $k\left(\frac{1}{4}x\right)$  when  $x = 4$ ?  
 68. Is  $x = \sqrt[3]{2t + 5}$  a solution to the equation  $3x^2 \frac{dx}{dt} = 2$ ? Why or why not?  
 69. Find a possible formula for a function  $m(x)$  such that  $m'(x) = x^5 \cdot e^{(x^6)}$ .

70. Given  $F(2) = 1$ ,  $F'(2) = 5$ ,  $F(4) = 3$ ,  $F'(4) = 7$  and  $G(4) = 2$ ,  $G'(4) = 6$ ,  $G(3) = 4$ ,  $G'(3) = 8$ , find:

- (a)  $H(4)$  if  $H(x) = F(G(x))$   
 (b)  $H'(4)$  if  $H(x) = F(G(x))$   
 (c)  $H(4)$  if  $H(x) = G(F(x))$   
 (d)  $H'(4)$  if  $H(x) = G(F(x))$   
 (e)  $H'(4)$  if  $H(x) = F(x)/G(x)$

71. Given  $y = f(x)$  with  $f(1) = 4$  and  $f'(1) = 3$ , find

- (a)  $g'(1)$  if  $g(x) = \sqrt{f(x)}$ .  
 (b)  $h'(1)$  if  $h(x) = f(\sqrt{x})$ .

In Problems 72–76, use Figures 3.19 and 3.20 and  $h(x) = f(g(x))$ .

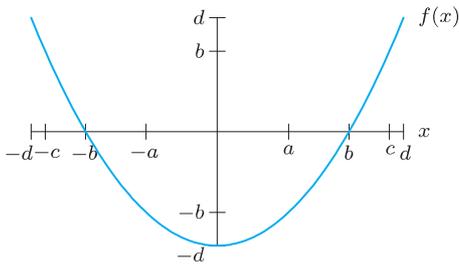


Figure 3.19

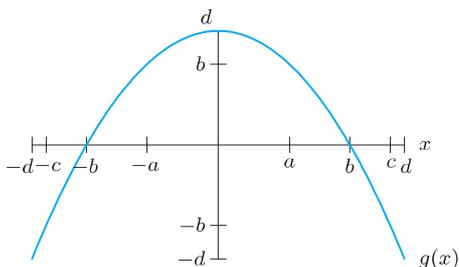


Figure 3.20

72. Evaluate  $h(0)$  and  $h'(0)$ .  
 73. At  $x = -c$ , is  $h$  positive, negative, or zero? Increasing or decreasing?  
 74. At  $x = a$ , is  $h$  increasing or decreasing?  
 75. What are the signs of  $h(d)$  and  $h'(d)$ ?  
 76. How does the value of  $h(x)$  change on the interval  $-d < x < -b$ ?  
 77. The world's population is about  $f(t) = 6e^{0.013t}$  billion, where  $t$  is time in years since 1999. Find  $f(0)$ ,  $f'(0)$ ,  $f(10)$ , and  $f'(10)$ . Using units, interpret your answers in terms of population.  
 78. On October 17, 2006, the US population was 300 million and growing exponentially. If the population was increasing at a rate of 2.9 million a year on that date, find a formula for the population as a function of time,  $t$ , in years since that date.

79. If you invest  $P$  dollars in a bank account at an annual interest rate of  $r\%$ , then after  $t$  years you will have  $B$  dollars, where

$$B = P \left( 1 + \frac{r}{100} \right)^t.$$

- (a) Find  $dB/dt$ , assuming  $P$  and  $r$  are constant. In terms of money, what does  $dB/dt$  represent?  
 (b) Find  $dB/dr$ , assuming  $P$  and  $t$  are constant. In terms of money, what does  $dB/dr$  represent?  
 80. The theory of relativity predicts that an object whose mass is  $m_0$  when it is at rest will appear heavier when moving at speeds near the speed of light. When the object is moving at speed  $v$ , its mass  $m$  is given by

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}, \quad \text{where } c \text{ is the speed of light.}$$

- (a) Find  $dm/dv$ .  
 (b) In terms of physics, what does  $dm/dv$  tell you?  
 81. The charge,  $Q$ , on a capacitor which starts discharging at time  $t = 0$  is given by

$$Q = \begin{cases} Q_0 & \text{for } t \leq 0 \\ Q_0 e^{-t/RC} & \text{for } t > 0, \end{cases}$$

where  $R$  and  $C$  are positive constants depending on the circuit and  $Q_0$  is the charge at  $t = 0$ , where  $Q_0 \neq 0$ . The current,  $I$ , flowing in the circuit is given by  $I = dQ/dt$ .

- (a) Find the current  $I$  for  $t < 0$  and for  $t > 0$ .  
 (b) Is it possible to define  $I$  at  $t = 0$ ?  
 (c) Is the function  $Q$  differentiable at  $t = 0$ ?  
 82. A particle is moving on the  $x$ -axis, where  $x$  is in centimeters. Its velocity,  $v$ , in cm/sec, when it is at the point with coordinate  $x$  is given by

$$v = x^2 + 3x - 2.$$

Find the acceleration of the particle when it is at the point  $x = 2$ . Give units in your answer.

83. A particle is moving on the  $x$ -axis. It has velocity  $v(x)$  when it is at the point with coordinate  $x$ . Show that its acceleration at that point is  $v(x)v'(x)$ .  
 84. A polynomial  $f$  is said to have a *zero of multiplicity*  $m$  at  $x = a$  if

$$f(x) = (x - a)^m h(x),$$

with  $h$  a polynomial such that  $h(a) \neq 0$ . Explain why a polynomial having a zero of multiplicity  $m$  at  $x = a$  satisfies  $f^{(p)}(a) = 0$ , for  $p = 1, 2, \dots, m - 1$ .

[Note:  $f^{(p)}$  is the  $p^{\text{th}}$  derivative.]

85. Find and simplify  $\frac{d^2}{dx^2} (f(g(x)))$ .  
 86. Find and simplify  $\frac{d^2}{dx^2} \left( \frac{f(x)}{g(x)} \right)$ .

## 3.5 THE TRIGONOMETRIC FUNCTIONS

### Derivatives of the Sine and Cosine

Since the sine and cosine functions are periodic, their derivatives must be periodic also. (Why?) Let's look at the graph of  $f(x) = \sin x$  in Figure 3.21 and estimate the derivative function graphically.

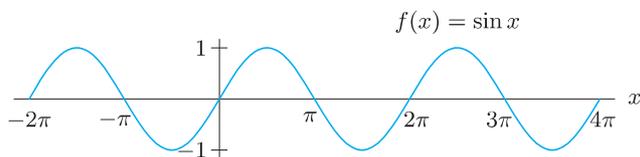


Figure 3.21: The sine function

First we might ask where the derivative is zero. (At  $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2$ , etc.) Then ask where the derivative is positive and where it is negative. (Positive for  $-\pi/2 < x < \pi/2$ ; negative for  $\pi/2 < x < 3\pi/2$ , etc.) Since the largest positive slopes are at  $x = 0, 2\pi$ , and so on, and the largest negative slopes are at  $x = \pi, 3\pi$ , and so on, we get something like the graph in Figure 3.22.

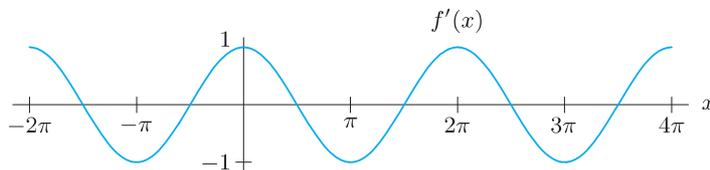


Figure 3.22: Derivative of  $f(x) = \sin x$

The graph of the derivative in Figure 3.22 looks suspiciously like the graph of the cosine function. This might lead us to conjecture, quite correctly, that the derivative of the sine is the cosine.

Of course, we cannot be sure, just from the graphs, that the derivative of the sine really is the cosine. However, for now we'll assume that the derivative of the sine *is* the cosine and confirm the result at the end of the section.

One thing we can do now is to check that the derivative function in Figure 3.22 has amplitude 1 (as it ought to if it is the cosine). That means we have to convince ourselves that the derivative of  $f(x) = \sin x$  is 1 when  $x = 0$ . The next example suggests that this is true when  $x$  is in radians.

**Example 1** Using a calculator set in radians, estimate the derivative of  $f(x) = \sin x$  at  $x = 0$ .

**Solution** Since  $f(x) = \sin x$ ,

$$f'(0) = \lim_{h \rightarrow 0} \frac{\sin(0+h) - \sin 0}{h} = \lim_{h \rightarrow 0} \frac{\sin h}{h}.$$

Table 3.5 contains values of  $(\sin h)/h$  which suggest that this limit is 1, so we estimate

$$f'(0) = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1.$$

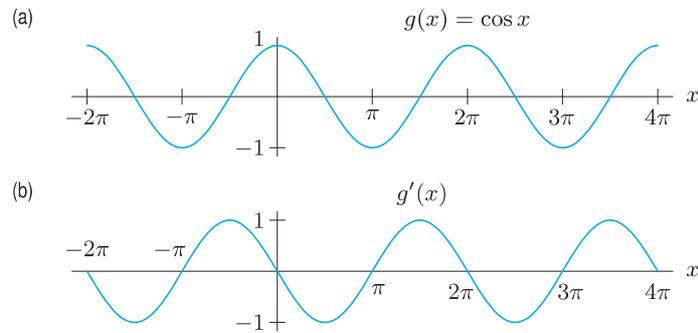
Table 3.5

$h$ (radians)	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
$(\sin h)/h$	0.99833	0.99998	1.0000	1.0000	1.0000	1.0000	0.99998	0.99833

**Warning:** It is important to notice that in the previous example  $h$  was in *radians*; any conclusions we have drawn about the derivative of  $\sin x$  are valid *only* when  $x$  is in radians. If you find the derivative with  $h$  in degrees, you get a different result.

**Example 2** Starting with the graph of the cosine function, sketch a graph of its derivative.

**Solution** The graph of  $g(x) = \cos x$  is in Figure 3.23(a). Its derivative is 0 at  $x = 0, \pm\pi, \pm2\pi$ , and so on; it is positive for  $-\pi < x < 0$ ,  $\pi < x < 2\pi$ , and so on; and it is negative for  $0 < x < \pi$ ,  $2\pi < x < 3\pi$ , and so on. The derivative is in Figure 3.23(b).



**Figure 3.23:**  $g(x) = \cos x$  and its derivative,  $g'(x)$

As we did with the sine, we use the graphs to make a conjecture. The derivative of the cosine in Figure 3.23(b) looks exactly like the graph of sine, except reflected about the  $x$ -axis. But how can we be sure that the derivative is  $-\sin x$ ?

**Example 3** Use the relation  $\frac{d}{dx}(\sin x) = \cos x$  to show that  $\frac{d}{dx}(\cos x) = -\sin x$ .

**Solution** Since the cosine function is the sine function shifted to the left by  $\pi/2$  (that is,  $\cos x = \sin(x + \pi/2)$ ), we expect the derivative of the cosine to be the derivative of the sine, shifted to the left by  $\pi/2$ . Differentiating using the chain rule:

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left(\sin\left(x + \frac{\pi}{2}\right)\right) = \cos\left(x + \frac{\pi}{2}\right).$$

But  $\cos(x + \pi/2)$  is the cosine shifted to the left by  $\pi/2$ , which gives a sine curve reflected about the  $x$ -axis. So we have

$$\frac{d}{dx}(\cos x) = \cos\left(x + \frac{\pi}{2}\right) = -\sin x.$$

At the end of this section and in Problems 53 and 54, we show that our conjectures for the derivatives of  $\sin x$  and  $\cos x$  are correct. Thus, we have:

For  $x$  in radians,  $\frac{d}{dx}(\sin x) = \cos x$  and  $\frac{d}{dx}(\cos x) = -\sin x$ .

**Example 4** Differentiate (a)  $2 \sin(3\theta)$ , (b)  $\cos^2 x$ , (c)  $\cos(x^2)$ , (d)  $e^{-\sin t}$ .

**Solution** Use the chain rule:

$$(a) \frac{d}{d\theta}(2 \sin(3\theta)) = 2 \frac{d}{d\theta}(\sin(3\theta)) = 2(\cos(3\theta)) \frac{d}{d\theta}(3\theta) = 2(\cos(3\theta))3 = 6 \cos(3\theta).$$

$$(b) \frac{d}{dx}(\cos^2 x) = \frac{d}{dx}((\cos x)^2) = 2(\cos x) \cdot \frac{d}{dx}(\cos x) = 2(\cos x)(-\sin x) = -2 \cos x \sin x.$$

$$(c) \frac{d}{dx}(\cos(x^2)) = -\sin(x^2) \cdot \frac{d}{dx}(x^2) = -2x \sin(x^2).$$

$$(d) \frac{d}{dt}(e^{-\sin t}) = e^{-\sin t} \frac{d}{dt}(-\sin t) = -(\cos t)e^{-\sin t}.$$

## Derivative of the Tangent Function

Since  $\tan x = \sin x / \cos x$ , we differentiate  $\tan x$  using the quotient rule. Writing  $(\sin x)'$  for  $d(\sin x)/dx$ , we have:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

For  $x$  in radians,  $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$ .

The graphs of  $f(x) = \tan x$  and  $f'(x) = 1/\cos^2 x$  are in Figure 3.24. Is it reasonable that  $f'$  is always positive? Are the asymptotes of  $f'$  where we expect?

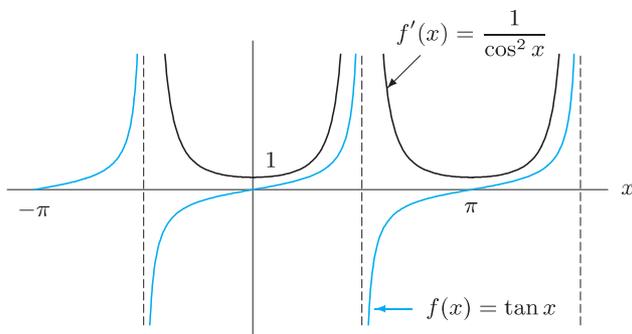


Figure 3.24: The function  $\tan x$  and its derivative

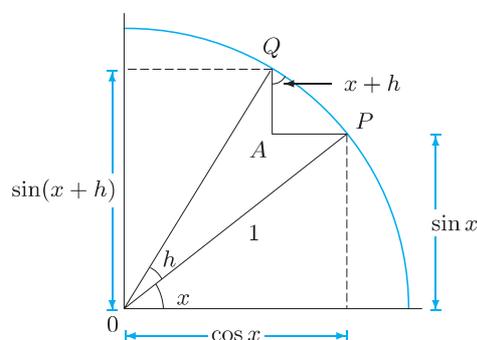


Figure 3.25: Unit circle showing  $\sin(x+h)$  and  $\sin x$

**Example 5** Differentiate (a)  $2 \tan(3t)$ , (b)  $\tan(1 - \theta)$ , (c)  $\frac{1 + \tan t}{1 - \tan t}$ .

**Solution** (a) Use the chain rule:

$$\frac{d}{dt}(2 \tan(3t)) = 2 \frac{1}{\cos^2(3t)} \frac{d}{dt}(3t) = \frac{6}{\cos^2(3t)}.$$

(b) Use the chain rule:

$$\frac{d}{d\theta}(\tan(1 - \theta)) = \frac{1}{\cos^2(1 - \theta)} \cdot \frac{d}{d\theta}(1 - \theta) = \frac{-1}{\cos^2(1 - \theta)}.$$

(c) Use the quotient rule:

$$\begin{aligned}\frac{d}{dt} \left( \frac{1 + \tan t}{1 - \tan t} \right) &= \frac{\left( \frac{d(1 + \tan t)}{dt} \right) (1 - \tan t) - (1 + \tan t) \frac{d(1 - \tan t)}{dt}}{(1 - \tan t)^2} \\ &= \frac{\frac{1}{\cos^2 t} (1 - \tan t) - (1 + \tan t) \left( -\frac{1}{\cos^2 t} \right)}{(1 - \tan t)^2} \\ &= \frac{2}{\cos^2 t \cdot (1 - \tan t)^2}.\end{aligned}$$

### Informal Justification of $\frac{d}{dx}(\sin x) = \cos x$

Consider the unit circle in Figure 3.25. To find the derivative of  $\sin x$ , we need to estimate

$$\frac{\sin(x+h) - \sin x}{h}.$$

In Figure 3.25, the quantity  $\sin(x+h) - \sin x$  is represented by the length  $QA$ . The arc  $QP$  is of length  $h$ , so

$$\frac{\sin(x+h) - \sin x}{h} = \frac{QA}{\text{Arc } QP}.$$

Now, if  $h$  is small,  $QAP$  is approximately a right triangle because the arc  $QP$  is almost a straight line. Furthermore, using geometry, we can show that angle  $AQP = x+h$ . For small  $h$ , we have

$$\frac{\sin(x+h) - \sin x}{h} = \frac{QA}{\text{Arc } QP} \approx \cos(x+h).$$

As  $h \rightarrow 0$ , the approximation gets better, so

$$\frac{d}{dx}(\sin x) = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \cos x.$$

Other derivations of this result are given in Problems 53 and 54 on page 145.

## Exercises and Problems for Section 3.5

### Exercises

- |  |  |                                     |
|--|--|-------------------------------------|
| 1. Construct a table of values for $\cos x$ , $x = 0, 0.1, 0.2, \dots, 0.6$ . Using the difference quotient, estimate the derivative at these points (use $h = 0.001$ ), and compare it with $(-\sin x)$ . | 10. $w = \sin(e^t)$                      | 11. $f(x) = e^{\cos x}$             |
|  | 12. $f(y) = e^{\sin y}$                  | 13. $z = \theta e^{\cos \theta}$    |
|  | 14. $R(\theta) = e^{\sin(3\theta)}$      | 15. $g(\theta) = \sin(\tan \theta)$ |
| Find the derivatives of the functions in Exercises 2–39. Assume $a$ is a constant.   | 16. $w(x) = \tan(x^2)$                   | 17. $f(x) = \sqrt{1 - \cos x}$      |
| 2. $r(\theta) = \sin \theta + \cos \theta$   | 3. $s(\theta) = \cos \theta \sin \theta$ | 18. $f(x) = \cos(\sin x)$           |
| 4. $z = \cos(4\theta)$   | 5. $f(x) = \sin(3x)$                     | 19. $f(x) = \tan(\sin x)$           |
| 6. $g(x) = \sin(2 - 3x)$   | 7. $R(x) = 10 - 3 \cos(\pi x)$           | 20. $k(x) = \sqrt{(\sin(2x))^3}$    |
| 8. $g(\theta) = \sin^2(2\theta) - \pi\theta$   | 9. $f(x) = x^2 \cos x$                   | 21. $f(x) = 2x \sin(3x)$            |
|  | 22. $y = e^\theta \sin(2\theta)$         | 23. $f(x) = e^{-2x} \cdot \sin x$   |

24.  $z = \sqrt{\sin t}$       25.  $y = \sin^5 \theta$       34.  $y = \sin(\sin x + \cos x)$       35.  $y = \sin(2x) \cdot \sin(3x)$
26.  $g(z) = \tan(e^z)$       27.  $z = \tan(e^{-3\theta})$
28.  $w = e^{-\sin \theta}$       29.  $h(t) = t \cos t + \tan t$       36.  $t(\theta) = \frac{\cos \theta}{\sin \theta}$       37.  $f(x) = \sqrt{\frac{1 - \sin x}{1 - \cos x}}$
30.  $f(\alpha) = \cos \alpha + 3 \sin \alpha$       31.  $k(\alpha) = \sin^5 \alpha \cos^3 \alpha$
32.  $f(\theta) = \theta^3 \cos \theta$       33.  $y = \cos^2 w + \cos(w^2)$       38.  $r(y) = \frac{y}{\cos y + a}$       39.  $G(x) = \frac{\sin^2 x + 1}{\cos^2 x + 1}$

### Problems

40. Is the graph of  $y = \sin(x^4)$  increasing or decreasing when  $x = 10$ ? Is it concave up or concave down?
41. Find the fiftieth derivative of  $y = \cos x$ .
42. Find a possible formula for the function  $q(x)$  such that
- $$q'(x) = \frac{e^x \cdot \sin x - e^x \cdot \cos x}{(\sin x)^2}.$$
43. Find a function  $F(x)$  satisfying  $F'(x) = \sin(4x)$ .
44. On page 33 the depth,  $y$ , in feet, of water in Boston harbor is given in terms of  $t$ , the number of hours since midnight, by
- $$y = 5 + 4.9 \cos\left(\frac{\pi}{6}t\right).$$
- (a) Find  $dy/dt$ . What does  $dy/dt$  represent, in terms of water level?
- (b) For  $0 \leq t \leq 24$ , when is  $dy/dt$  zero? (Figure 1.51 on page 33 may be helpful.) Explain what it means (in terms of water level) for  $dy/dt$  to be zero.
45. A boat at anchor is bobbing up and down in the sea. The vertical distance,  $y$ , in feet, between the sea floor and the boat is given as a function of time,  $t$ , in minutes, by
- $$y = 15 + \sin(2\pi t).$$
- (a) Find the vertical velocity,  $v$ , of the boat at time  $t$ .
- (b) Make rough sketches of  $y$  and  $v$  against  $t$ .
46. The voltage,  $V$ , in volts, in an electrical outlet is given as a function of time,  $t$ , in seconds, by the function  $V = 156 \cos(120\pi t)$ .
- (a) Give an expression for the rate of change of voltage with respect to time.
- (b) Is the rate of change ever zero? Explain.
- (c) What is the maximum value of the rate of change?
47. The function  $y = A \sin\left(\left(\sqrt{\frac{k}{m}}\right)t\right)$  represents the oscillations of a mass  $m$  at the end of a spring. The constant  $k$  measures the stiffness of the spring.
- (a) Find a time at which the mass is farthest from its equilibrium position. Find a time at which the mass is moving fastest. Find a time at which the mass is accelerating fastest.
- (b) What is the period,  $T$ , of the oscillation?
- (c) Find  $dT/dm$ . What does the sign of  $dT/dm$  tell you?
48. With  $t$  in years, the population of a herd of deer is represented by
- $$P(t) = 4000 + 500 \sin\left(2\pi t - \frac{\pi}{2}\right).$$
- (a) How does this population vary with time? Graph  $P(t)$  for one year.
- (b) When in the year the population is a maximum? What is that maximum? Is there a minimum? If so, when?
- (c) When is the population growing fastest? When is it decreasing fastest?
- (d) How fast is the population changing on July 1?
49. The metal bar of length  $l$  in Figure 3.26 has one end attached at the point  $P$  to a circle of radius  $a$ . Point  $Q$  at the other end can slide back and forth along the  $x$ -axis.
- (a) Find  $x$  as a function of  $\theta$ .
- (b) Assume lengths are in centimeters and the angular speed ( $d\theta/dt$ ) is 2 radians/second counterclockwise. Find the speed at which the point  $Q$  is moving when
- (i)  $\theta = \pi/2$ ,      (ii)  $\theta = \pi/4$ .

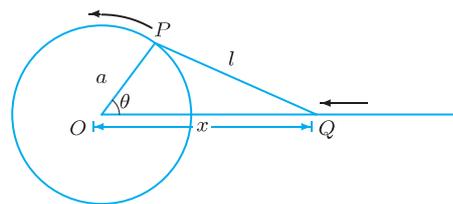


Figure 3.26

50. Find the equations of the tangent lines to the graph of  $f(x) = \sin x$  at  $x = 0$  and at  $x = \pi/3$ . Use each tangent line to approximate  $\sin(\pi/6)$ . Would you expect these results to be equally accurate, since they are taken equally far away from  $x = \pi/6$  but on opposite sides? If the accuracy is different, can you account for the difference?

51. If  $k \geq 1$ , the graphs of  $y = \sin x$  and  $y = ke^{-x}$  intersect for  $x \geq 0$ . Find the smallest value of  $k$  for which the graphs are tangent. What are the coordinates of the point of tangency?
52. Find  $d^2x/dt^2$  as a function of  $x$  if  $dx/dt = x \sin x$ .
53. We will use the following identities to calculate the derivatives of  $\sin x$  and  $\cos x$ :

$$\begin{aligned} \sin(a + b) &= \sin a \cos b + \sin b \cos a \\ \cos(a + b) &= \cos a \cos b - \sin a \sin b. \end{aligned}$$

- (a) Use the definition of the derivative to show that if  $f(x) = \sin x$ ,

$$f'(x) = \sin x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h}.$$

- (b) Estimate the limits in part (a) with your calculator to explain why  $f'(x) = \cos x$ .
- (c) If  $g(x) = \cos x$ , use the definition of the derivative to show that  $g'(x) = -\sin x$ .
54. In this problem you will calculate the derivative of  $\tan \theta$  rigorously (and without using the derivatives of  $\sin \theta$  or  $\cos \theta$ ). You will then use your result for  $\tan \theta$  to calculate the derivatives of  $\sin \theta$  and  $\cos \theta$ . Figure 3.27 shows  $\tan \theta$  and  $\Delta(\tan \theta)$ , which is the change in  $\tan \theta$ , namely  $\tan(\theta + \Delta\theta) - \tan \theta$ .

- (a) By paying particular attention to how the two figures relate and using the fact that

$$\text{Area of Sector OAQ} \leq \text{Area of Triangle OQR} \leq \text{Area of Sector OBR}$$

explain why

$$\frac{\Delta\theta}{2\pi} \cdot \frac{\pi}{(\cos \theta)^2} \leq \frac{\Delta(\tan \theta)}{2} \leq \frac{\Delta\theta}{2\pi} \cdot \frac{\pi}{(\cos(\theta + \Delta\theta))^2}.$$

[Hint: A sector of a circle with angle  $\alpha$  at the center has area  $\alpha/(2\pi)$  times the area of the whole circle.]

- (b) Use part (a) to show as  $\Delta\theta \rightarrow 0$  that

$$\frac{\Delta \tan \theta}{\Delta \theta} \rightarrow \left( \frac{1}{\cos \theta} \right)^2,$$

and hence that  $\frac{d(\tan \theta)}{d\theta} = \left( \frac{1}{\cos \theta} \right)^2$ .

- (c) Derive the identity  $(\tan \theta)^2 + 1 = \left( \frac{1}{\cos \theta} \right)^2$ . Then differentiate both sides of this identity with respect to  $\theta$ , using the chain rule and the result of part (b) to show that  $\frac{d}{d\theta}(\cos \theta) = -\sin \theta$ .
- (d) Differentiate both sides of the identity  $(\sin \theta)^2 + (\cos \theta)^2 = 1$  and use the result of part (c) to show that  $\frac{d}{d\theta}(\sin \theta) = \cos \theta$ .

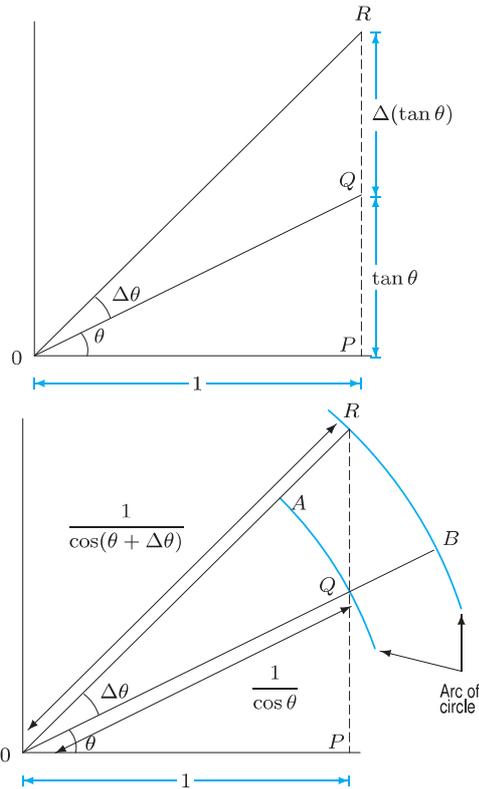


Figure 3.27:  $\tan \theta$  and  $\Delta(\tan \theta)$

## 3.6 THE CHAIN RULE AND INVERSE FUNCTIONS

In this section we will use the chain rule to calculate the derivatives of fractional powers, logarithms, exponentials, and the inverse trigonometric functions.<sup>4</sup> The same method is used to obtain a formula for the derivative of a general inverse function.

### Finding the Derivative of an Inverse Function: Derivative of $x^{1/2}$

Earlier we calculated the derivative of  $x^n$  with  $n$  an integer, but we have been using the result for non-integral values of  $n$  as well. We now confirm that the power rule holds for  $n = 1/2$  by

<sup>4</sup>It requires a separate justification, not given here, that these functions are differentiable.

calculating the derivative of  $f(x) = x^{1/2}$  using the chain rule. Since

$$[f(x)]^2 = x,$$

the derivative of  $[f(x)]^2$  and the derivative of  $x$  must be equal, so

$$\frac{d}{dx} [f(x)]^2 = \frac{d}{dx} (x).$$

We can use the chain rule with  $f(x)$  as the inside function to obtain:

$$\frac{d}{dx} [f(x)]^2 = 2f(x) \cdot f'(x) = 1.$$

Solving for  $f'(x)$  gives

$$f'(x) = \frac{1}{2f(x)} = \frac{1}{2x^{1/2}},$$

or

$$\frac{d}{dx} (x^{1/2}) = \frac{1}{2x^{1/2}} = \frac{1}{2} x^{-1/2}.$$

A similar calculation can be used to obtain the derivative of  $x^{1/n}$  where  $n$  is a positive integer.

## Derivative of $\ln x$

We use the chain rule to differentiate an identity involving  $\ln x$ . Since  $e^{\ln x} = x$ , we have

$$\begin{aligned} \frac{d}{dx} (e^{\ln x}) &= \frac{d}{dx} (x), \\ e^{\ln x} \cdot \frac{d}{dx} (\ln x) &= 1. \quad (\text{Since } e^x \text{ is outside function and } \ln x \text{ is inside function}) \end{aligned}$$

Solving for  $d(\ln x)/dx$  gives

$$\frac{d}{dx} (\ln x) = \frac{1}{e^{\ln x}} = \frac{1}{x},$$

so

$$\boxed{\frac{d}{dx} (\ln x) = \frac{1}{x}.}$$

**Example 1** Differentiate (a)  $\ln(x^2 + 1)$  (b)  $t^2 \ln t$  (c)  $\sqrt{1 + \ln(1 - y)}$ .

**Solution** (a) Using the chain rule:

$$\frac{d}{dx} (\ln(x^2 + 1)) = \frac{1}{x^2 + 1} \frac{d}{dx} (x^2 + 1) = \frac{2x}{x^2 + 1}.$$

(b) Using the product rule:

$$\frac{d}{dt} (t^2 \ln t) = \frac{d}{dt} (t^2) \cdot \ln t + t^2 \frac{d}{dt} (\ln t) = 2t \ln t + t^2 \cdot \frac{1}{t} = 2t \ln t + t.$$

(c) Using the chain rule:

$$\begin{aligned} \frac{d}{dy} (\sqrt{1 + \ln(1 - y)}) &= \frac{d}{dy} (1 + \ln(1 - y))^{1/2} \\ &= \frac{1}{2} (1 + \ln(1 - y))^{-1/2} \cdot \frac{d}{dy} (1 + \ln(1 - y)) \quad (\text{Using the chain rule}) \\ &= \frac{1}{2\sqrt{1 + \ln(1 - y)}} \cdot \frac{1}{1 - y} \cdot \frac{d}{dy} (1 - y) \quad (\text{Using the chain rule again}) \\ &= \frac{-1}{2(1 - y)\sqrt{1 + \ln(1 - y)}}. \end{aligned}$$

## Derivative of $a^x$

In Section 3.2, we saw that the derivative of  $a^x$  is proportional to  $a^x$ . Now we see another way of calculating the constant of proportionality. We use the identity

$$\ln(a^x) = x \ln a.$$

Differentiating both sides, using  $\frac{d}{dx}(\ln x) = \frac{1}{x}$  and the chain rule, and remembering that  $\ln a$  is a constant, we obtain:

$$\frac{d}{dx}(\ln a^x) = \frac{1}{a^x} \cdot \frac{d}{dx}(a^x) = \ln a.$$

Solving gives the result we obtained earlier:

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

## Derivatives of Inverse Trigonometric Functions

In Section 1.5 we defined  $\arcsin x$  as the angle between  $-\pi/2$  and  $\pi/2$  (inclusive) whose sine is  $x$ . Similarly,  $\arctan x$  as the angle strictly between  $-\pi/2$  and  $\pi/2$  whose tangent is  $x$ . To find  $\frac{d}{dx}(\arctan x)$  we use the identity  $\tan(\arctan x) = x$ . Differentiating using the chain rule gives

$$\frac{1}{\cos^2(\arctan x)} \cdot \frac{d}{dx}(\arctan x) = 1,$$

so

$$\frac{d}{dx}(\arctan x) = \cos^2(\arctan x).$$

Using the identity  $1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$ , and replacing  $\theta$  by  $\arctan x$ , we have

$$\cos^2(\arctan x) = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1 + x^2}.$$

Thus we have

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.$$

By a similar argument, we obtain the result:

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}.$$

**Example 2** Differentiate (a)  $\arctan(t^2)$  (b)  $\arcsin(\tan \theta)$ .

**Solution** Use the chain rule:

$$(a) \frac{d}{dt}(\arctan(t^2)) = \frac{1}{1 + (t^2)^2} \cdot \frac{d}{dt}(t^2) = \frac{2t}{1 + t^4}.$$

$$(b) \frac{d}{dt}(\arcsin(\tan \theta)) = \frac{1}{\sqrt{1 - (\tan \theta)^2}} \cdot \frac{d}{d\theta}(\tan \theta) = \frac{1}{\sqrt{1 - \tan^2 \theta}} \cdot \frac{1}{\cos^2 \theta}.$$

## Derivative of a General Inverse Function

Each of the previous results gives the derivative of an inverse function. In general, if a function  $f$  has a differentiable inverse,  $f^{-1}$ , we find its derivative by differentiating  $f(f^{-1}(x)) = x$  by the chain rule:

$$\begin{aligned}\frac{d}{dx}(f(f^{-1}(x))) &= 1 \\ f'(f^{-1}(x)) \cdot \frac{d}{dx}(f^{-1}(x)) &= 1\end{aligned}$$

so

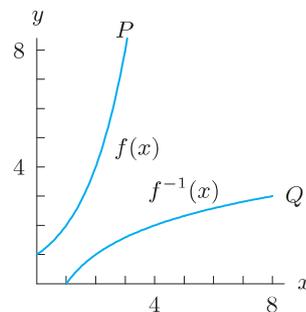
$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}.$$

Thus, the derivative of the inverse is the reciprocal of the derivative of the original function, but evaluated at the point  $f^{-1}(x)$  instead of the point  $x$ .

- Example 3** Figure 3.28 shows  $f(x)$  and  $f^{-1}(x)$ . Using Table 3.6, find
- (i)  $f(2)$       (ii)  $f^{-1}(2)$       (iii)  $f'(2)$       (iv)  $(f^{-1})'(2)$
  - The equation of the tangent lines at the points  $P$  and  $Q$ .
  - What is the relationship between the two tangent lines?

**Table 3.6**

$x$	$f(x)$	$f'(x)$
0	1	0.7
1	2	1.4
2	4	2.8
3	8	5.5



**Figure 3.28**

- Solution**
- Reading from the table, we have
    - $f(2) = 4$ .
    - $f^{-1}(2) = 1$ .
    - $f'(2) = 2.8$ .
    - To find the derivative of the inverse function, we use

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{1.4} = 0.714.$$

Notice that the derivative of  $f^{-1}$  is the reciprocal of the derivative of  $f$ . However, the derivative of  $f^{-1}$  is evaluated at 2, while the derivative of  $f$  is evaluated at 1, where  $f^{-1}(2) = 1$  and  $f(1) = 2$ .

- At the point  $P$ , we have  $f(3) = 8$  and  $f'(3) = 5.5$ , so the equation of the tangent line at  $P$  is

$$y - 8 = 5.5(x - 3).$$

At the point  $Q$ , we have  $f^{-1}(8) = 3$ , so the slope at  $Q$  is

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(3)} = \frac{1}{5.5}.$$

Thus, the equation of the tangent line at  $Q$  is

$$y - 3 = \frac{1}{5.5}(x - 8).$$

- (c) The two tangent lines have reciprocal slopes, and the points  $(3, 8)$  and  $(8, 3)$  are reflections of one another in the line  $y = x$ . Thus, the two tangent lines are reflections of one another in the line  $y = x$ .

## Exercises and Problems for Section 3.6

### Exercises

For Exercises 1–33, find the derivative. It may be to your advantage to simplify before differentiating. Assume  $a$ ,  $b$ ,  $c$ , and  $k$  are constants.

1.  $f(t) = \ln(t^2 + 1)$
2.  $f(x) = \ln(1 - x)$
3.  $f(x) = \ln(e^{2x})$
4.  $f(x) = e^{\ln(e^{2x^2+3})}$
5.  $f(x) = \ln(1 - e^{-x})$
6.  $f(\alpha) = \ln(\sin \alpha)$
7.  $f(x) = \ln(e^x + 1)$
8.  $y = x \ln x - x + 2$
9.  $j(x) = \ln(e^{ax} + b)$
10.  $h(w) = w^3 \ln(10w)$
11.  $f(x) = \ln(e^{7x})$
12.  $f(x) = e^{(\ln x)+1}$
13.  $f(w) = \ln(\cos(w - 1))$
14.  $f(t) = \ln(e^{\ln t})$
15.  $f(y) = \arcsin(y^2)$
16.  $g(t) = \arctan(3t - 4)$
17.  $g(\alpha) = \sin(\arcsin \alpha)$
18.  $g(t) = e^{\arctan(3t^2)}$
19.  $g(t) = \cos(\ln t)$
20.  $h(z) = z^{\ln 2}$
21.  $h(w) = w \arcsin w$
22.  $f(x) = e^{\ln(kx)}$
23.  $r(t) = \arcsin(2t)$
24.  $j(x) = \cos(\sin^{-1} x)$
25.  $f(x) = \cos(\arctan 3x)$
26.  $f(z) = \frac{1}{\ln z}$
27.  $f(x) = \frac{x}{1 + \ln x}$
28.  $y = 2x(\ln x + \ln 2) - 2x + e$
29.  $f(x) = \ln(\sin x + \cos x)$
30.  $f(t) = \ln(\ln t) + \ln(\ln 2)$
31.  $T(u) = \arctan\left(\frac{u}{1+u}\right)$
32.  $a(t) = \ln\left(\frac{1 - \cos t}{1 + \cos t}\right)^4$
33.  $f(x) = \cos(\arcsin(x + 1))$

### Problems

For Problems 34–37, let  $h(x) = f(g(x))$  and  $k(x) = g(f(x))$ . Use Figure 3.29 to estimate the derivatives.

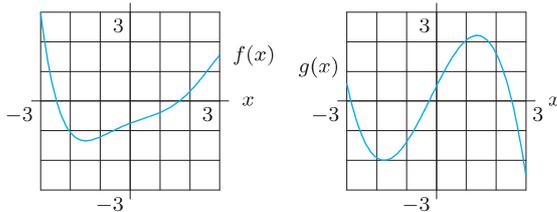


Figure 3.29

34.  $h'(1)$
35.  $k'(1)$
36.  $h'(2)$
37.  $k'(2)$
38. On what intervals is  $\ln(x^2 + 1)$  concave up?
39. Use the chain rule to obtain the formula for  $\frac{d}{dx}(\arcsin x)$ .
40. Using the chain rule, find  $\frac{d}{dx}(\log x)$ . (Recall  $\log x = \log_{10} x$ .)
41. To compare the acidity of different solutions, chemists use the pH (which is a single number, not the product of  $p$  and  $H$ ). The pH is defined in terms of the concentration,  $x$ , of hydrogen ions in the solution as
 
$$\text{pH} = -\log x.$$
 Find the rate of change of pH with respect to hydrogen ion concentration when the pH is 2. [Hint: Use the result of Problem 40.]
42. A firm estimates that the total revenue,  $R$ , in dollars, received from the sale of  $q$  goods is given by
 
$$R = \ln(1 + 1000q^2).$$
 The marginal revenue,  $MR$ , is the rate of change of the total revenue as a function of quantity. Calculate the marginal revenue when  $q = 10$ .

43. (a) Find the equation of the tangent line to  $y = \ln x$  at  $x = 1$ .  
 (b) Use it to calculate approximate values for  $\ln(1.1)$  and  $\ln(2)$ .  
 (c) Using a graph, explain whether the approximate values are smaller or larger than the true values. Would the same result have held if you had used the tangent line to estimate  $\ln(0.9)$  and  $\ln(0.5)$ ? Why?
44. (a) Find the equation of the best quadratic approximation to  $y = \ln x$  at  $x = 1$ . The best quadratic approximation has the same first and second derivatives as  $y = \ln x$  at  $x = 1$ .  
 (b) Use a computer or calculator to graph the approximation and  $y = \ln x$  on the same set of axes. What do you notice?  
 (c) Use your quadratic approximation to calculate approximate values for  $\ln(1.1)$  and  $\ln(2)$ .
45. (a) For  $x > 0$ , find and simplify the derivative of  $f(x) = \arctan x + \arctan(1/x)$ .  
 (b) What does your result tell you about  $f$ ?
46. Imagine you are zooming in on the graph of each of the following functions near the origin:

$$\begin{array}{ll} y = x & y = \sqrt{x} \\ y = x^2 & y = \sin x \\ y = x \sin x & y = \tan x \\ y = \sqrt{x/(x+1)} & y = x^3 \\ y = \ln(x+1) & y = \frac{1}{2} \ln(x^2+1) \\ y = 1 - \cos x & y = \sqrt{2x-x^2} \end{array}$$

Which of them look the same? Group together those functions which become indistinguishable, and give the equations of the lines they look like.

In Problems 47–50, use Figure 3.30 to find a point  $x$  where  $h(x) = n(m(x))$  has the given derivative.

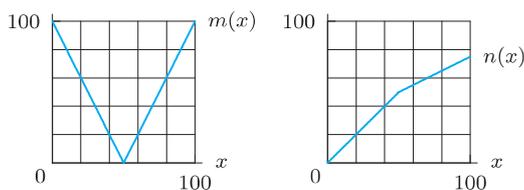


Figure 3.30

47.  $h'(x) = -2$                       48.  $h'(x) = 2$   
 49.  $h'(x) = 1$                         50.  $h'(x) = -1$

In Problems 51–53, use Figure 3.31 to calculate the derivative.

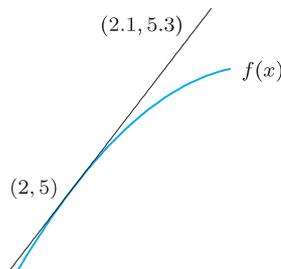


Figure 3.31

51.  $h'(2)$  if  $h(x) = (f(x))^3$   
 52.  $k'(2)$  if  $k(x) = (f(x))^{-1}$   
 53.  $g'(5)$  if  $g(x) = f^{-1}(x)$
54. (a) Given that  $f(x) = x^3$ , find  $f'(2)$ .  
 (b) Find  $f^{-1}(x)$ .  
 (c) Use your answer from part (b) to find  $(f^{-1})'(8)$ .  
 (d) How could you have used your answer from part (a) to find  $(f^{-1})'(8)$ ?
55. (a) For  $f(x) = 2x^5 + 3x^3 + x$ , find  $f'(x)$ .  
 (b) How can you use your answer to part (a) to determine if  $f(x)$  is invertible?  
 (c) Find  $f(1)$ .  
 (d) Find  $f'(1)$ .  
 (e) Find  $(f^{-1})'(6)$ .
56. Given that  $f$  and  $g$  are differentiable everywhere,  $g$  is the inverse of  $f$ , and that  $f(3) = 4$ ,  $f'(3) = 6$ ,  $f'(4) = 7$ , find  $g'(4)$ .
57. Use the table and the fact that  $f(x)$  is invertible and differentiable everywhere to find  $(f^{-1})'(3)$ .

$x$	$f(x)$	$f'(x)$
3	1	7
6	2	10
9	3	5

58. Let  $P = f(t)$  give the US population<sup>5</sup> in millions in year  $t$ .
- (a) What does the statement  $f(2005) = 296$  tell you about the US population?  
 (b) Find and interpret  $f^{-1}(296)$ . Give units.  
 (c) What does the statement  $f'(2005) = 2.65$  tell you about the population? Give units.  
 (d) Evaluate and interpret  $(f^{-1})'(296)$ . Give units.

<sup>5</sup>Data from [www.census.gov/Press-Release/www/releases/archives/population/006142.html](http://www.census.gov/Press-Release/www/releases/archives/population/006142.html), accessed May 27, 2007.

59. Figure 3.32 shows the number of motor vehicles,<sup>6</sup>  $f(t)$ , in millions, registered in the world  $t$  years after 1965. With units, estimate and interpret

- (a)  $f(20)$                       (b)  $f'(20)$   
 (c)  $f^{-1}(500)$               (d)  $(f^{-1})'(500)$

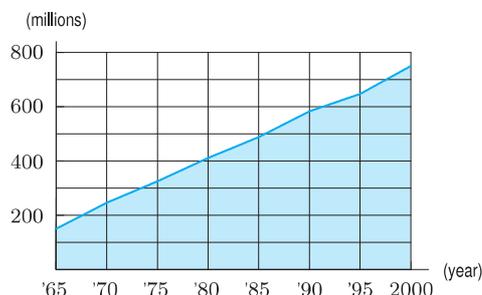


Figure 3.32

60. Using Figure 3.33, where  $f'(2) = 2.1$ ,  $f'(4) = 3.0$ ,  $f'(6) = 3.7$ ,  $f'(8) = 4.2$ , find  $(f^{-1})'(8)$ .

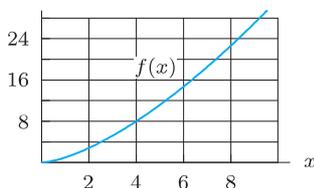


Figure 3.33

61. An increasing function  $f(x)$  has the value  $f(10) = 5$ . Explain how you know that the calculations  $f'(10) = 8$  and  $(f^{-1})'(5) = 8$  cannot both be correct.

62. An invertible function  $f(x)$  has values in the table. Evaluate

- (a)  $f'(a) \cdot (f^{-1})'(A)$     (b)  $f'(b) \cdot (f^{-1})'(B)$   
 (c)  $f'(c) \cdot (f^{-1})'(C)$

$x$	$a$	$b$	$c$	$d$
$f(x)$	$A$	$B$	$C$	$D$

63. If  $f$  is continuous, invertible, and defined for all  $x$ , why must at least one of the statements  $(f^{-1})'(10) = 8$ ,  $(f^{-1})'(20) = -6$  be wrong?

64. (a) Calculate  $\lim_{h \rightarrow 0} (\ln(1+h)/h)$  by identifying the limit as the derivative of  $\ln(1+x)$  at  $x=0$ .  
 (b) Use the result of part (a) to show that  $\lim_{h \rightarrow 0} (1+h)^{1/h} = e$ .  
 (c) Use the result of part (b) to calculate the related limit,  $\lim_{n \rightarrow \infty} (1+1/n)^n$ .

## 3.7 IMPLICIT FUNCTIONS

In earlier chapters, most functions were written in the form  $y = f(x)$ ; here  $y$  is said to be an *explicit* function of  $x$ . An equation such as

$$x^2 + y^2 = 4$$

is said to give  $y$  as an *implicit* function of  $x$ . Its graph is the circle in Figure 3.34. Since there are  $x$ -values which correspond to two  $y$ -values,  $y$  is not a function of  $x$  on the whole circle. Solving gives

$$y = \pm \sqrt{4 - x^2},$$

where  $y = \sqrt{4 - x^2}$  represents the top half of the circle and  $y = -\sqrt{4 - x^2}$  represents the bottom half. So  $y$  is a function of  $x$  on the top half, and  $y$  is a different function of  $x$  on the bottom half.

But let's consider the circle as a whole. The equation does represent a curve which has a tangent line at each point. The slope of this tangent can be found by differentiating the equation of the circle with respect to  $x$ :

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4).$$

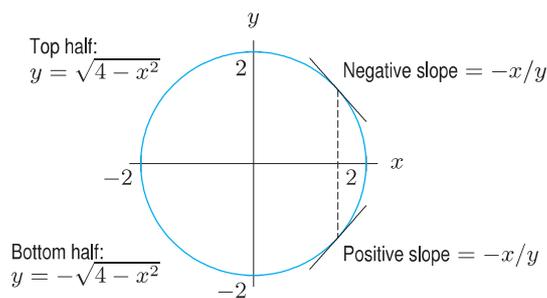
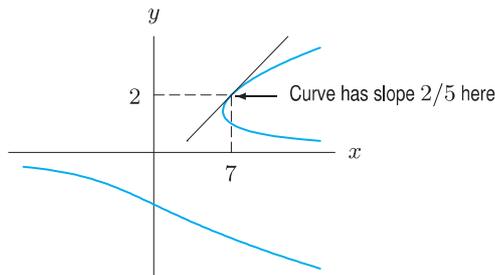
If we think of  $y$  as a function of  $x$  and use the chain rule, we get

$$2x + 2y \frac{dy}{dx} = 0.$$

Solving gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$

<sup>6</sup>www.earth-policy.org, accessed May 18, 2007.

Figure 3.34: Graph of  $x^2 + y^2 = 4$ Figure 3.35: Graph of  $y^3 - xy = -6$  and its tangent line at  $(7, 2)$ 

The derivative here depends on both  $x$  and  $y$  (instead of just on  $x$ ). This is because for many  $x$ -values there are two  $y$ -values, and the curve has a different slope at each one. Figure 3.34 shows that for  $x$  and  $y$  both positive, we are on the top right quarter of the curve and the slope is negative (as the formula predicts). For  $x$  positive and  $y$  negative, we are on the bottom right quarter of the curve and the slope is positive (as the formula predicts).

Differentiating the equation of the circle has given us the slope of the curve at all points except  $(2, 0)$  and  $(-2, 0)$ , where the tangent is vertical. In general, this process of *implicit differentiation* leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

**Example 1** Make a table of  $x$  and approximate  $y$ -values for the equation  $y^3 - xy = -6$  near  $x = 7, y = 2$ . Your table should include the  $x$ -values 6.8, 6.9, 7.0, 7.1, and 7.2.

**Solution** We would like to solve for  $y$  in terms of  $x$ , but we cannot isolate  $y$  by factoring. There is a formula for solving cubics, somewhat like the quadratic formula, but it is too complicated to be useful here. Instead, first observe that  $x = 7, y = 2$  does satisfy the equation. (Check this!) Now find  $dy/dx$  by implicit differentiation:

$$\begin{aligned} \frac{d}{dx}(y^3) - \frac{d}{dx}(xy) &= \frac{d}{dx}(-6) \\ 3y^2 \frac{dy}{dx} - 1 \cdot y - x \frac{dy}{dx} &= 0 \quad (\text{Differentiating with respect to } x) \\ 3y^2 \frac{dy}{dx} - x \frac{dy}{dx} &= y \\ (3y^2 - x) \frac{dy}{dx} &= y \quad (\text{Factoring out } \frac{dy}{dx}) \\ \frac{dy}{dx} &= \frac{y}{3y^2 - x}. \end{aligned}$$

When  $x = 7$  and  $y = 2$ , we have

$$\frac{dy}{dx} = \frac{2}{12 - 7} = \frac{2}{5}.$$

(See Figure 3.35.) The equation of the tangent line at  $(7, 2)$  is

$$y - 2 = \frac{2}{5}(x - 7)$$

or

$$y = 0.4x - 0.8.$$

Since the tangent lies very close to the curve near the point  $(7, 2)$ , we use the equation of the tangent line to calculate the following approximate  $y$ -values:

$x$	6.8	6.9	7.0	7.1	7.2
Approximate $y$	1.92	1.96	2.00	2.04	2.08

Notice that although the equation  $y^3 - xy = -6$  leads to a curve which is difficult to deal with algebraically, it still looks like a straight line locally.

**Example 2** Find all points where the tangent line to  $y^3 - xy = -6$  is either horizontal or vertical.

**Solution** From the previous example,  $\frac{dy}{dx} = \frac{y}{3y^2 - x}$ . The tangent is horizontal when the numerator of  $dy/dx$  equals 0, so  $y = 0$ . Since we also must satisfy  $y^3 - xy = -6$ , we get  $0^3 - x \cdot 0 = -6$ , which is impossible. We conclude that there are no points on the curve where the tangent line is horizontal.

The tangent is vertical when the denominator of  $dy/dx$  is 0, giving  $3y^2 - x = 0$ . Thus,  $x = 3y^2$  at any point with a vertical tangent line. Again, we must also satisfy  $y^3 - xy = -6$ , so

$$\begin{aligned}y^3 - (3y^2)y &= -6, \\-2y^3 &= -6, \\y &= \sqrt[3]{3} \approx 1.442.\end{aligned}$$

We can then find  $x$  by substituting  $y = \sqrt[3]{3}$  in  $y^3 - xy = -6$ . We get  $3 - x(\sqrt[3]{3}) = -6$ , so  $x = 9/(\sqrt[3]{3}) \approx 6.240$ . So the tangent line is vertical at  $(6.240, 1.442)$ .

Using implicit differentiation and the expression for  $dy/dx$  to locate the points where the tangent is vertical or horizontal, as in the previous example, is a first step in obtaining an overall picture of the curve  $y^3 - xy = -6$ . However, filling in the rest of the graph, even roughly, by using the sign of  $dy/dx$  to tell us where the curve is increasing or decreasing can be difficult.

## Exercises and Problems for Section 3.7

### Exercises

For Exercises 1–18, find  $dy/dx$ . Assume  $a, b, c$  are constants.

1.  $x^2 + y^2 = \sqrt{7}$
2.  $x^2 + xy - y^3 = xy^2$
3.  $xy + x + y = 5$
4.  $x^2y - 2y + 5 = 0$
5.  $\sqrt{x} = 5\sqrt{y}$
6.  $\sqrt{x} + \sqrt{y} = 25$
7.  $xy - x - 3y - 4 = 0$
8.  $6x^2 + 4y^2 = 36$
9.  $ax^2 - by^2 = c^2$
10.  $\ln x + \ln(y^2) = 3$
11.  $x \ln y + y^3 = \ln x$
12.  $\sin(xy) = 2x + 5$
13.  $\cos^2 y + \sin^2 y = y + 2$
14.  $e^{\cos y} = x^3 \arctan y$
15.  $\arctan(x^2y) = xy^2$
16.  $e^{x^2} + \ln y = 0$
17.  $(x - a)^2 + y^2 = a^2$
18.  $x^{2/3} + y^{2/3} = a^{2/3}$

In Exercises 19–22, find the slope of the tangent to the curve at the point specified.

19.  $x^2 + y^2 = 1$  at  $(0, 1)$
20.  $\sin(xy) = x$  at  $(1, \pi/2)$
21.  $x^3 + 2xy + y^2 = 4$  at  $(1, 1)$
22.  $x^3 + 5x^2y + 2y^2 = 4y + 11$  at  $(1, 2)$

For Exercises 23–27, find the equations of the tangent lines to the following curves at the indicated points.

23.  $xy^2 = 1$  at  $(1, -1)$
24.  $\ln(xy) = 2x$  at  $(1, e^2)$
25.  $y^2 = \frac{x^2}{xy - 4}$  at  $(4, 2)$
26.  $y = \frac{x}{y + a}$  at  $(0, 0)$
27.  $x^{2/3} + y^{2/3} = a^{2/3}$  at  $(a, 0)$

## Problems

28. (a) Find  $dy/dx$  given that  $x^2 + y^2 - 4x + 7y = 15$ .  
 (b) Under what conditions on  $x$  and/or  $y$  is the tangent line to this curve horizontal? Vertical?
29. (a) Find the slope of the tangent line to the ellipse  $\frac{x^2}{25} + \frac{y^2}{9} = 1$  at the point  $(x, y)$ .  
 (b) Are there any points where the slope is not defined?
30. (a) Find the equations of the tangent lines to the circle  $x^2 + y^2 = 25$  at the points where  $x = 4$ .  
 (b) Find the equations of the normal lines to this circle at the same points. (The normal line is perpendicular to the tangent line at that point.)  
 (c) At what point do the two normal lines intersect?
31. (a) If  $x^3 + y^3 - xy^2 = 5$ , find  $dy/dx$ .  
 (b) Using your answer to part (a), make a table of approximate  $y$ -values of points on the curve near  $x = 1$ ,  $y = 2$ . Include  $x = 0.96, 0.98, 1, 1.02, 1.04$ .  
 (c) Find the  $y$ -value for  $x = 0.96$  by substituting  $x = 0.96$  in the original equation and solving for  $y$  using a computer or calculator. Compare with your answer in part (b).  
 (d) Find all points where the tangent line is horizontal or vertical.
32. Find the equation of the tangent line to the curve  $y = x^2$  at  $x = 1$ . Show that this line is also a tangent to a circle centered at  $(8, 0)$  and find the equation of this circle.
33. At pressure  $P$  atmospheres, a certain fraction  $f$  of a gas decomposes. The quantities  $P$  and  $f$  are related, for some positive constant  $K$ , by the equation
- $$\frac{4f^2P}{1-f^2} = K.$$
- (a) Find  $df/dP$ .  
 (b) Show that  $df/dP < 0$  always. What does this mean in practical terms?
34. Sketch the circles  $y^2 + x^2 = 1$  and  $y^2 + (x - 3)^2 = 4$ . There is a line with positive slope that is tangent to both circles. Determine the points at which this tangent line touches each circle.
35. Show that the power rule for derivatives applies to rational powers of the form  $y = x^{m/n}$  by raising both sides to the  $n^{\text{th}}$  power and using implicit differentiation.
36. For constants  $a, b, n, R$ , Van der Waal's equation relates the pressure,  $P$ , to the volume,  $V$ , of a fixed quantity of a gas at constant temperature  $T$ :
- $$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT.$$
- Find the rate of change of volume with pressure,  $dV/dP$ .

## 3.8 HYPERBOLIC FUNCTIONS

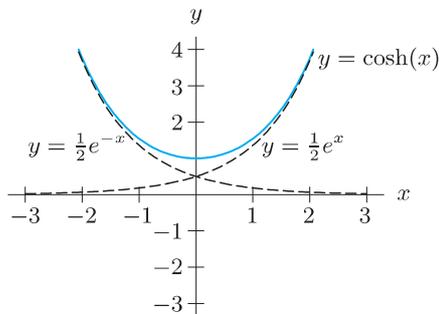
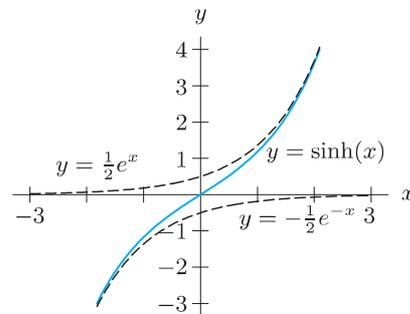
There are two combinations of  $e^x$  and  $e^{-x}$  which are used so often in engineering that they are given their own name. They are the *hyperbolic sine*, abbreviated  $\sinh$ , and the *hyperbolic cosine*, abbreviated  $\cosh$ . They are defined as follows:

## Hyperbolic Functions

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

## Properties of Hyperbolic Functions

The graphs of  $\cosh x$  and  $\sinh x$  are given in Figures 3.36 and 3.37 together with the graphs of multiples of  $e^x$  and  $e^{-x}$ . The graph of  $\cosh x$  is called a *catenary*; it is the shape of a hanging cable.

Figure 3.36: Graph of  $y = \cosh x$ Figure 3.37: Graph of  $y = \sinh x$

The graphs suggest that the following results hold:

$$\begin{aligned} \cosh 0 &= 1 & \sinh 0 &= 0 \\ \cosh(-x) &= \cosh x & \sinh(-x) &= -\sinh x \end{aligned}$$

To show that the hyperbolic functions really do have these properties, we use their formulas.

**Example 1** Show that (a)  $\cosh(0) = 1$  (b)  $\cosh(-x) = \cosh x$

**Solution** (a) Substituting  $x = 0$  into the formula for  $\cosh x$  gives

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1 + 1}{2} = 1.$$

(b) Substituting  $-x$  for  $x$  gives

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x.$$

Thus, we know that  $\cosh x$  is an even function.

**Example 2** Describe and explain the behavior of  $\cosh x$  as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ .

**Solution** From Figure 3.36, it appears that as  $x \rightarrow \infty$ , the graph of  $\cosh x$  resembles the graph of  $\frac{1}{2}e^x$ . Similarly, as  $x \rightarrow -\infty$ , the graph of  $\cosh x$  resembles the graph of  $\frac{1}{2}e^{-x}$ . This behavior is explained by using the formula for  $\cosh x$  and the facts that  $e^{-x} \rightarrow 0$  as  $x \rightarrow \infty$  and  $e^x \rightarrow 0$  as  $x \rightarrow -\infty$ :

$$\text{As } x \rightarrow \infty, \quad \cosh x = \frac{e^x + e^{-x}}{2} \rightarrow \frac{1}{2}e^x.$$

$$\text{As } x \rightarrow -\infty, \quad \cosh x = \frac{e^x + e^{-x}}{2} \rightarrow \frac{1}{2}e^{-x}.$$

### Identities Involving $\cosh x$ and $\sinh x$

The reason the hyperbolic functions have names that remind us of the trigonometric functions is that they share similar properties. A familiar identity for trigonometric functions is

$$(\cos x)^2 + (\sin x)^2 = 1.$$

To discover an analogous identity relating  $(\cosh x)^2$  and  $(\sinh x)^2$ , we first calculate

$$\begin{aligned} (\cosh x)^2 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2e^x e^{-x} + e^{-2x}}{4} = \frac{e^{2x} + 2 + e^{-2x}}{4} \\ (\sinh x)^2 &= \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} - 2e^x e^{-x} + e^{-2x}}{4} = \frac{e^{2x} - 2 + e^{-2x}}{4}. \end{aligned}$$

If we add these expressions, the resulting right-hand side contains terms involving both  $e^{2x}$  and  $e^{-2x}$ . If, however, we subtract the expressions for  $(\cosh x)^2$  and  $(\sinh x)^2$ , we obtain a simple result:

$$(\cosh x)^2 - (\sinh x)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1.$$

Thus, writing  $\cosh^2 x$  for  $(\cosh x)^2$  and  $\sinh^2 x$  for  $(\sinh x)^2$ , we have the identity

$$\cosh^2 x - \sinh^2 x = 1$$

## The Hyperbolic Tangent

Extending the analogy to the trigonometric functions, we define

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

## Derivatives of Hyperbolic Functions

We calculate the derivatives using the fact that  $\frac{d}{dx}(e^x) = e^x$ . The results are again reminiscent of the trigonometric functions. For example,

$$\frac{d}{dx}(\cosh x) = \frac{d}{dx} \left( \frac{e^x + e^{-x}}{2} \right) = \frac{e^x - e^{-x}}{2} = \sinh x.$$

We find  $\frac{d}{dx}(\sinh x)$  similarly, giving the following results:

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sinh x) = \cosh x$$

**Example 3** Compute the derivative of  $\tanh x$ .

**Solution** Using the quotient rule gives

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx} \left( \frac{\sinh x}{\cosh x} \right) = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = \frac{1}{\cosh^2 x}.$$

## Exercises and Problems for Section 3.8

### Exercises

Find the derivatives of the functions in Exercises 1–11.

1.  $y = \cosh(2x)$

2.  $y = \sinh(3z + 5)$

7.  $f(t) = \cosh(e^{t^2})$

8.  $y = \tanh(3 + \sinh x)$

3.  $f(t) = \cosh(\sinh t)$

4.  $f(t) = t^3 \sinh t$

9.  $f(y) = \sinh(\sinh(3y))$

10.  $g(\theta) = \ln(\cosh(1 + \theta))$

5.  $g(t) = \cosh^2 t$

6.  $y = \cosh(3t) \sinh(4t)$

11.  $f(t) = \cosh^2 t - \sinh^2 t$

- 12. Show that  $\sinh 0 = 0$ .
- 13. Show that  $\sinh(-x) = -\sinh(x)$ .
- 14. Show that  $d(\sinh x)/dx = \cosh x$ .

Simplify the expressions in Exercises 15–16.

- 15.  $\sinh(\ln t)$
- 16.  $\cosh(\ln t)$

**Problems**

- 17. Describe and explain the behavior of  $\sinh x$  as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ .
- 18. Is there an identity analogous to  $\sin(2x) = 2 \sin x \cos x$  for the hyperbolic functions? Explain.
- 19. Is there an identity analogous to  $\cos(2x) = \cos^2 x - \sin^2 x$  for the hyperbolic functions? Explain.

Prove the identities in Problems 20–21.

- 20.  $\sinh(A + B) = \sinh A \cosh B + \sinh B \cosh A$
- 21.  $\cosh(A + B) = \cosh A \cosh B + \sinh B \sinh A$

In Problems 22–25, find the limit of the function as  $x \rightarrow \infty$ .

- 22.  $\frac{\cosh(2x)}{\sinh(3x)}$
- 23.  $\frac{\sinh(2x)}{\cosh(3x)}$
- 24.  $\frac{e^{2x}}{\sinh(2x)}$
- 25.  $\frac{\sinh(x^2)}{\cosh(x^2)}$

- 26. For what values of  $k$  is  $\lim_{x \rightarrow \infty} \frac{\sinh kx}{\cosh 2x}$  finite?
- 27. For what values of  $k$  is  $\lim_{x \rightarrow \infty} e^{-3x} \cosh kx$  finite?
- 28. (a) Using a calculator or computer, sketch the graph of  $y = 2e^x + 5e^{-x}$  for  $-3 \leq x \leq 3, 0 \leq y \leq 20$ . Observe that it looks like the graph of  $y = \cosh x$ . Approximately where is its minimum?  
 (b) Show algebraically that  $y = 2e^x + 5e^{-x}$  can be written in the form  $y = A \cosh(x - c)$ . Calculate the values of  $A$  and  $c$ . Explain what this tells you about the graph in part (a).
- 29. The following problem is a generalization of Problem 28. Show that any function of the form

$$y = Ae^x + Be^{-x}, \quad A > 0, B > 0,$$

can be written, for some  $K$  and  $c$ , in the form

$$y = K \cosh(x - c).$$

What does this tell you about the graph of  $y = Ae^x + Be^{-x}$ ?

- 30. The cable between the two towers of a power line hangs in the shape of the curve

$$y = \frac{T}{w} \cosh\left(\frac{wx}{T}\right),$$

where  $T$  is the tension in the cable at its lowest point and  $w$  is the weight of the cable per unit length. This curve is called a *catenary*.

- (a) Suppose the cable stretches between the points  $x = -T/w$  and  $x = T/w$ . Find an expression for the “sag” in the cable. (That is, find the difference between the height of the cable at the highest and lowest points.)
- (b) Show that the shape of the cable satisfies the equation

$$\frac{d^2y}{dx^2} = \frac{w}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

- 31. The Saint Louis arch can be approximated by using a function of the form  $y = b - a \cosh(x/a)$ . Putting the origin on the ground in the center of the arch and the  $y$ -axis upward, find an approximate equation for the arch given the dimensions shown in Figure 3.38. (In other words, find  $a$  and  $b$ .)

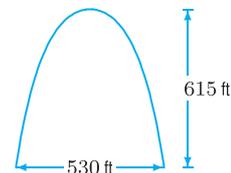


Figure 3.38

- 32. (a) Find  $\tanh 0$ .
- (b) For what values of  $x$  is  $\tanh x$  positive? Negative? Explain your answer algebraically.
- (c) On what intervals is  $\tanh x$  increasing? Decreasing? Use derivatives to explain your answer.
- (d) Find  $\lim_{x \rightarrow \infty} \tanh x$  and  $\lim_{x \rightarrow -\infty} \tanh x$ . Show this information on a graph.
- (e) Does  $\tanh x$  have an inverse? Justify your answer using derivatives.

## 3.9 LINEAR APPROXIMATION AND THE DERIVATIVE

### The Tangent Line Approximation

When we zoom in on the graph of a differentiable function, it looks like a straight line. In fact, the graph is not exactly a straight line when we zoom in; however, its deviation from straightness is so small that it can't be detected by the naked eye. Let's examine what this means. The straight line that we think we see when we zoom in on the graph of  $f(x)$  at  $x = a$  has slope equal to the derivative,  $f'(a)$ , so the equation is

$$y = f(a) + f'(a)(x - a).$$

The fact that the graph looks like a line means that  $y$  is a good approximation to  $f(x)$ . (See Figure 3.39.) This suggests the following definition:

#### The Tangent Line Approximation

Suppose  $f$  is differentiable at  $a$ . Then, for values of  $x$  near  $a$ , the tangent line approximation to  $f(x)$  is

$$f(x) \approx f(a) + f'(a)(x - a).$$

The expression  $f(a) + f'(a)(x - a)$  is called the *local linearization* of  $f$  near  $x = a$ . We are thinking of  $a$  as fixed, so that  $f(a)$  and  $f'(a)$  are constant.

The **error**,  $E(x)$ , in the approximation is defined by

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

It can be shown that the tangent line approximation is the best linear approximation to  $f$  near  $a$ . See Problem 38.

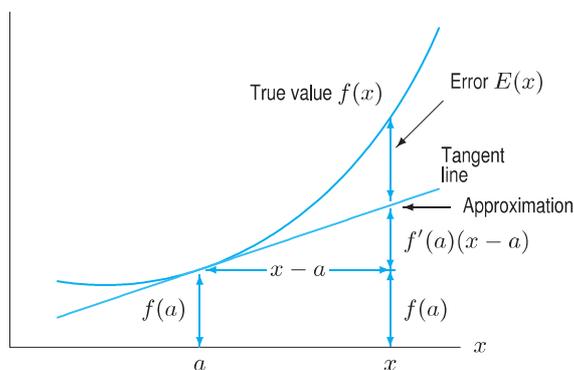


Figure 3.39: The tangent line approximation and its error

**Example 1** What is the tangent line approximation for  $f(x) = \sin x$  near  $x = 0$ ?

**Solution** The tangent line approximation of  $f$  near  $x = 0$  is

$$f(x) \approx f(0) + f'(0)(x - 0).$$

If  $f(x) = \sin x$ , then  $f'(x) = \cos x$ , so  $f(0) = \sin 0 = 0$  and  $f'(0) = \cos 0 = 1$ , and the approximation is

$$\sin x \approx x.$$

This means that, near  $x = 0$ , the function  $f(x) = \sin x$  is well approximated by the function  $y = x$ . If we zoom in on the graphs of the functions  $\sin x$  and  $x$  near the origin, we won't be able to tell them apart. (See Figure 3.40.)

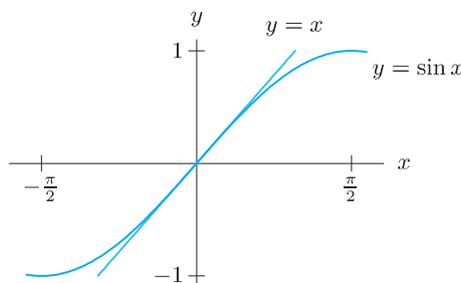


Figure 3.40: Tangent line approximation to  $y = \sin x$

**Example 2** What is the local linearization of  $e^{kx}$  near  $x = 0$ ?

**Solution** If  $f(x) = e^{kx}$ , then  $f(0) = 1$  and, by the chain rule,  $f'(x) = ke^{kx}$ , so  $f'(0) = ke^{k \cdot 0} = k$ . Thus

$$f(x) \approx f(0) + f'(0)(x - 0)$$

becomes

$$e^{kx} \approx 1 + kx.$$

This is the tangent line approximation to  $e^{kx}$  near  $x = 0$ . In other words, if we zoom in on the functions  $f(x) = e^{kx}$  and  $y = 1 + kx$  near the origin, we won't be able to tell them apart.

## Estimating the Error in the Approximation

Let us look at the error,  $E(x)$ , which is the difference between  $f(x)$  and the local linearization. (Look back at Figure 3.39.) The fact that the graph of  $f$  looks like a line as we zoom in means that not only is  $E(x)$  small for  $x$  near  $a$ , but also that  $E(x)$  is small relative to  $(x - a)$ . To demonstrate this, we prove the following theorem about the ratio  $E(x)/(x - a)$ .

### Theorem 3.6: Differentiability and Local Linearity

Suppose  $f$  is differentiable at  $x = a$  and  $E(x)$  is the error in the tangent line approximation, that is:

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$

Then

$$\lim_{x \rightarrow a} \frac{E(x)}{x - a} = 0.$$

**Proof** Using the definition of  $E(x)$ , we have

$$\frac{E(x)}{x-a} = \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a).$$

Taking the limit as  $x \rightarrow a$  and using the definition of the derivative, we see that

$$\lim_{x \rightarrow a} \frac{E(x)}{x-a} = \lim_{x \rightarrow a} \left( \frac{f(x) - f(a)}{x-a} - f'(a) \right) = f'(a) - f'(a) = 0.$$

Theorem 3.6 says that  $E(x)$  approaches 0 faster than  $(x-a)$ . For the function in Example 3, we see that  $E(x) \approx k(x-a)^2$  for constant  $k$  if  $x$  is near  $a$ .

**Example 3** Let  $E(x)$  be the error in the tangent line approximation to  $f(x) = x^3 - 5x + 3$  for  $x$  near 2.

- (a) What does a table of values for  $E(x)/(x-2)$  suggest about  $\lim_{x \rightarrow 2} E(x)/(x-2)$ ?  
 (b) Make another table to see that  $E(x) \approx k(x-2)^2$ . Estimate the value of  $k$ . Check that a possible value is  $k = f''(2)/2$ .

**Solution** (a) Since  $f(x) = x^3 - 5x + 3$ , we have  $f'(x) = 3x^2 - 5$ , and  $f''(x) = 6x$ . Thus,  $f(2) = 1$  and  $f'(2) = 3 \cdot 2^2 - 5 = 7$ , so the tangent line approximation for  $x$  near 2 is

$$f(x) \approx f(2) + f'(2)(x-2)$$

$$f(x) \approx 1 + 7(x-2).$$

Thus,

$$E(x) = \text{True value} - \text{Approximation} = (x^3 - 5x + 3) - (1 + 7(x-2)).$$

The values of  $E(x)/(x-2)$  in Table 3.7 suggest that  $E(x)/(x-2)$  approaches 0 as  $x \rightarrow 2$ .

- (b) Notice that if  $E(x) \approx k(x-2)^2$ , then  $E(x)/(x-2)^2 \approx k$ . Thus we make Table 3.8 showing values of  $E(x)/(x-2)^2$ . Since the values are all approximately 6, we guess that  $k = 6$  and  $E(x) \approx 6(x-2)^2$ .

Since  $f''(2) = 12$ , our value of  $k$  satisfies  $k = f''(2)/2$ .

**Table 3.7**

$x$	$E(x)/(x-2)$
2.1	0.61
2.01	0.0601
2.001	0.006001
2.0001	0.00060001

**Table 3.8**

$x$	$E(x)/(x-2)^2$
2.1	6.1
2.01	6.01
2.001	6.001
2.0001	6.0001

The relationship between  $E(x)$  and  $f''(x)$  that appears in Example 3 holds more generally. If  $f(x)$  satisfies certain conditions, it can be shown that the error in the tangent line approximation behaves near  $x = a$  as

$$E(x) \approx \frac{f''(a)}{2}(x-a)^2.$$

This is part of a general pattern for obtaining higher order approximations called Taylor polynomials, which are studied in Chapter 10.

### Why Differentiability Makes A Graph Look Straight

We use the properties of the error  $E(x)$  to understand why differentiability makes a graph look straight when we zoom in.

**Example 4** Consider the graph of  $f(x) = \sin x$  near  $x = 0$ , and its linear approximation computed in Example 1. Show that there is an interval around 0 with the property that the distance from  $f(x) = \sin x$  to the linear approximation is less than  $0.1|x|$  for all  $x$  in the interval.

**Solution** The linear approximation of  $f(x) = \sin x$  near 0 is  $y = x$ , so we write

$$\sin x = x + E(x).$$

Since  $\sin x$  is differentiable at  $x = 0$ , Theorem 3.6 tells us that

$$\lim_{x \rightarrow 0} \frac{E(x)}{x} = 0.$$

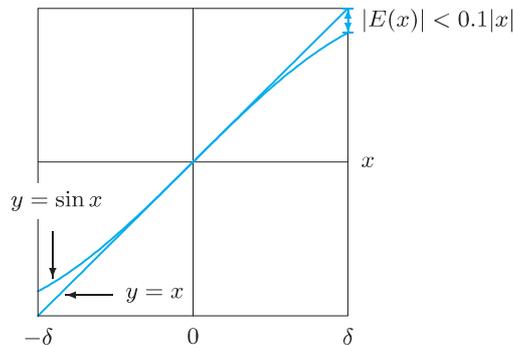
If we take  $\epsilon = 1/10$ , then the definition of limit guarantees that there is a  $\delta > 0$  such that

$$\left| \frac{E(x)}{x} \right| < 0.1 \quad \text{for all } |x| < \delta.$$

In other words, for  $x$  in the interval  $(-\delta, \delta)$ , we have  $|x| < \delta$ , so

$$|E(x)| < 0.1|x|.$$

(See Figure 3.41.)



**Figure 3.41:** Graph of  $y = \sin x$  and its linear approximation  $y = x$ , showing a window in which the magnitude of the error,  $|E(x)|$ , is less than  $0.1|x|$  for all  $x$  in the window

We can generalize from this example to explain why differentiability makes the graph of  $f$  look straight when viewed over a small graphing window. Suppose  $f$  is differentiable at  $x = a$ . Then we know  $\lim_{x \rightarrow a} \left| \frac{E(x)}{x - a} \right| = 0$ . So, for any  $\epsilon > 0$ , we can find a  $\delta$  small enough so that

$$\left| \frac{E(x)}{x - a} \right| < \epsilon, \quad \text{for } a - \delta < x < a + \delta.$$

So, for any  $x$  in the interval  $(a - \delta, a + \delta)$ , we have

$$|E(x)| < \epsilon|x - a|.$$

Thus, the error,  $E(x)$ , is less than  $\epsilon$  times  $|x - a|$ , the distance between  $x$  and  $a$ . So, as we zoom in on the graph by choosing smaller  $\epsilon$ , the deviation,  $|E(x)|$ , of  $f$  from its tangent line shrinks, even relative to the scale on the  $x$ -axis. So, zooming makes a differentiable function look straight.

## Exercises and Problems for Section 3.9

## Exercises

- Find the local linearization of  $f(x) = x^2$  near  $x = 1$ .
- Find the tangent line approximation for  $\sqrt{1+x}$  near  $x = 0$ .
- What is the tangent line approximation to  $e^x$  near  $x = 0$ ?
- Find the tangent line approximation to  $1/x$  near  $x = 1$ .
- Show that  $1 - x/2$  is the tangent line approximation to  $1/\sqrt{1+x}$  near  $x = 0$ .
- Show that  $e^{-x} \approx 1 - x$  near  $x = 0$ .
- What is the local linearization of  $e^{x^2}$  near  $x = 1$ ?
- Local linearization gives values too small for the function  $x^2$  and too large for the function  $\sqrt{x}$ . Draw pictures to explain why.
- Using a graph like Figure 3.40, estimate to one decimal place the magnitude of the error in approximating  $\sin x$  by  $x$  for  $-1 \leq x \leq 1$ . Is the approximation an over- or an underestimate?
- For  $x$  near 0, local linearization gives
 
$$e^x \approx 1 + x.$$
 Using a graph, decide if the approximation is an over- or underestimate, and estimate to one decimal place the magnitude of the error for  $-1 \leq x \leq 1$ .

## Problems

- (a) Find the best linear approximation,  $L(x)$ , to  $f(x) = e^x$  near  $x = 0$ .  
 (b) What is the sign of the error,  $E(x) = f(x) - L(x)$  for  $x$  near 0?  
 (c) Find the true value of the function at  $x = 1$ . What is the error? (Give decimal answers.) Illustrate with a graph.  
 (d) Before doing any calculations, explain which you expect to be larger,  $E(0.1)$  or  $E(1)$ , and why.  
 (e) Find  $E(0.1)$ .

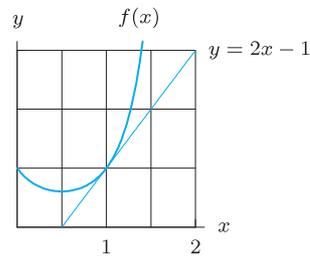


Figure 3.42

- (a) Graph  $f(x) = x^3 - 3x^2 + 3x + 1$ .  
 (b) Find and add to your sketch the local linearization to  $f(x)$  at  $x = 2$ .  
 (c) Mark on your sketch the true value of  $f(1.5)$ , the tangent line approximation to  $f(1.5)$  and the error in the approximation.
- (a) Find the tangent line approximation to  $\cos x$  at  $x = \pi/4$ .  
 (b) Use a graph to explain how you know whether the tangent line approximation is an under- or overestimate for  $0 \leq x \leq \pi/2$ .  
 (c) To one decimal place, estimate the error in the approximation for  $0 \leq x \leq \pi/2$ .
- (a) Show that  $1+kx$  is the local linearization of  $(1+x)^k$  near  $x = 0$ .  
 (b) Someone claims that the square root of 1.1 is about 1.05. Without using a calculator, do you think that this estimate is about right?  
 (c) Is the actual number above or below 1.05?
- Figure 3.42 shows  $f(x)$  and its local linearization at  $x = a$ . What is the value of  $a$ ? Of  $f(a)$ ? Is the approximation an under- or overestimate? Use the linearization to approximate the value of  $f(1.2)$ .

The equations in Problems 16–17 have a solution near  $x = 0$ . By replacing the left side of the equation by its linearization, find an approximate value for the solution.

$$16. e^x + x = 2 \qquad 17. x + \ln(1+x) = 0.2$$

- (a) Explain why the following equation has a solution near 0:

$$e^t = 0.02t + 1.098.$$

- (b) Replace  $e^t$  by its linearization near 0. Solve the new equation to get an approximate solution to the original equation.

- The speed of sound in dry air is

$$f(T) = 331.3 \sqrt{1 + \frac{T}{273.15}} \text{ meters/second}$$

where  $T$  is the temperature in  $^\circ$  Celsius. Find a linear function that approximates the speed of sound for temperatures near  $0^\circ$  C.

- Air pressure at sea level is 30 inches of mercury. At an altitude of  $h$  feet above sea level, the air pressure,  $P$ , in inches of mercury, is given by

$$P = 30e^{-3.23 \times 10^{-5} h}$$

- (a) Sketch a graph of  $P$  against  $h$ .  
 (b) Find the equation of the tangent line at  $h = 0$ .  
 (c) A rule of thumb used by travelers is that air pressure drops about 1 inch for every 1000-foot increase in height above sea level. Write a formula for the air pressure given by this rule of thumb.  
 (d) What is the relation between your answers to parts (b) and (c)? Explain why the rule of thumb works.  
 (e) Are the predictions made by the rule of thumb too large or too small? Why?
21. Writing  $g$  for the acceleration due to gravity, the period,  $T$ , of a pendulum of length  $l$  is given by

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

- (a) Show that if the length of the pendulum changes by  $\Delta l$ , the change in the period,  $\Delta T$ , is given by
- $$\Delta T \approx \frac{T}{2l}\Delta l.$$
- (b) If the length of the pendulum increases by 2%, by what percent does the period change?
22. Suppose now the length of the pendulum in Problem 21 remains constant, but that the acceleration due to gravity changes.

- (a) Use the method of the preceding problem to relate  $\Delta T$  approximately to  $\Delta g$ , the change in  $g$ .  
 (b) If  $g$  increases by 1%, find the percent change in  $T$ .
23. Suppose  $f$  has a continuous positive second derivative for all  $x$ . Which is larger,  $f(1+\Delta x)$  or  $f(1)+f'(1)\Delta x$ ? Explain.
24. Suppose  $f'(x)$  is a differentiable decreasing function for all  $x$ . In each of the following pairs, which number is the larger? Give a reason for your answer.
- (a)  $f'(5)$  and  $f'(6)$   
 (b)  $f''(5)$  and 0  
 (c)  $f(5+\Delta x)$  and  $f(5)+f'(5)\Delta x$

Problems 25–27 investigate the motion of a projectile shot from a cannon. The fixed parameters are the acceleration of gravity,  $g = 9.8 \text{ m/sec}^2$ , and the muzzle velocity,  $v_0 = 500 \text{ m/sec}$ , at which the projectile leaves the cannon. The angle  $\theta$ , in degrees, between the muzzle of the cannon and the ground can vary.

25. The range of the projectile is

$$f(\theta) = \frac{v_0^2}{g} \sin \frac{\pi\theta}{90} \sin \frac{\pi\theta}{90} = 25510 \sin \frac{\pi\theta}{90} \sin \frac{\pi\theta}{90} \text{ meters.}$$

- (a) Find the range with  $\theta = 20^\circ$ .  
 (b) Find a linear function of  $\theta$  that approximates the range for angles near  $20^\circ$ .  
 (c) Find the range and its approximation from part (b) for  $21^\circ$ .

26. The time that the projectile stays in the air is

$$t(\theta) = \frac{2v_0}{g} \sin \frac{\pi\theta}{180} = 102 \sin \frac{\pi\theta}{180} \text{ seconds.}$$

- (a) Find the time in the air for  $\theta = 20^\circ$ .  
 (b) Find a linear function of  $\theta$  that approximates the time in the air for angles near  $20^\circ$ .  
 (c) Find the time in air and its approximation from part (b) for  $21^\circ$ .
27. At its highest point the projectile reaches a peak altitude given by

$$h(\theta) = \frac{v_0^2}{2g} \sin^2 \frac{\pi\theta}{180} = 12755 \sin^2 \frac{\pi\theta}{180} \text{ meters.}$$

- (a) Find the peak altitude for  $\theta = 20^\circ$ .  
 (b) Find a linear function of  $\theta$  that approximates the peak altitude for angles near  $20^\circ$ .  
 (c) Find the peak altitude and its approximation from part (b) for  $21^\circ$ .

In Problems 28–32, find a formula for the error  $E(x)$  in the tangent line approximation to the function near  $x = a$ . Using a table of values for  $E(x)/(x-a)$  near  $x = a$ , find a value of  $k$  such that  $E(x)/(x-a) \approx k(x-a)$ . Check that, approximately,  $k = f''(a)/2$  and that  $E(x) \approx (f''(a)/2)(x-a)^2$ .

28.  $f(x) = x^4$ ,  $a = 1$       29.  $f(x) = \cos x$ ,  $a = 0$   
 30.  $f(x) = e^x$ ,  $a = 0$       31.  $f(x) = \sqrt{x}$ ,  $a = 1$   
 32.  $f(x) = \ln x$ ,  $a = 1$

33. Multiply the local linearization of  $e^x$  near  $x = 0$  by itself to obtain an approximation for  $e^{2x}$ . Compare this with the actual local linearization of  $e^{2x}$ . Explain why these two approximations are consistent, and discuss which one is more accurate.

34. (a) Show that  $1-x$  is the local linearization of  $\frac{1}{1+x}$  near  $x = 0$ .  
 (b) From your answer to part (a), show that near  $x = 0$ ,

$$\frac{1}{1+x^2} \approx 1-x^2.$$

- (c) Without differentiating, what do you think the derivative of  $\frac{1}{1+x^2}$  is at  $x = 0$ ?

35. From the local linearizations of  $e^x$  and  $\sin x$  near  $x = 0$ , write down the local linearization of the function  $e^x \sin x$ . From this result, write down the derivative of  $e^x \sin x$  at  $x = 0$ . Using this technique, write down the derivative of  $e^x \sin x/(1+x)$  at  $x = 0$ .

36. Use local linearization to derive the product rule,

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$

[Hint: Use the definition of the derivative and the local linearizations  $f(x+h) \approx f(x)+f'(x)h$  and  $g(x+h) \approx g(x)+g'(x)h$ .]

37. Derive the chain rule using local linearization. [Hint: In other words, differentiate  $f(g(x))$ , using  $g(x+h) \approx g(x) + g'(x)h$  and  $f(z+k) \approx f(z) + f'(z)k$ .]
38. Consider a function  $f$  and a point  $a$ . Suppose there is a number  $L$  such that the linear function  $g$

$$g(x) = f(a) + L(x - a)$$

is a good approximation to  $f$ . By good approximation, we mean that

$$\lim_{x \rightarrow a} \frac{E_L(x)}{x - a} = 0,$$

where  $E_L(x)$  is the approximation error defined by

$$f(x) = g(x) + E_L(x) = f(a) + L(x - a) + E_L(x).$$

Show that  $f$  is differentiable at  $x = a$  and that  $f'(a) = L$ . Thus the tangent line approximation is the only good linear approximation.

39. Consider the graph of  $f(x) = x^2$  near  $x = 1$ . Find an interval around  $x = 1$  with the property that throughout any smaller interval, the graph of  $f(x) = x^2$  never differs from its local linearization at  $x = 1$  by more than  $0.1|x - 1|$ .

## 3.10 THEOREMS ABOUT DIFFERENTIABLE FUNCTIONS

### A Relationship Between Local and Global: The Mean Value Theorem

We often want to infer a global conclusion (for example,  $f$  is increasing on an interval) from local information (for example,  $f'$  is positive at each point on an interval.) The following theorem relates the average rate of change of a function on an interval (global information) to the instantaneous rate of change at a point in the interval (local information).

#### Theorem 3.7: The Mean Value Theorem

If  $f$  is continuous on  $a \leq x \leq b$  and differentiable on  $a < x < b$ , then there exists a number  $c$ , with  $a < c < b$ , such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In other words,  $f(b) - f(a) = f'(c)(b - a)$ .

To understand this theorem geometrically, look at Figure 3.43. Join the points on the curve where  $x = a$  and  $x = b$  with a secant line and observe that

$$\text{Slope of secant line} = \frac{f(b) - f(a)}{b - a}.$$

Now consider the tangent lines drawn to the curve at each point between  $x = a$  and  $x = b$ . In general, these lines have different slopes. For the curve shown in Figure 3.43, the tangent line at  $x = a$  is flatter than the secant line. Similarly, the tangent line at  $x = b$  is steeper than the secant line. However, there appears to be at least one point between  $a$  and  $b$  where the slope of the tangent line to the curve is precisely the same as the slope of the secant line. Suppose this occurs at  $x = c$ . Then

$$\text{Slope of tangent line} = f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem tells us that the point  $x = c$  exists, but it does not tell us how to find  $c$ . Problems 43 and 44 in Section 4.2 show how the Mean Value Theorem can be derived.

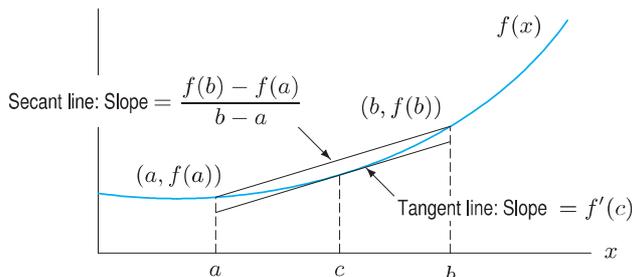


Figure 3.43: The point  $c$  with  $f'(c) = \frac{f(b) - f(a)}{b - a}$

If  $f$  satisfies the conditions of the Mean Value Theorem on  $a < x < b$  and  $f(a) = f(b) = 0$ , the Mean Value Theorem tells us that there is a point  $c$ , with  $a < c < b$ , such that  $f'(c) = 0$ . This result is called **Rolle's Theorem**.

### The Increasing Function Theorem

We say that a function  $f$  is *increasing* on an interval if, for any two numbers  $x_1$  and  $x_2$  in the interval such that  $x_1 < x_2$ , we have  $f(x_1) < f(x_2)$ . If instead we have  $f(x_1) \leq f(x_2)$ , we say  $f$  is *nondecreasing*.

#### Theorem 3.8: The Increasing Function Theorem

Suppose that  $f$  is continuous on  $a \leq x \leq b$  and differentiable on  $a < x < b$ .

- If  $f'(x) > 0$  on  $a < x < b$ , then  $f$  is increasing on  $a \leq x \leq b$ .
- If  $f'(x) \geq 0$  on  $a < x < b$ , then  $f$  is nondecreasing on  $a \leq x \leq b$ .

**Proof** Suppose  $a \leq x_1 < x_2 \leq b$ . By the Mean Value Theorem, there is a number  $c$ , with  $x_1 < c < x_2$ , such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

If  $f'(c) > 0$ , this says  $f(x_2) - f(x_1) > 0$ , which means  $f$  is increasing. If  $f'(c) \geq 0$ , this says  $f(x_2) - f(x_1) \geq 0$ , which means  $f$  is nondecreasing.

It may seem that something as simple as the Increasing Function Theorem should follow immediately from the definition of the derivative, and you may be surprised that the Mean Value Theorem is needed.

### The Constant Function Theorem

If  $f$  is constant on an interval, then we know that  $f'(x) = 0$  on the interval. The following theorem is the converse.

#### Theorem 3.9: The Constant Function Theorem

Suppose that  $f$  is continuous on  $a \leq x \leq b$  and differentiable on  $a < x < b$ . If  $f'(x) = 0$  on  $a < x < b$ , then  $f$  is constant on  $a \leq x \leq b$ .

**Proof** The proof is the same as for the Increasing Function Theorem, only in this case  $f'(c) = 0$  so  $f(x_2) - f(x_1) = 0$ . Thus  $f(x_2) = f(x_1)$  for  $a \leq x_1 < x_2 \leq b$ , so  $f$  is constant.

A proof of the Constant Function Theorem using the Increasing Function Theorem is given in Problems 17 and 24.

## The Racetrack Principle

### Theorem 3.10: The Racetrack Principle<sup>7</sup>

Suppose that  $g$  and  $h$  are continuous on  $a \leq x \leq b$  and differentiable on  $a < x < b$ , and that  $g'(x) \leq h'(x)$  for  $a < x < b$ .

- If  $g(a) = h(a)$ , then  $g(x) \leq h(x)$  for  $a \leq x \leq b$ .
- If  $g(b) = h(b)$ , then  $g(x) \geq h(x)$  for  $a \leq x \leq b$ .

The Racetrack Principle has the following interpretation. We can think of  $g(x)$  and  $h(x)$  as the positions of two racehorses at time  $x$ , with horse  $h$  always moving faster than horse  $g$ . If they start together, horse  $h$  is ahead during the whole race. If they finish together, horse  $g$  was ahead during the whole race.

**Proof** Consider the function  $f(x) = h(x) - g(x)$ . Since  $f'(x) = h'(x) - g'(x) \geq 0$ , we know that  $f$  is nondecreasing by the Increasing Function Theorem. So  $f(x) \geq f(a) = h(a) - g(a) = 0$ . Thus  $g(x) \leq h(x)$  for  $a \leq x \leq b$ . This proves the first part of the Racetrack Principle. Problem 23 asks for a proof of the second part.

**Example 1** Explain graphically why  $e^x \geq 1 + x$  for all values of  $x$ . Then use the Racetrack Principle to prove the inequality.

**Solution** The graph of the function  $y = e^x$  is concave up everywhere and the equation of its tangent line at the point  $(0, 1)$  is  $y = x + 1$ . (See Figure 3.44.) Since the graph always lies above its tangent, we have the inequality

$$e^x \geq 1 + x.$$

Now we prove the inequality using the Racetrack Principle. Let  $g(x) = 1 + x$  and  $h(x) = e^x$ . Then  $g(0) = h(0) = 1$ . Furthermore,  $g'(x) = 1$  and  $h'(x) = e^x$ . Hence  $g'(x) \leq h'(x)$  for  $x \geq 0$ . So by the Racetrack Principle, with  $a = 0$ , we have  $g(x) \leq h(x)$ , that is,  $1 + x \leq e^x$ .

For  $x \leq 0$  we have  $h'(x) \leq g'(x)$ . So by the Racetrack Principle, with  $b = 0$ , we have  $g(x) \leq h(x)$ , that is,  $1 + x \leq e^x$ .

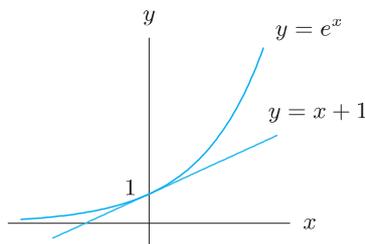


Figure 3.44: Graph showing that  $e^x \geq 1 + x$

<sup>7</sup>Based on the Racetrack Principle in *Calculus & Mathematics*, by William Davis, Horacio Porta, Jerry Uhl (Reading: Addison Wesley, 1994).

## Exercises and Problems for Section 3.10

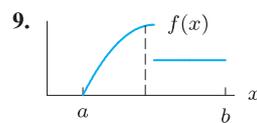
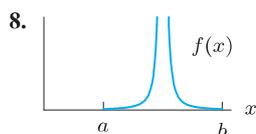
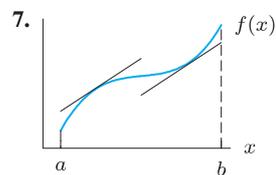
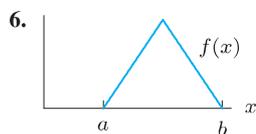
## Exercises

Decide if the statements in Exercises 1–5 are true or false. Give an explanation for your answer.

- Let  $f(x) = [x]$ , the greatest integer less than or equal to  $x$ . Then  $f'(x) = 0$ , so  $f(x)$  is constant by the Constant Function Theorem.
- The Racetrack Principle can be used to justify the statement that if two horses start a race at the same time, the horse that wins must have been moving faster than the other throughout the race.
- Two horses start a race at the same time and one runs slower than the other throughout the race. The Racetrack Principle can be used to justify the fact that the slower horse loses the race.
- If  $a < b$  and  $f'(x)$  is positive on  $[a, b]$  then  $f(a) < f(b)$ .
- If  $f(x)$  is increasing and differentiable on the interval  $[a, b]$ , then  $f'(x) > 0$  on  $[a, b]$ .

Do the functions graphed in Exercises 6–9 appear to satisfy

the hypotheses of the Mean Value Theorem on the interval  $[a, b]$ ? Do they satisfy the conclusion?



## Problems

10. Applying the Mean Value Theorem with  $a = 2$ ,  $b = 7$  to the function in Figure 3.45 leads to  $c = 4$ . What is the equation of the tangent line at 4?

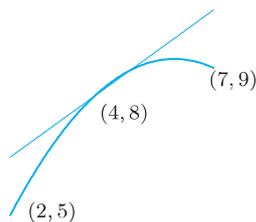


Figure 3.45

11. Applying the Mean Value Theorem with  $a = 3$ ,  $b = 13$  to the function in Figure 3.46 leads to the point  $c$  shown. What is the value of  $f'(c)$ ? What can you say about the values of  $f'(x_1)$  and  $f'(x_2)$ ?

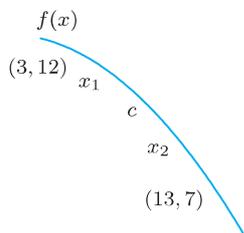


Figure 3.46

12. Let  $p(x) = x^5 + 8x^4 - 30x^3 + 30x^2 - 31x + 22$ . What is the relationship between  $p(x)$  and  $f(x) = 5x^4 + 32x^3 - 90x^2 + 60x - 31$ ? What do the values of  $p(1)$  and  $p(2)$  tell you about the values of  $f(x)$ ?

13. Let  $p(x)$  be a seventh degree polynomial with 7 distinct zeros. How many zeros does  $p'(x)$  have?

14. Use the Racetrack Principle and the fact that  $\sin 0 = 0$  to show that  $\sin x \leq x$  for all  $x \geq 0$ .

15. Use the Racetrack Principle to show that  $\ln x \leq x - 1$ .

16. Use the fact that  $\ln x$  and  $e^x$  are inverse functions to show that the inequalities  $e^x \geq 1 + x$  and  $\ln x \leq x - 1$  are equivalent for  $x > 0$ .

17. State a Decreasing Function Theorem, analogous to the Increasing Function Theorem. Deduce your theorem from the Increasing Function Theorem. [Hint: Apply the Increasing Function Theorem to  $-f$ .]

Use one of the theorems in this section to prove the statements in Problems 18–21.

18. If  $f'(x) \leq 1$  for all  $x$  and  $f(0) = 0$ , then  $f(x) \leq x$  for all  $x \geq 0$ .

19. If  $f''(t) \leq 3$  for all  $t$  and  $f(0) = f'(0) = 0$ , then  $f(t) \leq \frac{3}{2}t^2$  for all  $t \geq 0$ .

20. If  $f'(x) = g'(x)$  for all  $x$  and  $f(5) = g(5)$ , then  $f(x) = g(x)$  for all  $x$ .
21. If  $f$  is differentiable and  $f(0) < f(1)$ , then there is a number  $c$ , with  $0 < c < 1$ , such that  $f'(c) > 0$ .
22. The position of a particle on the  $x$ -axis is given by  $s = f(t)$ ; its initial position and velocity are  $f(0) = 3$  and  $f'(0) = 4$ . The acceleration is bounded by  $5 \leq f''(t) \leq 7$  for  $0 \leq t \leq 2$ . What can we say about the position  $f(2)$  of the particle at  $t = 2$ ?
23. Suppose that  $g$  and  $h$  are continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Prove that if  $g'(x) \leq h'(x)$  for  $a < x < b$  and  $g(b) = h(b)$ , then  $h(x) \leq g(x)$  for  $a \leq x \leq b$ .
24. Deduce the Constant Function Theorem from the Increasing Function Theorem and the Decreasing Function Theorem. (See Problem 17.)
25. Prove that if  $f'(x) = g'(x)$  for all  $x$  in  $(a, b)$ , then there is a constant  $C$  such that  $f(x) = g(x) + C$  on  $(a, b)$ . [Hint: Apply the Constant Function Theorem to  $h(x) = f(x) - g(x)$ .]
26. Suppose that  $f'(x) = f(x)$  for all  $x$ . Prove that  $f(x) = Ce^x$  for some constant  $C$ . [Hint: Consider  $f(x)/e^x$ .]
27. Suppose that  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and that  $m \leq f'(x) \leq M$  on  $(a, b)$ . Use the Racetrack Principle to prove that  $f(x) - f(a) \leq M(x - a)$  for all  $x$  in  $[a, b]$ , and that  $m(x - a) \leq f(x) - f(a)$  for all  $x$  in  $[a, b]$ . Conclude that  $m \leq (f(b) - f(a))/(b - a) \leq M$ . This is called the Mean Value Inequality. In words: If the instantaneous rate of change of  $f$  is between  $m$  and  $M$  on an interval, so is the average rate of change of  $f$  over the interval.
28. Suppose that  $f''(x) \geq 0$  for all  $x$  in  $(a, b)$ . We will show the graph of  $f$  lies above the tangent line at  $(c, f(c))$  for any  $c$  with  $a < c < b$ .
- (a) Use the Increasing Function Theorem to prove that  $f'(c) \leq f'(x)$  for  $c \leq x < b$  and that  $f'(x) \leq f'(c)$  for  $a < x \leq c$ .
- (b) Use (a) and the Racetrack Principle to conclude that  $f(c) + f'(c)(x - c) \leq f(x)$ , for  $a < x < b$ .

## CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Derivatives of elementary functions**

Power, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric, and hyperbolic functions.

- **Derivatives of sums, differences, and constant multiples**

- **Product and quotient rules**

- **Chain rule**

Differentiation of implicitly defined functions, inverse functions.

- **Tangent line approximation, local linearity**

- **Hyperbolic functions**

- **Theorems about differentiable functions**

Mean value theorem, increasing function theorem, constant function theorem, Racetrack Principle.

## REVIEW EXERCISES AND PROBLEMS FOR CHAPTER THREE

### Exercises

Find derivatives for the functions in Exercises 1–74. Assume  $a, b, c$ , and  $k$  are constants.

- |  |  |  |  |
|--|--|--|--|
| 1. $w = (t^2 + 1)^{100}$                             | 2. $y = e^{3w/2}$                      | 15. $f(t) = \cos^2(3t + 5)$                                  | 16. $M(\alpha) = \tan^2(2 + 3\alpha)$          |
| 3. $f(t) = 2te^t - \frac{1}{\sqrt{t}}$               | 4. $g(t) = \frac{4}{3 + \sqrt{t}}$     | 17. $s(\theta) = \sin^2(3\theta - \pi)$                      | 18. $h(t) = \ln(e^{-t} - t)$                   |
| 5. $h(t) = \frac{4 - t}{4 + t}$                      | 6. $f(x) = \frac{x^3}{9}(3 \ln x - 1)$ | 19. $p(\theta) = \frac{\sin(5 - \theta)}{\theta^2}$          | 20. $w(\theta) = \frac{\theta}{\sin^2 \theta}$ |
| 7. $f(x) = \frac{x^2 + 3x + 2}{x + 1}$               | 8. $g(\theta) = e^{\sin \theta}$       | 21. $f(\theta) = \frac{1}{1 + e^{-\theta}}$                  | 22. $g(w) = \frac{1}{2^w + e^w}$               |
| 9. $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2})$ | 10. $f(\theta) = \ln(\cos \theta)$     | 23. $g(x) = \frac{x^2 + \sqrt{x} + 1}{x^{3/2}}$              | 24. $h(z) = \sqrt{\frac{\sin(2z)}{\cos(2z)}}$  |
| 11. $f(y) = \ln(\ln(2y^3))$                          | 12. $g(x) = x^k + k^x$                 | 25. $\frac{q(\theta)}{\sqrt{4\theta^2 - \sin^2(2\theta)}} =$ | 26. $w = 2^{-4z} \sin(\pi z)$                  |
| 13. $y = e^{-\pi} + \pi^{-e}$                        | 14. $z = \sin^3 \theta$                | 27. $s(x) = \arctan(2 - x)$                                  | 28. $r(\theta) = e^{(e^\theta + e^{-\theta})}$ |

29.  $m(n) = \sin(e^n)$       30.  $k(\alpha) = e^{\tan(\sin \alpha)}$       60.  $f(z) = (\ln 3)z^2 + (\ln 4)e^z$   
 31.  $g(t) = t \cos(\sqrt{t}e^t)$       32.  $f(r) = (\tan 2 + \tan r)^e$       61.  $g(x) = 2x - \frac{1}{\sqrt[3]{x}} + 3^x - e$   
 33.  $h(x) = xe^{\tan x}$       34.  $y = e^{2x} \sin^2(3x)$       62.  $f(x) = (3x^2 + \pi)(e^x - 4)$   
 35.  $g(x) = \tan^{-1}(3x^2 + 1)$       36.  $y = 2^{\sin x} \cos x$       63.  $f(\theta) = \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta$   
 37.  $h(x) = \ln e^{ax}$       38.  $k(x) = \ln e^{ax} + \ln b$       64.  $y = \sqrt{\cos(5\theta) + \sin^2(6\theta)}$   
 39.  $f(\theta) = e^{k\theta} - 1$       40.  $f(t) = e^{-4kt} \sin t$       65.  $r(\theta) = \sin((3\theta - \pi)^2)$   
 41.  $H(t) = (at^2 + b)e^{-ct}$       42.  $g(\theta) = \sqrt{a^2 - \sin^2 \theta}$       66.  $y = (x^2 + 5)^3 (3x^3 - 2)^2$   
 43.  $f(x) = a^{5x}$       44.  $f(x) = \frac{a^2 - x^2}{a^2 + x^2}$       67.  $N(\theta) = \tan(\arctan(k\theta))$   
 45.  $w(r) = \frac{ar^2}{b + r^3}$       46.  $f(s) = \frac{a^2 - s^2}{\sqrt{a^2 + s^2}}$       68.  $h(t) = e^{kt}(\sin at + \cos bt)$   
 47.  $y = \arctan\left(\frac{2}{x}\right)$       48.  $r(t) = \ln\left(\sin\left(\frac{t}{k}\right)\right)$       69.  $f(x) = (2 - 4x - 3x^2)(6x^e - 3\pi)$   
 49.  $g(w) = \frac{5}{(a^2 - w^2)^2}$       50.  $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$       70.  $f(t) = (\sin(2t) - \cos(3t))^4$   
 51.  $g(u) = \frac{e^{au}}{a^2 + b^2}$       52.  $y = \frac{e^{ax} - e^{-ax}}{e^{ax} + e^{-ax}}$       71.  $s(y) = \sqrt[3]{\cos^2 y + 3 + \sin^2 y}$   
 53.  $g(t) = \frac{\ln(kt) + t}{\ln(kt) - t}$       54.  $z = \frac{e^{t^2} + t}{\sin(2t)}$       72.  $f(x) = (4 - x^2 + 2x^3)(6 - 4x + x^7)$   
 55.  $f(t) = \sin \sqrt{e^t + 1}$       56.  $g(y) = e^{2e^{(y^3)}}$       73.  $h(x) = \left(\frac{1}{x} - \frac{1}{x^2}\right)(2x^3 + 4)$   
 57.  $g(x) = -\frac{1}{2}(x^5 + 2x - 9)$       74.  $f(z) = \sqrt{5z} + 5\sqrt{z} + \frac{5}{\sqrt{z}} - \sqrt{\frac{5}{z}} + \sqrt{5}$
- For Exercises 75–76, assume that  $y$  is a differentiable function of  $x$  and find  $dy/dx$ .
75.  $x^3 + y^3 - 4x^2y = 0$   
 76.  $\sin(ay) + \cos(bx) = xy$
77. Find the slope of the curve  $x^2 + 3y^2 = 7$  at  $(2, -1)$ .  
 78. Assume  $y$  is a differentiable function of  $x$  and that  $y + \sin y + x^2 = 9$ . Find  $dy/dx$  at the point  $x = 3, y = 0$ .  
 79. Find the equations for the lines tangent to the graph of  $xy + y^2 = 4$  where  $x = 3$ .

### Problems

80. If  $f(t) = 2t^3 - 4t^2 + 3t - 1$ , find  $f'(t)$  and  $f''(t)$ .  
 81. If  $f(x) = 13 - 8x + \sqrt{2}x^2$  and  $f'(r) = 4$ , find  $r$ .  
 82. If  $f(x) = 4x^3 + 6x^2 - 23x + 7$ , find the intervals on which  $f'(x) \geq 1$ .  
 83. If  $f(x) = (3x + 8)(2x - 5)$ , find  $f'(x)$  and  $f''(x)$ .

For Problems 84–89, use Figure 3.47.

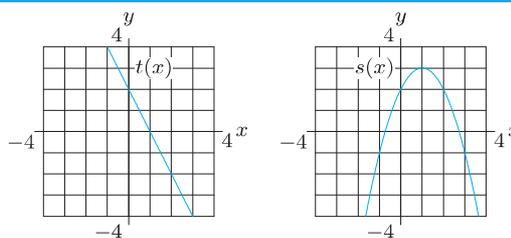


Figure 3.47

84. Let  $h(x) = t(x)s(x)$  and  $p(x) = t(x)/s(x)$ . Estimate:  
 (a)  $h'(1)$       (b)  $h'(0)$       (c)  $p'(0)$   
 85. Let  $r(x) = s(t(x))$ . Estimate  $r'(0)$ .

86. Let  $h(x) = s(s(x))$ . Estimate:  
 (a)  $h'(1)$  (b)  $h'(2)$
87. Estimate all values of  $x$  for which the tangent line to  $y = s(s(x))$  is horizontal.
88. Let  $h(x) = x^2t(x)$  and  $p(x) = t(x^2)$ . Estimate:  
 (a)  $h'(-1)$  (b)  $p'(-1)$
89. Find an approximate equation for the tangent line to  $r(x) = s(t(x))$  at  $x = 1$ .

In Problems 90–92, use Figure 3.48 to evaluate the derivatives.

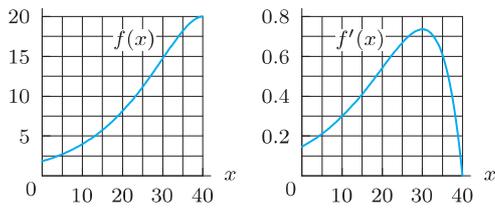


Figure 3.48

90.  $(f^{-1})'(5)$     91.  $(f^{-1})'(10)$     92.  $(f^{-1})'(15)$

93. Suppose  $W$  is proportional to  $r^3$ . The derivative  $dW/dr$  is proportional to what power of  $r$ ?

94. Using the information in the table about  $f$  and  $g$ , find:

- (a)  $h(4)$  if  $h(x) = f(g(x))$   
 (b)  $h'(4)$  if  $h(x) = f(g(x))$   
 (c)  $h(4)$  if  $h(x) = g(f(x))$   
 (d)  $h'(4)$  if  $h(x) = g(f(x))$   
 (e)  $h'(4)$  if  $h(x) = g(x)/f(x)$   
 (f)  $h'(4)$  if  $h(x) = f(x)g(x)$

$x$	1	2	3	4
$f(x)$	3	2	1	4
$f'(x)$	1	4	2	3
$g(x)$	2	1	4	3
$g'(x)$	4	2	3	1

95. Given:  $r(2) = 4$ ,  $s(2) = 1$ ,  $s(4) = 2$ ,  $r'(2) = -1$ ,  $s'(2) = 3$ ,  $s'(4) = 3$ . Compute the following derivatives, or state what additional information you would need to be able to do so.

- (a)  $H'(2)$  if  $H(x) = r(x) \cdot s(x)$   
 (b)  $H'(2)$  if  $H(x) = \sqrt{r(x)}$   
 (c)  $H'(2)$  if  $H(x) = r(s(x))$   
 (d)  $H'(2)$  if  $H(x) = s(r(x))$

96. If  $g(2) = 3$  and  $g'(2) = -4$ , find  $f'(2)$  for the following:

- (a)  $f(x) = x^2 - 4g(x)$     (b)  $f(x) = \frac{x}{g(x)}$   
 (c)  $f(x) = x^2g(x)$     (d)  $f(x) = (g(x))^2$   
 (e)  $f(x) = x \sin(g(x))$     (f)  $f(x) = x^2 \ln(g(x))$

97. For parts (a)–(f) of Problem 96, determine the equation of the line tangent to  $f$  at  $x = 2$ .

98. Imagine you are zooming in on the graphs of the following functions near the origin:

$$y = \arcsin x \quad y = \sin x - \tan x \quad y = x - \sin x$$

$$y = \arctan x \quad y = \frac{\sin x}{1 + \sin x} \quad y = \frac{x^2}{x^2 + 1}$$

$$y = \frac{1 - \cos x}{\cos x} \quad y = \frac{x}{x^2 + 1} \quad y = \frac{\sin x}{x} - 1$$

$$y = -x \ln x \quad y = e^x - 1 \quad y = x^{10} + \sqrt[10]{x}$$

$$y = \frac{x}{x + 1}$$

Which of them look the same? Group together those functions which become indistinguishable, and give the equation of the line they look like. [Note:  $(\sin x)/x - 1$  and  $-x \ln x$  never quite make it to the origin.]

99. The graphs of  $\sin x$  and  $\cos x$  intersect once between  $0$  and  $\pi/2$ . What is the angle between the two curves at the point where they intersect? (You need to think about how the angle between two curves should be defined.)

In Problems 100–101, show that the curves meet at least once and determine whether the curves are perpendicular at the point of intersection.

100.  $y = 1 + x - x^2$  and  $y = 1 - x + x^2$

101.  $y = 1 - x^3/3$  and  $y = x - 1$

102. For some constant  $b$  and  $x > 0$ , let  $y = x \ln x - bx$ . Find the equation of the tangent line to this graph at the point at which the graph crosses the  $x$ -axis.

In Problems 103–105, find the limit as  $x \rightarrow -\infty$ .

103.  $\frac{\cosh(2x)}{\sinh(3x)}$     104.  $\frac{e^{-2x}}{\sinh(2x)}$     105.  $\frac{\sinh(x^2)}{\cosh(x^2)}$

106. Consider the function  $f(x) = \sqrt{x}$ .

- (a) Find and sketch  $f(x)$  and the tangent line approximation to  $f(x)$  near  $x = 4$ .  
 (b) Compare the true value of  $f(4.1)$  with the value obtained by using the tangent line approximation.

- (c) Compare the true and approximate values of  $f(16)$ .  
 (d) Using a graph, explain why the tangent line approximation is a good one when  $x = 4.1$  but not when  $x = 16$ .

107. Figure 3.49 shows the tangent line approximation to  $f(x)$  near  $x = a$ .

- (a) Find  $a$ ,  $f(a)$ ,  $f'(a)$ .  
 (b) Estimate  $f(2.1)$  and  $f(1.98)$ . Are these under or overestimates? Which estimate would you expect to be more accurate and why?

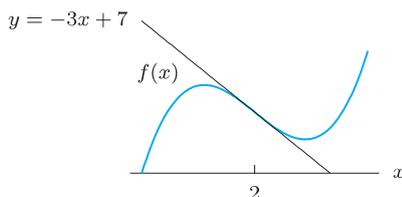


Figure 3.49

108. Some antique furniture increased very rapidly in price over the past decade. For example, the price of a particular rocking chair is well approximated by

$$V = 75(1.35)^t,$$

where  $V$  is in dollars and  $t$  is in years since 2000. Find the rate, in dollars per year, at which the price is increasing at time  $t$ .

109. The acceleration due to gravity,  $g$ , at a distance  $r$  from the center of the earth is given by

$$g = \frac{GM}{r^2},$$

where  $M$  is the mass of the earth and  $G$  is a constant.

- (a) Find  $dg/dr$ .  
 (b) What is the practical interpretation (in terms of acceleration) of  $dg/dr$ ? Why would you expect it to be negative?  
 (c) You are told that  $M = 6 \cdot 10^{24}$  and  $G = 6.67 \cdot 10^{-20}$  where  $M$  is in kilograms and  $r$  in kilometers. What is the value of  $dg/dr$  at the surface of the earth ( $r = 6400$  km)?  
 (d) What does this tell you about whether or not it is reasonable to assume  $g$  is constant near the surface of the earth?
110. The distance,  $s$ , of a moving body from a fixed point is given as a function of time,  $t$ , by  $s = 20e^{t/2}$ .
- (a) Find the velocity,  $v$ , of the body as a function of  $t$ .  
 (b) Find a relationship between  $v$  and  $s$ , then show that  $s$  satisfies the differential equation  $s' = \frac{1}{2}s$ .
111. The depth of the water,  $y$ , in meters, in the Bay of Fundy, Canada, is given as a function of time,  $t$ , in hours after midnight, by the function
- $$y = 10 + 7.5 \cos(0.507t).$$
- How quickly is the tide rising or falling (in meters/hour) at each of the following times?
- (a) 6:00 am                      (b) 9:00 am  
 (c) Noon                              (d) 6:00 pm
112. A yam is put in a hot oven, maintained at a constant temperature  $200^\circ\text{C}$ . At time  $t = 30$  minutes, the temperature  $T$  of the yam is  $120^\circ$  and is increasing at an (instantaneous) rate of  $2^\circ/\text{min}$ . Newton's law of cooling (or, in our case, warming) implies that the temperature at time  $t$  is given by
- $$T(t) = 200 - ae^{-bt}.$$
- Find  $a$  and  $b$ .
113. An object is oscillating at the end of a spring. Its position, in centimeters, relative to a fixed point, is given as a function of time,  $t$ , in seconds, by
- $$y = y_0 \cos(2\pi\omega t), \quad \text{with } \omega \text{ a constant.}$$
- (a) Find an expression for the velocity and acceleration of the object.  
 (b) How do the amplitudes of the position, velocity, and acceleration functions compare? How do the periods of these functions compare?  
 (c) Show that the function  $y$  satisfies the differential equation
- $$\frac{d^2y}{dt^2} + 4\pi^2\omega^2y = 0.$$
114. The total number of people,  $N$ , who have contracted a disease by a time  $t$  days after its outbreak is given by
- $$N = \frac{1,000,000}{1 + 5,000e^{-0.1t}}.$$
- (a) In the long run, how many people get the disease?  
 (b) Is there any day on which more than a million people fall sick? Half a million? Quarter of a million? (Note: You do not have to find on what days these things happen.)
115. The world population was 6.7 billion at the beginning of 2008. An exponential model predicts the population to be  $P(t) = 6.7e^{kt}$  billion  $t$  years after 2008, where  $k$  is the continuous annual growth rate.
- (a) How long does the model predict it will take for the population to reach 10 billion, as a function  $f(k)$ ?  
 (b) One current estimate is  $k = 0.012 = 1.2\%$ . How long will it take for the population to reach 10 billion if  $k$  has this value?  
 (c) For continuous growth rates near  $1.2\%$ , find a linear function of  $k$  that approximates the time for the world population to reach 10 billion.  
 (d) Find the time to reach 10 billion and its approximation from part (c) if the continuous growth rate is  $1.0\%$ .

116. The acceleration due to gravity,  $g$ , is given by

$$g = \frac{GM}{r^2},$$

where  $M$  is the mass of the earth,  $r$  is the distance from the center of the earth, and  $G$  is the universal gravitational constant.

- (a) Show that when  $r$  changes by  $\Delta r$ , the change in the acceleration due to gravity,  $\Delta g$ , is given by

$$\Delta g \approx -2g \frac{\Delta r}{r}.$$

- (b) What is the significance of the negative sign?  
 (c) What is the percent change in  $g$  when moving from sea level to the top of Pike's Peak (4.315 km)? Assume the radius of the earth is 6400 km.
117. At a particular location,  $f(p)$  is the number of gallons of gas sold when the price is  $p$  dollars per gallon.

- (a) What does the statement  $f(2) = 4023$  tell you about gas sales?  
 (b) Find and interpret  $f^{-1}(4023)$ .  
 (c) What does the statement  $f'(2) = -1250$  tell you about gas sales?  
 (d) Find and interpret  $(f^{-1})'(4023)$
118. If  $f$  is increasing and  $f(20) = 10$ , which of the two options, (a) or (b), must be wrong?  
 (a)  $f'(10)(f^{-1})'(20) = 1$ .

(b)  $f'(20)(f^{-1})'(10) = 2$ .

119. If  $f$  is decreasing and  $f(20) = 10$ , which of the following must be incorrect?

(a)  $(f^{-1})'(20) = -3$ .      (b)  $(f^{-1})'(10) = 12$ .

120. Find the  $n^{\text{th}}$  derivative of the following functions:

(a)  $\ln x$       (b)  $xe^x$       (c)  $e^x \cos x$

121. The derivative  $f'$  gives the (absolute) rate of change of a quantity  $f$ , and  $f'/f$  gives the relative rate of change of the quantity. In this problem, we show that the product rule is equivalent to an additive rule for relative rates of change. Assume  $h = f \cdot g$  with  $f \neq 0$  and  $g \neq 0$ .

- (a) Show that the additive rule

$$\frac{f'}{f} + \frac{g'}{g} = \frac{h'}{h}$$

implies the product rule, by multiplying through by  $h$  and using the fact that  $h = f \cdot g$ .

- (b) Show that the product rule implies the additive rule in part (a), by starting with the product rule and dividing through by  $h = f \cdot g$ .

122. The relative rate of change of a function  $f$  is defined to be  $f'/f$ . Find an expression for the relative rate of change of a quotient  $f/g$  in terms of the relative rates of change of the functions  $f$  and  $g$ .

### CAS Challenge Problems

123. (a) Use a computer algebra system to differentiate  $(x+1)^x$  and  $(\sin x)^x$ .  
 (b) Conjecture a rule for differentiating  $(f(x))^x$ , where  $f$  is any differentiable function.  
 (c) Apply your rule to  $g(x) = (\ln x)^x$ . Does your answer agree with the answer given by the computer algebra system?  
 (d) Prove your conjecture by rewriting  $(f(x))^x$  in the form  $e^{h(x)}$ .

For Problems 124–126,

- (a) Use a computer algebra system to find and simplify the

derivative of the given function.

- (b) Without a computer algebra system, use differentiation rules to calculate the derivative. Make sure that the answer simplifies to the same answer as in part (a).  
 (c) Explain how you could have predicted the derivative by using algebra before taking the derivative.

124.  $f(x) = \sin(\arcsin x)$

125.  $g(r) = 2^{-2r} 4^r$

126.  $h(t) = \ln(1 - 1/t) + \ln(t/(t - 1))$

### CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–14 true or false? Give an explanation for your answer.

- The derivative of a polynomial is always a polynomial.
- The derivative of  $\pi/x^2$  is  $-\pi/x$ .
- The derivative of  $\tan \theta$  is periodic.
- The graph of  $\ln(x^2)$  is concave up for  $x > 0$ .
- If  $f'(x)$  is defined for all  $x$ , then  $f(x)$  is defined for all  $x$ .

6. If  $f'(2) = 3.1$  and  $g'(2) = 7.3$ , then the graph of  $f(x) + g(x)$  has slope 10.4 at  $x = 2$ .

7. Let  $f$  and  $g$  be two functions whose second derivatives are defined. Then

$$(fg)'' = fg'' + f''g.$$

8. If a function is periodic, with period  $c$ , then so is its derivative.

9. If  $y$  satisfies the equation  $y^2 + xy - 1 = 0$ , then  $dy/dx$  exists everywhere.
10. The function  $\tanh x$  is odd, that is,  $\tanh(-x) = -\tanh x$ .
11. The 100<sup>th</sup> derivative of  $\sinh x$  is  $\cosh x$ .
12.  $\sinh x + \cosh x = e^x$ .
13. The function  $\sinh x$  is periodic.
14. The function  $\sinh^2 x$  is concave down everywhere.
15. If  $f(x)$  is defined for all  $x$ , then  $f'(x)$  is defined for all  $x$ .
16. If  $f(x)$  is increasing, then  $f'(x)$  is increasing.
17. The only functions whose fourth derivatives are equal to  $\cos t$  are of the form  $\cos t + C$ , where  $C$  is any constant.
18. If  $f(x)$  has an inverse function,  $g(x)$ , then the derivative of  $g(x)$  is  $1/f'(x)$ .
19.  $(fg)'(x)$  is never equal to  $f'(x)g'(x)$ .
20. If the function  $f(x)/g(x)$  is defined but not differentiable at  $x = 1$ , then either  $f(x)$  or  $g(x)$  is not differentiable at  $x = 1$ .
21. If the derivative of  $f(g(x))$  is equal to the derivative of  $f(x)$  for all  $x$ , then  $g(x) = x$  for all  $x$ .
- (b) If  $\lim_{x \rightarrow a} (f(x) - L(x)) = 0$ , then  $L$  is the local linearization for  $f$  near  $x = a$ .
27. Which of the following would be a counterexample to the product rule?
- (a) Two differentiable functions  $f$  and  $g$  satisfying  $(fg)' = f'g'$ .
- (b) A differentiable function  $f$  such that  $(xf(x))' = xf'(x) + f(x)$ .
- (c) A differentiable function  $f$  such that  $(f(x)^2)' = 2f(x)$ .
- (d) Two differentiable functions  $f$  and  $g$  such that  $f'(a) = 0$  and  $g'(a) = 0$  and  $fg$  has positive slope at  $x = a$ .

Are the statements in Problems 15–21 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

Are the statements in Problems 28–31 true or false for a function  $f$  whose domain is all real numbers? If a statement is true, explain how you know. If a statement is false, give a counterexample.

22.  $f(x) + g(x)$  is concave up for all  $x$ .
23.  $f(x)g(x)$  is concave up for all  $x$ .
24.  $f(x) - g(x)$  cannot be concave up for all  $x$ .
25.  $f(g(x))$  is concave up for all  $x$ .
26. Let  $f$  be a differentiable function and let  $L$  be the linear function  $L(x) = f(a) + k(x - a)$  for some constant  $a$ . Decide whether the following statements are true or false for all constants  $k$ . Explain your answer.
- (a)  $L$  is the local linearization for  $f$  near  $x = a$ .
28. If  $f'(x) \geq 0$  for all  $x$ , then  $f(a) \leq f(b)$  whenever  $a \leq b$ .
29. If  $f'(x) \leq g'(x)$  for all  $x$ , then  $f(x) \leq g(x)$  for all  $x$ .
30. If  $f'(x) = g'(x)$  for all  $x$ , then  $f(x) = g(x)$  for all  $x$ .
31. If  $f'(x) \leq 1$  for all  $x$  and  $f(0) = 0$ , then  $f(x) \leq x$  for all  $x$ .

Suppose that  $f''$  and  $g''$  exist and that  $f$  and  $g$  are concave up for all  $x$ . Are the statements in Problems 22–25 true or false for all such  $f$  and  $g$ ? If a statement is true, explain how you know. If a statement is false, give a counterexample.

In Problems 32–34, give an example of function(s) with the given properties.

32. A continuous function  $f$  on the interval  $[-1, 1]$  that does not satisfy the conclusion of the Mean Value Theorem.
33. A function  $f$  that is differentiable on the interval  $(0, 2)$ , but does not satisfy the conclusion of the Mean Value Theorem on the interval  $[0, 2]$ .
34. A function that is differentiable on  $(0, 1)$  and not continuous on  $[0, 1]$ , but which satisfies the conclusion of the Mean Value Theorem.

## PROJECTS FOR CHAPTER THREE

### 1. Rule of 70

The “Rule of 70” is a rule of thumb to estimate how long it takes money in a bank to double. Suppose the money is in an account earning  $i\%$  annual interest, compounded yearly. The Rule of 70 says that the time it takes the amount of money to double is approximately  $70/i$  years, assuming  $i$  is small. Find the local linearization of  $\ln(1 + x)$ , and use it to explain why this rule works.

### 2. Newton’s Method

Read about how to find roots using bisection and Newton’s method in Appendices A and C.

- (a) What is the smallest positive zero of the function  $f(x) = \sin x$ ? Apply Newton's method, with initial guess  $x_0 = 3$ , to see how fast it converges to  $\pi = 3.1415926536 \dots$
- (i) Compute the first two approximations,  $x_1$  and  $x_2$ ; compare  $x_2$  with  $\pi$ .
  - (ii) Newton's method works very well here. Explain why. To do this, you will have to outline the basic idea behind Newton's method.
  - (iii) Estimate the location of the zero using bisection, starting with the interval  $[3, 4]$ . How does bisection compare to Newton's method in terms of accuracy?
- (b) Newton's method can be very sensitive to your initial estimate,  $x_0$ . For example, consider finding a zero of  $f(x) = \sin x - \frac{2}{3}x$ .
- (i) Use Newton's method with the following initial estimates to find a zero:

$$x_0 = 0.904, \quad x_0 = 0.905, \quad x_0 = 0.906.$$

- (ii) What happens?