

Chapter Five

KEY CONCEPT: THE DEFINITE INTEGRAL

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5.1 HOW DO WE MEASURE DISTANCE TRAVELED?

For positive constant velocities, we can find the distance a moving object travels using the formula

$$\text{Distance} = \text{Velocity} \times \text{Time}.$$

In this section we see how to estimate the distance when the velocity is not a constant.

A Thought Experiment: How Far Did the Car Go?

Velocity Data Every Two Seconds

A car is moving with increasing velocity. Table 5.1 shows the velocity every two seconds:

Table 5.1 *Velocity of car every two seconds*

Time (sec)	0	2	4	6	8	10
Velocity (ft/sec)	20	30	38	44	48	50

How far has the car traveled? Since we don't know how fast the car is moving at every moment, we can't calculate the distance exactly, but we can make an estimate. The velocity is increasing, so the car is going at least 20 ft/sec for the first two seconds. Since $\text{Distance} = \text{Velocity} \times \text{Time}$, the car goes at least $20 \cdot 2 = 40$ feet during the first two seconds. Likewise, it goes at least $30 \cdot 2 = 60$ feet during the next two seconds, and so on. During the ten-second period it goes at least

$$20 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 = 360 \text{ feet.}$$

Thus, 360 feet is an underestimate of the total distance traveled during the ten seconds.

To get an overestimate, we can reason this way: During the first two seconds, the car's velocity is at most 30 ft/sec, so it moved at most $30 \cdot 2 = 60$ feet. In the next two seconds it moved at most $38 \cdot 2 = 76$ feet, and so on. Therefore, over the ten-second period it moved at most

$$30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 + 50 \cdot 2 = 420 \text{ feet.}$$

Therefore,

$$360 \text{ feet} \leq \text{Total distance traveled} \leq 420 \text{ feet.}$$

There is a difference of 60 feet between the upper and lower estimates.

Velocity Data Every One Second

What if we want a more accurate estimate? Then we make more frequent velocity measurements, say every second, as in Table 5.2.

As before, we get a lower estimate for each second by using the velocity at the beginning of that second. During the first second the velocity is at least 20 ft/sec, so the car travels at least $20 \cdot 1 = 20$ feet. During the next second the car moves at least 26 feet, and so on. We have

$$\begin{aligned} \text{New lower estimate} &= 20 \cdot 1 + 26 \cdot 1 + 30 \cdot 1 + 34 \cdot 1 + 38 \cdot 1 \\ &\quad + 41 \cdot 1 + 44 \cdot 1 + 46 \cdot 1 + 48 \cdot 1 + 49 \cdot 1 \\ &= 376 \text{ feet.} \end{aligned}$$

Table 5.2 *Velocity of car every second*

Time (sec)	0	1	2	3	4	5	6	7	8	9	10
Velocity (ft/sec)	20	26	30	34	38	41	44	46	48	49	50

Notice that this lower estimate is greater than the old lower estimate of 360 feet.

We get a new upper estimate by considering the velocity at the end of each second. During the first second the velocity is at most 26 ft/sec, so the car moves at most $26 \cdot 1 = 26$ feet; in the next second it moves at most 30 feet, and so on.

$$\begin{aligned} \text{New upper estimate} &= 26 \cdot 1 + 30 \cdot 1 + 34 \cdot 1 + 38 \cdot 1 + 41 \cdot 1 \\ &\quad + 44 \cdot 1 + 46 \cdot 1 + 48 \cdot 1 + 49 \cdot 1 + 50 \cdot 1 \\ &= 406 \text{ feet.} \end{aligned}$$

This is less than the old upper estimate of 420 feet. Now we know that

$$376 \text{ feet} \leq \text{Total distance traveled} \leq 406 \text{ feet.}$$

The difference between upper and lower estimates is now 30 feet, half of what it was before. By halving the interval of measurement, we have halved the difference between the upper and lower estimates.

Visualizing Distance on the Velocity Graph: Two Second Data

We can represent both upper and lower estimates on a graph of the velocity. The graph also shows how changing the time interval between velocity measurements changes the accuracy of our estimates.

The velocity can be graphed by plotting the two-second data in Table 5.1 and drawing a curve through the data points. (See Figure 5.1.) The area of the first dark rectangle is $20 \cdot 2 = 40$, the lower estimate of the distance moved during the first two seconds. The area of the second dark rectangle is $30 \cdot 2 = 60$, the lower estimate for the distance moved in the next two seconds. The total area of the dark rectangles represents the lower estimate for the total distance moved during the ten seconds.

If the dark and light rectangles are considered together, the first area is $30 \cdot 2 = 60$, the upper estimate for the distance moved in the first two seconds. The second area is $38 \cdot 2 = 76$, the upper estimate for the next two seconds. The upper estimate for the total distance is represented by the sum of the areas of the dark and light rectangles. Therefore, the area of the light rectangles alone represents the difference between the two estimates.

To visualize the difference between the two estimates, look at Figure 5.1 and imagine the light rectangles all pushed to the right and stacked on top of each other. This gives a rectangle of width 2 and height 30. The height, 30, is the difference between the initial and final values of the velocity: $30 = 50 - 20$. The width, 2, is the time interval between velocity measurements.

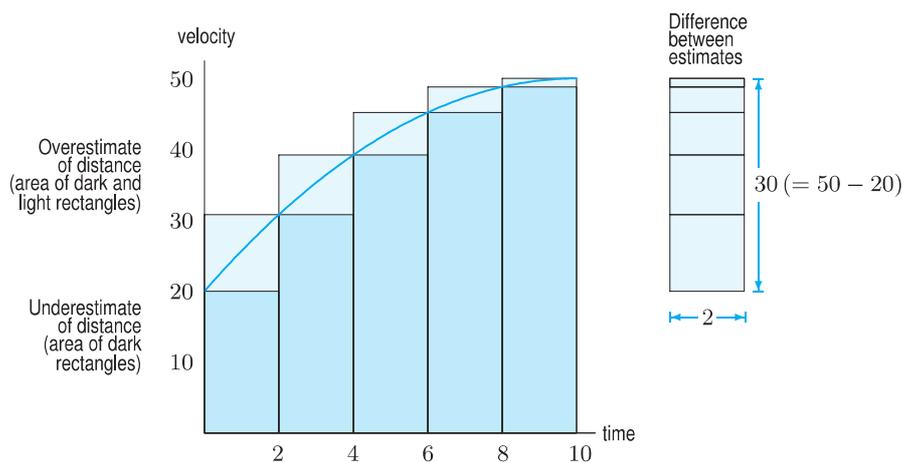


Figure 5.1: Velocity measured every 2 seconds

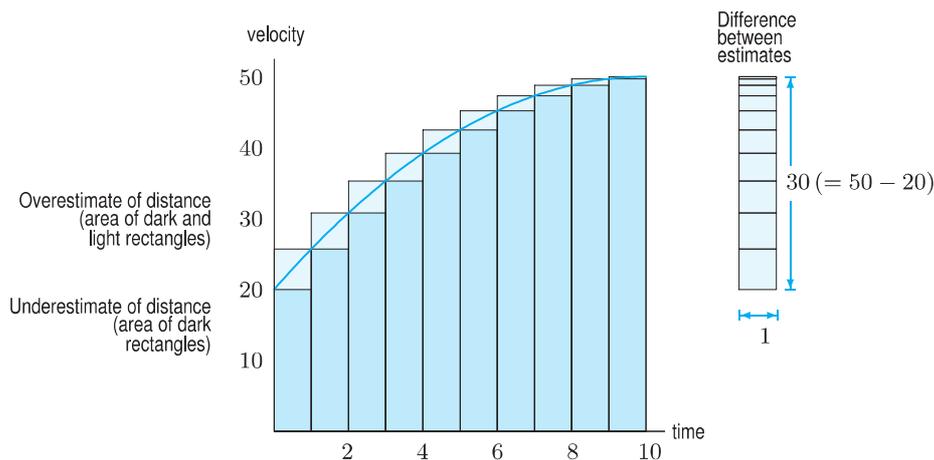


Figure 5.2: Velocity measured every second

Visualizing Distance on the Velocity Graph: One Second Data

Figure 5.2 shows the velocities measured every second. The area of the dark rectangles again represents the lower estimate, and the area of the dark and light rectangles together represent the upper estimate. As before, the difference between the two estimates is represented by the area of the light rectangles. This difference can be calculated by stacking the light rectangles vertically, giving a rectangle of the same height as before but of half the width. Its area is therefore half what it was before. Again, the height of this stack is $50 - 20 = 30$, but its width is now 1.

Example 1 What would be the difference between the upper and lower estimates if the velocity were given every tenth of a second? Every hundredth of a second? Every thousandth of a second?

Solution Every tenth of a second: Difference between estimates = $(50 - 20)(1/10) = 3$ feet.
 Every hundredth of a second: Difference between estimates = $(50 - 20)(1/100) = 0.3$ feet.
 Every thousandth of a second: Difference between estimates = $(50 - 20)(1/1000) = 0.03$ feet.

Example 2 How frequently must the velocity be recorded in order to estimate the total distance traveled to within 0.1 feet?

Solution The difference between the velocity at the beginning and end of the observation period is $50 - 20 = 30$. If the time between the measurements is h , then the difference between the upper and lower estimates is $(30)h$. We want

$$(30)h < 0.1,$$

or

$$h < \frac{0.1}{30} = 0.0033.$$

So if the measurements are made less than 0.0033 seconds apart, the distance estimate is accurate to within 0.1 feet.

Visualizing Distance on the Velocity Graph: Area Under Curve

As we make more frequent velocity measurements, the rectangles used to estimate the distance traveled fit the curve more closely. See Figures 5.3 and 5.4. In the limit, as the number of subdivisions increases, we see that the distance traveled is given by the area between the velocity curve and the horizontal axis. See Figure 5.5. In general:

If the velocity is positive, the total distance traveled is the area under the velocity curve.

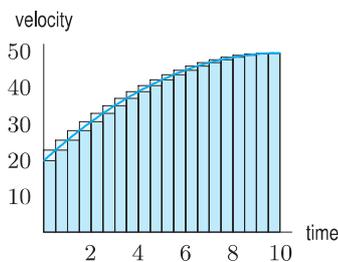


Figure 5.3: Velocity measured every 1/2 second

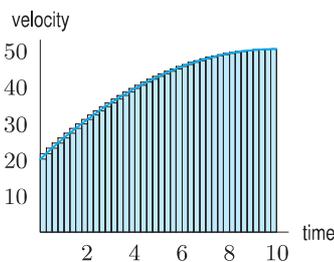


Figure 5.4: Velocity measured every 1/4 second

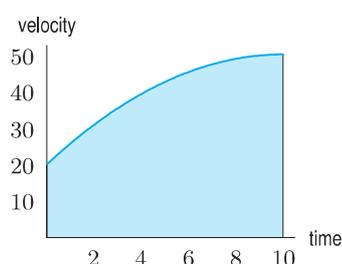


Figure 5.5: Distance traveled is area under curve

Example 3 With time t in seconds, the velocity of a bicycle, in feet per second, is given by $v(t) = 5t$. How far does the bicycle travel in 3 seconds?

Solution The velocity is linear. See Figure 5.6. The distance traveled is the area between the line $v(t) = 5t$ and the t -axis. Since this region is a triangle of height 15 and base 3

$$\text{Distance traveled} = \text{Area of triangle} = \frac{1}{2} \cdot 15 \cdot 3 = 22.5 \text{ feet.}$$

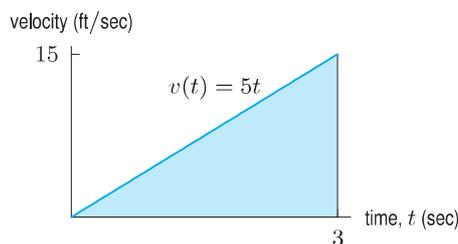


Figure 5.6: Shaded area represents distance traveled

Negative Velocity and Change in Position

In the thought experiment, the velocity is positive and our sums represent distance traveled. What if the velocity is sometimes negative?

Example 4 A particle moves along the y -axis with velocity 30 cm/sec for 5 seconds and velocity -10 cm/sec for the next 5 seconds. Positive velocity indicates upward motion; negative velocity represents downward motion. What is represented by the sum

$$30 \cdot 5 + (-10) \cdot 5?$$

Solution The first term in the sum represents an upward motion of $30 \cdot 5 = 150$ centimeters. The second term represents a motion of $(-10) \cdot 5 = -50$ centimeters, that is, 50 centimeters downward. Thus, the sum represents a change in position of $150 - 50 = 100$ centimeters upward.

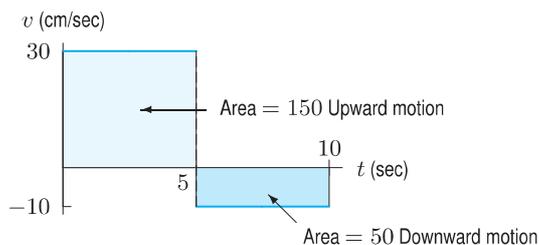


Figure 5.7: Difference in areas gives change in position

Figure 5.7 shows velocity versus time. The area of the rectangle above the t -axis represents upward distance, while the area of the rectangle below the t -axis represents downward distance.

In general, if the velocity can be negative as well as positive, the limit of the sums represents change in position, rather than distance.

Left and Right Sums

We now write the estimates for the distance traveled by the car in new notation. Let $v = f(t)$ denote any nonnegative velocity function. We want to find the distance traveled between times $t = a$ and $t = b$. We take measurements of $f(t)$ at equally spaced times $t_0, t_1, t_2, \dots, t_n$, with time $t_0 = a$ and time $t_n = b$. The time interval between any two consecutive measurements is

$$\Delta t = \frac{b - a}{n},$$

where Δt means the change, or increment, in t .

During the first time interval, from t_0 and t_1 , the velocity can be approximated by $f(t_0)$, so the distance traveled is approximately

$$f(t_0)\Delta t.$$

During the second time interval, the velocity is about $f(t_1)$, so the distance traveled is about

$$f(t_1)\Delta t.$$

Continuing in this way and adding all the estimates, we get an estimate for the total distance traveled. In the last interval, the velocity is approximately $f(t_{n-1})$, so the last term is $f(t_{n-1})\Delta t$:

$$\begin{array}{l} \text{Total distance traveled} \\ \text{between } t = a \text{ and } t = b \end{array} \approx f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{n-1})\Delta t.$$

This is called a *left-hand sum* because we used the value of velocity from the left end of each time interval. It is represented by the sum of the areas of the rectangles in Figure 5.8.

We can also calculate a *right-hand sum* by using the value of the velocity at the right end of each time interval. In that case the estimate for the first interval is $f(t_1)\Delta t$, for the second interval it is $f(t_2)\Delta t$, and so on. The estimate for the last interval is now $f(t_n)\Delta t$, so

$$\begin{array}{l} \text{Total distance traveled} \\ \text{between } t = a \text{ and } t = b \end{array} \approx f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t + \cdots + f(t_n)\Delta t.$$

The right-hand sum is represented by the area of the rectangles in Figure 5.9.

If f is an increasing function, as in Figures 5.8 and 5.9, the left-hand sum is an underestimate and the right-hand sum is an overestimate of the total distance traveled. If f is decreasing, as in Figure 5.10, then the roles of the two sums are reversed.

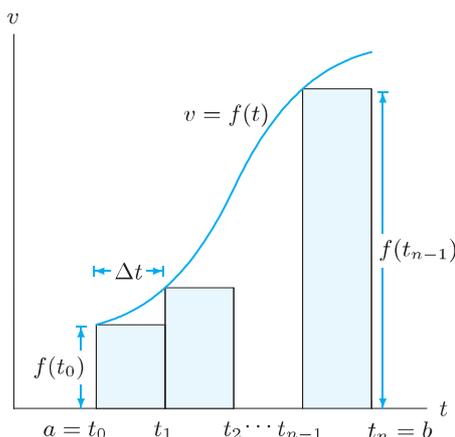


Figure 5.8: Left-hand sums

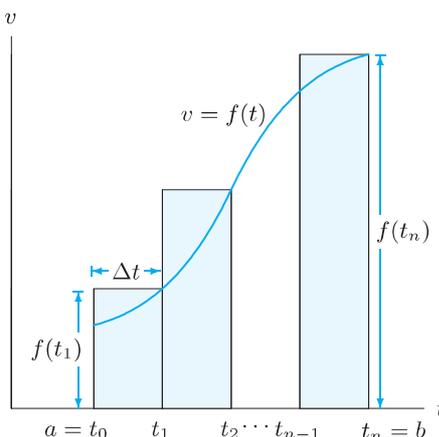


Figure 5.9: Right-hand sums

Accuracy of Estimates

For either increasing or decreasing velocity functions, the exact value of the distance traveled lies somewhere between the two estimates. Thus, the accuracy of our estimate depends on how close these two sums are. For a function which is increasing throughout or decreasing throughout the interval $[a, b]$:

$$\left| \begin{array}{l} \text{Difference between} \\ \text{upper and lower estimates} \end{array} \right| = \left| \begin{array}{l} \text{Difference between} \\ f(a) \text{ and } f(b) \end{array} \right| \cdot \Delta t = |f(b) - f(a)| \cdot \Delta t.$$

(Absolute values make the differences nonnegative.) In Figure 5.10, the area of the light rectangles is the difference between estimates. By making the time interval, Δt , between measurements small enough, we can make this difference between lower and upper estimates as small as we like.

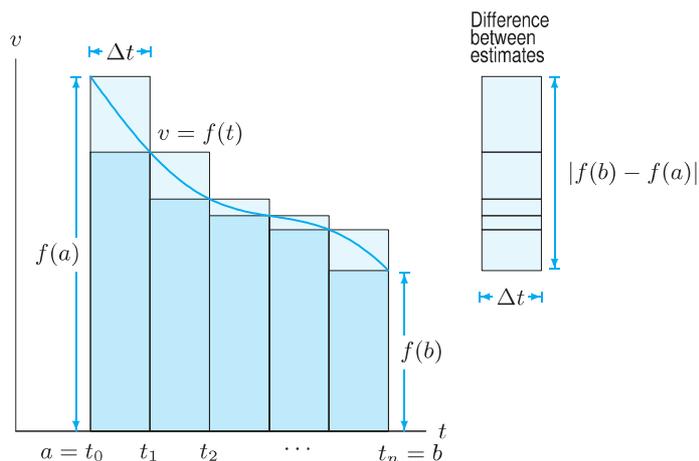


Figure 5.10: Left and right sums if f is decreasing

Exercises and Problems for Section 5.1

Exercises

- The velocity $v(t)$ in Table 5.3 is increasing, $0 \leq t \leq 12$.
 - Find an upper estimate for the total distance traveled using
 - $n = 4$
 - $n = 2$
 - Which of the two answers in part (a) is more accurate? Why?
 - Find a lower estimate of the total distance traveled using $n = 4$.

Table 5.3

t	0	3	6	9	12
$v(t)$	34	37	38	40	45

- The velocity $v(t)$ in Table 5.4 is decreasing, $2 \leq t \leq 12$. Using $n = 5$ subdivisions to approximate the total distance traveled, find
 - An upper estimate
 - A lower estimate

Table 5.4

t	2	4	6	8	10	12
$v(t)$	44	42	41	40	37	35

- A car comes to a stop six seconds after the driver applies the brakes. While the brakes are on, the velocities recorded are in Table 5.5.

Table 5.5

Time since brakes applied (sec)	0	2	4	6
Velocity (ft/sec)	88	45	16	0

- Give lower and upper estimates for the distance the car traveled after the brakes were applied.
 - On a sketch of velocity against time, show the lower and upper estimates of part (a).
- Figure 5.11 shows the velocity, v , of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.

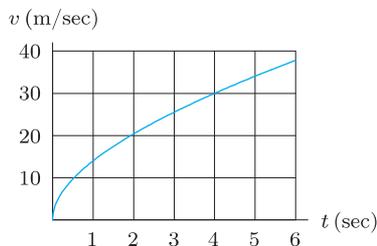


Figure 5.11

- Figure 5.12 shows the velocity of a particle, in cm/sec, along the t -axis for $-3 \leq t \leq 3$.
 - Describe the motion in words: Is the particle changing direction or always moving in the same direction? Is the particle speeding up or slowing down?
 - Make over and underestimates of the distance traveled for $-3 \leq t \leq 3$.

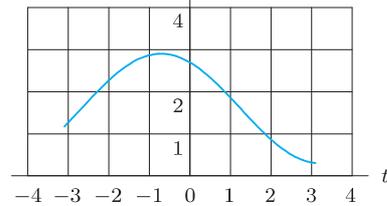


Figure 5.12

- Use the expressions for left and right sums on page 260 and Table 5.6.
 - If $n = 4$, what is Δt ? What are t_0, t_1, t_2, t_3, t_4 ? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?
 - Find the left and right sums using $n = 4$.
 - If $n = 2$, what is Δt ? What are t_0, t_1, t_2 ? What are $f(t_0), f(t_1), f(t_2)$?
 - Find the left and right sums using $n = 2$.

Table 5.6

t	15	17	19	21	23
$f(t)$	10	13	18	20	30

- Use the expressions for left and right sums on page 260 and Table 5.7.
 - If $n = 4$, what is Δt ? What are t_0, t_1, t_2, t_3, t_4 ? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?
 - Find the left and right sums using $n = 4$.
 - If $n = 2$, what is Δt ? What are t_0, t_1, t_2 ? What are $f(t_0), f(t_1), f(t_2)$?
 - Find the left and right sums using $n = 2$.

Table 5.7

t	0	4	8	12	16
$f(t)$	25	23	22	20	17

- At time, t , in seconds, your velocity, v , in meters/second, is given by

$$v(t) = 1 + t^2 \quad \text{for } 0 \leq t \leq 6.$$

Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the upper and lower estimates, and then average the two.

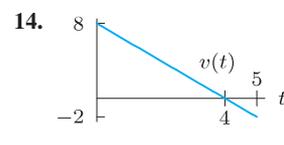
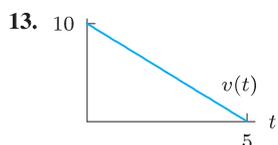
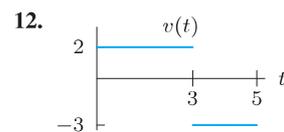
9. For time, t , in hours, $0 \leq t \leq 1$, a bug is crawling at a velocity, v , in meters/hour given by

$$v = \frac{1}{1+t}.$$

Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

10. The velocity of a car is $f(t) = 5t$ meters/sec. Use a graph of $f(t)$ to find the exact distance traveled by the car, in meters, from $t = 0$ to $t = 10$ seconds.

Exercises 11–14 show the velocity, in cm/sec, of a particle moving along the x -axis. Compute the particle's change in position, left (negative) or right (positive), between times $t = 0$ and $t = 5$ seconds.



Problems

15. A student is speeding down Route 11 in his fancy red Porsche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him. The “black box” in the Porsche records the car's speed every two seconds, producing the following table. The speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.

Time since brakes applied (sec)	0	2	4	6	8	10
Speed (ft/sec)	100	80	50	25	10	0

- (a) What is your best estimate of the total distance the student's car traveled before coming to rest?
 (b) Which one of the following statements can you justify from the information given?
 (i) The car stopped before getting to the skunk.
 (ii) The “black box” data is inconclusive. The skunk may or may not have been hit.
 (iii) The skunk was hit by the car.
16. Roger runs a marathon. His friend Jeff rides behind him on a bicycle and clocks his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:

Time since start (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

- (a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the first half hour.
 (b) Give upper and lower estimates for the distance Roger ran in total during the entire hour and a half.

- (c) How often would Jeff have needed to measure Roger's speed in order to find lower and upper estimates within 0.1 mile of the actual distance he ran?

In Problems 17–20, find the difference between the upper and lower estimates of the distance traveled at velocity $f(t)$ on the interval $a \leq t \leq b$ for n subdivisions.

17. $f(t) = 5t + 8$, $a = 1$, $b = 3$, $n = 100$
 18. $f(t) = 25 - t^2$, $a = 1$, $b = 4$, $n = 500$
 19. $f(t) = \sin t$, $a = 0$, $b = \pi/2$, $n = 100$
 20. $f(t) = e^{-t^2/2}$, $a = 0$, $b = 2$, $n = 20$
21. The velocity of a particle moving along the x -axis is given by $f(t) = 6 - 2t$ cm/sec. Use a graph of $f(t)$ to find the exact change in position of the particle from time $t = 0$ to $t = 4$ seconds.
22. A baseball thrown directly upward at 96 ft/sec has velocity $v(t) = 96 - 32t$ ft/sec at time t seconds.
 (a) Graph the velocity from $t = 0$ to $t = 6$.
 (b) When does the baseball reach the peak of its flight? How high does it go?
 (c) How high is the baseball at time $t = 5$?
23. A car initially going 50 ft/sec brakes at a constant rate (constant negative acceleration), coming to a stop in 5 seconds.
 (a) Graph the velocity from $t = 0$ to $t = 5$.
 (b) How far does the car travel?
 (c) How far does the car travel if its initial velocity is doubled, but it brakes at the same constant rate?
24. An object has zero initial velocity and a constant acceleration of 32 ft/sec². Find a formula for its velocity as a function of time. Use left and right sums with $\Delta t = 1$ to find upper and lower bounds on the distance that the object travels in four seconds. Find the precise distance using the area under the curve.

25. A woman drives 10 miles, accelerating uniformly from rest to 60 mph. Graph her velocity versus time. How long does it take for her to reach 60 mph?
26. Two cars start at the same time and travel in the same direction along a straight road. Figure 5.13 gives the velocity, v , of each car as a function of time, t . Which car:
- Attains the larger maximum velocity?
 - Stops first?
 - Travels farther?
27. Two cars travel in the same direction along a straight road. Figure 5.14 shows the velocity, v , of each car at time t . Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/hr.
- For approximately how long does each car travel?
 - Estimate car A 's maximum velocity.
 - Approximately how far does each car travel?

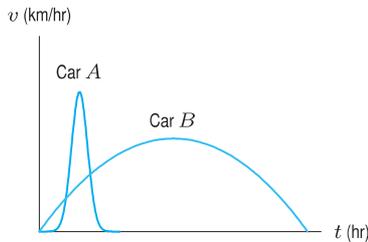


Figure 5.13

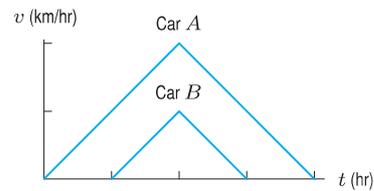


Figure 5.14

5.2 THE DEFINITE INTEGRAL

In Section 5.1, we saw how distance traveled can be approximated by a sum of areas of rectangles. We also saw how the approximation improves as the width of the rectangles gets smaller. In this section, we construct these sums for any function f , whether or not it represents a velocity.

Sigma Notation

Suppose $f(t)$ is a continuous function for $a \leq t \leq b$. We divide the interval from a to b into n equal subdivisions, and we call the width of an individual subdivision Δt , so

$$\Delta t = \frac{b - a}{n}.$$

Let $t_0, t_1, t_2, \dots, t_n$ be endpoints of the subdivisions. Both the left-hand and right-hand sums can be written more compactly using *sigma*, or summation, notation. The symbol \sum is a capital sigma, or Greek letter “S.” We write

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^n f(t_i)\Delta t.$$

The \sum tells us to add terms of the form $f(t_i)\Delta t$. The “ $i = 1$ ” at the base of the sigma sign tells us to start at $i = 1$, and the “ n ” at the top tells us to stop at $i = n$.

In the left-hand sum we start at $i = 0$ and stop at $i = n - 1$, so we write

$$\text{Left-hand sum} = f(t_0)\Delta t + f(t_1)\Delta t + \cdots + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t.$$

Taking the Limit to Obtain the Definite Integral

Now we take the limit of these sums as n goes to infinity. If f is continuous for $a \leq t \leq b$, the limits of the left- and right-hand sums exist and are equal. The *definite integral* is the limit of these sums. A formal definition of the definite integral is given in the online supplement to the text at www.wiley.com/college/hugheshallett.

Suppose f is continuous for $a \leq t \leq b$. The **definite integral** of f from a to b , written

$$\int_a^b f(t) dt,$$

is the limit of the left-hand or right-hand sums with n subdivisions of $a \leq t \leq b$ as n gets arbitrarily large. In other words,

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} (\text{Left-hand sum}) = \lim_{n \rightarrow \infty} \left(\sum_{i=0}^{n-1} f(t_i) \Delta t \right)$$

and

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} (\text{Right-hand sum}) = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(t_i) \Delta t \right).$$

Each of these sums is called a *Riemann sum*, f is called the *integrand*, and a and b are called the *limits of integration*.

The “ \int ” notation comes from an old-fashioned “S,” which stands for “sum” in the same way that \sum does. The “ dt ” in the integral comes from the factor Δt . Notice that the limits on the \sum symbol are 0 and $n - 1$ for the left-hand sum, and 1 and n for the right-hand sum, whereas the limits on the \int sign are a and b .

Computing a Definite Integral

In practice, we often approximate definite integrals numerically using a calculator or computer. They use programs which compute sums for larger and larger values of n , and eventually give a value for the integral. Some (but not all) definite integrals can be computed exactly. However, any definite integral can be approximated numerically.

In the next example, we see how numerical approximation works. For each value of n , we show an over- and an under-estimate for the integral $\int_1^2 (1/t) dt$. As we increase the value of n , the over- and under-estimates get closer together, trapping the value of the integral between them. By increasing the value of n sufficiently, we can calculate the integral to any desired accuracy.

Example 1 Calculate the left-hand and right-hand sums with $n = 2$ for $\int_1^2 \frac{1}{t} dt$. What is the relation between the left- and right-hand sums for $n = 10$ and $n = 250$ and the integral?

Solution Here $a = 1$ and $b = 2$, so for $n = 2$, $\Delta t = (2 - 1)/2 = 0.5$. Therefore, $t_0 = 1$, $t_1 = 1.5$ and $t_2 = 2$. (See Figure 5.15.) We have

$$\text{Left-hand sum} = f(1)\Delta t + f(1.5)\Delta t = 1(0.5) + \frac{1}{1.5}(0.5) = 0.8333,$$

$$\text{Right-hand sum} = f(1.5)\Delta t + f(2)\Delta t = \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) = 0.5833.$$

In Figure 5.15 we see that the left-hand sum is bigger than the area under the curve and the right-hand sum is smaller. So the area under the curve $f(t) = 1/t$ from $t = 1$ to $t = 2$ is between them:

$$0.5833 < \int_1^2 \frac{1}{t} dt < 0.8333.$$

Since $1/t$ is decreasing, when $n = 10$ in Figure 5.16 we again see that the left-hand sum is larger than the area under the curve, and the right-hand sum smaller. A calculator or computer gives

$$0.6688 < \int_1^2 \frac{1}{t} dt < 0.7188.$$

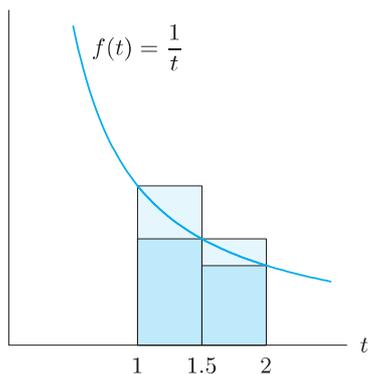


Figure 5.15: Approximating $\int_1^2 \frac{1}{t} dt$ with $n = 2$

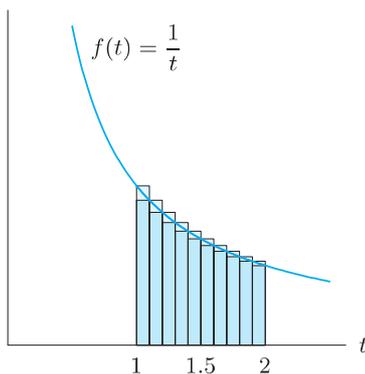


Figure 5.16: Approximating $\int_1^2 \frac{1}{t} dt$ with $n = 10$

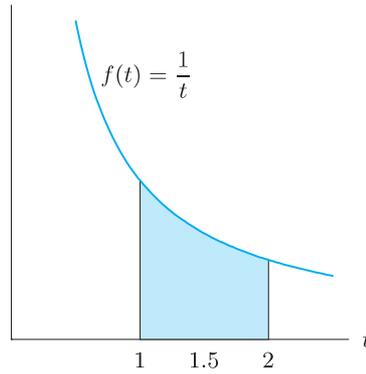


Figure 5.17: Shaded area is exact value of $\int_1^2 \frac{1}{t} dt$

The left- and right-hand sums trap the exact value of the integral between them. As the subdivisions become finer, the left- and right-hand sums get closer together.

When $n = 250$, a calculator or computer gives

$$0.6921 < \int_1^2 \frac{1}{t} dt < 0.6941.$$

So, to two decimal places, we can say that

$$\int_1^2 \frac{1}{t} dt \approx 0.69.$$

The exact value is known to be $\int_1^2 \frac{1}{t} dt = \ln 2 = 0.693147 \dots$. See Figure 5.17.

The Definite Integral as an Area

If $f(x)$ is positive we can interpret each term $f(x_0)\Delta x, f(x_1)\Delta x, \dots$ in a left- or right-hand Riemann sum as the area of a rectangle. See Figure 5.18. As the width Δx of the rectangles approaches zero, the rectangles fit the curve of the graph more exactly, and the sum of their areas gets closer and closer to the area under the curve shaded in Figure 5.19. This suggests that:

When $f(x) \geq 0$ and $a < b$:

$$\text{Area under graph of } f \text{ and above } x\text{-axis} \\ \text{between } a \text{ and } b = \int_a^b f(x) dx.$$

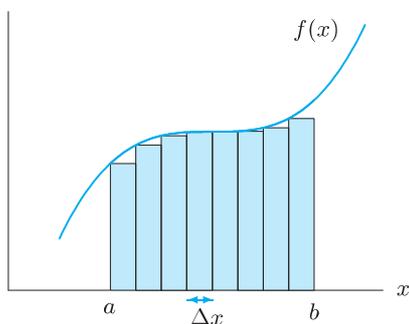


Figure 5.18: Area of rectangles approximating the area under the curve

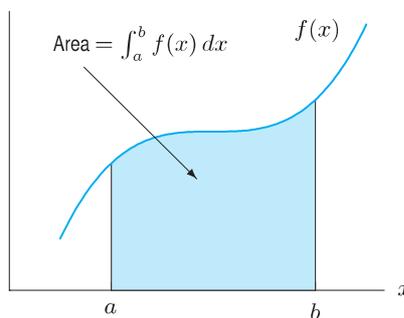


Figure 5.19: The definite integral $\int_a^b f(x) dx$

Example 2 Consider the integral $\int_{-1}^1 \sqrt{1-x^2} dx$.

- (a) Interpret the integral as an area, and find its exact value.
 (b) Estimate the integral using a calculator or computer. Compare your answer to the exact value.

Solution (a) The integral is the area under the graph of $y = \sqrt{1-x^2}$ between -1 and 1 . See Figure 5.20. Rewriting this equation as $x^2 + y^2 = 1$, we see that the graph is a semicircle of radius 1 and area $\pi/2$.
 (b) A calculator gives the value of the integral as 1.5707963

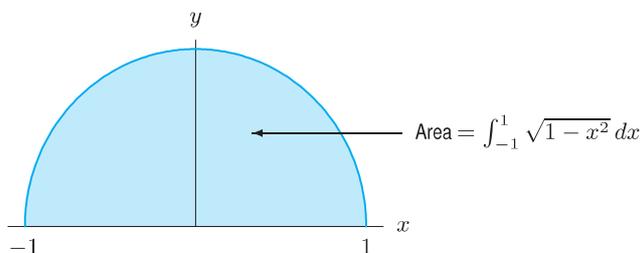


Figure 5.20: Area interpretation of $\int_{-1}^1 \sqrt{1-x^2} dx$

When $f(x)$ is Not Positive

We have assumed in drawing Figure 5.19 that the graph of $f(x)$ lies above the x -axis. If the graph lies below the x -axis, then each value of $f(x)$ is negative, so each $f(x)\Delta x$ is negative, and the area gets counted negatively. In that case, the definite integral is the negative of the area.

When $f(x)$ is positive for some x values and negative for others, and $a < b$:

$\int_a^b f(x) dx$ is the sum of areas above the x -axis, counted positively, and areas below the x -axis, counted negatively.

Example 3 How does the definite integral $\int_{-1}^1 (x^2 - 1) dx$ relate to the area between the parabola $y = x^2 - 1$ and the x -axis?

Solution A calculator gives $\int_{-1}^1 (x^2 - 1) dx = -1.33$. The parabola lies below the axis between $x = -1$ and $x = 1$. (See Figure 5.21.) So the area between the parabola and the x -axis is approximately 1.33.

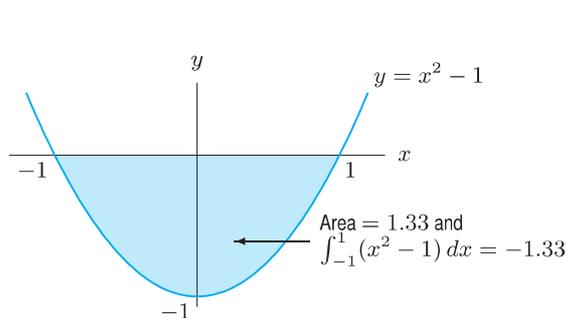


Figure 5.21: Integral $\int_{-1}^1 (x^2 - 1) dx$ is negative of shaded area

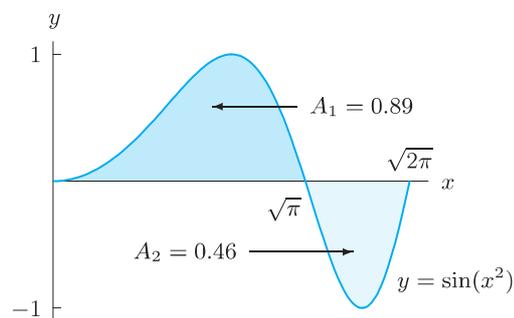


Figure 5.22: Integral $\int_0^{\sqrt{2\pi}} \sin(x^2) dx = A_1 - A_2$

Example 4 Interpret the definite integral $\int_0^{\sqrt{2\pi}} \sin(x^2) dx$ in terms of areas.

Solution The integral is the area above the x -axis, A_1 , minus the area below the x -axis, A_2 . See Figure 5.22. Estimating the integral with a calculator gives

$$\int_0^{\sqrt{2\pi}} \sin(x^2) dx = 0.43.$$

The graph of $y = \sin(x^2)$ crosses the x -axis where $x^2 = \pi$, that is, at $x = \sqrt{\pi}$. The next crossing is at $x = \sqrt{2\pi}$. Breaking the integral into two parts and calculating each one separately gives

$$\int_0^{\sqrt{\pi}} \sin(x^2) dx = 0.89 \quad \text{and} \quad \int_{\sqrt{\pi}}^{\sqrt{2\pi}} \sin(x^2) dx = -0.46.$$

So $A_1 = 0.89$ and $A_2 = 0.46$. Then, as we would expect,

$$\int_0^{\sqrt{2\pi}} \sin(x^2) dx = A_1 - A_2 = 0.89 - 0.46 = 0.43.$$

More General Riemann Sums

Left- and right-hand sums are special cases of Riemann sums. For a general Riemann sum we allow subdivisions to have different lengths. Also, instead of evaluating f only at the left or right endpoint of each subdivision, we allow it to be evaluated anywhere in the subdivision. Thus, a general Riemann sum has the form

$$\sum_{i=1}^n \text{Value of } f(t) \text{ at some point in } i^{\text{th}} \text{ subdivision} \times \text{Length of } i^{\text{th}} \text{ subdivision}.$$

(See Figure 5.23.) As before, we let t_0, t_1, \dots, t_n be the endpoints of the subdivisions, so the length of the i -th subdivision is $\Delta t_i = t_i - t_{i-1}$. For each i we choose a point c_i in the i -th subinterval at which to evaluate f , leading to the following definition:

A general Riemann sum for f on the interval $[a, b]$ is a sum of the form

$$\sum_{i=1}^n f(c_i) \Delta t_i,$$

where $a = t_0 < t_1 < \dots < t_n = b$, and, for $i = 1, \dots, n$, $\Delta t_i = t_i - t_{i-1}$, and $t_{i-1} \leq c_i \leq t_i$.

If f is continuous, we can make a general Riemann sum as close as we like to the value of the definite integral by making the interval lengths small enough. Thus, in approximating definite integrals or in proving theorems about them, we can use general Riemann sums rather than left- or right-hand sums. Generalized Riemann sums are especially useful in establishing properties of the definite integral; see www.wiley.com/college/hugheshallett.

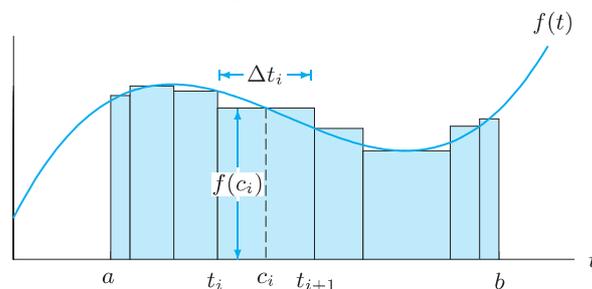


Figure 5.23: A general Riemann sum approximating $\int_a^b f(t) dt$

Exercises and Problems for Section 5.2

Exercises

1. Figure 5.24 shows a Riemann sum approximation with n subdivisions to $\int_a^b f(x) dx$.

- (a) Is it a left- or right-hand approximation? Would the other one be larger or smaller?
 (b) What are a , b , n and Δx ?

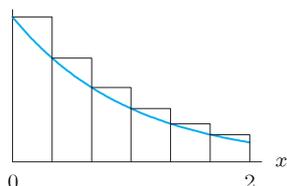


Figure 5.24

2. Using Figure 5.25, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \leq t \leq 8$. Calculate the value of each sum.

- (a) Left-hand sum with $\Delta t = 4$
 (b) Right-hand sum with $\Delta t = 4$
 (c) Left-hand sum with $\Delta t = 2$
 (d) Right-hand sum with $\Delta t = 2$

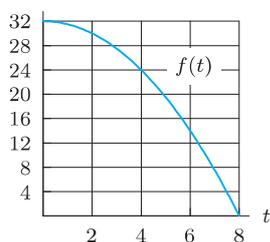


Figure 5.25

3. Use Figure 5.26 to estimate $\int_0^3 f(x) dx$.

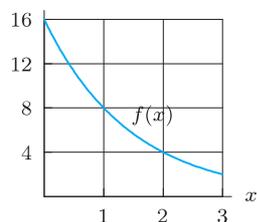


Figure 5.26

4. Use Figure 5.27 to estimate $\int_{-10}^{15} f(x) dx$.

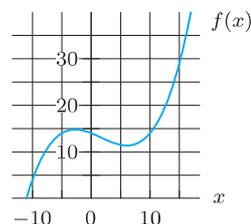


Figure 5.27

5. Use Figure 5.28 to estimate $\int_{-15}^{20} f(x) dx$.

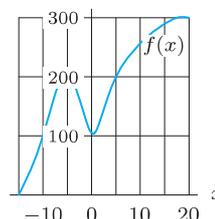


Figure 5.28

6. The graph of a function $f(t)$ is given in Figure 5.29. Which of the following four numbers could be an estimate of $\int_0^1 f(t) dt$ accurate to two decimal places? Explain how you chose your answer.

- (a) -98.35 (b) 71.84
 (c) 100.12 (d) 93.47

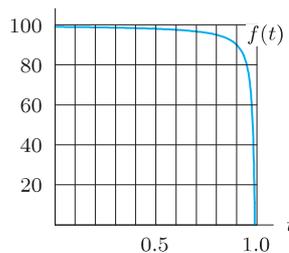


Figure 5.29

7. Use the table to estimate $\int_0^{40} f(x) dx$. What values of n and Δx did you use?

x	0	10	20	30	40
$f(x)$	350	410	435	450	460

8. Use the table to estimate $\int_0^{15} f(x) dx$.

x	0	3	6	9	12	15
$f(x)$	50	48	44	36	24	8

9. Use the table to estimate $\int_0^{12} f(x) dx$.

x	0	3	6	9	12
$f(x)$	32	22	15	11	9

10. Write out the terms of the right-hand sum with $n = 5$ that could be used to approximate $\int_3^7 \frac{1}{1+x} dx$. Do not evaluate the terms or the sum.

In Exercises 11–13, use a calculator or a computer to find the value of the definite integral.

11. $\int_0^3 2^x dx$ 12. $\int_0^1 \sin(t^2) dt$ 13. $\int_{-1}^1 e^{-x^2} dx$

14. (a) What is the area between the graph of $f(x)$ in Figure 5.30 and the x -axis, between $x = 0$ and $x = 5$?

- (b) What is $\int_0^5 f(x) dx$?

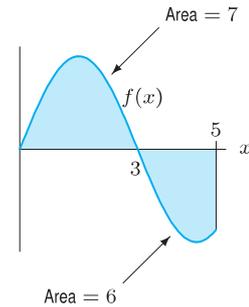


Figure 5.30

In Exercises 15–21, find the area of the regions between the curve and the horizontal axis

15. Under the curve $y = \cos t$ for $0 \leq t \leq \pi/2$.
 16. Under $y = 6x^3 - 2$ for $5 \leq x \leq 10$.
 17. Under $y = \ln x$ for $1 \leq x \leq 4$.
 18. Under the curve $y = \cos \sqrt{x}$ for $0 \leq x \leq 2$.
 19. Under $y = 2 \cos(t/10)$ for $1 \leq t \leq 2$.
 20. Under the curve $y = 7 - x^2$ and above the x -axis.
 21. Above the curve $y = x^4 - 8$ and below the x -axis.

Problems

22. (a) On a sketch of $y = \ln x$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate it.
 (b) On another sketch, represent the right Riemann sum with $n = 2$ approximating $\int_1^2 \ln x dx$. Write out the terms in the sum, but do not evaluate it.
 (c) Which sum is an overestimate? Which sum is an underestimate?
23. (a) Draw the rectangles that give the left-hand sum approximation to $\int_0^\pi \sin x dx$ with $n = 2$.
 (b) Repeat part (a) for $\int_{-\pi}^0 \sin x dx$.
 (c) From your answers to parts (a) and (b), what is the value of the left-hand sum approximation to $\int_{-\pi}^\pi \sin x dx$ with $n = 4$?
24. (a) Use a calculator or computer to find $\int_0^6 (x^2 + 1) dx$. Represent this value as the area under a curve.
 (b) Estimate $\int_0^6 (x^2 + 1) dx$ using a left-hand sum with $n = 3$. Represent this sum graphically on a sketch of $f(x) = x^2 + 1$. Is this sum an overestimate or underestimate of the true value found in part (a)?
 (c) Estimate $\int_0^6 (x^2 + 1) dx$ using a right-hand sum with $n = 3$. Represent this sum on your sketch. Is this sum an overestimate or underestimate?
25. Estimate $\int_1^2 x^2 dx$ using left- and right-hand sums with four subdivisions. How far from the true value of the integral could your estimate be?
26. Without computing the sums, find the difference between the right- and left-hand Riemann sums if we use $n = 500$ subintervals to approximate $\int_{-1}^1 (2x^3 + 4) dx$.
27. (a) Graph $f(x) = x(x+2)(x-1)$.
 (b) Find the total area between the graph and the x -axis between $x = -2$ and $x = 1$.
 (c) Find $\int_{-2}^1 f(x) dx$ and interpret it in terms of areas.
28. Compute the definite integral $\int_0^4 \cos \sqrt{x} dx$ and interpret the result in terms of areas.
29. Without computation, decide if $\int_0^{2\pi} e^{-x} \sin x dx$ is positive or negative. [Hint: Sketch $e^{-x} \sin x$.]
30. (a) Graph $f(x) = \begin{cases} 1-x & 0 \leq x \leq 1 \\ x-1 & 1 < x \leq 2 \end{cases}$
 (b) Find $\int_0^2 f(x) dx$.
 (c) Calculate the 4-term left Riemann sum approximation to the definite integral. How does the approximation compare to the exact value?

31. Use Figure 5.31 to find the values of

- (a) $\int_a^b f(x) dx$ (b) $\int_b^c f(x) dx$
 (c) $\int_a^c f(x) dx$ (d) $\int_a^c |f(x)| dx$

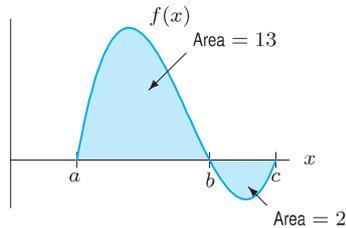


Figure 5.31

32. Given $\int_{-2}^0 f(x) dx = 4$ and Figure 5.32, estimate:

- (a) $\int_0^2 f(x) dx$ (b) $\int_{-2}^2 f(x) dx$
 (c) The total shaded area.

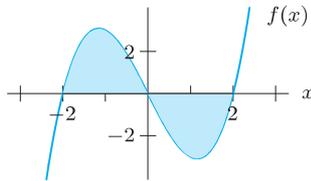


Figure 5.32

33. (a) Using Figure 5.33, find $\int_{-3}^0 f(x) dx$.
 (b) If the area of the shaded region is A , estimate $\int_{-3}^4 f(x) dx$.

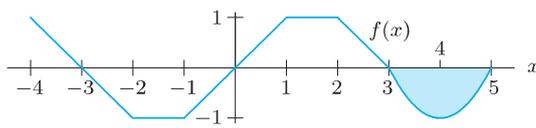


Figure 5.33

34. Write a few sentences in support of or in opposition to the following statement:

“If a left-hand sum underestimates a definite integral by a certain amount, then the corresponding right-hand sum will overestimate the integral by the same amount.”

35. Sketch the graph of a function f (you do not need to give a formula for f) on an interval $[a, b]$ with the property that with $n = 2$ subdivisions,

$$\int_a^b f(x) dx < \text{Left-hand sum} < \text{Right-hand sum}.$$

36. Three terms of a left-hand sum used to approximate a definite integral $\int_a^b f(x) dx$ are as follows.

$$\left(2 + 0 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3} + \left(2 + 1 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3} + \left(2 + 2 \cdot \frac{4}{3}\right)^2 \cdot \frac{4}{3}.$$

Find possible values for a and b and a possible formula for $f(x)$.

37. Consider the integral $\int_1^2 (1/t) dt$ in Example 1. By dividing the interval $1 \leq t \leq 2$ into 10 equal parts, we can show that

$$0.1 \left[\frac{1}{1.1} + \frac{1}{1.2} + \dots + \frac{1}{2} \right] \leq \int_1^2 \frac{1}{t} dt$$

and

$$\int_1^2 \frac{1}{t} dt \leq 0.1 \left[\frac{1}{1} + \frac{1}{1.1} + \dots + \frac{1}{1.9} \right].$$

(a) Now divide the interval $1 \leq t \leq 2$ into n equal parts to show that

$$\sum_{r=1}^n \frac{1}{n+r} < \int_1^2 \frac{1}{t} dt < \sum_{r=0}^{n-1} \frac{1}{n+r}.$$

(b) Show that the difference between the upper and lower sums in part (a) is $1/(2n)$.

(c) The exact value of $\int_1^2 (1/t) dt$ is $\ln 2$. How large should n be to approximate $\ln 2$ with an error of at most $5 \cdot 10^{-6}$, using one of the sums in part (a)?

5.3 THE FUNDAMENTAL THEOREM AND INTERPRETATIONS

The Notation and Units for the Definite Integral

Just as the Leibniz notation dy/dx for the derivative reminds us that the derivative is the limit of a ratio of differences, the notation for the definite integral helps us recall the meaning of the integral. The symbol

$$\int_a^b f(x) dx$$

reminds us that an integral is a limit of sums of terms of the form “ $f(x)$ times a small difference of x .” Officially, dx is not a separate entity, but a part of the whole integral symbol. Just as

one thinks of d/dx as a single symbol meaning “the derivative with respect to x of . . . ,” one can think of $\int_a^b \dots dx$ as a single symbol meaning “the integral of . . . with respect to x .”

However, many scientists and mathematicians informally think of dx as an “infinitesimally” small bit of x multiplied by $f(x)$. This viewpoint is often the key to interpreting the meaning of a definite integral. For example, if $f(t)$ is the velocity of a moving particle at time t , then $f(t) dt$ may be thought of informally as velocity \times time, giving the distance traveled by the particle during a small bit of time dt . The integral $\int_a^b f(t) dt$ may then be thought of as the sum of all these small distances, giving us the net change in position of the particle between $t = a$ and $t = b$. The notation for the integral suggests units for the value of the integral. Since the terms being added up are products of the form “ $f(x)$ times a difference in x ,” the unit of measurement for $\int_a^b f(x) dx$ is the product of the units for $f(x)$ and the units for x . For example, if $f(t)$ is velocity measured in meters/second and t is time measured in seconds, then

$$\int_a^b f(t) dt$$

has units of (meters/sec) \times (sec) = meters. This is what we expect, since the value of this integral represents change in position.

As another example, graph $y = f(x)$ with the same units of measurement of length along the x - and y -axes, say cm. Then $f(x)$ and x are measured in the same units, so

$$\int_a^b f(x) dx$$

is measured in square units of $\text{cm} \times \text{cm} = \text{cm}^2$. Again, this is what we would expect since in this context the integral represents an area.

The Fundamental Theorem of Calculus

We have seen that change in position can be calculated as the limit of Riemann sums of the velocity function $v = f(t)$. Thus, change in position is given by the definite integral $\int_a^b f(t) dt$. If we let $F(t)$ denote the position function, then the change in position can also be written as $F(b) - F(a)$. Thus we have:

$$\int_a^b f(t) dt = \begin{array}{l} \text{Change in position from} \\ t = a \text{ to } t = b \end{array} = F(b) - F(a)$$

We also know that the position F and velocity f are related using derivatives: $F'(t) = f(t)$. Thus, we have uncovered a connection between the integral and derivative, which is so important it is called the Fundamental Theorem of Calculus. It applies to any function F with a continuous derivative $f = F'$.

Theorem 5.1: The Fundamental Theorem of Calculus¹

If f is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t) dt = F(b) - F(a).$$

To understand the Fundamental Theorem of Calculus, think of $f(t) = F'(t)$ as the rate of change of the quantity $F(t)$. To calculate the total change in $F(t)$ between times $t = a$ and $t = b$, we divide the interval $a \leq t \leq b$ into n subintervals, each of length Δt . For each small interval, we

¹This result is sometimes called the First Fundamental Theorem of Calculus, to distinguish it from the Second Fundamental Theorem of Section 6.4.

estimate the change in $F(t)$, written ΔF , and add these. In each subinterval we assume the rate of change of $F(t)$ is approximately constant, so that we can say

$$\Delta F \approx \text{Rate of change of } F \times \text{Time elapsed.}$$

For the first subinterval, from t_0 to t_1 , the rate of change of $F(t)$ is approximately $F'(t_0)$, so

$$\Delta F \approx F'(t_0) \Delta t.$$

Similarly, for the second interval

$$\Delta F \approx F'(t_1) \Delta t.$$

Summing over all the subintervals, we get

$$\begin{aligned} \text{Total change in } F(t) \\ \text{between } t = a \text{ and } t = b \end{aligned} &= \sum_{i=0}^{n-1} \Delta F \approx \sum_{i=0}^{n-1} F'(t_i) \Delta t.$$

We have approximated the change in $F(t)$ as a left-hand sum.

However, the total change in $F(t)$ between the times $t = a$ and $t = b$ is simply $F(b) - F(a)$. Taking the limit as n goes to infinity converts the Riemann sum to a definite integral and suggests the following interpretation of the Fundamental Theorem of Calculus:²

$$F(b) - F(a) = \begin{array}{l} \text{Total change in } F(t) \\ \text{between } t = a \text{ and } t = b \end{array} = \int_a^b F'(t) dt.$$

In words, the definite integral of a rate of change gives the total change.

This argument does not, however, constitute a proof of the Fundamental Theorem. The errors in the various approximations must be investigated using the definition of limit. A proof is given in Section 6.4 where we learn how to construct antiderivatives using the Second Fundamental Theorem of Calculus.

Example 1 If $F'(t) = f(t)$ and $f(t)$ is velocity in miles/hour, with t in hours, what are the units of $\int_a^b f(t) dt$ and $F(b) - F(a)$?

Solution Since the units of $f(t)$ are miles/hour and the units of t are hours, the units of $\int_a^b f(t) dt$ are (miles/hour) \times hours = miles. Since F measures change in position, the units of $F(b) - F(a)$ are miles. As expected, the units of $\int_a^b f(t) dt$ and $F(b) - F(a)$ are the same.

The Definite Integral of a Rate of Change: Applications of the Fundamental Theorem

Many applications are based on the Fundamental Theorem, which tells us that the definite integral of a rate of change gives the total change.

Example 2 Let $F(t)$ represent a bacteria population which is 5 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the total increase in the bacteria population during the first hour, and the population at $t = 1$.

Solution Since the rate at which the population is growing is $F'(t) = 2^t$, we have

$$\text{Change in population} = F(1) - F(0) = \int_0^1 2^t dt.$$

Using a calculator to evaluate the integral,

$$\text{Change in population} = \int_0^1 2^t dt = 1.44 \text{ million bacteria.}$$

²We could equally well have used a right-hand sum, since the definite integral is their common limit.

Since $F(0) = 5$, the population at $t = 1$ is given by

$$\text{Population} = F(1) = F(0) + \int_0^1 2^t dt = 5 + 1.44 = 6.44 \text{ million.}$$

The following example shows how representing a quantity as a definite integral, and thereby as an area, can be helpful even if we don't evaluate the integral.

Example 3 Two cars start from rest at a traffic light and accelerate for several minutes. Figure 5.34 shows their velocities as a function of time.

- (a) Which car is ahead after one minute? (b) Which car is ahead after two minutes?

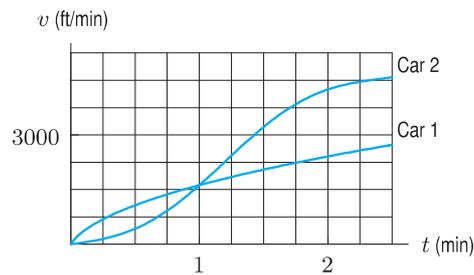


Figure 5.34: Velocities of two cars in Example 3. Which is ahead when?

- Solution**
- (a) For the first minute car 1 goes faster than car 2, and therefore car 1 must be ahead at the end of one minute.
- (b) At the end of two minutes the situation is less clear, since car 1 was going faster for the first minute and car 2 for the second. However, if $v = f(t)$ is the velocity of a car after t minutes, then we know that

$$\text{Distance traveled in two minutes} = \int_0^2 f(t) dt,$$

since the integral of velocity is distance traveled. This definite integral may also be interpreted as the area under the graph of f between 0 and 2. Since the area representing the distance traveled by car 2 is clearly larger than the area for car 1 (see Figure 5.34), we know that car 2 has traveled farther than car 1.

Example 4 Biological activity in water is reflected in the rate at which carbon dioxide, CO_2 , is added or removed. Plants take CO_2 out of the water during the day for photosynthesis and put CO_2 into the water at night. Animals put CO_2 into the water all the time as they breathe. Figure 5.35 shows the rate of change of the CO_2 level in a pond.³ At dawn, there were 2.600 mmol of CO_2 per liter of water.

- (a) At what time was the CO_2 level lowest? Highest?
- (b) Estimate how much CO_2 enters the pond during the night ($t = 12$ to $t = 24$).
- (c) Estimate the CO_2 level at dusk (twelve hours after dawn).
- (d) Does the CO_2 level appear to be approximately in equilibrium?

³Data from R. J. Beyers, *The Pattern of Photosynthesis and Respiration in Laboratory Microsystems* (Mem. 1st. Ital. Idrobiol., 1965).

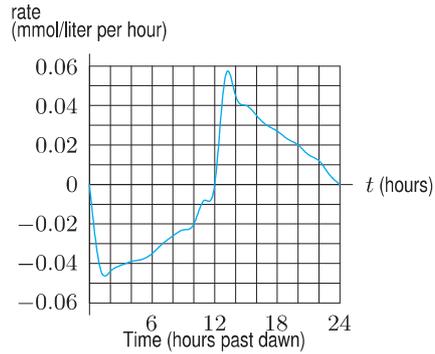


Figure 5.35: Rate at which CO_2 enters a pond over a 24-hour period

Solution

Let $f(t)$ be the rate at which CO_2 is entering the water at time t and let $F(t)$ be the concentration of CO_2 in the water at time t , so $F'(t) = f(t)$.

- (a) From Figure 5.35, we see $f(t)$ is negative for $0 \leq t \leq 12$, so the CO_2 level is decreasing during this interval (daytime). Since $f(t)$ is positive for $12 < t < 24$, the CO_2 level is increasing during this interval (night). The CO_2 is lowest at $t = 12$ (dusk) and highest at $t = 0$ and $t = 24$ (dawn).
- (b) We want to calculate the total change in the CO_2 level in the pond, $F(24) - F(12)$. By the Fundamental Theorem of Calculus,

$$F(24) - F(12) = \int_{12}^{24} f(t) dt.$$

We use values of $f(t)$ from the graph (displayed in Table 5.8) to construct a left Riemann sum approximation to this integral with $n = 6$, $\Delta t = 2$:

$$\begin{aligned} \int_{12}^{24} f(t) dt &\approx f(12) \cdot 2 + f(14) \cdot 2 + f(16) \cdot 2 + \cdots + f(22) \cdot 2 \\ &\approx (0.000)2 + (0.045)2 + (0.035)2 + \cdots + (0.012)2 = 0.278. \end{aligned}$$

Thus, between $t = 12$ and $t = 24$,

$$\text{Change in } \text{CO}_2 \text{ level} = F(24) - F(12) = \int_{12}^{24} f(t) dt \approx 0.278 \text{ mmol/liter.}$$

- (c) To find the CO_2 level at $t = 12$, we use the Fundamental Theorem to estimate the change in CO_2 level during the day:

$$F(12) - F(0) = \int_0^{12} f(t) dt$$

Using a left Riemann sum as in part (c), we have

$$F(12) - F(0) = \int_0^{12} f(t) dt \approx -0.328.$$

Since initially there were $F(0) = 2.600$ mmol/liter, we have

$$F(12) = F(0) - 0.328 = 2.272 \text{ mmol/liter.}$$

- (d) The amount of CO_2 removed during the day is represented by the area of the region below the t -axis; the amount of CO_2 added during the night is represented by the area above the t -axis. These areas look approximately equal, so the CO_2 level is approximately in equilibrium.

Table 5.8 Rate, $f(t)$, at which CO_2 is entering or leaving water (read from Figure 5.35)

t	$f(t)$										
0	0.000	4	-0.039	8	-0.026	12	0.000	16	0.035	20	0.020
2	-0.044	6	-0.035	10	-0.020	14	0.045	18	0.027	22	0.012

Using Riemann sums to estimate these areas, we find that about 0.278 mmol/l of CO_2 was released into the pond during the night and about 0.328 mmol/l of CO_2 was absorbed from the pond during the day. These quantities are sufficiently close that the difference could be due to measurement error, or to errors from the Riemann sum approximation.

The Definite Integral as an Average

We know how to find the average of n numbers: Add them and divide by n . But how do we find the average value of a continuously varying function? Let us consider an example. Suppose $f(t)$ is the temperature at time t , measured in hours since midnight, and that we want to calculate the average temperature over a 24-hour period. One way to start is to average the temperatures at n equally spaced times, t_1, t_2, \dots, t_n , during the day.

$$\text{Average temperature} \approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{n}.$$

The larger we make n , the better the approximation. We can rewrite this expression as a Riemann sum over the interval $0 \leq t \leq 24$ if we use the fact that $\Delta t = 24/n$, so $n = 24/\Delta t$:

$$\begin{aligned} \text{Average temperature} &\approx \frac{f(t_1) + f(t_2) + \cdots + f(t_n)}{24/\Delta t} \\ &= \frac{f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t}{24} \\ &= \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t. \end{aligned}$$

As $n \rightarrow \infty$, the Riemann sum tends toward an integral, and $1/24$ of the sum also approximates the average temperature better. It makes sense, then, to write

$$\text{Average temperature} = \lim_{n \rightarrow \infty} \frac{1}{24} \sum_{i=1}^n f(t_i)\Delta t = \frac{1}{24} \int_0^{24} f(t) dt.$$

We have found a way of expressing the average temperature over an interval in terms of an integral. Generalizing for any function f , if $a < b$, we define

$$\text{Average value of } f \text{ from } a \text{ to } b = \frac{1}{b-a} \int_a^b f(x) dx.$$

How to Visualize the Average on a Graph

The definition of average value tells us that

$$(\text{Average value of } f) \cdot (b - a) = \int_a^b f(x) dx.$$

Let's interpret the integral as the area under the graph of f . Then the average value of f is the height of a rectangle whose base is $(b - a)$ and whose area is the same as the area under the graph of f . (See Figure 5.36.)

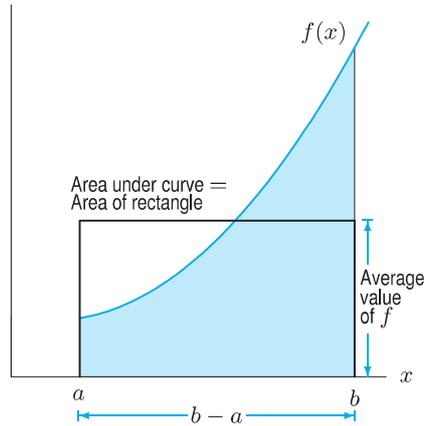


Figure 5.36: Area and average value

Example 5 Suppose that $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time measured in days and $t = 0$ corresponds to January 1, 2008. Interpret $\int_0^{90} C(t) dt$ and $\frac{1}{90-0} \int_0^{90} C(t) dt$.

Solution The units for the integral $\int_0^{90} C(t) dt$ are (dollars/day) \times (days) = dollars. The integral represents the total cost in dollars to heat your house for the first 90 days of 2008, namely the months of January, February, and March. The second expression is measured in (1/days)(dollars) or dollars per day, the same units as $C(t)$. It represents the average cost per day to heat your house during the first 90 days of 2008.

Example 6 On page 10, we saw that the population of Nevada could be modeled by the function

$$P = f(t) = 2.020(1.036)^t,$$

where P is in millions of people and t is in years since 2000. Use this function to predict the average population of Nevada between the years 2000 and 2020.

Solution We want the average value of $f(t)$ between $t = 0$ and $t = 20$. This is given by

$$\text{Average population} = \frac{1}{20-0} \int_0^{20} f(t) dt = \frac{1}{20}(58.748) = 2.937.$$

We used a calculator to evaluate the integral. The average population of Nevada between 2000 and 2020 is predicted to be about 2.9 million people.

Exercises and Problems for Section 5.3

Exercises

1. If $f(t)$ is measured in dollars per year and t is measured in years, what are the units of $\int_a^b f(t) dt$?
2. If $f(t)$ is measured in meters/second² and t is measured in seconds, what are the units of $\int_a^b f(t) dt$?
3. If $f(x)$ is measured in pounds and x is measured in feet, what are the units of $\int_a^b f(x) dx$?

In Exercises 4–7, explain in words what the integral represents and give units.

4. $\int_1^3 v(t) dt$, where $v(t)$ is velocity in meters/sec and t is time in seconds.
5. $\int_0^6 a(t) dt$, where $a(t)$ is acceleration in km/hr² and t is time in hours.
6. $\int_{2000}^{2004} f(t) dt$, where $f(t)$ is the rate at which the world's population is growing in year t , in billion people per year.
7. $\int_0^5 s(x) dx$, where $s(x)$ is rate of change of salinity (salt concentration) in gm/liter per cm in sea water, where x is depth below the surface of the water in cm.
8. For the two cars in Example 3, page 274, estimate:
 - (a) The distances moved by car 1 and car 2 during the first minute.
 - (b) The time at which the two cars have gone the same distance.

In Exercises 9–12, find the average value of the function over the given interval.

9. $g(t) = 1 + t$ over $[0, 2]$
10. $g(t) = e^t$ over $[0, 10]$
11. $f(x) = 2$ over $[a, b]$
12. $f(x) = 4x + 7$ over $[1, 3]$

13. How do the units for the average value of f relate to the units for $f(x)$ and the units for x ?
14. Oil leaks out of a tanker at a rate of $r = f(t)$ gallons per minute, where t is in minutes. Write a definite integral expressing the total quantity of oil which leaks out of the tanker in the first hour.
15. Water is leaking out of a tank at a rate of $R(t)$ gallons/hour, where t is measured in hours.
 - (a) Write a definite integral that expresses the total amount of water that leaks out in the first two hours.
 - (b) In Figure 5.37, shade the region whose area represents the total amount of water that leaks out in the first two hours.
 - (c) Give an upper and lower estimate of the total amount of water that leaks out in the first two hours.

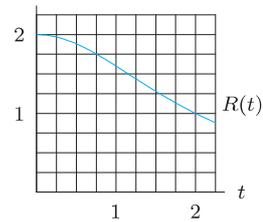


Figure 5.37

Problems

16. The rate at which the world's oil is being consumed is continuously increasing. Suppose the rate of oil consumption (in billions of barrels per year) is given by the function $r = f(t)$, where t is measured in years and $t = 0$ is the start of 2004.
 - (a) Write a definite integral which represents the total quantity of oil used between the start of 2004 and the start of 2009.
 - (b) Suppose $r = 32e^{0.05t}$. Using a left-hand sum with five subdivisions, find an approximate value for the total quantity of oil used between the start of 2004 and the start of 2009.
 - (c) Interpret each of the five terms in the sum from part (b) in terms of oil consumption.
17. As coal deposits are depleted, it becomes necessary to strip-mine larger areas for each ton of coal. Figure 5.38 shows the number of acres of land per million tons of coal that will be defaced during strip-mining as a function of the number of million tons removed, starting from the present day.
 - (a) Estimate the total number of acres defaced in extracting the next 4 million tons of coal (measured from the present day). Draw four rectangles under the curve, and compute their area.

- (b) Reestimate the number of acres defaced using rectangles above the curve.
- (c) Use your answers to parts (a) and (b) to get a better estimate of the actual number of acres defaced.

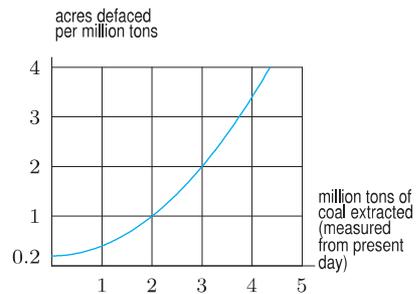


Figure 5.38

18. After a spill of radioactive iodine, measurements showed the ambient radiation levels at the site of the spill to be four times the maximum acceptable limit. The level of radiation from an iodine source decreases according to the formula

$$R(t) = R_0 e^{-0.004t}$$

where R is the radiation level (in millirems/hour) at time t in hours and R_0 is the initial radiation level (at $t = 0$).

- (a) How long will it take for the site to reach an acceptable level of radiation?
 - (b) How much total radiation (in millirems) will have been emitted by that time, assuming the maximum acceptable limit is 0.6 millirems/hour?
19. In 2005, the population of Mexico was growing at 1% a year. Assuming that this growth rate continues into the future and that t is in years since 2005, the Mexican population, P , in millions, will be given by

$$P = 103(1.01)^t.$$

- (a) Predict the average population of Mexico between 2005 and 2055.
 - (b) Find the average of the population in 2005 and the predicted population in 2055.
 - (c) Explain, in terms of the concavity of the graph of P why your answer to part (b) is larger or smaller than your answer to part (a).
20. The following table gives the emissions, E , of nitrogen oxides in millions of metric tons per year in the US.⁴ Let t be the number of years since 1970 and $E = f(t)$.

Year	1970	1975	1980	1985	1990	1995	2000
E	26.9	26.4	27.1	25.8	25.5	25.0	22.6

- (a) What are the units and meaning of $\int_0^{30} f(t)dt$?
 - (b) Estimate $\int_0^{30} f(t)dt$.
21. Coal gas is produced at a gasworks. Pollutants in the gas are removed by scrubbers, which become less and less efficient as time goes on. The following measurements, made at the start of each month, show the rate at which pollutants are escaping (in tons/month) in the gas:

Time (months)	0	1	2	3	4	5	6
Rate pollutants escape	5	7	8	10	13	16	20

- (a) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the first month.
- (b) Make an overestimate and an underestimate of the total quantity of pollutants that escape during the six months.
- (c) How often would measurements have to be made to find overestimates and underestimates which differ by less than 1 ton from the exact quantity of pollutants that escaped during the first six months?

22. When an aircraft attempts to climb as rapidly as possible, its climb rate decreases with altitude. (This occurs because the air is less dense at higher altitudes.) The table shows performance data for a single-engine aircraft.

Altitude (1000 ft)	0	1	2	3	4	5
Climb rate (ft/min)	925	875	830	780	730	685
Altitude (1000 ft)	6	7	8	9	10	
Climb rate (ft/min)	635	585	535	490	440	

- (a) Calculate upper and lower estimates for the time required for this aircraft to climb from sea level to 10,000 ft.
 - (b) If climb rate data were available in increments of 500 ft, what would be the difference between a lower and upper estimate of climb time based on 20 subdivisions?
23. A two-day environmental clean up started at 9 am on the first day. The number of workers fluctuated as shown in Figure 5.39. If the workers were paid \$10 per hour, how much was the total personnel cost of the clean up?

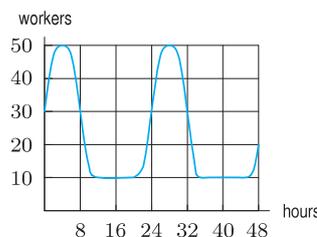


Figure 5.39

24. Suppose in Problem 23 that the workers were paid \$10 per hour for work during the time period 9 am to 5 pm and were paid \$15 per hour for work during the rest of the day. What would the total personnel costs of the clean up have been under these conditions?
25. A warehouse charges its customers \$5 per day for every 10 cubic feet of space used for storage. Figure 5.40 records the storage used by one company over a month. How much will the company have to pay?

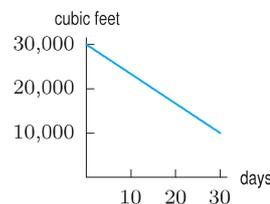


Figure 5.40

⁴The World Almanac and Book of Facts 2005, p. 177 (New York: World Almanac Books).

26. A cup of coffee at 90°C is put into a 20°C room when $t = 0$. The coffee's temperature is changing at a rate of $r(t) = -7e^{-0.1t}$ $^\circ\text{C}$ per minute, with t in minutes. Estimate the coffee's temperature when $t = 10$.
27. The amount of waste a company produces, W , in tons per week, is approximated by $W = 3.75e^{-0.008t}$, where t is in weeks since January 1, 2005. Waste removal for the company costs $\$15/\text{ton}$. How much does the company pay for waste removal during the year 2005?
28. Let $F(x) = \int_0^x 2t \, dt$. Then $F(x)$ is the area under the line $y = 2t$ from the origin to x .
- Construct a table showing the values of F for $x = 0, 1, 2, 3, 4, 5$.
 - Is F increasing or decreasing when $x > 0$? Concave up or down? Explain.
 - When $t < 0$, the line $y = 2t$ is below the t -axis (the horizontal axis). Explain why $F(-1)$ is positive.
29. A force F parallel to the x -axis is given by the graph in Figure 5.41. Estimate the work, W , done by the force, where $W = \int_0^{16} F(x) \, dx$.

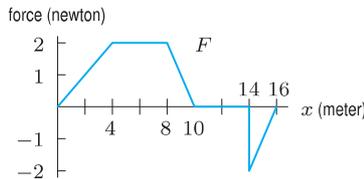


Figure 5.41

30. (a) Using Figures 5.42 and 5.43, find the average value on $0 \leq x \leq 2$ of
- $f(x)$
 - $g(x)$
 - $f(x) \cdot g(x)$
- (b) Is the following statement true? Explain your answer.

$$\text{Average}(f) \cdot \text{Average}(g) = \text{Average}(f \cdot g)$$

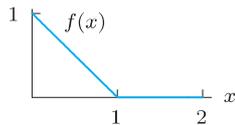


Figure 5.42

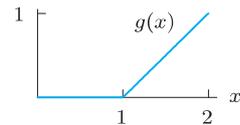


Figure 5.43

31. (a) Using Figure 5.44, find $\int_1^6 f(x) \, dx$.
- (b) What is the average value of f on $[1, 6]$?

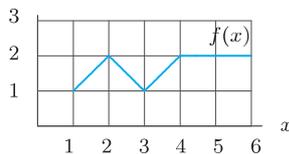


Figure 5.44

32. (a) Using Figure 5.45, estimate $\int_{-3}^3 f(x) \, dx$.
- (b) Which of the following average values of $f(x)$ is larger?
- Between $x = -3$ and $x = 3$
 - Between $x = 0$ and $x = 3$

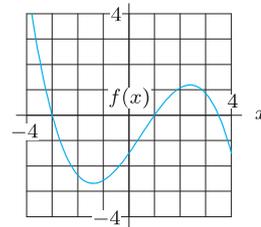


Figure 5.45

33. For the even function f in Figure 5.46, write an expression involving one or more definite integrals that denotes:
- The average value of f for $0 \leq x \leq 5$.
 - The average value of $|f|$ for $0 \leq x \leq 5$.

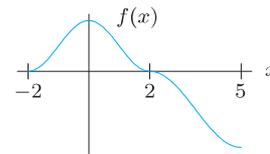


Figure 5.46

34. For the even function f in Figure 5.46, consider the average value of f over the following intervals:
- $0 \leq x \leq 1$
 - $0 \leq x \leq 2$
 - $0 \leq x \leq 5$
 - $-2 \leq x \leq 2$
- For which interval is the average value of f least?
 - For which interval is the average value of f greatest?
 - For which pair of intervals are the average values equal?
35. (a) Without computing any integrals, explain why the average value of $f(x) = \sin x$ on $[0, \pi]$ must be between 0.5 and 1.
- (b) Compute this average.
36. (a) What is the average value of $f(x) = \sqrt{1-x^2}$ over the interval $0 \leq x \leq 1$?
- (b) How can you tell whether this average value is more or less than 0.5 without doing any calculations?

37. Figure 5.47 shows the rate of change of the quantity of water in a water tower, in liters per day, during the month of April. If the tower had 12,000 liters of water in it on April 1, estimate the quantity of water in the tower on April 30.

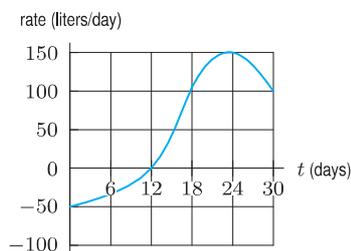


Figure 5.47

38. A bicyclist pedals along a straight road with velocity, v , given in Figure 5.48. She starts 5 miles from a lake; positive velocities take her away from the lake and negative velocities take her toward the lake. When is the cyclist farthest from the lake, and how far away is she then?

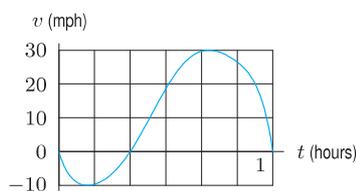


Figure 5.48

39. The value, V , of a Tiffany lamp, worth \$225 in 1975, increases at 15% per year. Its value in dollars t years after 1975 is given by

$$V = 225(1.15)^t.$$

Find the average value of the lamp over the period 1975–2010.

40. A car speeds up at a constant rate from 10 to 70 mph over a period of half an hour. Its fuel efficiency (in miles per gallon) increases with speed; values are in the table. Make lower and upper estimates of the quantity of fuel used during the half hour.

Speed (mph)	10	20	30	40	50	60	70
Fuel efficiency (mpg)	15	18	21	23	24	25	26

41. The number of hours, H , of daylight in Madrid as a function of date is approximated by the formula

$$H = 12 + 2.4 \sin[0.0172(t - 80)],$$

where t is the number of days since the start of the year. Find the average number of hours of daylight in Madrid:

- (a) in January (b) in June (c) over a year
(d) Explain why the relative magnitudes of your answers to parts (a), (b), and (c) are reasonable.

42. Height velocity graphs are used by endocrinologists to follow the progress of children with growth deficiencies. Figure 5.49 shows the height velocity curves of an average boy and an average girl between ages 3 and 18.

- (a) Which curve is for girls and which is for boys? Explain how you can tell.
(b) About how much does the average boy grow between ages 3 and 10?
(c) The growth spurt associated with adolescence and the onset of puberty occurs between ages 12 and 15 for the average boy and between ages 10 and 12.5 for the average girl. Estimate the height gained by each average child during this growth spurt.
(d) When fully grown, about how much taller is the average man than the average woman? (The average boy and girl are about the same height at age 3.)

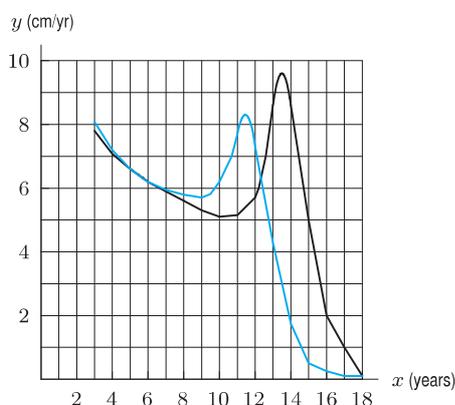


Figure 5.49

43. In Chapter 2, the average velocity over the time interval $a \leq t \leq b$ was defined to be $(s(b) - s(a))/(b - a)$, where $s(t)$ is the position function. Use the Fundamental Theorem of Calculus to show that the average value of the velocity function $v(t)$, on the interval $a \leq t \leq b$ is also $(s(b) - s(a))/(b - a)$.

44. If you jump out of an airplane and your parachute fails to open, your downward velocity t seconds after the jump is approximated, for $g = 9.8 \text{ m/sec}^2$ and $k = 0.2 \text{ sec}$, by

$$v(t) = \frac{g}{k}(1 - e^{-kt}).$$

- (a) Write an expression for the distance you fall in T seconds.
(b) If you jump from 5000 meters above the ground, write an equation whose solution is the number of seconds you fall before hitting the ground.
(c) Estimate the solution to the equation in part (b).

5.4 THEOREMS ABOUT DEFINITE INTEGRALS

Properties of the Definite Integral

For the definite integral $\int_a^b f(x) dx$, we have so far only considered the case $a < b$. We now allow $a \geq b$. We still set $x_0 = a$, $x_n = b$, and $\Delta x = (b - a)/n$. As before, we have $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$.

Theorem 5.2: Properties of Limits of Integration

If a , b , and c are any numbers and f is a continuous function, then

- $\int_b^a f(x) dx = - \int_a^b f(x) dx$.
- $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$.

In words:

- The integral from b to a is the negative of the integral from a to b .
- The integral from a to c plus the integral from c to b is the integral from a to b .

By interpreting the integrals as areas, we can justify these results for $f \geq 0$. In fact, they are true for all functions for which the integrals make sense. For a formal proof, see the online supplement at www.wiley.com/college/hugheshallett.

Why is $\int_b^a f(x) dx = - \int_a^b f(x) dx$?

By definition, both integrals are approximated by sums of the form $\sum f(x_i) \Delta x$. The only difference in the sums for $\int_b^a f(x) dx$ and $\int_a^b f(x) dx$ is that in the first $\Delta x = (a - b)/n = -(b - a)/n$ and in the second $\Delta x = (b - a)/n$. Since everything else about the sums is the same, we must have $\int_b^a f(x) dx = - \int_a^b f(x) dx$.

Why is $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$?

Suppose $a < c < b$. Figure 5.50 suggests that $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$ since the area under f from a to c plus the area under f from c to b together make up the whole area under f from a to b .

This property holds for all numbers a , b , and c , not just those satisfying $a < c < b$. (See Figure 5.51.) For example, the area under f from 3 to 6 is equal to the area from 3 to 8 *minus* the area from 6 to 8, so

$$\int_3^6 f(x) dx = \int_3^8 f(x) dx - \int_6^8 f(x) dx = \int_3^8 f(x) dx + \int_8^6 f(x) dx.$$

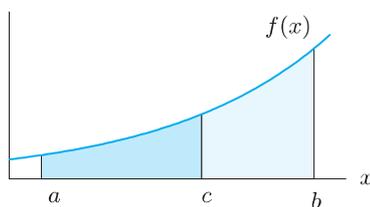


Figure 5.50: Additivity of the definite integral ($a < c < b$)

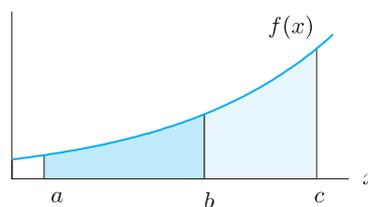


Figure 5.51: Additivity of the definite integral ($a < b < c$)

Example 1 Given that $\int_0^{1.25} \cos(x^2) dx = 0.98$ and $\int_0^1 \cos(x^2) dx = 0.90$, what are the values of the following integrals? (See Figure 5.52.)

(a) $\int_1^{1.25} \cos(x^2) dx$ (b) $\int_{-1}^1 \cos(x^2) dx$ (c) $\int_{1.25}^{-1} \cos(x^2) dx$

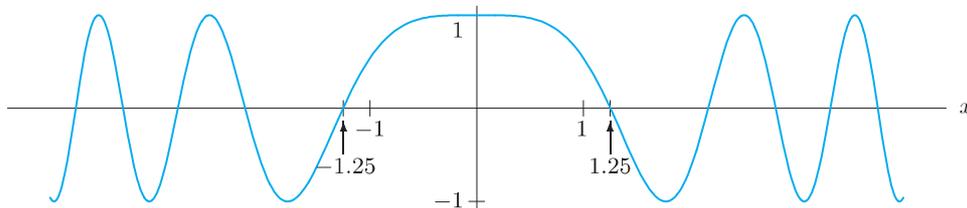


Figure 5.52: Graph of $f(x) = \cos(x^2)$

Solution (a) Since, by the additivity property,

$$\int_0^{1.25} \cos(x^2) dx = \int_0^1 \cos(x^2) dx + \int_1^{1.25} \cos(x^2) dx,$$

we get

$$0.98 = 0.90 + \int_1^{1.25} \cos(x^2) dx,$$

so

$$\int_1^{1.25} \cos(x^2) dx = 0.08.$$

(b) By the additivity property, we have

$$\int_{-1}^1 \cos(x^2) dx = \int_{-1}^0 \cos(x^2) dx + \int_0^1 \cos(x^2) dx.$$

By the symmetry of $\cos(x^2)$ about the y -axis,

$$\int_{-1}^0 \cos(x^2) dx = \int_0^1 \cos(x^2) dx = 0.90.$$

So

$$\int_{-1}^1 \cos(x^2) dx = 0.90 + 0.90 = 1.80.$$

(c) Using both properties in Theorem 5.2, we have

$$\begin{aligned} \int_{1.25}^{-1} \cos(x^2) dx &= -\int_{-1}^{1.25} \cos(x^2) dx = -\left(\int_{-1}^0 \cos(x^2) dx + \int_0^{1.25} \cos(x^2) dx\right) \\ &= -(0.90 + 0.98) = -1.88. \end{aligned}$$

Theorem 5.3: Properties of Sums and Constant Multiples of the Integrand

Let f and g be continuous functions and let c be a constant.

- $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$
- $\int_a^b cf(x) dx = c \int_a^b f(x) dx.$

In words:

- The integral of the sum (or difference) of two functions is the sum (or difference) of their integrals.
- The integral of a constant times a function is that constant times the integral of the function.

Why Do these Properties Hold?

Both can be visualized by thinking of the definite integral as the limit of a sum of areas of rectangles.

For property 1, suppose that f and g are positive on the interval $[a, b]$ so that the area under $f(x) + g(x)$ is approximated by the sum of the areas of rectangles like the one shaded in Figure 5.53. The area of this rectangle is

$$(f(x_i) + g(x_i))\Delta x = f(x_i)\Delta x + g(x_i)\Delta x.$$

Since $f(x_i)\Delta x$ is the area of a rectangle under the graph of f , and $g(x_i)\Delta x$ is the area of a rectangle under the graph of g , the area under $f(x) + g(x)$ is the sum of the areas under $f(x)$ and $g(x)$.

For property 2, notice that multiplying a function by c stretches or shrinks the graph in the vertical direction by a factor of c . Thus, it stretches or shrinks the height of each approximating rectangle by c , and hence multiplies the area by c .

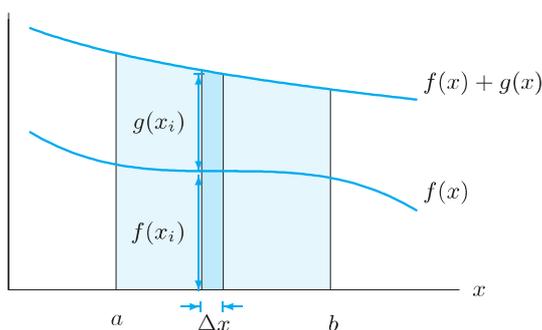


Figure 5.53: Area = $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

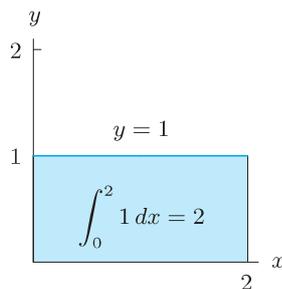
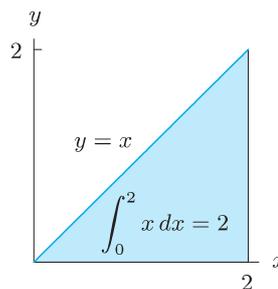
Example 2 Evaluate the definite integral $\int_0^2 (1 + 3x) dx$ exactly.

Solution We can break this integral up as follows:

$$\int_0^2 (1 + 3x) dx = \int_0^2 1 dx + \int_0^2 3x dx = \int_0^2 1 dx + 3 \int_0^2 x dx.$$

From Figures 5.54 and 5.55 and the area interpretation of the integral, we see that

$$\int_0^2 1 dx = \begin{array}{l} \text{Area of} \\ \text{rectangle} \end{array} = 2 \quad \text{and} \quad \int_0^2 x dx = \begin{array}{l} \text{Area of} \\ \text{triangle} \end{array} = \frac{1}{2} \cdot 2 \cdot 2 = 2.$$

Figure 5.54: Area representing $\int_0^2 1 dx$ Figure 5.55: Area representing $\int_0^2 x dx$

Therefore,

$$\int_0^2 (1 + 3x) dx = \int_0^2 1 dx + 3 \int_0^2 x dx = 2 + 3 \cdot 2 = 8.$$

Area Between Curves

Theorem 5.3 enables us to find the area of a region between curves. We have the following result:

If the graph of $f(x)$ lies above the graph of $g(x)$ for $a \leq x \leq b$, then

$$\text{Area between } f \text{ and } g \text{ for } a \leq x \leq b = \int_a^b (f(x) - g(x)) dx.$$

Example 3 Find the area of the shaded region in Figure 5.56.

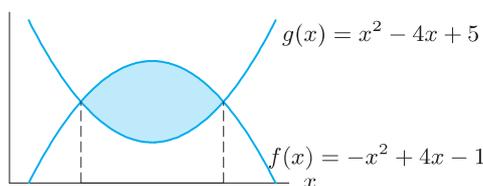


Figure 5.56: Area between two parabolas

Solution The curves cross where

$$\begin{aligned} x^2 - 4x + 5 &= -x^2 + 4x - 1 \\ 2x^2 - 8x + 6 &= 0 \\ 2(x - 1)(x - 3) &= 0 \\ x &= 1, 3. \end{aligned}$$

Since $f(x) = -x^2 + 4x - 1$ is above $g(x) = x^2 - 4x + 5$ for x between 1 and 3, we find the shaded area by subtraction:

$$\begin{aligned} \text{Area} &= \int_1^3 f(x) dx - \int_1^3 g(x) dx = \int_1^3 (f(x) - g(x)) dx \\ &= \int_1^3 ((-x^2 + 4x - 1) - (x^2 - 4x + 5)) dx \\ &= \int_1^3 (-2x^2 + 8x - 6) dx = 2.667. \end{aligned}$$

Using Symmetry to Evaluate Integrals

Symmetry can be useful in evaluating definite integrals. An even function is symmetric about the y -axis. An odd function is symmetric about the origin. Figures 5.57 and 5.58 suggest the following results:

$$\text{If } f \text{ is even, then } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx. \quad \text{If } g \text{ is odd, then } \int_{-a}^a g(x) dx = 0.$$

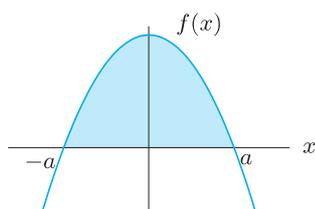


Figure 5.57: For an even function,
 $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

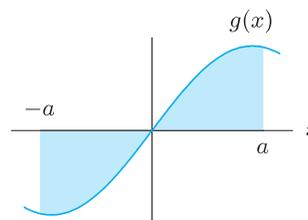


Figure 5.58: For an odd function,
 $\int_{-a}^a g(x) dx = 0$

Example 4 Given that $\int_0^\pi \sin t dt = 2$, find (a) $\int_{-\pi}^\pi \sin t dt$ (b) $\int_{-\pi}^\pi |\sin t| dt$

Solution Graphs of $\sin t$ and $|\sin t|$ are in Figures 5.59 and 5.60.

(a) Since $\sin t$ is an odd function

$$\int_{-\pi}^\pi \sin t dt = 0.$$

(b) Since $|\sin t|$ is an even function

$$\int_{-\pi}^\pi |\sin t| dt = 2 \int_0^\pi |\sin t| dt = 4.$$

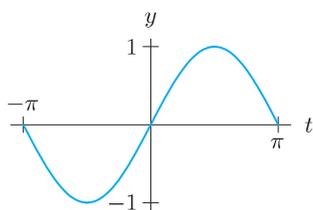


Figure 5.59

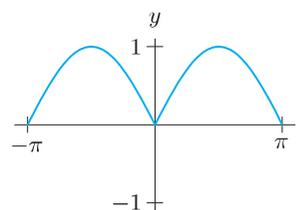


Figure 5.60

Using the Fundamental Theorem to Compute Integrals

The Fundamental Theorem provides an exact way of computing certain definite integrals.

Example 5 Compute $\int_1^3 2x dx$ by two different methods.

Solution Using left- and right-hand sums, we can approximate this integral as accurately as we want. With $n = 100$, for example, the left-sum is 7.96 and the right sum is 8.04. Using $n = 500$ we learn

$$7.992 < \int_1^3 2x dx < 8.008.$$

The Fundamental Theorem, on the other hand, allows us to compute the integral exactly. We take $f(x) = 2x$. We know that if $F(x) = x^2$, then $F'(x) = 2x$. So we use $f(x) = 2x$ and $F(x) = x^2$ and obtain

$$\int_1^3 2x \, dx = F(3) - F(1) = 3^2 - 1^2 = 8.$$

Comparing Integrals

Suppose we have constants m and M such that $m \leq f(x) \leq M$ for $a \leq x \leq b$. We say f is *bounded above* by M and *bounded below* by m . Then the graph of f lies between the horizontal lines $y = m$ and $y = M$. So the definite integral lies between $m(b - a)$ and $M(b - a)$. See Figure 5.61.

Suppose $f(x) \leq g(x)$ for $a \leq x \leq b$, as in Figure 5.62. Then the definite integral of f is less than or equal to the definite integral of g . This leads us to the following results:

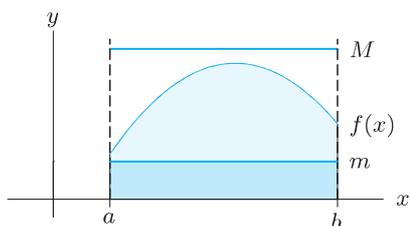


Figure 5.61: The area under the graph of f lies between the areas of the rectangles

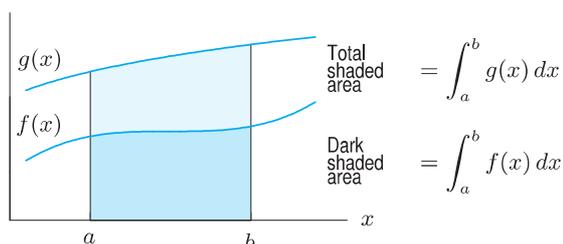


Figure 5.62: If $f(x) \leq g(x)$ then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$

Theorem 5.4: Comparison of Definite Integrals

Let f and g be continuous functions.

1. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b - a) \leq \int_a^b f(x) \, dx \leq M(b - a)$.
2. If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.

Example 6 Explain why $\int_0^{\sqrt{\pi}} \sin(x^2) \, dx \leq \sqrt{\pi}$.

Solution Since $\sin(x^2) \leq 1$ for all x (see Figure 5.63), part 2 of Theorem 5.4 gives

$$\int_0^{\sqrt{\pi}} \sin(x^2) \, dx \leq \int_0^{\sqrt{\pi}} 1 \, dx = \sqrt{\pi}.$$

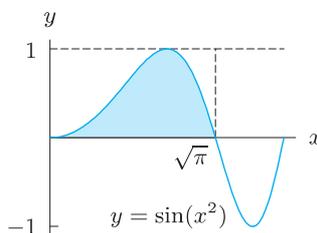


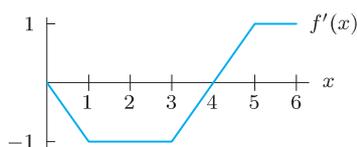
Figure 5.63: Graph showing that $\int_0^{\sqrt{\pi}} \sin(x^2) \, dx < \sqrt{\pi}$

Exercises and Problems for Section 5.4

Exercises

1. (a) Suppose $f'(x) = \sin(x^2)$ and $f(0) = 2$. Use a graph of $f'(x)$ to decide which is larger:
 (i) $f(0)$ or $f(1)$ (ii) $f(2)$ or $f(2.5)$
 (b) Estimate $f(b)$ for $b = 0, 1, 2, 3$.
2. The graph of a derivative $f'(x)$ is shown in Figure 5.64. Fill in the table of values for $f(x)$ given that $f(0) = 2$.

x	0	1	2	3	4	5	6
$f(x)$	2						

Figure 5.64: Graph of f' , not f

3. Figure 5.65 shows f . If $F' = f$ and $F(0) = 0$, find $F(b)$ for $b = 1, 2, 3, 4, 5, 6$.

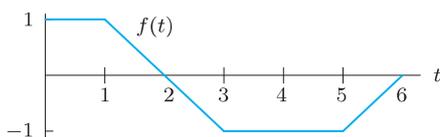


Figure 5.65

Find the area of the regions in Exercises 4–11.

4. Under $y = e^x$ and above $y = 1$ for $0 \leq x \leq 2$.
 5. Under $y = 5 \ln(2x)$ and above $y = 3$ for $3 \leq x \leq 5$.
 6. Between $y = x^2$ and $y = x^3$ for $0 \leq x \leq 1$.
 7. Between $y = x^{1/2}$ and $y = x^{1/3}$ for $0 \leq x \leq 1$.
 8. Between $y = \sin x + 2$ and $y = 0.5$ for $6 \leq x \leq 10$.
 9. Between $y = \cos t$ and $y = \sin t$ for $0 \leq t \leq \pi$.
 10. Between $y = e^{-x}$ and $y = 4(x - x^2)$.
 11. Between $y = e^{-x}$ and $y = \ln x$ for $1 \leq x \leq 2$.

In Exercises 12–17, let $f(t) = F'(t)$. Write the integral $\int_a^b f(t) dt$ and evaluate it using the Fundamental Theorem of Calculus.

12. $F(t) = t^2$; $a = 1, b = 3$
 13. $F(t) = 3t^2 + 4t$; $a = 2, b = 5$
 14. $F(t) = \ln t$; $a = 1, b = 5$
 15. $F(t) = \sin t$; $a = 0, b = \pi/2$
 16. $F(t) = 7 \cdot 4^t$; $a = 2, b = 3$
 17. $F(t) = \tan t$; $a = 0, b = \pi$

Problems

18. (a) If $F(t) = t(\ln t) - t$, find $F'(t)$.
 (b) Find $\int_{10}^{12} \ln t dt$ two ways:
 (i) Numerically.
 (ii) Using the Fundamental Theorem of Calculus.
19. (a) If $F(x) = e^{x^2}$, find $F'(x)$.
 (b) Find $\int_0^1 2xe^{x^2} dx$ two ways:
 (i) Numerically.
 (ii) Using the Fundamental Theorem of Calculus.
20. (a) If $F(x) = \sin x$, find $F'(x)$.
 (b) Find $\int_0^{\pi/2} \cos x dx$ two ways:
 (i) Numerically.
 (ii) Using the Fundamental Theorem of Calculus.

Let $\int_a^b f(x) dx = 8$, $\int_a^b (f(x))^2 dx = 12$, $\int_a^b g(t) dt = 2$, and $\int_a^b (g(t))^2 dt = 3$. Find the integrals in Problems 21–26.

21. $\int_a^b (f(x) + g(x)) dx$ 22. $\int_a^b cf(z) dz$

23. $\int_a^b ((f(x))^2 - (g(x))^2) dx$

24. $\int_a^b (f(x))^2 dx - (\int_a^b f(x) dx)^2$

25. $\int_a^b (c_1g(x) + c_2f(x))^2 dx$

26. $\int_{a+5}^{b+5} f(x-5) dx$

27. In Exercises 6 and 7 we calculated the areas between $y = x^2$ and $y = x^3$ and between $y = x^{1/2}$ and $y = x^{1/3}$ on $0 \leq x \leq 1$. Explain why you would expect these two areas to be equal.

In Problems 28–31, find $\int_2^5 f(x) dx$.

28. $f(x)$ is odd and $\int_{-2}^5 f(x) dx = 8$

29. $f(x)$ is even, $\int_{-2}^2 f(x) dx = 6$, and $\int_{-5}^5 f(x) dx = 14$

30. $\int_2^5 (3f(x) + 4) dx = 18$

31. $\int_2^4 2f(x) dx = 8$ and $\int_5^4 f(x) dx = 1$

32. (a) Using a graph, decide if the area under $y = e^{-x^2/2}$ between 0 and 1 is more or less than 1.
 (b) Find the area.

33. Without any computation, find the values of

(a) $\int_{-2}^2 \sin x \, dx$, (b) $\int_{-\pi}^{\pi} x^{113} \, dx$.

34. Without computation, show that $2 \leq \int_0^2 \sqrt{1+x^3} \, dx \leq 6$.

35. Without calculating the integral, explain why the following statements are false.

(a) $\int_{-2}^1 e^{x^2} \, dx = -3$ (b) $\int_{-1}^1 \left| \frac{\cos(x+2)}{1+\tan^2 x} \right| \, dx = 0$

36. Using Figure 5.66, write $\int_0^3 f(x) \, dx$ in terms of $\int_{-1}^1 f(x) \, dx$ and $\int_1^3 f(x) \, dx$.

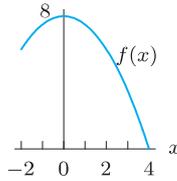


Figure 5.66

37. Using the graph of f in Figure 5.67, arrange the following quantities in increasing order, from least to greatest.

- (i) $\int_0^1 f(x) \, dx$
- (ii) $\int_1^2 f(x) \, dx$
- (iii) $\int_0^2 f(x) \, dx$
- (iv) $\int_2^3 f(x) \, dx$
- (v) $-\int_1^2 f(x) \, dx$
- (vi) The number 0
- (vii) The number 20
- (viii) The number -10

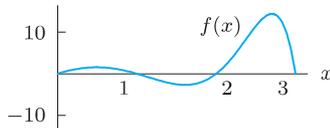


Figure 5.67

38. Find the shaded area in Figure 5.68.

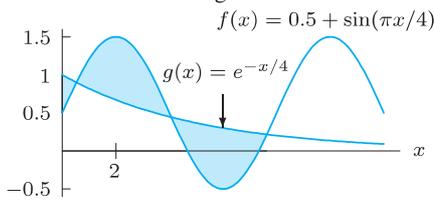


Figure 5.68

39. (a) Let $\int_0^3 f(x) \, dx = 6$. What is the average value of $f(x)$ on the interval $x = 0$ to $x = 3$?
- (b) If $f(x)$ is even, what is $\int_{-3}^3 f(x) \, dx$? What is the average value of $f(x)$ on the interval $x = -3$ to $x = 3$?
- (c) If $f(x)$ is odd, what is $\int_{-3}^3 f(x) \, dx$? What is the average value of $f(x)$ on the interval $x = -3$ to $x = 3$?

40. (a) Use Figure 5.69 to explain why $\int_{-3}^3 x e^{-x^2} \, dx = 0$.
- (b) Find the left-hand sum approximation with $n = 3$ to $\int_0^3 x e^{-x^2} \, dx$. Give your answer to four decimal places.
- (c) Repeat part (b) for $\int_{-3}^0 x e^{-x^2} \, dx$.
- (d) Do your answers to parts (b) and (c) add to 0? Explain.

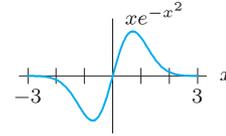


Figure 5.69

41. (a) For any continuous function f , is $\int_1^2 f(x) \, dx + \int_2^3 f(x) \, dx = \int_1^3 f(x) \, dx$?
- (b) For any function f , add the left-hand sum approximation with 10 subdivisions to $\int_1^2 f(x) \, dx$ to the left-hand sum approximation with 10 subdivisions to $\int_2^3 f(x) \, dx$. Do you get the left sum approximations with 10 subdivisions to $\int_1^3 f(x) \, dx$? If not, interpret the result as a different Riemann Sum.

Problems 42–43 concern the graph of f' in Figure 5.70.

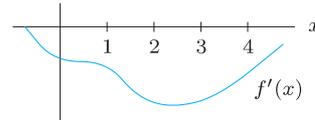


Figure 5.70: Graph of f' , not f

42. Which is greater, $f(0)$ or $f(1)$?
43. List the following in increasing order: $\frac{f(4) - f(2)}{2}$, $f(3) - f(2)$, $f(4) - f(3)$.

For Problems 44–47, mark the following quantities on a copy of the graph of f in Figure 5.71.

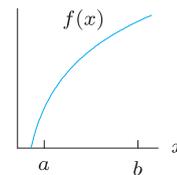


Figure 5.71

44. A length representing $f(b) - f(a)$.
45. A slope representing $\frac{f(b) - f(a)}{b - a}$.
46. An area representing $F(b) - F(a)$, where $F' = f$.
47. A length roughly approximating $\frac{F(b) - F(a)}{b - a}$, where $F' = f$.

48. Figure 5.72 shows the *standard normal distribution* from statistics, which is given by

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Statistics books often contain tables such as the following, which show the area under the curve from 0 to b for various values of b .

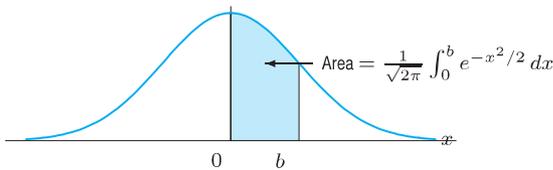


Figure 5.72

b	$\frac{1}{\sqrt{2\pi}} \int_0^b e^{-x^2/2} dx$
1	0.3413
2	0.4772
3	0.4987
4	0.5000

Use the information given in the table and the symmetry of the curve about the y -axis to find:

(a) $\frac{1}{\sqrt{2\pi}} \int_1^3 e^{-x^2/2} dx$ (b) $\frac{1}{\sqrt{2\pi}} \int_{-2}^3 e^{-x^2/2} dx$

49. Use the property $\int_b^a f(x) dx = -\int_a^b f(x) dx$ to show that $\int_a^a f(x) dx = 0$.
50. The average value of $y = v(x)$ equals 4 for $1 \leq x \leq 6$, and equals 5 for $6 \leq x \leq 8$. What is the average value of $v(x)$ for $1 \leq x \leq 8$?

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Definite integral as limit of right or left sums**
- **Fundamental Theorem of Calculus**
- **Interpretations of the definite integral**
Area, total change from rate of change, change in position given velocity, $(b - a) \cdot$ average value.
- **Properties of the definite integral**

Properties involving integrand, properties involving limits, comparison between integrals.

- **Working with the definite integral**
Estimate definite integral from graph, table of values, or formula. Units of the definite integral.
- **Theorems about definite integrals**

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER FIVE

Exercises

- A village wishes to measure the quantity of water that is piped to a factory during a typical morning. A gauge on the water line gives the flow rate (in cubic meters per hour) at any instant. The flow rate is about $100 \text{ m}^3/\text{hr}$ at 6 am and increases steadily to about $280 \text{ m}^3/\text{hr}$ at 9 am.
 - Using only this information, give your best estimate of the total volume of water used by the factory between 6 am and 9 am.
 - How often should the flow rate gauge be read to obtain an estimate of this volume to within 6 m^3 ?
- A car comes to a stop five seconds after the driver applies the brakes. While the brakes are on, the velocities in the table are recorded.
 - Give lower and upper estimates of the distance the car traveled after the brakes were applied.
 - On a sketch of velocity against time, show the lower and upper estimates of part (a).
 - Find the difference between the estimates. Explain how this difference can be visualized on the graph in part (b).

Time since brakes applied (sec)	0	1	2	3	4	5
Velocity (ft/sec)	88	60	40	25	10	0

- You jump out of an airplane. Before your parachute opens you fall faster and faster, but your acceleration decreases as you fall because of air resistance. The table gives your acceleration, a (in m/sec^2), after t seconds.

t	0	1	2	3	4	5
a	9.81	8.03	6.53	5.38	4.41	3.61

- Give upper and lower estimates of your speed at $t = 5$.
- Get a new estimate by taking the average of your upper and lower estimates. What does the concavity of the graph of acceleration tell you about your new estimate?

4. Use Figure 5.73 to estimate $\int_0^{20} f(x) dx$.

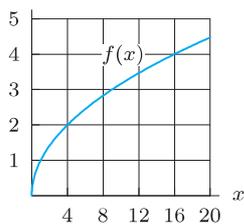


Figure 5.73

5. Using Figure 5.74, estimate $\int_{-3}^5 f(x) dx$.

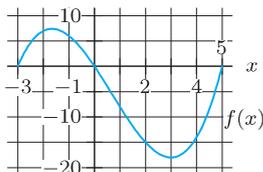


Figure 5.74

6. Using the table, estimate $\int_0^{100} f(t) dt$.

t	0	20	40	60	80	100
$f(t)$	1.2	2.8	4.0	4.7	5.1	5.2

Find the area of the regions in Exercises 7–13.

- Between the parabola $y = 4 - x^2$ and the x -axis.
- Between $y = x^2 - 9$ and the x -axis.
- Under one arch of $y = \sin x$ and above the x -axis.
- Between the line $y = 1$ and one arch of $y = \sin \theta$.
- Between $y = -x^2 + 5x - 4$ and the x -axis, $0 \leq x \leq 3$.
- Between $y = \cos x + 7$ and $y = \ln(x - 3)$, $5 \leq x \leq 7$.

Problems

20. Statisticians sometimes use values of the function

$$F(b) = \int_0^b e^{-x^2} dx.$$

- What is $F(0)$?
 - Does the value of F increase or decrease as b increases? (Assume $b \geq 0$.)
 - Estimate $F(1)$, $F(2)$, and $F(3)$.
21. (a) If $F(t) = \frac{1}{2} \sin^2 t$, find $F'(t)$.
- (b) Find $\int_{0.2}^{0.4} \sin t \cos t dt$ two ways:
- Numerically.

(ii) Using the Fundamental Theorem of Calculus.

- Above the curve $y = -e^x + e^{2(x-1)}$ and below the x -axis, for $x \geq 0$.
- A car going 80 ft/sec (about 55 mph) brakes to a stop in 8 seconds. Its velocity is recorded every 2 seconds and is given in the following table.
 - Give your best estimate of the distance traveled by the car during the 8 seconds.
 - To estimate the distance traveled accurate to within 20 feet, how often should you record the velocity?

t (seconds)	0	2	4	6	8
$v(t)$ (ft/sec)	80	52	28	10	0

15. A car accelerates smoothly from 0 to 60 mph in 10 seconds with the velocity given in Figure 5.75. Estimate how far the car travels during the 10-second period.

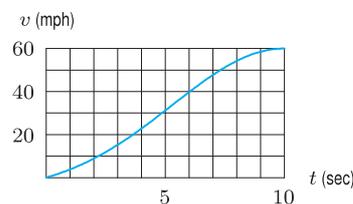


Figure 5.75

- Your velocity is $v(t) = \ln(t^2 + 1)$ ft/sec for t in seconds, $0 \leq t \leq 3$. Estimate the distance traveled during this time.
- Your velocity is $v(t) = \sin(t^2)$ mph for $0 \leq t \leq 1.1$. Estimate the distance traveled during this time.

In Exercises 18–19, let $f(t) = F'(t)$. Write the integral $\int_a^b f(t) dt$ and evaluate it using the Fundamental Theorem of Calculus.

- $F(t) = t^4$, $a = -1$, $b = 1$
- $F(t) = 3t^4 - 5t^3 + 5t$; $a = -2$, $b = 1$

- If $\int_2^5 (2f(x) + 3) dx = 17$, find $\int_2^5 f(x) dx$.
- If $f(x)$ is odd and $\int_{-2}^3 f(x) dx = 30$, find $\int_2^3 f(x) dx$.
- If $f(x)$ is even and $\int_{-2}^2 (f(x) - 3) dx = 8$, find $\int_0^2 f(x) dx$.
- If the average value of f on the interval $2 \leq x \leq 5$ is 4, find $\int_2^5 (3f(x) + 2) dx$.
- Find $\int_{-1}^1 |x| dx$ geometrically.

27. Without any computation, find $\int_{-\pi/4}^{\pi/4} x^3 \cos x^2 dx$.
28. (a) Sketch a graph of $f(x) = \sin(x^2)$ and mark on it the points $x = \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}, \sqrt{4\pi}$.
 (b) Use your graph to decide which of the four numbers

$$\int_0^{\sqrt{n\pi}} \sin(x^2) dx \quad n = 1, 2, 3, 4$$

is largest. Which is smallest? How many of the numbers are positive?

29. Two trains travel along parallel tracks. The velocity, v , of the trains as functions of time t are shown in Figure 5.76.
- (a) Describe in words the trips taken by each train.
 (b) Estimate the ratio of the following quantities for Train A to Train B:
 (i) Maximum velocity (ii) Time traveled
 (iii) Distance traveled

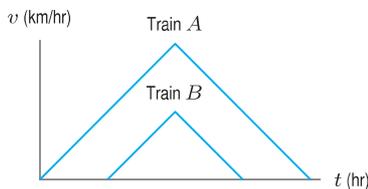


Figure 5.76

30. Annual coal production in the US (in quadrillion BTU per year) is given in the table.⁵ Estimate the total amount of coal produced in the US between 1960 and 1990. If $r = f(t)$ is the rate of coal production t years since 1960, write an integral to represent the 1960–1990 coal production.

Year	1960	1965	1970	1975	1980	1985	1990
Rate	10.82	13.06	14.61	14.99	18.60	19.33	22.46

31. An old rowboat has sprung a leak. Water is flowing into the boat at a rate, $r(t)$, given in the following table.

t minutes	0	5	10	15
$r(t)$ liters/min	12	20	24	16

- (a) Compute upper and lower estimates for the volume of water that has flowed into the boat during the 15 minutes.
 (b) Draw a graph to illustrate the lower estimate.
32. Figure 5.77 gives your velocity during a trip starting from home. Positive velocities take you away from home and negative velocities take you toward home. Where are you

at the end of the 5 hours? When are you farthest from home? How far away are you at that time?

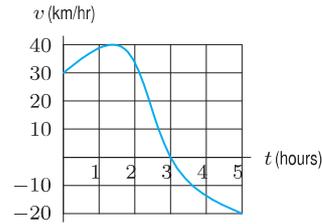


Figure 5.77

33. A bicyclist is pedaling along a straight road for one hour with a velocity v shown in Figure 5.78. She starts out five kilometers from the lake and positive velocities take her toward the lake. [Note: The vertical lines on the graph are at 10 minute ($1/6$ hour) intervals.]

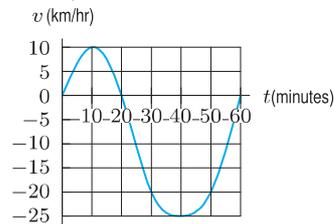


Figure 5.78

- (a) Does the cyclist ever turn around? If so, at what time(s)?
 (b) When is she going the fastest? How fast is she going then? Toward the lake or away?
 (c) When is she closest to the lake? Approximately how close to the lake does she get?
 (d) When is she farthest from the lake? Approximately how far from the lake is she then?

34. Figure 5.79 shows the rate, $f(x)$, in thousands of algae per hour, at which a population of algae is growing, where x is in hours.

- (a) Estimate the average value of the rate over the interval $x = -1$ to $x = 3$.
 (b) Estimate the total change in the population over the interval $x = -3$ to $x = 3$.

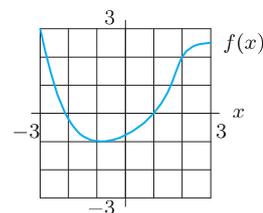


Figure 5.79

⁵World Almanac, 1995.

35. A bar of metal is cooling from 1000°C to room temperature, 20°C . The temperature, H , of the bar t minutes after it starts cooling is given, in $^\circ\text{C}$, by

$$H = 20 + 980e^{-0.1t}.$$

- (a) Find the temperature of the bar at the end of one hour.
 (b) Find the average value of the temperature over the first hour.
 (c) Is your answer to part (b) greater or smaller than the average of the temperatures at the beginning and the end of the hour? Explain this in terms of the concavity of the graph of H .
36. Water is pumped out of a holding tank at a rate of $5 - 5e^{-0.12t}$ liters/minute, where t is in minutes since the pump is started. If the holding tank contains 1000 liters of water when the pump is started, how much water does it hold one hour later?
37. The graph of a continuous function f is given in Figure 5.80. Rank the following integrals in ascending numerical order. Explain your reasons.

- (i) $\int_0^2 f(x) dx$ (ii) $\int_0^1 f(x) dx$
 (iii) $\int_0^2 (f(x))^{1/2} dx$ (iv) $\int_0^2 (f(x))^2 dx$.

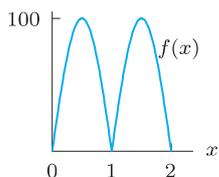


Figure 5.80

38. Using Figure 5.81, list the following integrals in increasing order (from smallest to largest). Which integrals are negative, which are positive? Give reasons.

- I. $\int_a^b f(x) dx$ II. $\int_a^c f(x) dx$ III. $\int_a^e f(x) dx$
 IV. $\int_b^e f(x) dx$ V. $\int_b^c f(x) dx$

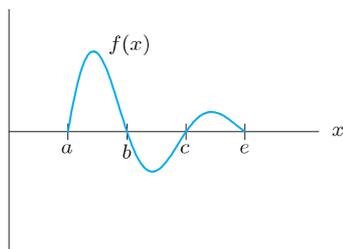


Figure 5.81

39. For the even function f graphed in Figure 5.82:

- (a) Suppose you know $\int_0^2 f(x) dx$. What is $\int_{-2}^2 f(x) dx$?

- (b) Suppose you know $\int_0^5 f(x) dx$ and $\int_2^5 f(x) dx$. What is $\int_0^2 f(x) dx$?
- (c) Suppose you know $\int_{-2}^5 f(x) dx$ and $\int_{-2}^2 f(x) dx$. What is $\int_0^5 f(x) dx$?

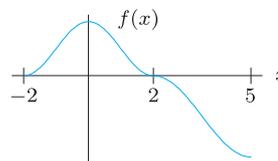


Figure 5.82

40. For the even function f graphed in Figure 5.82:

- (a) Suppose you know $\int_{-2}^2 f(x) dx$ and $\int_0^5 f(x) dx$. What is $\int_2^5 f(x) dx$?
- (b) Suppose you know $\int_{-2}^5 f(x) dx$ and $\int_{-2}^0 f(x) dx$. What is $\int_2^5 f(x) dx$?
- (c) Suppose you know $\int_2^5 f(x) dx$ and $\int_{-2}^5 f(x) dx$. What is $\int_0^2 f(x) dx$?

41. The graphs in Figure 5.83 represent the velocity, v , of a particle moving along the x -axis for time $0 \leq t \leq 5$. The vertical scales of all graphs are the same. Identify the graph showing which particle:

- (a) Has a constant acceleration.
 (b) Ends up farthest to the left of where it started.
 (c) Ends up the farthest from its starting point.
 (d) Experiences the greatest initial acceleration.
 (e) Has the greatest average velocity.
 (f) Has the greatest average acceleration.

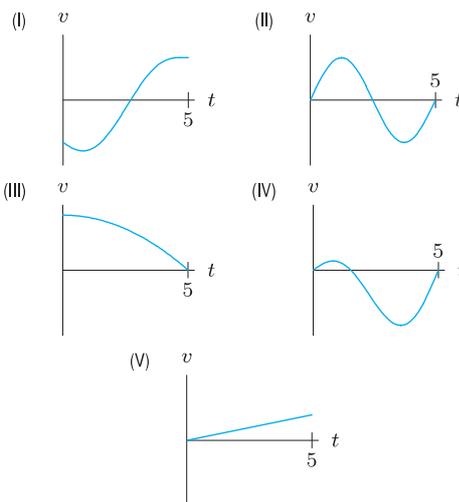


Figure 5.83

42. Assume $w, b_1,$ and b_2 are positive constants, with $b_2 > b_1$. Compute the following integral using its geometric interpretation.

$$\int_0^w \left(b_1 + \frac{b_2 - b_1}{w} x \right) dx$$

43. Water is run into a large tank through a hose at a constant rate. After 5 minutes a hole is opened in the bottom of the tank, and water starts to flow out. Initially the flow rate through the hole is twice as great as the rate through the hose, but as the water level in the tank goes down, the flow rate through the hole decreases; after another 10 minutes the water level in the tank appears to be constant. Plot graphs of the flow rates through the hose and through the hole against time on the same pair of axes. Show how the volume of water in the tank at any time can be interpreted as an area (or the difference between two areas) on the graph. In particular, interpret the steady-state volume of water in the tank.⁶
44. Figure 5.84 shows thrust-time curves for two model rockets. The thrust or force, F , of the engine (in newtons) is plotted against time, t , (in seconds). The total impulse of the rocket's engine is defined as the definite integral of F with respect to t . The total impulse is a measure of the strength of the engine.

- (a) For approximately how many seconds is the thrust of rocket B greater than 10 newtons?
 (b) Estimate the total impulse for model rocket A .
 (c) What are the units for the total impulse calculated in part (b)?
 (d) Which rocket has the largest total impulse?
 (e) Which rocket has the largest maximum thrust?

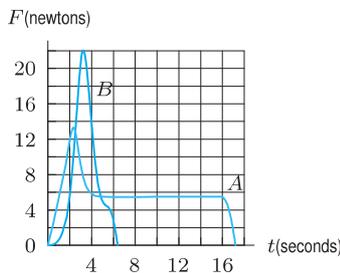


Figure 5.84

45. The Glen Canyon Dam at the top of the Grand Canyon prevents natural flooding. In 1996, scientists decided an artificial flood was necessary to restore the environmental balance. Water was released through the dam at a controlled rate⁷ shown in Figure 5.85. The figure also shows the rate of flow of the last natural flood in 1957.

- (a) At what rate was water passing through the dam in 1996 before the artificial flood?

- (b) At what rate was water passing down the river in the pre-flood season in 1957?
 (c) Estimate the maximum rates of discharge for the 1996 and 1957 floods.
 (d) Approximately how long did the 1996 flood last? How long did the 1957 flood last?
 (e) Estimate how much additional water passed down the river in 1996 as a result of the artificial flood.
 (f) Estimate how much additional water passed down the river in 1957 as a result of the flood.

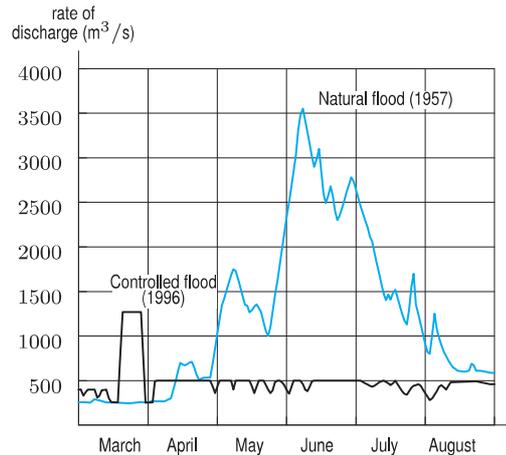


Figure 5.85

46. The Montgolfier brothers (Joseph and Etienne) were eighteenth-century pioneers in the field of hot-air ballooning. Had they had the appropriate instruments, they might have left us a record, like that shown in Figure 5.86, of one of their early experiments. The graph shows their vertical velocity, v , with upward as positive.

- (a) Over what intervals was the acceleration positive? Negative?
 (b) What was the greatest altitude achieved, and at what time?
 (c) At what time was the upward acceleration greatest?
 (d) At what time was the deceleration greatest?
 (e) What might have happened during this flight to explain the answer to part (d)?
 (f) This particular flight ended on top of a hill. How do you know that it did, and what was the height of the hill above the starting point?

⁶From *Calculus: The Analysis of Functions*, by Peter D. Taylor (Toronto: Wall & Emerson, Inc., 1992)

⁷Adapted from M. Collier, R. Webb, E. Andrews, "Experimental Flooding in Grand Canyon" in *Scientific American* (January 1997).

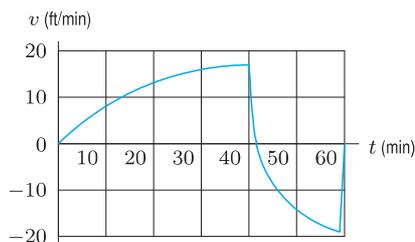


Figure 5.86

47. A mouse moves back and forth in a straight tunnel, attracted to bits of cheddar cheese alternately introduced to and removed from the ends (right and left) of the tunnel. The graph of the mouse's velocity, v , is given in Figure 5.87, with positive velocity corresponding to motion toward the right end. Assuming that the mouse starts ($t = 0$) at the center of the tunnel, use the graph to estimate the time(s) at which:

- The mouse changes direction.
- The mouse is moving most rapidly to the right; to the left.
- The mouse is farthest to the right of center; farthest to the left.
- The mouse's speed (i.e., the magnitude of its velocity) is decreasing.
- The mouse is at the center of the tunnel.

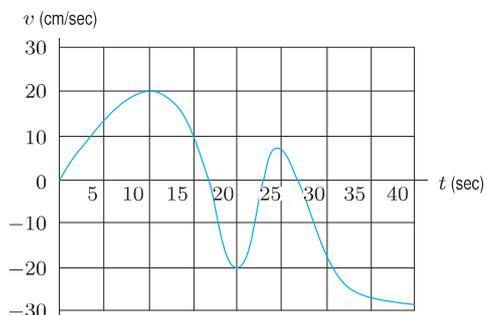


Figure 5.87

48. Pollution is being dumped into a lake at a rate which is increasing at a constant rate from 10 kg/year to 50 kg/year

until a total of 270 kg has been dumped. Sketch a graph of the rate at which pollution is being dumped in the lake against time. How long does it take until 270 kg has been dumped?

49. Using Figure 5.88, list from least to greatest,

- $f'(1)$.
- The average value of $f(x)$ on $0 \leq x \leq a$.
- The average value of the rate of change of $f(x)$, for $0 \leq x \leq a$.
- $\int_0^a f(x) dx$.

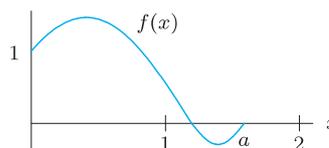


Figure 5.88

50. The number of days a cold-blooded organism, such as an insect, takes to mature depends on the surrounding temperature, H . Each organism has a minimum temperature H_{\min} below which no development takes place.⁸ For an interval of time, Δt , on which the temperature is constant, the increase in maturity of the organism can be measured by the number of degree-days, ΔS , where t is in days and

$$\Delta S = (H - H_{\min})\Delta t.$$

- If H varies with time, so $H = f(t)$, write an integral that represents the total number of degree-days, S , required if development to maturity takes T days.
- An organism, which has $H_{\min} = 15^\circ\text{C}$, requires 125 degree-days to develop to maturity. Estimate the development time if the temperature, $H^\circ\text{C}$, at time t days is in Table 5.9.

Table 5.9

t	1	2	3	4	5	6	7	8	9	10	11	12
H	20	22	27	28	27	31	29	30	28	25	24	26

CAS Challenge Problems

51. Consider the definite integral $\int_0^1 x^4 dx$.
- Write an expression for a right-hand Riemann sum approximation for this integral using n subdivisions. Express each x_i , $i = 1, 2, \dots, n$, in terms of i .
 - Use a computer algebra system to obtain a formula

for the sum you wrote in part (a) in terms of n .

- Take the limit of this expression for the sum as $n \rightarrow \infty$, thereby finding the exact value of this integral.

52. Repeat Problem 51, using the definite integral $\int_0^1 x^5 dx$.

⁸Information from <http://www.ento.vt.edu/~sharov/PopEcol/popecol.html> (Accessed Nov. 18, 2003).

For Problems 53–55, you will write a Riemann sum approximating a definite integral and use a computer algebra system to find a formula for the Riemann sum. By evaluating the limit of this sum as the number of subdivisions approaches infinity, you will obtain the definite integral.

53. (a) Using summation notation, write the left-hand Riemann sum with n subdivisions for $\int_1^2 t \, dt$.
 (b) Use a computer algebra system to find a formula for the Riemann sum.
 (c) Evaluate the limit of the sum as n approaches infinity.
 (d) Calculate directly the area under the graph of $y = t$ between $t = 1$ and $t = 2$, and compare it with your answer to part (c).
54. (a) Using summation notation, write the left-hand Riemann sum with n subdivisions for $\int_1^2 t^2 \, dt$.
 (b) Use a computer algebra system to find a formula for the Riemann sum.
 (c) Evaluate the limit of the sum as n approaches infinity.
 (d) What is the area under the graph of $y = t^2$ between $t = 1$ and $t = 2$?
55. (a) Using summation notation, write the right-hand Rie-

- mann sum with n subdivisions for $\int_0^\pi \sin x \, dx$.
 (b) Use a computer algebra system to find a formula for the Riemann sum. [Note: Not all computer algebra systems can evaluate this sum.]
 (c) Use a computer algebra system to evaluate the limit of the sum as n approaches infinity.
 (d) Confirm your answer to part (c) by calculating the definite integral with the computer algebra system.

In Problems 56–57:

- (a) Use a computer algebra system to compute the given definite integral.
 (b) From your answer to part (a) and the Fundamental Theorem of Calculus, guess a function whose derivative is the integrand. Check your guess using the computer algebra system.

[Hint: Make sure that the constants a , b , and c do not have previously assigned values in your computer algebra system.]

56. $\int_a^b \sin(cx) \, dx$

57. $\int_a^c \frac{x}{1+bx^2} \, dx, \quad b > 0$

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–5 true or false? Give an explanation for your answer.

- The units for an integral of a function $f(x)$ are the same as the units for $f(x)$.
- For an increasing function, the left-hand sum on a given interval with a given number of subdivisions is always less than the right-hand sum.
- For a decreasing function, the difference between the left-hand sum and right-hand sum is halved when the number of subdivisions is doubled.
- For a given function on a given interval, the difference between the left-hand sum and right-hand sum gets smaller as the number of subdivisions gets larger.
- On the interval $a \leq t \leq b$, the integral of the velocity is the total distance traveled from $t = a$ to $t = b$.

In Problems 6–22, are the statements true for all continuous functions $f(x)$ and $g(x)$? Give an explanation for your answer.

- If $\int_0^2 (f(x) + g(x)) \, dx = 10$ and $\int_0^2 f(x) \, dx = 3$, then $\int_0^2 g(x) \, dx = 7$.
- If $\int_0^2 (f(x) + g(x)) \, dx = 10$, then $\int_0^2 f(x) \, dx = 3$ and $\int_0^2 g(x) \, dx = 7$.
- If $\int_0^2 f(x) \, dx = 6$, then $\int_0^4 f(x) \, dx = 12$.

- If $\int_0^2 f(x) \, dx = 6$ and $g(x) = 2f(x)$, then $\int_0^2 g(x) \, dx = 12$.
- If $\int_0^2 f(x) \, dx = 6$ and $h(x) = f(5x)$, then $\int_0^2 h(x) \, dx = 30$.
- If $a = b$, then $\int_a^b f(x) \, dx = 0$.
- If $a \neq b$, then $\int_a^b f(x) \, dx \neq 0$.
- $\int_1^2 f(x) \, dx + \int_2^3 g(x) \, dx = \int_1^3 (f(x) + g(x)) \, dx$.
- $\int_{-1}^1 f(x) \, dx = 2 \int_0^1 f(x) \, dx$.
- $\int_0^2 f(x) \, dx \leq \int_0^3 f(x) \, dx$.
- $\int_0^2 f(x) \, dx = \int_0^2 f(t) \, dt$.
- If $\int_2^6 f(x) \, dx \leq \int_2^6 g(x) \, dx$, then $f(x) \leq g(x)$ for $2 \leq x \leq 6$.
- If $f(x) \leq g(x)$ on the interval $[a, b]$, then the average value of f is less than or equal to the average value of g on the interval $[a, b]$.
- The average value of f on the interval $[0, 10]$ is the average of the average value of f on $[0, 5]$ and the average value of f on $[5, 10]$.
- If $a < c < d < b$, then the average value of f on the interval $[c, d]$ is less than the average value of f on the interval $[a, b]$.

21. Suppose that A is the average value of f on the interval $[1, 4]$ and B is the average value of f on the interval $[4, 9]$. Then the average value of f on $[1, 9]$ is the weighted average $(3/8)A + (5/8)B$.
22. On the interval $[a, b]$, the average value of $f(x) + g(x)$ is the average value of $f(x)$ plus the average value of $g(x)$.
- In Problems 23–25 decide whether the statement is true or false. Justify your answer.
23. The average value of the product, $f(x)g(x)$, of two functions on an interval equals the product of the average values of $f(x)$ and $g(x)$ on the interval.
24. A 4-term left-hand Riemann sum approximation cannot give the exact value of a definite integral.
25. If $f(x)$ is decreasing and $g(x)$ is increasing, then $\int_a^b f(x) dx \neq \int_a^b g(x) dx$.
26. The maximum value taken on by $f(x)$ for $0 \leq x \leq 10$ is 1. In addition $\int_0^{10} f(x) dx = 5$.
27. The maximum value taken on by $f(x)$ for $0 \leq x \leq 10$ is 5. In addition $\int_0^{10} f(x) dx = 1$.
28. Which of the following statements follow directly from the rule
- $$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx?$$
- (a) If $\int_a^b (f(x) + g(x)) dx = 5 + 7$, then $\int_a^b f(x) dx = 5$ and $\int_a^b g(x) dx = 7$.
- (b) If $\int_a^b f(x) dx = \int_a^b g(x) dx = 7$, then $\int_a^b (f(x) + g(x)) dx = 14$.
- (c) If $h(x) = f(x) + g(x)$, then $\int_a^b (h(x) - g(x)) dx = \int_a^b h(x) dx - \int_a^b g(x) dx$.

In Problems 26–27, graph a continuous function $f(x) \geq 0$ on

PROJECTS FOR CHAPTER FIVE

1. The Car and the Truck

A car starts at noon and travels along a straight road with the velocity shown in Figure 5.89. A truck starts at 1 pm from the same place and travels along the same road at a constant velocity of 50 mph.

- (a) How far away is the car when the truck starts?
- (b) How fast is the distance between the car and the truck increasing or decreasing at 3 pm? What is the practical significance (in terms of the distance between the car and the truck) of the fact that the car's velocity is maximized at about 2 pm?
- (c) During the period when the car is ahead of the truck, when is the distance between them greatest, and what is that greatest distance?
- (d) When does the truck overtake the car, and how far have both traveled then?
- (e) Suppose the truck starts at noon. (Everything else remains the same.) Sketch a new graph showing the velocities of both car and truck against time.
- (f) How many times do the two graphs in part (e) intersect? What does each intersection mean in terms of the distance between the two?

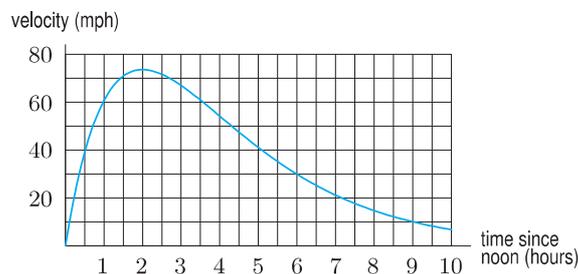


Figure 5.89: Velocity of car starting at noon

2. An Orbiting Satellite

A NASA satellite orbits the earth every 90 minutes. During an orbit, the satellite's electric power comes either from solar array wings, when these are illuminated by the sun, or from batteries. The batteries discharge whenever the satellite uses more electricity than the solar array can provide or whenever the satellite is in the shadow of the earth (where the solar array cannot be used). If the batteries are overused, however, they can be damaged.⁹

You are to determine whether the batteries could be damaged in either of the following operations. You are told that the battery capacity is 50 ampere-hours. If the total battery discharge does not exceed 40% of battery capacity, the batteries will not be damaged.

- (a) Operation 1 is performed by the satellite while orbiting the earth. At the beginning of a given 90-minute orbit, the satellite performs a 15-minute maneuver which requires more current than the solar array can deliver, causing the batteries to discharge. The maneuver causes a sinusoidally varying battery discharge of period 30 minutes with a maximum discharge of ten amperes at 7.5 minutes. For the next 45 minutes the solar array meets the total satellite current demand, and the batteries do not discharge. During the last 30 minutes, the satellite is in the shadow of the earth and the batteries supply the total current demand of 30 amperes.
- The battery current in amperes is a function of time. Plot the function, showing the current in amperes as a function of time for the 90-minute orbit. Write a formula (or formulas) for the battery current function.
 - Calculate the total battery discharge (in units of ampere-hours) for the 90-minute orbit for Operation 1.
 - What is your recommendation regarding the advisability of Operation 1?
- (b) Operation 2 is simulated at NASA's laboratory in Houston. The following graph was produced by the laboratory simulation of the current demands on the battery during the 90-minute orbit required for Operation 2.

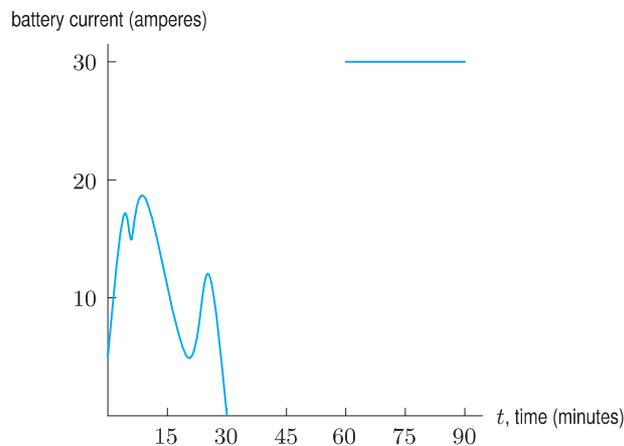


Figure 5.90: Battery discharge simulation graph for Operation 2

- Calculate the total battery discharge (in units of ampere-hours) for the 90-minute orbit for Operation 2.
- What is your recommendation regarding the advisability of Operation 2?

⁹Adapted from Amy C. R. Gerson, "Electrical Engineering: Space Systems," in *She Does Math! Real Life Problems from Women on the Job*, ed. Marla Parker, p. 61 (Washington, DC: Mathematical Association of America, 1995).