Chapter Six

CONSTRUCTING ANTIDERIVATIVES

Contents

6.1 ANTIDERIVATIVES GRAPHICALLY AND NUMERICALLY

The Family of Antiderivatives

If the derivative of F is f, we call F an *antiderivative* of f. For example, since the derivative of x^2 is $2x$, we say that

$$x^2 \text{ is an antiderivative of } 2x.$$

Notice that $2x$ has many antiderivatives, since x^2+1, x^2+2, and x^2+3, all have derivative $2x$. In fact, if C is any constant, we have

$$\frac{d}{dx}(x^2+C) = 2x + 0 = 2x,$$

so any function of the form x^2+C is an antiderivative of $2x$. The function $f(x) = 2x$ has a *family of antiderivatives*.

Let us look at another example. If v is the velocity of a car and s is its position, then $v = ds/dt$ and s is an antiderivative of v. As before, $s+C$ is an antiderivative of v for any constant C. In terms of the car, adding C to s is equivalent to adding C to the odometer reading. Adding a constant to the odometer reading simply means measuring distance from a different point, which does not alter the car's velocity.

Visualizing Antiderivatives Using Slopes

Suppose we have the graph of f', and we want to sketch an approximate graph of f. We are looking for the graph of f whose slope at any point is equal to the value of f' there. Where f' is above the x-axis, f is increasing; where f' is below the x-axis, f is decreasing. If f' is increasing, f is concave up; if f' is decreasing, f is concave down.

Example 1 The graph of f' is given in Figure 6.1. Sketch a graph of f in the cases when $f(0) = 0$ and $f(0) = 1$.

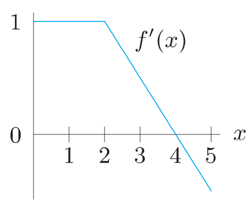


Figure 6.1: Graph of f'

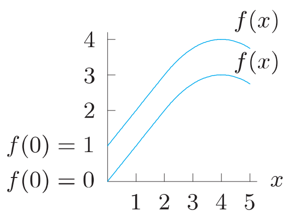


Figure 6.2: Two different f's which have the same derivative f'

Solution For $0 \le x \le 2$, the function f has a constant slope of 1, so the graph of f is a straight line. For $2 \le x \le 4$, the function f is increasing but more and more slowly; it has a maximum at $x = 4$ and decreases thereafter. (See Figure 6.2.) The solutions with $f(0) = 0$ and $f(0) = 1$ start at different points on the vertical axis but have the same shape.

Example 2 Sketch a graph of the antiderivative F of $f(x) = e^{-x^2}$ satisfying $F(0) = 0$.

Solution The graph of $f(x) = e^{-x^2}$ is shown in Figure 6.3. The slope of the antiderivative $F(x)$ is given by $f(x)$. Since $f(x)$ is always positive, the antiderivative $F(x)$ is always increasing. Since $f(x)$ is increasing for negative x, we know that $F(x)$ is concave up for negative x. Since $f(x)$ is decreasing

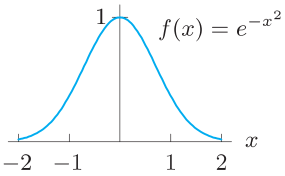


Figure 6.3: Graph of $f(x) = e^{-x^2}$

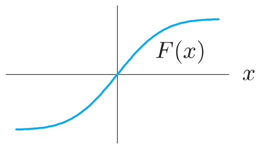


Figure 6.4: An antiderivative $F(x)$ of $f(x) = e^{-x^2}$

for positive x, we know that $F(x)$ is concave down for positive x. Since $f(x) \to 0$ as $x \to \pm\infty$, the graph of $F(x)$ levels off at both ends. See Figure 6.4.

Example 3 For the function f' given in Figure 6.5, sketch a graph of three antiderivative functions f, one with $f(0) = 0$, one with $f(0) = 1$, and one with $f(0) = 2$.

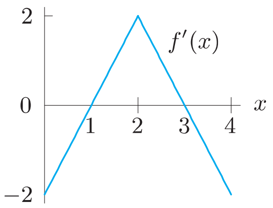


Figure 6.5: Slope function, f'

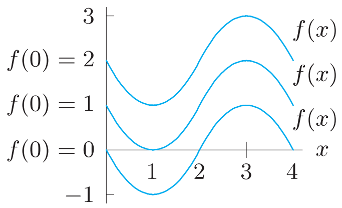


Figure 6.6: Antiderivatives f

Solution To graph f, start at the point on the vertical axis specified by the initial condition and move with slope given by the value of f' in Figure 6.5. Different initial conditions lead to different graphs for f, but for a given x-value they all have the same slope (because the value of f' is the same for each). Thus, the different f curves are obtained from one another by a vertical shift. See Figure 6.6.

- Where f' is positive $(1 < x < 3)$, we see f is increasing; where f' is negative $(0 < x < 1$ or $3 < x < 4)$, we see f is decreasing.
- Where f' is increasing $(0 < x < 2)$, we see f is concave up; where f' is decreasing $(2 < x < 4)$, we see f is concave down.
- Where $f' = 0$, we see f has a local maximum at $x = 3$ and a local minimum at $x = 1$.
- Where f' has a maximum $(x = 2)$, we see f has a point of inflection.

Computing Values of an Antiderivative Using the Fundamental Theorem

A graph of f' shows where f is increasing and where f is decreasing. We can calculate the actual value of the function f by using the Fundamental Theorem of Calculus (Theorem 5.1 on page 272): If f' is continuous, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Example 4 Figure 6.7 is the graph of the derivative $f'(x)$ of a function $f(x)$. It is given that $f(0) = 100$. Sketch the graph of $f(x)$, showing all critical points and inflection points of f and giving their coordinates.

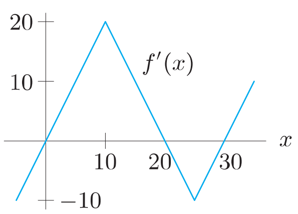


Figure 6.7: Graph of derivative

Solution

The critical points of f occur at $x = 0$, $x = 20$, and $x = 30$, where $f'(x) = 0$. The inflection points of f occur at $x = 10$ and $x = 25$, where $f'(x)$ has a maximum or minimum. To find the coordinates of the critical points and inflection points of f, we evaluate $f(x)$ for $x = 0, 10, 20, 25, 30$. Using the Fundamental Theorem, we can express the values of $f(x)$ in terms of definite integrals. We evaluate the definite integrals using the areas of triangular regions under the graph of $f'(x)$, remembering that areas below the x-axis are subtracted. (See Figure 6.8.)

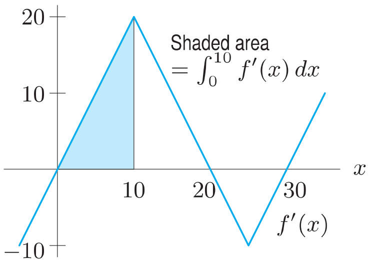


Figure 6.8: Finding $f(10) = f(0) + \int_0^{10} f'(x)\,dx$

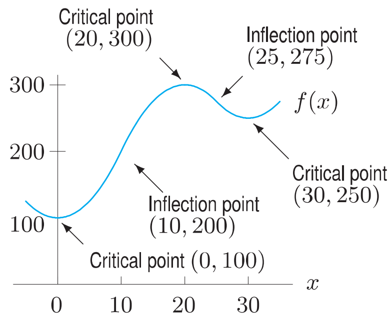


Figure 6.9: Graph of $f(x)$

Since $f(0) = 100$, the Fundamental Theorem gives us the following values of f, which are marked in Figure 6.9.

$$f(10) = f(0) + \int_0^{10} f'(x)\,dx = 100 + (\text{Shaded area in Figure 6.8}) = 100 + \frac{1}{2}(10)(20) = 200,$$

$$f(20) = f(10) + \int_{10}^{20} f'(x)\,dx = 200 + \frac{1}{2}(10)(20) = 300,$$

$$f(25) = f(20) + \int_{20}^{25} f'(x)\,dx = 300 - \frac{1}{2}(5)(10) = 275,$$

$$f(30) = f(25) + \int_{25}^{30} f'(x)\,dx = 275 - \frac{1}{2}(5)(10) = 250.$$

Example 5

Suppose $F'(t) = t\cos t$ and $F(0) = 2$. Find $F(b)$ at the points $b = 0, 0.1, 0.2, \ldots, 1.0$.

Solution

We apply the Fundamental Theorem with $f(t) = t\cos t$ and $a = 0$ to get values for $F(b)$:

$$F(b) - F(0) = \int_0^b F'(t)\,dt = \int_0^b t\cos t\,dt.$$

Since $F(0) = 2$, we have

$$F(b) = 2 + \int_0^b t\cos t\,dt.$$

Calculating the definite integral $\int_0^b t\cos t\,dt$ numerically for $b = 0, 0.1, 0.2, \ldots, 1.0$ gives the values for F in Table 6.1:

Table 6.1 *Approximate values for F*

b	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F(b)$	2.000	2.005	2.020	2.044	2.077	2.117	2.164	2.216	2.271	2.327	2.382

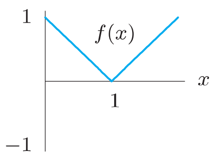
Notice that $F(b)$ appears to be increasing between $b = 0$ and $b = 1$. This could have been predicted from the fact that $t\cos t$, the derivative of $F(t)$, is positive for t between 0 and 1.

Exercises and Problems for Section 6.1

Exercises

In Exercises 1–4, sketch two functions F such that $F' = f$. In one case let $F(0) = 0$ and in the other, let $F(0) = 1$.

1.



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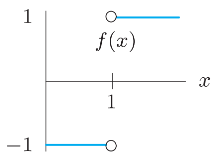

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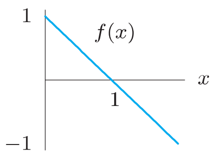

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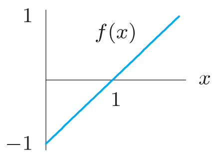


5. Use Figure 6.10 and the fact that $P = 2$ when $t = 0$ to find values of P when $t = 1,\ 2,\ 3,\ 4$ and 5.

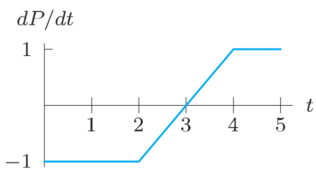


Figure 6.10

6. Given the values of the derivative $f'(x)$ in the table and that $f(0) = 100$, estimate $f(x)$ for $x = 2, 4, 6$.

x	0	2	4	6
$f'(x)$	10	18	23	25

7. Estimate $f(x)$ for $x = 2,\ 4,\ 6$, using the given values of $f'(x)$ and the fact that $f(0) = 50$.

x	0	2	4	6
$f'(x)$	17	15	10	2

8. **(a)** Using Figure 6.11, estimate $\int_0^7 f(x)dx$.
(b) If F is an antiderivative of the same function f and $F(0) = 25$, estimate $F(7)$.

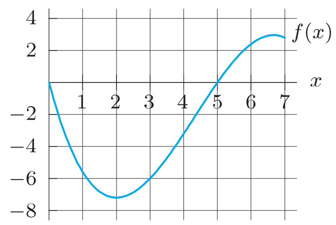
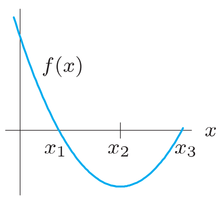


Figure 6.11

Problems

In Problems 9–12, sketch two functions F with $F'(x) = f(x)$. In one, let $F(0) = 0$; in the other, let $F(0) = 1$. Mark x_1, x_2, and x_3 on the x-axis of your graph. Identify local maxima, minima, and inflection points of $F(x)$.

9.



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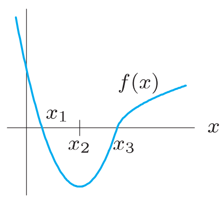

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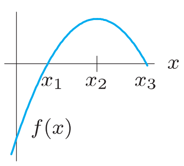
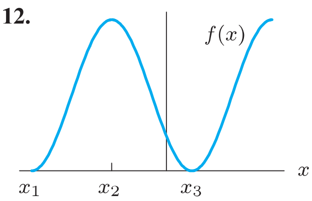


12.



13. A particle moves back and forth along the x-axis. Figure 6.12 approximates the velocity of the particle as a function of time. Positive velocities represent movement to the right and negative velocities represent movement to the left. The particle starts at the point $x = 5$. Graph the distance of the particle from the origin, with distance measured in kilometers and time in hours.

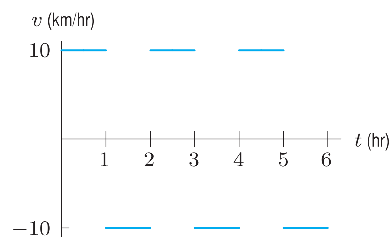


Figure 6.12

14. Assume f' is given by the graph in Figure 6.13. Suppose f is continuous and that $f(3) = 0$.

(a) Sketch a graph of f.
(b) Find $f(0)$ and $f(7)$.
(c) Find $\int_0^7 f'(x)\,dx$ in two different ways.

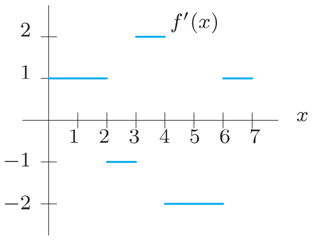


Figure 6.13

15. Use Figure 6.14 and the fact that $F(2) = 3$ to sketch the graph of $F(x)$. Label the values of at least four points.

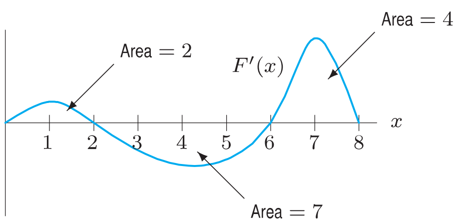


Figure 6.14

16. Using Figure 6.15, sketch a graph of an antiderivative $G(t)$ of $g(t)$ satisfying $G(0) = 5$. Label each critical point of $G(t)$ with its coordinates.

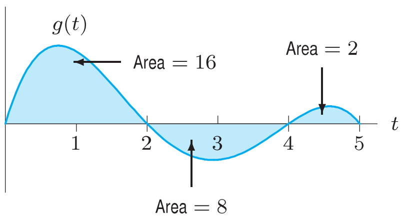


Figure 6.15

17. Using the graph of g' in Figure 6.16 and the fact that $g(0) = 50$, sketch the graph of $g(x)$. Give the coordinates of all critical points and inflection points of g.

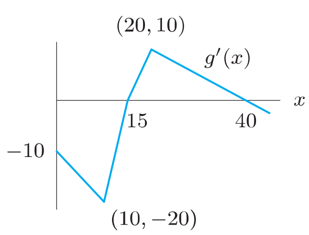


Figure 6.16

18. The vertical velocity of a cork bobbing up and down on the waves in the sea is given by Figure 6.17. Upward is considered positive. Describe the motion of the cork at each of the labeled points. At which point(s), if any, is the acceleration zero? Sketch a graph of the height of the cork above the sea floor as a function of time.

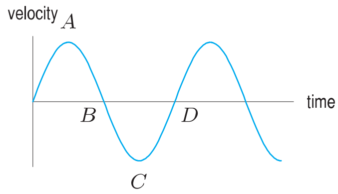


Figure 6.17

19. Figure 6.18 shows the rate of change of the concentration of adrenaline, in micrograms per milliliter per minute, in a person's body. Sketch a graph of the concentration of adrenaline, in micrograms per milliliter, in the body as a function of time, in minutes.

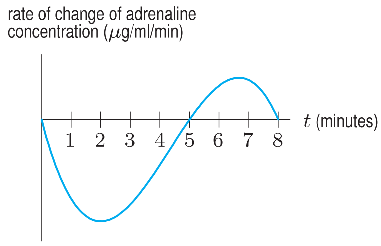


Figure 6.18

20. Urologists are physicians who specialize in the health of the bladder. In a common diagnostic test, urologists monitor the emptying of the bladder using a device that produces two graphs. In one of the graphs the flow rate (in milliliters per second) is measured as a function of time (in seconds). In the other graph, the volume emptied from the bladder is measured (in milliliters) as a function of time (in seconds). See Figure 6.19.

(a) Which graph is the flow rate and which is the volume?
(b) Which one of these graphs is an antiderivative of the other?

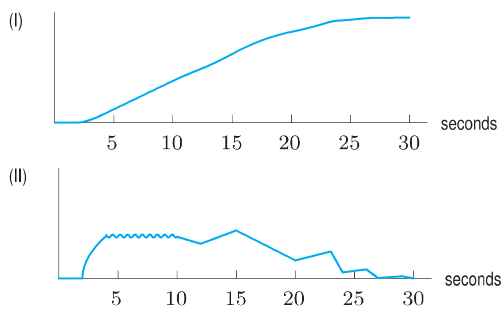


Figure 6.19

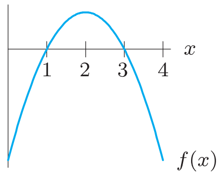
In Problems 21–22, a graph of f is given. Let $F'(x) = f(x)$.

(a) What are the critical points of $F(x)$?

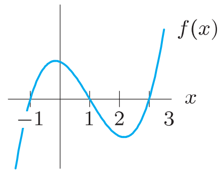
(b) Which critical points are local maxima, which are local minima, and which are neither?

(c) Sketch a possible graph of $F(x)$.

21.



22.



23. Use a graph of $f(x) = 2\sin(x^2)$ to determine where an antiderivative, F, of this function reaches its maximum on $0 \le x \le 3$. If $F(1) = 5$, find the maximum value attained by F.

24. The graph of f'' is given in Figure 6.20. Draw graphs of f and f', assuming both go through the origin, and use them to decide at which of the labeled x-values:

(a) $f(x)$ is greatest.
(b) $f(x)$ is least.
(c) $f'(x)$ is greatest.
(d) $f'(x)$ is least.
(e) $f''(x)$ is greatest.
(f) $f''(x)$ is least.

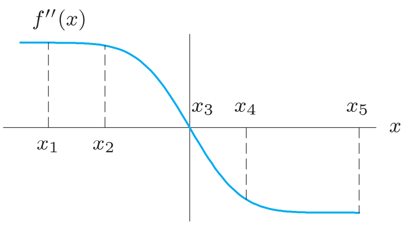


Figure 6.20: Graph of f''

25. Two functions, $f(x)$ and $g(x)$, are shown in Figure 6.21. Let F and G be antiderivatives of f and g, respectively. On the same axes, sketch graphs of the antiderivatives $F(x)$ and $G(x)$ satisfying $F(0) = 0$ and $G(0) = 0$. Compare F and G, including a discussion of zeros and x- and y-coordinates of critical points.

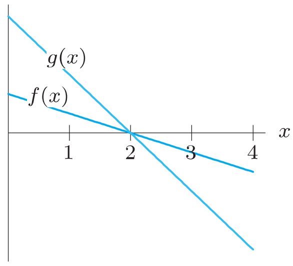


Figure 6.21

26. The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph in Figure 6.22 represents the flow of water in and out of the Quabbin Reservoir throughout 2007.

(a) Sketch a graph of the quantity of water in the reservoir, as a function of time.

(b) When, in the course of 2007, was the quantity of water in the reservoir largest? Smallest? Mark and label these points on the graph you drew in part (a).

(c) When was the quantity of water increasing most rapidly? Decreasing most rapidly? Mark and label these times on both graphs.

(d) By July 2008 the quantity of water in the reservoir was about the same as in January 2007. Draw plausible graphs for the flow into and the flow out of the reservoir for the first half of 2008.

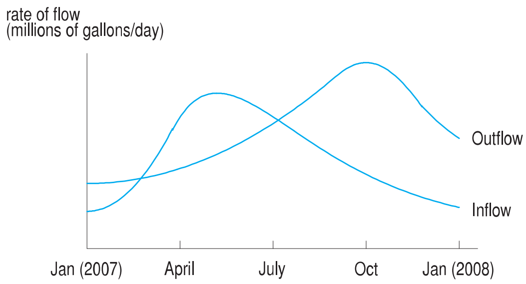


Figure 6.22

6.2 CONSTRUCTING ANTIDERIVATIVES ANALYTICALLY

What Is an Antiderivative of $f(x) = 0$?

A function whose derivative is zero everywhere on an interval must have a horizontal tangent line at every point of its graph, and the only way this can happen is if the function is constant. Alternatively, if we think of the derivative as a velocity, and if the velocity is always zero, then the object is standing still; the position function is constant. A rigorous proof of this result using the definition of the derivative is surprisingly subtle. (See the Constant Function Theorem on page 165.)

> If $F'(x) = 0$ on an interval, then $F(x) = C$ on this interval, for some constant C.

What Is the Most General Antiderivative of f ?

We know that if a function f has an antiderivative F, then it has a family of antiderivatives of the form $F(x) + C$, where C is any constant. You might wonder if there are any others. To decide, suppose that we have two functions F and G with $F' = f$ and $G' = f$: that is, F and G are both antiderivatives of the same function f. Since $F' = G'$ we have $(F-G)' = 0$. But this means that we must have $F - G = C$, so $F(x) = G(x) + C$, where C is a constant. Thus, any two antiderivatives of the same function differ only by a constant.

> If F and G are both antiderivatives of f on an interval, then $F(x) = G(x) + C$.

The Indefinite Integral

All antiderivatives of $f(x)$ are of the form $F(x) + C$. We introduce a notation for the general antiderivative that looks like the definite integral without the limits and is called the *indefinite integral*:

$$\int f(x)\,dx = F(x) + C.$$

It is important to understand the difference between

$$\int_a^b f(x)\,dx \qquad \text{and} \qquad \int f(x)\,dx.$$

The first is a number and the second is a family of *functions*. The word "integration" is frequently used for the process of finding the antiderivative as well as of finding the definite integral. The context usually makes clear which is intended.

What is an Antiderivative of $f(x) = k$?

If k is a constant, the derivative of kx is k, so we have

An antiderivative of k is kx.

Using the indefinite integral notation, we have

> If k is constant,
>
> $$\int k\,dx = kx + C.$$

Finding Antiderivatives

Finding antiderivatives of functions is like taking square roots of numbers: if we pick a number at random, such as 7 or 493, we may have trouble finding its square root without a calculator. But if we happen to pick a number such as 25 or 64, which we know is a perfect square, then we can find its square root exactly. Similarly, if we pick a function which we recognize as a derivative, then we can find its antiderivative easily.

For example, to find an antiderivative of $f(x) = x$, notice that $2x$ is the derivative of x^2; this tells us that x^2 is an antiderivative of $2x$. If we divide by 2, then we guess that

$$\text{An antiderivative of } x \text{ is } \frac{x^2}{2}.$$

To check this statement, take the derivative of $x^2/2$:

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2} \cdot \frac{d}{dx} x^2 = \frac{1}{2} \cdot 2x = x.$$

What about an antiderivative of x^2? The derivative of x^3 is $3x^2$, so the derivative of $x^3/3$ is $3x^2/3 = x^2$. Thus,

$$\text{An antiderivative of } x^2 \text{ is } \frac{x^3}{3}.$$

The pattern looks like

$$\text{An antiderivative of } x^n \text{ is } \frac{x^{n+1}}{n+1}.$$

(We assume $n \neq -1$, or we would have $x^0/0$, which does not make sense.) It is easy to check this formula by differentiation:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{n+1} = x^n.$$

In indefinite integral notation, we have shown that

$$\boxed{\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.}$$

What about when $n = -1$? In other words, what is an antiderivative of $1/x$? Fortunately, we know a function whose derivative is $1/x$, namely, the natural logarithm. Thus, since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

we know that

$$\int \frac{1}{x} \, dx = \ln x + C, \quad \text{for } x > 0.$$

If $x < 0$, then $\ln x$ is not defined, so it can't be an antiderivative of $1/x$. In this case, we can try $\ln(-x)$:

$$\frac{d}{dx} \ln(-x) = (-1)\frac{1}{-x} = \frac{1}{x}$$

so

$$\int \frac{1}{x} \, dx = \ln(-x) + C, \quad \text{for } x < 0.$$

This means $\ln x$ is an antiderivative of $1/x$ if $x > 0$, and $\ln(-x)$ is an antiderivative of $1/x$ if $x < 0$. Since $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$ we can collapse these two formulas into:

$$\text{An antiderivative of } \frac{1}{x} \text{ is } \ln|x|.$$

Therefore

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C.}$$

Since the exponential function is its own derivative, it is also its own antiderivative; thus

$$\boxed{\int e^x\,dx = e^x + C.}$$

Also, antiderivatives of the sine and cosine are easy to guess. Since

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x,$$

we get

$$\boxed{\int \cos x\,dx = \sin x + C} \qquad \text{and} \qquad \boxed{\int \sin x\,dx = -\cos x + C.}$$

Example 1 Find $\int (3x + x^2)\,dx$.

Solution We know that $x^2/2$ is an antiderivative of x and that $x^3/3$ is an antiderivative of x^2, so we expect

$$\int (3x + x^2)\,dx = 3\left(\frac{x^2}{2}\right) + \frac{x^3}{3} + C.$$

You should always check your antiderivatives by differentiation—it's easy to do. Here

$$\frac{d}{dx}\left(\frac{3}{2}x^2 + \frac{x^3}{3} + C\right) = \frac{3}{2}\cdot 2x + \frac{3x^2}{3} = 3x + x^2.$$

The preceding example illustrates that the sum and constant multiplication rules of differentiation work in reverse:

Theorem 6.1: Properties of Antiderivatives: Sums and Constant Multiples

In indefinite integral notation,

1. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
2. $\int cf(x)\,dx = c\int f(x)\,dx.$

In words,

1. An antiderivative of the sum (or difference) of two functions is the sum (or difference) of their antiderivatives.
2. An antiderivative of a constant times a function is the constant times an antiderivative of the function.

Example 2 Find $\int (\sin x + 3\cos x)\, dx$.

Solution We break the antiderivative into two terms:

$$\int (\sin x + 3\cos x)\, dx = \int \sin x\, dx + 3\int \cos x\, dx = -\cos x + 3\sin x + C.$$

Check by differentiating:

$$\frac{d}{dx}(-\cos x + 3\sin x + C) = \sin x + 3\cos x.$$

Using Antiderivatives to Compute Definite Integrals

As we saw in Section 5.3, the Fundamental Theorem of Calculus gives us a way of calculating definite integrals. Denoting $F(b) - F(a)$ by $F(x)\big|_a^b$, the theorem says that if $F' = f$ and f is continuous, then

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b = F(b) - F(a),$$

To find $\int_a^b f(x)\, dx$, we first find F, and then calculate $F(b) - F(a)$. This method of computing definite integrals gives an exact answer. However, the method only works in situations where we can find the antiderivative $F(x)$. This is not always easy; for example, none of the functions we have encountered so far is an antiderivative of $\sin(x^2)$.

Example 3 Compute $\int_1^2 3x^2\, dx$ using the Fundamental Theorem.

Solution Since $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$,

$$\int_1^2 3x^2\, dx = F(x)\Big|_1^2 = F(2) - F(1),$$

gives

$$\int_1^2 3x^2\, dx = x^3\Big|_1^2 = 2^3 - 1^3 = 7.$$

Notice in this example we used the antiderivative x^3, but $x^3 + C$ works just as well because the constant C cancels out:

$$\int_1^2 3x^2\, dx = (x^3 + C)\Big|_1^2 = (2^3 + C) - (1^3 + C) = 7.$$

Example 4 Compute $\int_0^{\pi/4} \frac{1}{\cos^2\theta}\, d\theta$ exactly.

Solution We use the Fundamental Theorem. Since $F(\theta) = \tan\theta$ is an antiderivative of $f(\theta) = 1/\cos^2\theta$, we get

$$\int_0^{\pi/4} \frac{1}{\cos^2\theta}\, d\theta = \tan\theta\Big|_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan(0) = 1.$$

Exercises and Problems for Section 6.2

Exercises

In Exercises 1–16, find an antiderivative.

1. $f(x) = 5$

2. $f(t) = 5t$

3. $f(x) = x^2$

4. $g(t) = t^2 + t$

5. $h(t) = \cos t$

6. $g(z) = \sqrt{z}$

7. $h(z) = \dfrac{1}{z}$

8. $r(t) = \dfrac{1}{t^2}$

9. $g(z) = \dfrac{1}{z^3}$

10. $f(z) = e^z$

11. $g(t) = \sin t$

12. $f(t) = 2t^2 + 3t^3 + 4t^4$

13. $p(t) = t^3 - \dfrac{t^2}{2} - t$

14. $q(y) = y^4 + \dfrac{1}{y}$

15. $f(x) = 5x - \sqrt{x}$

16. $f(t) = \dfrac{t^2+1}{t}$

In Exercises 17–28, find the general antiderivative.

17. $f(t) = 6t$

18. $h(x) = x^3 - x$

19. $f(x) = x^2 - 4x + 7$

20. $r(t) = t^3 + 5t - 1$

21. $f(z) = z + e^z$

22. $g(t) = \sqrt{t}$

23. $g(x) = \sin x + \cos x$

24. $h(x) = 4x^3 - 7$

25. $p(t) = 2 + \sin t$

26. $p(t) = \dfrac{1}{\sqrt{t}}$

27. $g(x) = \dfrac{5}{x^3}$

28. $h(t) = \dfrac{7}{\cos^2 t}$

In Exercises 29–36, find an antiderivative $F(x)$ with $F'(x) = f(x)$ and $F(0) = 0$. Is there only one possible solution?

29. $f(x) = 3$

30. $f(x) = 2x$

31. $f(x) = -7x$

32. $f(x) = \dfrac{1}{4}x$

33. $f(x) = x^2$

34. $f(x) = \sqrt{x}$

35. $f(x) = 2 + 4x + 5x^2$

36. $f(x) = \sin x$

Find the indefinite integrals in Exercises 37–52.

37. $\displaystyle\int 5x\,dx$

38. $\displaystyle\int x^3\,dx$

39. $\displaystyle\int \sin\theta\,d\theta$

40. $\displaystyle\int (x^3 - 2)\,dx$

41. $\displaystyle\int \left(t^2 + \frac{1}{t^2}\right) dt$

42. $\displaystyle\int 4\sqrt{w}\,dw$

43. $\displaystyle\int (x^2 + 5x + 8)\,dx$

44. $\displaystyle\int \frac{4}{t^2}\,dt$

45. $\displaystyle\int (4t + 7)\,dt$

46. $\displaystyle\int \cos\theta\,d\theta$

47. $\displaystyle\int 5e^z\,dz$

48. $\displaystyle\int \left(x + \frac{1}{\sqrt{x}}\right) dx$

49. $\displaystyle\int \sin t\,dt$

50. $\displaystyle\int (\pi + x^{11})\,dx$

51. $\displaystyle\int \left(t\sqrt{t} + \frac{1}{t\sqrt{t}}\right) dt$

52. $\displaystyle\int \left(\frac{y^2-1}{y}\right)^2 dy$

In Exercises 53–63, evaluate the definite integrals exactly [as in $\ln(3\pi)$], using the Fundamental Theorem, and numerically [$\ln(3\pi) \approx 2.243$]:

53. $\displaystyle\int_0^3 (x^2 + 4x + 3)\,dx$

54. $\displaystyle\int_1^3 \frac{1}{t}\,dt$

55. $\displaystyle\int_0^{\pi/4} \sin x\,dx$

56. $\displaystyle\int_0^2 3e^x\,dx$

57. $\displaystyle\int_2^5 (x^3 - \pi x^2)\,dx$

58. $\displaystyle\int_0^1 \sin\theta\,d\theta$

59. $\displaystyle\int_1^2 \frac{1+y^2}{y}\,dy$

60. $\displaystyle\int_0^2 \left(\frac{x^3}{3} + 2x\right) dx$

61. $\displaystyle\int_0^{\pi/4} (\sin t + \cos t)\,dt$

62. $\displaystyle\int_0^1 2e^x\,dx$

63. $\displaystyle\int_{-3}^{-1} \frac{2}{r^3}\,dr$

Problems

64. Use the Fundamental Theorem to find the area under $f(x) = x^2$ between $x = 1$ and $x = 4$.

65. Calculate the exact area between the x-axis and the graph of $y = 7 - 8x + x^2$.

66. Find the exact area below the curve $y = x^3(1 - x)$ and above the x-axis.

67. Find the exact area enclosed by the curve $y = x^2(1-x)^2$ and the x-axis.

68. Find the exact area between the curves $y = x^2$ and $x = y^2$.

69. Calculate the exact area above the graph of $y = \sin\theta$ and below the graph of $y = \cos\theta$ for $0 \le \theta \le \pi/4$.

70. Find the exact area between $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$ for $0 \le \theta \le 2\pi$.

71. Find the exact value of the area between the graphs of $y = \cos x$ and $y = e^x$ for $0 \le x \le 1$.

72. Find the exact value of the area between the graphs of $y = \sinh x$, $y = \cosh x$, for $-1 \le x \le 1$.

73. Use the Fundamental Theorem to determine the value of b if the area under the graph of $f(x) = 8x$ between $x = 1$ and $x = b$ is equal to 192. Assume $b > 1$.

74. Find the exact positive value of c if the area between the graph of $y = x^2 - c^2$ and the x-axis is 36.

75. Use the Fundamental Theorem to find the average value of $f(x) = x^2 + 1$ on the interval $x = 0$ to $x = 10$. Illustrate your answer on a graph of $f(x)$.

76. The average value of the function $v(x) = 6/x^2$ on the interval $[1, c]$ is equal to 1. Find the value of c.

77. **(a)** What is the average value of $f(t) = \sin t$ over $0 \le t \le 2\pi$? Why is this a reasonable answer?
(b) Find the average of $f(t) = \sin t$ over $0 \le t \le \pi$.

78. The origin and the point (a, a) are at opposite corners of a square. Calculate the ratio of the areas of the two parts into which the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ divides the square.

79. If A_n is the area between the curves $y = x$ and $y = x^n$, show that $A_n \to \frac{1}{2}$ as $n \to \infty$ and explain this result graphically.

80. **(a)** Explain why you can rewrite x^x as $x^x = e^{x \ln x}$ for $x > 0$.
(b) Use your answer to part (a) to find $\frac{d}{dx}(x^x)$.
(c) Find $\int x^x(1 + \ln x)dx$.
(d) Find $\int_1^2 x^x(1 + \ln x)dx$ exactly using part (c). Check your answer numerically.

81. Gasoline is pumped into a cylindrical tank, standing vertically, at a decreasing rate given at time t minutes by

$$r(t) = 120 - 6t \text{ ft}^3/\text{min} \quad \text{for } 0 \le t \le 10.$$

The tank has radius 5 ft and is empty when $t = 0$. Find the depth of water in the tank at $t = 4$.

82. A store has an inventory of Q units of a product at time $t = 0$. The store sells the product at the steady rate of Q/A units per week, and it exhausts the inventory in A weeks.

(a) Find a formula $f(t)$ for the amount of product in inventory at time t. Graph $f(t)$.
(b) Find the average inventory level during the period $0 \le t \le A$. Explain why your answer is reasonable.

83. For $0 \le t \le 10$ seconds, a car moves along a straight line with velocity

$$v(t) = 2 + 10t \text{ ft/sec.}$$

(a) Graph $v(t)$ and find the total distance the car has traveled between $t = 0$ and $t = 10$ seconds using the formula for the area of a trapezoid.
(b) Find the function $s(t)$ that gives the position of the car as a function of time. Explain the meaning of any new constants.
(c) Use your function $s(t)$ to find the total distance traveled by the car between $t = 0$ and $t = 10$ seconds. Compare with your answer in part (a).
(d) Explain how your answers to parts (a) and (c) relate to the Fundamental Theorem of Calculus.

84. In drilling an oil well, the total cost, C, consists of fixed costs (independent of the depth of the well) and marginal costs, which depend on depth; drilling becomes more expensive, per meter, deeper into the earth. Suppose the fixed costs are 1,000,000 riyals (the riyal is the unit of currency of Saudi Arabia), and the marginal costs are

$$C'(x) = 4000 + 10x \text{ riyals/meter,}$$

where x is the depth in meters. Find the total cost of drilling a well x meters deep.

85. One of the earliest pollution problems brought to the attention of the Environmental Protection Agency (EPA) was the case of the Sioux Lake in eastern South Dakota. For years a small paper plant located nearby had been discharging waste containing carbon tetrachloride (CCl_4) into the waters of the lake. At the time the EPA learned of the situation, the chemical was entering at a rate of 16 cubic yards/year.

The agency ordered the installation of filters designed to slow (and eventually stop) the flow of CCl_4 from the mill. Implementation of this program took exactly three years, during which the flow of pollutant was steady at 16 cubic yards/year. Once the filters were installed, the flow declined. If t is time measured in years since the EPA learned of the situation, between the time the filters were installed and the time the flow stopped, the rate of flow was well approximated by

$$\text{Rate (in cubic yards/year)} = t^2 - 14t + 49.$$

(a) Graph the rate of CCl_4 flow into the lake as a function of time, beginning at the time the EPA first learned of the situation.

(b) How many years elapsed between the time the EPA learned of the situation and the time the pollution flow stopped entirely?

(c) How much CCl_4 entered the waters during the time shown in the graph in part (a)?

6.3 DIFFERENTIAL EQUATIONS

In Chapter 2 we saw that velocity is the derivative of distance and that acceleration is the derivative of velocity. In this section we analyze the motion of an object falling freely under the influence of gravity. This involves going "backward" from acceleration to velocity to position. Chapter 11 investigates such problems in more detail.

Motion With Constant Velocity

Let's briefly consider a familiar problem: An object moving in a straight line with constant velocity. Imagine a car moving at 50 mph. How far does it go in a given time? The answer is given by

$$\text{Distance} = \text{Rate} \times \text{Time}$$

or

$$s = 50t$$

where s is the distance of the car (in miles) from a fixed reference point and t is the time in hours. Alternatively, we can describe the motion by writing the equation

$$\frac{ds}{dt} = 50.$$

This is called a *differential equation* for the *function* s. The solution to this equation is the antiderivative

$$s = 50t + C.$$

The equation $s = 50t + C$ tells us that $s = C$ when $t = 0$. Thus, the constant C represents the initial distance, s_0, of the car from the reference point.

Uniformly Accelerated Motion

Now we consider an object moving with *constant acceleration* along a straight line, or *uniformly accelerated motion*. It has been known since Galileo's time that an object moving under the influence of gravity (ignoring air resistance) has constant acceleration, g. In the most frequently used units, its value is approximately

$$g = 9.8 \text{ m/sec}^2, \quad \text{or} \quad g = 32 \text{ ft/sec}^2.$$

Thus, if v is the upward velocity and t is the time,

$$\frac{dv}{dt} = -g.$$

The negative sign represents the fact that velocity is measured upward, whereas gravity acts downward.

Example 1 A stone is dropped from a 100-foot-high building. Find, as functions of time, its position and velocity. When does it hit the ground, and how fast is it going at that time?

Solution Suppose t is measured in seconds from when the stone was dropped. If we measure distance, s, in feet above the ground, then the velocity, v, is in ft/sec upward, and the acceleration due to gravity is 32 ft/sec^2 downward, so

$$\frac{dv}{dt} = -32.$$

From what we know about antiderivatives, we must have

$$v = -32t + C$$

where C is some constant. Since $v = C$ when $t = 0$, the constant C represents the initial velocity, v_0. The fact that the stone is dropped rather than thrown off the top of the building tells us that the initial velocity is zero, so $v_0 = 0$. Substituting gives

$$0 = -32(0) + C \quad \text{so} \quad C = 0.$$

Thus,

$$v = -32t.$$

But now we can write

$$v = \frac{ds}{dt} = -32t.$$

The general antiderivative of $-32t$ is

$$s = -16t^2 + K,$$

where K is another constant.

Since the stone starts at the top of the building, $s = 100$ when $t = 0$. Substituting gives

$$100 = -16(0^2) + K, \quad \text{so} \quad K = 100,$$

and therefore

$$s = -16t^2 + 100.$$

Thus, we have found both v and s as functions of time.

The stone hits the ground when $s = 0$, so we must solve

$$0 = -16t^2 + 100$$

giving $t^2 = 100/16$ or $t = \pm 10/4 = \pm 2.5$ sec. Since t must be positive, $t = 2.5$ sec. At that time, $v = -32(2.5) = -80$ ft/sec. (The velocity is negative because we are considering up as positive and down as negative.)

Example 2 An object is thrown vertically upward with a speed of 10 m/sec from a height of 2 meters. Find the highest point it reaches and when it hits the ground.

Solution We must find the position as a function of time. In this example, the velocity is in m/sec, so we use $g = 9.8$ m/sec^2. Measuring distance in meters upward from the ground, we have

$$\frac{dv}{dt} = -9.8.$$

As before, v is a function whose derivative is constant, so

$$v = -9.8t + C.$$

Since the initial velocity is 10 m/sec upward, we know that $v = 10$ when $t = 0$. Substituting gives

$$10 = -9.8(0) + C \quad \text{so} \quad C = 10.$$

Thus,

$$v = -9.8t + 10.$$

To find s, we use

$$v = \frac{ds}{dt} = -9.8t + 10$$

and look for a function that has $-9.8t+10$ as its derivative. The general antiderivative of $-9.8t+10$ is

$$s = -4.9t^2 + 10t + K,$$

where K is any constant. To find K, we use the fact that the object starts at a height of 2 meters, so $s = 2$ when $t = 0$. Substituting gives

$$2 = -4.9(0)^2 + 10(0) + K, \quad \text{so} \quad K = 2,$$

and therefore

$$s = -4.9t^2 + 10t + 2.$$

The object reaches its highest point when the velocity is 0, so at that time

$$v = -9.8t + 10 = 0.$$

This occurs when

$$t = \frac{10}{9.8} \approx 1.02 \text{ sec.}$$

When $t = 1.02$ seconds,

$$s = -4.9(1.02)^2 + 10(1.02) + 2 \approx 7.10 \text{ m.}$$

So the maximum height reached is 7.10 meters. The object reaches the ground when $s = 0$:

$$0 = -4.9t^2 + 10t + 2.$$

Solving this using the quadratic formula gives

$$t \approx -0.18 \text{ and } t \approx 2.22 \text{ sec.}$$

Since the time at which the object hits the ground must be positive, $t \approx 2.22$ seconds.

Antiderivatives and Differential Equations

We solved the problem of uniformly accelerated motion by working backward from the derivative of a function to the function itself. If f is a known function, finding the *general solution* to the differential equation

$$\frac{dy}{dx} = f(x)$$

means finding the general antiderivative $y = F(x) + C$ with $F'(x) = f(x)$.

Example 3 Find and graph the general solution of the differential equation

$$\frac{dy}{dx} = \sin x + 2.$$

Solution We are asking for a function whose derivative is $\sin x + 2$. One antiderivative of $\sin x + 2$ is

$$y = -\cos x + 2x.$$

The general solution is therefore

$$y = -\cos x + 2x + C,$$

where C is any constant. Figure 6.23 shows several curves in this family.

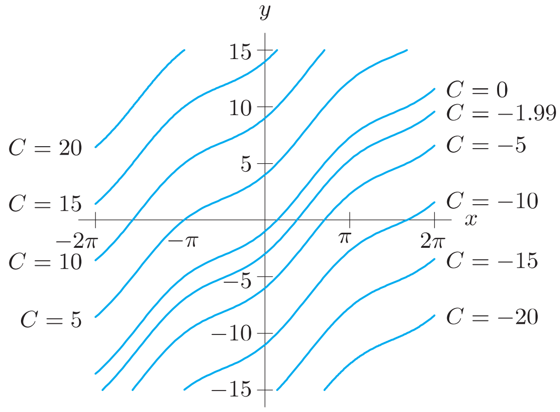


Figure 6.23: Solution curves of $dy/dx = \sin x + 2$

How Can We Pick One Solution to the Differential Equation $\dfrac{dy}{dx} = f(x)$?

Picking one antiderivative is equivalent to selecting a value of C. This requires knowing an extra piece of information, such as the initial velocity or initial position. In general, we pick a particular curve in the family of solutions by specifying that the curve should pass through a given point (x_0, y_0). The differential equation plus an extra condition

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0$$

is called an *initial value problem*. An initial value problem usually has a unique solution. Note that $y(x_0) = y_0$ is shorthand for $y = y_0$ when $x = x_0$.

Example 4 Find the solution of the initial value problem

$$\frac{dy}{dx} = \sin x + 2, \quad y(3) = 5.$$

Solution We have already seen that the general solution to the differential equation is $y = -\cos x + 2x + C$. The initial condition allows us to determine the constant C. Substituting $y(3) = 5$ gives

$$5 = y(3) = -\cos 3 + 2 \cdot 3 + C,$$

so C is given by

$$C = 5 + \cos 3 - 6 \approx -1.99.$$

Thus, the (unique) solution is

$$y = -\cos x + 2x - 1.99.$$

Figure 6.23 shows this particular solution, marked $C = -1.99$.

Exercises and Problems for Section 6.3

Exercises

In Exercises 1–4, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = x^3 + 5$

2. $\dfrac{dy}{dx} = 8x + \dfrac{1}{x}$

3. $\dfrac{dW}{dt} = 4\sqrt{t}$

4. $\dfrac{dr}{dp} = 3\sin p$

5. Show that $y = x + \sin x - \pi$ satisfies the initial value problem

$$\frac{dy}{dx} = 1 + \cos x, \quad y(\pi) = 0.$$

In Exercises 6–9, find the solution of the initial value problem.

6. $\dfrac{dy}{dx} = 6x^2 + 4x, \quad y(2) = 10$

7. $\dfrac{dP}{dt} = 10e^t, \quad P(0) = 25$

8. $\dfrac{ds}{dt} = -32t + 100, \quad s = 50$ when $t = 0$

9. $\dfrac{dq}{dz} = 2 + \sin z, \quad q = 5$ when $z = 0$.

10. Show that $y = xe^{-x} + 2$ is a solution of the initial value problem

$$\frac{dy}{dx} = (1-x)e^{-x}, \quad y(0) = 2.$$

Problems

11. **(a)** Find the general solution of the differential equation $dy/dx = 2x + 1$.
(b) Sketch a graph of at least three solutions.
(c) Find the solution satisfying $y(1) = 5$. Graph this solution with the others from part (b).

12. A tomato is thrown upward from a bridge 25 m above the ground at 40 m/sec.

(a) Give formulas for the acceleration, velocity, and height of the tomato at time t.
(b) How high does the tomato go, and when does it reach its highest point?
(c) How long is it in the air?

13. Figure 6.24 is a graph of

$$f(x) = \begin{cases} -x+1, & \text{for } 0 \le x \le 1; \\ x-1, & \text{for } 1 < x \le 2. \end{cases}$$

(a) Find a function F such that $F' = f$ and $F(1) = 1$.
(b) Use geometry to show the area under the graph of f above the x-axis between $x = 0$ and $x = 2$ is equal to $F(2) - F(0)$.
(c) Use parts (a) and (b) to check the Fundamental Theorem of Calculus.

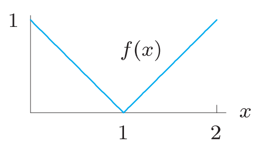


Figure 6.24

14. Ice is forming on a pond at a rate given by

$$\frac{dy}{dt} = k\sqrt{t},$$

where y is the thickness of the ice in inches at time t measured in hours since the ice started forming, and k is a positive constant. Find y as a function of t.

15. A firm's marginal cost function is $MC = 3q^2 + 6q + 9$.

(a) Write a differential equation for the total cost, $C(q)$.
(b) Find the total cost function if the fixed costs are 400.

16. A revenue $R(p)$ is obtained by a farmer from selling grain at price p dollars/unit. The marginal revenue is given by $R'(p) = 25 - 2p$.

(a) Find $R(p)$. Assume the revenue is zero when the price is zero.
(b) For what prices does the revenue increase as the price increases? For what prices does the revenue decrease as price increases?

17. A water balloon launched from the roof of a building at time $t = 0$ has vertical velocity $v(t) = -32t + 40$ feet/sec at time t seconds, with $v > 0$ corresponding to upward motion.

(a) If the roof of the building is 30 feet above the ground, find an expression for the height of the water balloon above the ground at time t.
(b) What is the average velocity of the balloon between $t = 1.5$ and $t = 3$ seconds?
(c) A 6-foot person is standing on the ground. How fast is the water balloon falling when it strikes the person on the top of the head?

18. If a car goes from 0 to 80 mph in six seconds with constant acceleration, what is that acceleration?

19. A car starts from rest at time $t = 0$ and accelerates at $-0.6t + 4$ meters/sec^2 for $0 \le t \le 12$. How long does it take for the car to go 100 meters?

20. A car going 80 ft/sec (about 55 mph) brakes to a stop in five seconds. Assume the deceleration is constant.

(a) Graph the velocity against time, t, for $0 \le t \le 5$ seconds.
(b) Represent, as an area on the graph, the total distance traveled from the time the brakes are applied until the car comes to a stop.
(c) Find this area and hence the distance traveled.
(d) Now find the total distance traveled using antidifferentiation.

21. A 727 jet needs to attain a speed of 200 mph to take off. If it can accelerate from 0 to 200 mph in 30 seconds, how long must the runway be? (Assume constant acceleration.)

22. A car going at 30 ft/sec decelerates at a constant 5 ft/sec^2.

(a) Draw up a table showing the velocity of the car every half second. When does the car come to rest?

(b) Using your table, find left and right sums which estimate the total distance traveled before the car comes to rest. Which is an overestimate, and which is an underestimate?

(c) Sketch a graph of velocity against time. On the graph, show an area representing the distance traveled before the car comes to rest. Use the graph to calculate this distance.

(d) Now find a formula for the velocity of the car as a function of time, and then find the total distance traveled by antidifferentiation. What is the relationship between your answer to parts (c) and (d) and your estimates in part (b)?

23. An object is shot vertically upward from the ground with an initial velocity of 160 ft/sec.

(a) At what rate is the velocity decreasing? Give units.
(b) Explain why the graph of velocity of the object against time (with upward positive) is a line.
(c) Using the starting velocity and your answer to part (b), find the time at which the object reaches the highest point.
(d) Use your answer to part (c) to decide when the object hits the ground.
(e) Graph the velocity against time. Mark on the graph when the object reaches its highest point and when it lands.
(f) Find the maximum height reached by the object by considering an area on the graph.
(g) Now express velocity as a function of time, and find the greatest height by antidifferentiation.

24. A stone thrown upward from the top of a 320-foot cliff at 128 ft/sec eventually falls to the beach below.

(a) How long does the stone take to reach its highest point?
(b) What is its maximum height?
(c) How long before the stone hits the beach?
(d) What is the velocity of the stone on impact?

25. On the moon, the acceleration due to gravity is about 1.6 m/sec^2 (compared to $g \approx 9.8$ m/sec^2 on earth). If you drop a rock on the moon (with initial velocity 0), find formulas for:

(a) Its velocity, $v(t)$, at time t.
(b) The distance, $s(t)$, it falls in time t.

26. **(a)** Imagine throwing a rock straight up in the air. What is its initial velocity if the rock reaches a maximum height of 100 feet above its starting point?
(b) Now imagine being transplanted to the moon and throwing a moon rock vertically upward with the same velocity as in part (a). How high will it go? (On the moon, $g = 5$ ft/sec^2.)

27. A cat, walking along the window ledge of a New York apartment, knocks off a flower pot, which falls to the street 200 feet below. How fast is the flower pot traveling when it hits the street? (Give your answer in ft/sec and in mph, given that 1 ft/sec $=$ 15/22 mph.)

28. An Acura NSX going at 70 mph stops in 157 feet. Find the acceleration, assuming it is constant.

6.4 SECOND FUNDAMENTAL THEOREM OF CALCULUS

Suppose f is an elementary function, that is, a combination of constants, powers of x, $\sin x$, $\cos x$, e^x, and $\ln x$. Then we have to be lucky to find an antiderivative F which is also an elementary function. But if we can't find F as an elementary function, how can we be sure that F exists at all? In this section we learn to use the definite integral to construct antiderivatives.

Construction of Antiderivatives Using the Definite Integral

Consider the function $f(x) = e^{-x^2}$. We would like to find a way of calculating values of its antiderivative, F, which is not an elementary function. However, we know from the Fundamental Theorem of Calculus that

$$F(b) - F(a) = \int_a^b e^{-t^2}\,dt.$$

Setting $a = 0$ and replacing b by x, we have

$$F(x) - F(0) = \int_0^x e^{-t^2}\,dt.$$

Suppose we want the antiderivative that satisfies $F(0) = 0$. Then we get

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

This is a formula for F. For any value of x, there is a unique value for $F(x)$, so F is a function. For any fixed x, we can calculate $F(x)$ numerically. For example,

$$F(2) = \int_0^2 e^{-t^2}\,dt = 0.88208\ldots.$$

Notice that our expression for F is not an elementary function; we have *created* a new function using the definite integral. The next theorem says that this method of constructing antiderivatives works in general. This means that if we define F by

$$F(x) = \int_a^x f(t)\,dt$$

then F must be an antiderivative of f.

Theorem 6.2: Construction Theorem for Antiderivatives

(Second Fundamental Theorem of Calculus) If f is a continuous function on an interval, and if a is any number in that interval, then the function F defined on the interval as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)\,dt.$$

Proof Our task is to show that F, defined by this integral, is an antiderivative of f. We want to show that $F'(x) = f(x)$. By the definition of the derivative,

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}.$$

To gain some geometric insight, let's suppose f is positive and h is positive. Then we can visualize

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad F(x+h) = \int_a^{x+h} f(t)\,dt$$

as areas, which leads to representing

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,dt$$

as a difference of two areas. From Figure 6.25, we see that $F(x+h) - F(x)$ is roughly the area of a rectangle of height $f(x)$ and width h (shaded darker in Figure 6.25), so we have

$$F(x+h) - F(x) \approx f(x)h,$$

hence

$$\frac{F(x+h) - F(x)}{h} \approx f(x).$$

More precisely, we can use Theorem 5.4 on comparing integrals on page 287 to conclude that

$$mh \le \int_x^{x+h} f(t)\,dt \le Mh,$$

where m is the greatest lower bound for f on the interval from x to $x+h$ and M is the least upper bound on that interval. (See Figure 6.26.) Hence

$$mh \le F(x+h) - F(x) \le Mh,$$

so

$$m \le \frac{F(x+h)-F(x)}{h} \le M.$$

Since f is continuous, both m and M approach $f(x)$ as h approaches zero. Thus

$$f(x) \le \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \le f(x).$$

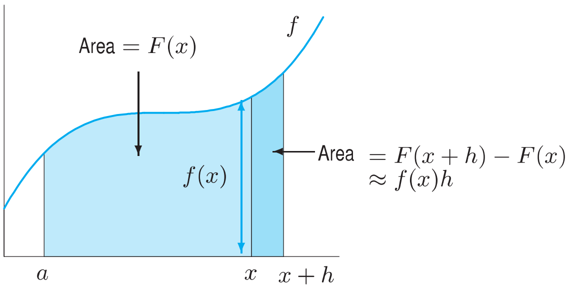


Figure 6.25: $F(x+h) - F(x)$ is area of roughly rectangular region

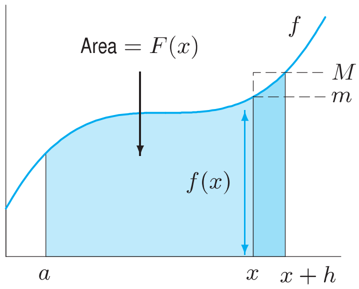


Figure 6.26: Upper and lower bounds for $F(x+h) - F(x)$

Thus both inequalities must actually be equalities, so we have the result we want:

$$f(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = F'(x).$$

Relationship between the Construction Theorem and the Fundamental Theorem of Calculus

If F is constructed as in Theorem 6.2, then we have just shown that $F' = f$. Suppose G is any other antiderivative of f, so $G' = f$, and therefore $F' = G'$. Since the derivative of $F - G$ is zero, the Constant Function Theorem on page 165 tells us that $F - G$ is a constant, so $F(x) = G(x) + C$.

Since we know $F(a) = 0$ (by the definition of F), we can write

$$\int_a^b f(t)\,dt = F(b) = F(b) - F(a) = (G(b) + C) - (G(a) + C) = G(b) - G(a).$$

This result, that the definite integral $\int_a^b f(t)\,dt$ can be evaluated using any antiderivative of f, is the (First) Fundamental Theorem of Calculus. Thus, we have shown that the First Fundamental Theorem of Calculus can be obtained from the Construction Theorem (the Second Fundamental Theorem of Calculus).

Using the Construction Theorem for Antiderivatives

The construction theorem enables us to write down antiderivatives of functions that do not have elementary antiderivatives. For example, an antiderivative of $(\sin x)/x$ is

$$F(x) = \int_0^x \frac{\sin t}{t}\,dt.$$

Notice that F is a function; we can calculate its values to any degree of accuracy. This function already has a name: it is called the *sine-integral*, and it is denoted $\mathrm{Si}(x)$.

Example 1 Construct a table of values of $\mathrm{Si}(x)$ for $x = 0, 1, 2, 3$.

Solution Using numerical methods, we calculate the values of $\mathrm{Si}(x) = \int_0^x \sin t/t\, dt$ given in Table 6.2. Since the integrand is undefined at $t = 0$, we took the lower limit as 0.00001 instead of 0.

Table 6.2 *A table of values of* $\mathrm{Si}(x)$

x	0	1	2	3
$\mathrm{Si}(x)$	0	0.95	1.61	1.85

The reason the sine-integral has a name is that some scientists and engineers use it all the time (for example, in optics). For them, it is just another common function like sine or cosine. Its derivative is given by

$$\frac{d}{dx}\,\mathrm{Si}(x) = \frac{\sin x}{x}.$$

Example 2 Find the derivative of $x\,\mathrm{Si}(x)$.

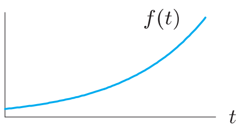
Solution Using the product rule,

$$\begin{aligned}\frac{d}{dx}\left(x\,\mathrm{Si}(x)\right) &= \left(\frac{d}{dx}x\right)\mathrm{Si}(x) + x\left(\frac{d}{dx}\,\mathrm{Si}(x)\right)\\ &= 1\cdot \mathrm{Si}(x) + x\frac{\sin x}{x}\\ &= \mathrm{Si}(x) + \sin x.\end{aligned}$$

Exercises and Problems for Section 6.4

Exercises

In Exercises 1–4, let $F(x) = \int_0^x f(t)\,dt$. Graph $F(x)$ as a function of x.

1.


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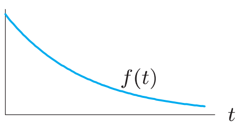

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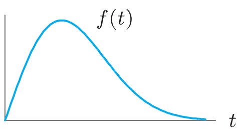

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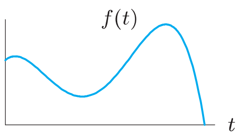


Find the derivatives in Exercises 5–12.

5. $\displaystyle \frac{d}{dx}\int_1^x (1+t)^{200}\,dt$

6. $\displaystyle \frac{d}{dx}\int_2^x \ln(t^2+1)\,dt$

7. $\displaystyle \frac{d}{dx}\int_0^x \cos(t^2)\,dt$

8. $\displaystyle \frac{d}{dt}\int_4^t \sin(\sqrt{x})\,dx$

9. $\displaystyle \frac{d}{dx}\int_{0.5}^x \arctan(t^2)\,dt$

10. $\displaystyle \frac{d}{dt}\int_t^\pi \cos(z^3)\,dz$

11. $\displaystyle \frac{d}{dx}\int_x^1 \ln t\,dt$

12. $\displaystyle \frac{d}{dx}\left[\mathrm{Si}(x^2)\right]$

13. For $x = 0, 0.5, 1.0, 1.5$, and 2.0, make a table of values for $I(x) = \int_0^x \sqrt{t^4+1}\,dt$.

14. Assume that $F'(t) = \sin t \cos t$ and $F(0) = 1$. Find $F(b)$ for $b = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$, and 3.

15. **(a)** Continue the table of values for $\mathrm{Si}(x) = \int_0^x (\sin t/t)\,dt$ on page 320 for $x = 4$ and $x = 5$.
(b) Why is $\mathrm{Si}(x)$ decreasing between $x = 4$ and $x = 5$?

In Exercises 16–18, write an expression for the function, $f(x)$, with the given properties.

16. $f'(x) = \sin(x^2)$ and $f(0) = 7$

17. $f'(x) = (\sin x)/x$ and $f(1) = 5$

18. $f'(x) = \mathrm{Si}(x)$ and $f(0) = 2$

Problems

For Problems 19–21, let $F(x) = \int_0^x \sin(t^2)\,dt$.

19. Approximate $F(x)$ for $x = 0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5$.

20. Using a graph of $F'(x)$, decide where $F(x)$ is increasing and where $F(x)$ is decreasing for $0 \le x \le 2.5$.

21. Does $F(x)$ have a maximum value for $0 \le x \le 2.5$? If so, at what value of x does it occur, and approximately what is that maximum value?

22. Use Figure 6.27 to sketch a graph of $F(x) = \int_0^x f(t)\,dt$. Label the points x_1, x_2, x_3.

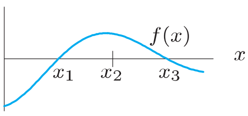


Figure 6.27

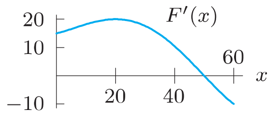


Figure 6.28

23. The graph of the derivative F' of some function F is given in Figure 6.28. If $F(20) = 150$, estimate the maximum value attained by F.

In Problems 24–25, find the value of the function with the given properties.

24. $F(1)$, where $F'(x) = e^{-x^2}$ and $F(0) = 2$

25. $G(-1)$, where $G'(x) = \cos(x^2)$ and $G(0) = -3$

26. Let $g(x) = \int_0^x f(t)\,dt$. Using Figure 6.29, find

(a) $g(0)$ **(b)** $g'(1)$

(c) The interval where g is concave up.

(d) The value of x where g takes its maximum on the interval $0 \le x \le 8$.

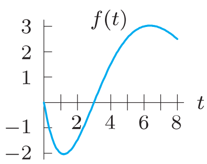


Figure 6.29

27. Let $F(x) = \int_0^x \sin(2t)\,dt$.

(a) Evaluate $F(\pi)$.

(b) Draw a sketch to explain geometrically why the answer to part (a) is correct.

(c) For what values of x is $F(x)$ positive? negative?

28. Let $F(x) = \int_2^x (1/\ln t)\,dt$ for $x \ge 2$.

(a) Find $F'(x)$.

(b) Is F increasing or decreasing? What can you say about the concavity of its graph?

(c) Sketch a graph of $F(x)$.

Calculate the derivatives in Problems 29–36.

29. $\dfrac{d}{dx}\displaystyle\int_2^{x^3} \sin(t^2)\,dt$

30. $\dfrac{d}{dt}\displaystyle\int_1^{\sin t} \cos(x^2)\,dx$

31. $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \ln(1+t^2)\,dt$

32. $\dfrac{d}{dt}\displaystyle\int_{2t}^{4} \sin(\sqrt{x})\,dx$

33. $\dfrac{d}{dx}\displaystyle\int_{\cos x}^{3} e^{t^2}\,dt$

34. $\dfrac{d}{dx}\displaystyle\int_{-x}^{x} e^{-t^4}\,dt$

35. $\dfrac{d}{dx}\displaystyle\int_{-x^2}^{x^2} e^{t^2}\,dt$

36. $\dfrac{d}{dt}\displaystyle\int_{e^t}^{t^3} \sqrt{1+x^2}\,dx$

37. Let $P(x) = \displaystyle\int_0^x \arctan(t^2)\,dt$.

(a) Evaluate $P(0)$ and determine if P is an even or an odd function.

(b) Is P increasing or decreasing?

(c) What can you say about concavity?

(d) Sketch a graph of $P(x)$.

38. Let $R(x) = \displaystyle\int_0^x \sqrt{1+t^2}\,dt$

(a) Evaluate $R(0)$ and determine if R is an even or an odd function.

(b) Is R increasing or decreasing?

(c) What can you say about concavity?

(d) Sketch a graph of $R(x)$.

(e) Show that $\lim_{x\to\infty}(R(x)/x^2)$ exists and find its value.

39. If $\int v(t-t_0)\,dt = s(t)+C$, where t_0 is a constant, what is $s'(t)$?

40. If $\int_{t_0}^t v(u)\,du = s(t)$, what is $\int_a^b v(t)\,dt$?

41. If $\int v(t)\,dt = s(t-t_0)+C$, where t_0 is a constant, what is $s'(t)$?

42. If $\int f(2x)\,dx = g(2x) + C$, what is $g'(x)$?

43. If $\int af(x)\,dx = g(ax) + C$, where a is a nonzero constant, what is $g'(x)$?

In Problems 44–47, find the given quantities. The *error function*, $\mathrm{erf}(x)$, is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

44. $\dfrac{d}{dx}(x\,\mathrm{erf}(x))$

45. $\dfrac{d}{dx}(\mathrm{erf}(\sqrt{x}))$

46. $\dfrac{d}{dx}\left(\int_0^{x^3} e^{-t^2}\,dt\right)$

47. $\dfrac{d}{dx}\left(\int_x^{x^3} e^{-t^2}\,dt\right)$

6.5 THE EQUATIONS OF MOTION

The problem of a body moving freely under the influence of gravity near the surface of the earth intrigued mathematicians and philosophers from Greek times onward and was finally solved by Galileo and Newton. The question to be answered was: How do the velocity and the position of the body vary with time? We define s to be the position, or height, of the body above a fixed point (often ground level); v then is the velocity of the body measured upward. We assume that the acceleration of the body is a constant, $-g$ (the negative sign means that the acceleration is downward), so

$$\text{Acceleration } = \frac{dv}{dt} = -g.$$

Thus, velocity is the antiderivative of $-g$:

$$v = -gt + C.$$

If the initial velocity is v_0, then $C = v_0$, so

$$v = -gt + v_0.$$

How about the position? We know that

$$\frac{ds}{dt} = v = -gt + v_0.$$

Therefore, we can find s by antidifferentiating again, giving:

$$s = -\frac{gt^2}{2} + v_0 t + C.$$

If the initial position is s_0, then we must have

$$s = -\frac{gt^2}{2} + v_0 t + s_0.$$

Our derivation of the formulas for the velocity and the position of the body took little effort. It hides an almost 2000-year struggle to understand the mechanics of falling bodies, from Aristotle's *Physics* to Galileo's *Dialogues Concerning Two New Sciences*.

Though it is an oversimplification of his ideas, we can say that Aristotle's conception of motion was primarily in terms of *change of position*. This seems entirely reasonable; it is what we commonly observe, and this view dominated discussions of motion for centuries. But it misses a subtlety that was brought to light by Descartes, Galileo, and, with a different emphasis, by Newton. That subtlety is now usually referred to as the *principle of inertia*.

This principle holds that a body traveling undisturbed at constant velocity in a straight line will continue in this motion indefinitely. Stated another way, it says that one cannot distinguish in any absolute sense (that is, by performing an experiment), between being at rest and moving with constant velocity in a straight line. If you are reading this book in a closed room and have no external reference points, there is no experiment that will tell you, one way or the other, whether you are at rest or whether you, the room, and everything in it are moving with constant velocity in a straight line. Therefore, as Newton saw, an understanding of motion should be based on *change of velocity* rather than change of position. Since acceleration is the rate of change of velocity, it is acceleration that must play a central role in the description of motion.

How does acceleration come about? How does the velocity change? Through the action of *forces*. Newton placed a new emphasis on the importance of forces. Newton's laws of motion do not say what a force *is*, they say how it *acts*. His first law is the principle of inertia, which says what happens in the *absence* of a force—there is no change in velocity. His second law says that a force acts to produce a change in velocity, that is, an acceleration. It states that $F = ma$, where m is the mass of the object, F is the net force, and a is the acceleration produced by this force.

Let's return to Galileo. He demonstrated that a body falling under the influence of gravity does so with constant acceleration. Furthermore, assuming we can neglect air resistance, this constant acceleration is independent of the mass of the body. This last fact was the outcome of Galileo's famous observation that a heavy ball and a light ball dropped off the Leaning Tower of Pisa hit the ground at the same time. Whether or not he actually performed this experiment, Galileo presented a very clear thought experiment in the *Dialogues* to prove the same point. (This point was counter to Aristotle's more common sense notion that the heavier ball would reach the ground first.) Galileo showed that the mass of the object did not appear as a variable in the equation of motion. Thus, the same constant acceleration equation applies to all bodies falling under the influence of gravity.

Nearly a hundred years after Galileo's experiment, Newton formulated his laws of motion and gravity, which led to a differential equation describing the motion of a falling body. According to Newton, acceleration is caused by force, and in the case of falling bodies, that force is the force of gravity. Newton's law of gravity says that the gravitational force between two bodies is attractive and given by

$$F = \frac{GMm}{r^2},$$

where G is the gravitational constant, m and M are the masses of the two bodies, and r is the distance between them. This is the famous *inverse square law*. For a falling body, we take M to be the mass of the earth and r to be the distance from the body to the center of the earth. So, actually, r changes as the body falls, but for anything we can easily observe (say, a ball dropped from the Tower of Pisa), it won't change significantly over the course of the motion. Hence, as an approximation, it is reasonable to assume that the force is constant. According to Newton's second law,

$$\text{Force} = \text{Mass} \times \text{Acceleration}.$$

Since the gravitational force is acting downward

$$-\frac{GMm}{r^2} = m\frac{d^2s}{dt^2}.$$

Hence,

$$\frac{d^2s}{dt^2} = -\frac{GM}{r^2} = \text{Constant}.$$

If we define $g = GM/r^2$, then

$$\frac{d^2s}{dt^2} = -g.$$

The fact that the mass cancels out of Newton's equations of motion reflects Galileo's experimental observation that the acceleration due to gravity is independent of the mass of the body.

Exercises and Problems for Section 6.5

Exercises

1. An object is thrown upward at time $t = 0$. After t seconds, its height is $y = -4.9t^2 + 7t + 1.5$ meters above the ground.

- **(a)** From what height was the object thrown?
- **(b)** What is the initial velocity of the object?
- **(c)** What is the acceleration due to gravity?

2. An object thrown in the air on a planet in a distant galaxy is at height $s = -25t^2 + 72t + 40$ feet at time t seconds after it is thrown. What is the acceleration due to gravity on this planet? With what velocity was the object thrown? From what height?

3. At time $t = 0$, a stone is thrown off a 250-meter cliff with velocity 20 meters/sec downward. Express its height, $h(t)$, in meters above the ground as a function of time, t, in seconds.

Problems

4. An object is dropped from a 400-foot tower. When does it hit the ground and how fast is it going at the time of the impact?

5. The object in Problem 4 falls off the same 400-foot tower. What would the acceleration due to gravity have to be to make it reach the ground in half the time?

6. A ball that is dropped from a window hits the ground in five seconds. How high is the window? (Give your answer in feet.)

7. On the moon the acceleration due to gravity is 5 ft/sec^2. An astronaut jumps into the air with an initial upward velocity of 10 ft/sec. How high does he go? How long is the astronaut off the ground?

8. Galileo was the first person to show that the distance traveled by a body falling from rest is proportional to the square of the time it has traveled, and independent of the mass of the body. Derive this result from the fact that the acceleration due to gravity is a constant.

9. While attempting to understand the motion of bodies under gravity, Galileo stated that:

> The time in which any space is traversed by a body starting at rest and uniformly accelerated is equal to the time in which that same space would be traversed by the same body moving at a uniform speed whose value is the mean of the highest velocity and the velocity just before acceleration began.

(a) Write Galileo's statement in symbols, defining all the symbols you use.
(b) Check Galileo's statement for a body dropped off a 100-foot building accelerating from rest under gravity until it hits the ground.
(c) Show why Galileo's statement is true in general.

10. The acceleration due to gravity 2 meters from the ground is 9.8 m/sec^2. What is the acceleration due to gravity 100 meters from the ground? At 100,000 meters? (The radius of the earth is $6.4 \cdot 10^6$ meters.)

11. A particle of mass, m, acted on by a force, F, moves in a straight line. Its acceleration, a, is given by Newton's Law:

$$F = ma.$$

The work, W, done by a constant force when the particle moves through a displacement, d, is

$$W = Fd.$$

The velocity, v, of the particle as a function of time, t, is given in Figure 6.30. What is the sign of the work done during each of the time intervals: $[0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$, $[t_3, t_4]$,$[t_2, t_4]$?

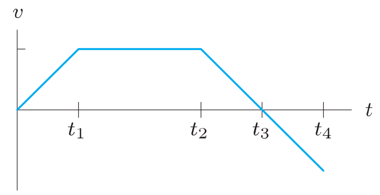


Figure 6.30

12. In his *Dialogues Concerning Two New Sciences*, Galileo wrote:

> The distances traversed during equal intervals of time by a body falling from rest stand to one another in the same ratio as the odd numbers beginning with unity.

Assume, as is now believed, that $s = -(gt^2)/2$, where s is the total distance traveled in time t, and g is the acceleration due to gravity.

(a) How far does a falling body travel in the first second (between $t = 0$ and $t = 1$)? During the second second (between $t = 1$ and $t = 2$)? The third second? The fourth second?
(b) What do your answers tell you about the truth of Galileo's statement?

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

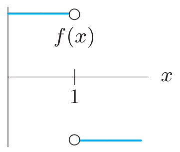
- **Constructing antiderivatives**
  Graphically, numerically, analytically.
- **The family of antiderivatives**
  The indefinite integral.
- **Differential equations**
  Initial value problems, uniform motion.
- **Construction theorem (Second Fundamental Theorem of Calculus)**
  Constructing antiderivatives using definite integrals.
- **Equations of motion**

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER SIX

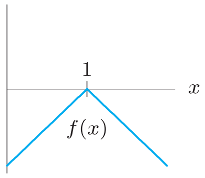
Exercises

In Exercises 1–2, sketch two functions F such that $F' = f$. In one case let $F(0) = 0$ and in the other, let $F(0) = 1$.

1.



2.



Find the indefinite integrals in Exercises 3–23.

3. $\int (5x+7)\,dx$

4. $\int \left(4t + \frac{1}{t}\right) dt$

5. $\int (2+\cos t)\,dt$

6. $\int 7e^x\,dx$

7. $\int (3e^x + 2\sin x)\,dx$

8. $\int (x+3)^2\,dx$

9. $\int \frac{8}{\sqrt{x}}\,dx$

10. $\int \left(\frac{3}{t} - \frac{2}{t^2}\right) dt$

11. $\int (e^x + 5)\,dx$

12. $\int \left(\sqrt{x^3} - \frac{2}{x}\right) dx$

13. $\int \frac{1}{\cos^2 x}\,dx$

14. $\int 2^x\,dx$

15. $\int (x+1)^2\,dx$

16. $\int (x+1)^3\,dx$

17. $\int (x+1)^9\,dx$

18. $\int \left(\frac{x+1}{x}\right) dx$

19. $\int \left(\frac{x^2+x+1}{x}\right) dx$

20. $\int \left(3\cos t + 3\sqrt{t}\right) dt$

21. $\int (3\cos x - 7\sin x)\,dx$

22. $\int \left(\frac{2}{x} + \pi\sin x\right) dx$

23. $\int (2e^x - 8\cos x)\,dx$

Find antiderivatives for the functions in Exercises 24–31. Check by differentiation.

24. $p(t) = \frac{1}{t}$

25. $f(x) = \frac{1}{x^2}$

26. $f(x) = \cos x$

27. $g(x) = \sin x$

28. $f(x) = e^x - 1$

29. $f(x) = 5e^x$

30. $h(t) = \frac{5}{t}$

31. $f(t) = t + \frac{1}{t}$

For Exercises 32–37, find an antiderivative $F(x)$ with $F'(x) = f(x)$ and $F(0) = 4$.

32. $f(x) = x^2$

33. $f(x) = x^3 + 6x^2 - 4$

34. $f(x) = \sqrt{x}$

35. $f(x) = e^x$

36. $f(x) = \sin x$

37. $f(x) = \cos x$

38. Use the Fundamental Theorem of Calculus to evaluate $\int_1^3 (6x^2 + 8x + 5)dx$.

39. Show that $y = x^n + A$ is a solution of the differential equation $y' = nx^{n-1}$ for any value of A.

40. Find the general solution of the differential equation $y' = 1/x$, where $x > 0$.

Problems

41. Use the Fundamental Theorem to find the area under $f(x) = x^2$ between $x = 0$ and $x = 3$.

42. Find the exact area of the region bounded by the x-axis and the graph of $y = x^3 - x$.

43. Calculate the exact area above the graph of $y = \frac{1}{2}\left(\frac{3}{\pi}x\right)^2$ and below the graph of $y = \cos x$. The curves intersect at $x = \pm\pi/3$.

44. Find the exact area of the shaded region in Figure 6.31 between $y = 3x^2 - 3$ and the x-axis.

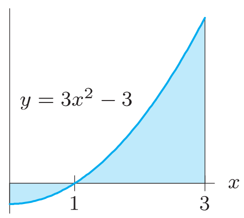


Figure 6.31

45. **(a)** Find the exact area between $f(x) = x^3 - 7x^2 + 10x$, the x-axis, $x = 0$, and $x = 5$.

(b) Find $\int_0^5 (x^3 - 7x^2 + 10x)\,dx$ exactly and interpret this integral in terms of areas.

46. Find the exact area between the curve $y = e^x - 2$ and the x-axis for $0 \le x \le 2$.

47. Consider the area between the curve $y = e^x - 2$ and the x-axis, between $x = 0$ and $x = c$ for $c > 0$. Find the value of c making the area above the axis equal to the area below the axis.

48. The area under $1/\sqrt{x}$ on the interval $1 \le x \le b$ is equal to 6. Find the value of b using the Fundamental Theorem.

49. Find the exact positive value of c which makes the area under the graph of $y = c(1 - x^2)$ and above the x-axis equal to 1.

50. Sketch the parabola $y = x(x - \pi)$ and the curve $y = \sin x$, showing their points of intersection. Find the exact area between the two graphs.

51. Find the exact average value of $f(x) = \sqrt{x}$ on the interval $0 \le x \le 9$. Illustrate your answer on a graph of $f(x) = \sqrt{x}$.

52. For t in years, $0 \le t \le 3$, the rate of discharge of a pollutant is estimated to be $t^2 - 14t + 49$ cubic meters per year. What is the total amount discharged during the three years?

53. If $A(r)$ represents the area of a circle of radius r and $C(r)$ represents its circumference, it can be shown that $A'(r) = C(r)$. Use the fact that $C(r) = 2\pi r$ to obtain the formula for $A(r)$.

54. If $V(r)$ represents the volume of a sphere of radius r and $S(r)$ represents its surface area, it can be shown that $V'(r) = S(r)$. Use the fact that $S(r) = 4\pi r^2$ to obtain the formula for $V(r)$.

55. For a function f, you are given the graph of the derivative f' in Figure 6.32 and that $f(0) = 50$.

(a) On the interval $0 \le t \le 5$, at what value of t does f appear to reach its maximum value? Its minimum value?
(b) Estimate these maximum and minimum values.
(c) Estimate $f(5) - f(0)$.

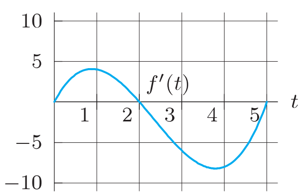


Figure 6.32

56. Assume f' is given by the graph in Figure 6.33. Suppose f is continuous and that $f(0) = 0$.

(a) Find $f(3)$ and $f(7)$.
(b) Find all x with $f(x) = 0$.
(c) Sketch a graph of f over the interval $0 \le x \le 7$.

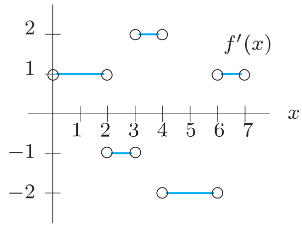
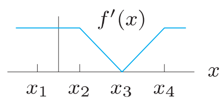


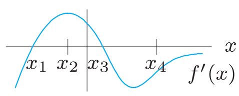
Figure 6.33

For Problems 57–58 the graph of $f'(x)$ is given. Sketch a possible graph for $f(x)$. Mark the points $x_1 \ldots x_4$ on your graph and label local maxima, local minima and points of inflection.

57.



58.



Calculate the derivatives in Problems 59–68.

59. $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{1+t^2}\,dt$

60. $\dfrac{d}{dt}\displaystyle\int_3^t \frac{1}{1+x^3}\,dx$

61. $\dfrac{d}{dx}\displaystyle\int_2^x \arccos(t^7)\,dt$

62. $\dfrac{d}{dt}\displaystyle\int_t^7 \log(x^6)\,dx$

63. $\dfrac{d}{dx}\displaystyle\int_0^{x^3} \sqrt{1+t^2}\,dt$

64. $\dfrac{d}{dt}\displaystyle\int_{-t}^0 \cos(x^2)\,dx$

65. $\dfrac{d}{dx}\displaystyle\int_5^{e^x} \cos(t^3)\,dt$

66. $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{17} \tan^3 t\,dt$

67. $\dfrac{d}{dt}\displaystyle\int_{t^5}^{\cos t} 4^{7x}\,dx$

68. $\dfrac{d}{dt}\displaystyle\int_{e^t}^{4\sin t} \frac{1+x}{1+x^2}\,dx$

69. Let $F(x) = \int_{\pi/2}^{x} (\sin t/t)\,dt$. Find the value(s) of x between $\pi/2$ and $3\pi/2$ for which $F(x)$ has a global maximum and global minimum.

70. The graphs of three functions are given in Figure 6.34. Determine which is f, which is f', and which is $\int_0^x f(t)\,dt$. Explain your answer.

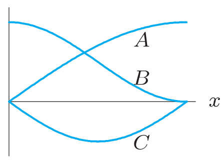


Figure 6.34

71. Write an expression for the function which is an antiderivative of e^{x^2} and which passes through the point **(a)** $(0, 3)$ **(b)** $(-1, 5)$

72. An object is attached to a coiled spring which is suspended from the ceiling of a room. The function $h(t)$ gives the height of the object above the floor of the room at time, t. The graph of $h'(t)$ is given in Figure 6.35. Sketch a possible graph of $h(t)$.

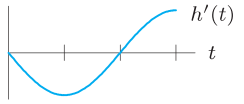


Figure 6.35

73. The acceleration, a, of a particle as a function of time is shown in Figure 6.36. Sketch graphs of velocity and position against time. The particle starts at rest at the origin.

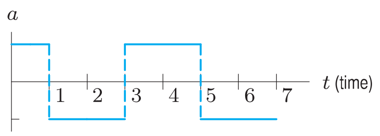


Figure 6.36

74. A car moves along a straight line with velocity, in feet/second, given by

$$v(t) = 6 - 2t \quad \text{for } t \geq 0.$$

(a) Describe the car's motion in words. (When is it moving forward, backward, and so on?)
(b) The car's position is measured from its starting point. When is it farthest forward? Backward?
(c) Find s, the car's position measured from its starting point, as a function of time.

75. The angular speed of a car engine increases from 1100 revs/min to 2500 revs/min in 6 sec.

(a) Assuming that it is constant, find the angular acceleration in revs/min^2.
(b) How many revolutions does the engine make in this time?

76. A helicopter rotor slows down at a constant rate from 350 revs/min to 260 revs/min in 1.5 minutes.

(a) Find the angular acceleration during this time interval. What are the units of this acceleration?
(b) Assuming the angular acceleration remains constant, how long does it take for the rotor to stop? (Measure time from the moment when speed was 350 revs/min.)
(c) How many revolutions does the rotor make between the time the angular speed was 350 revs/min and stopping?

77. An object is thrown vertically upward with a velocity of 80 ft/sec.

(a) Make a table showing its velocity every second.
(b) When does it reach its highest point? When does it hit the ground?
(c) Using your table, write left and right sums which under- and overestimate the height the object attains.
(d) Use antidifferentiation to find the greatest height it reaches.

78. A car, initially moving at 60 mph, has a constant deceleration and stops in a distance of 200 feet. What is its deceleration? (Give your answer in ft/sec^2. Note that 1 mph $= 22/15$ ft/sec.)

79. The birth rate, B, in births per hour, of a bacteria population is given in Figure 6.37. The curve marked D gives the death rate, in deaths per hour, of the same population.

(a) Explain what the shape of each of these graphs tells you about the population.
(b) Use the graphs to find the time at which the net rate of increase of the population is at a maximum.
(c) At time $t = 0$ the population has size N. Sketch the graph of the total number born by time t. Also sketch the graph of the number alive at time t. Estimate the time at which the population is a maximum.

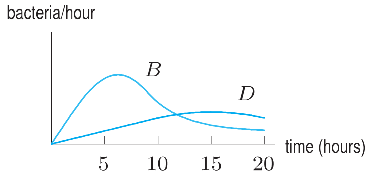


Figure 6.37

80. Water flows at a constant rate into the left side of the W-shaped container in Figure 6.38. Sketch a graph of the height, H, of the water in the left side of the container as a function of time, t. The container starts empty.

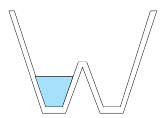


Figure 6.38

CAS Challenge Problems

81. **(a)** Set up a right-hand Riemann sum for $\int_a^b x^3 dx$ using n subdivisions. What is Δx? Express each x_i, for $i = 1,2,\ldots,n$, in terms of i.
(b) Use a computer algebra system to find an expression for the Riemann sum in part (a); then find the limit of this expression as $n \to \infty$.
(c) Simplify the final expression and compare the result to that obtained using the Fundamental Theorem of Calculus.

82. **(a)** Use a computer algebra system to find $\int e^{2x}\,dx$, $\int e^{3x}\,dx$, and $\int e^{3x+5}\,dx$.
(b) Using your answers to part (a), conjecture a formula for $\int e^{ax+b}\,dx$, where a and b are constants.
(c) Check your formula by differentiation. Explain which differentiation rules you are using.

83. **(a)** Use a computer algebra system to find $\int \sin(3x)\,dx$, $\int \sin(4x)\,dx$, and $\int \sin(3x-2)\,dx$.
(b) Using your answers to part (a), conjecture a formula for $\int \sin(ax+b)\,dx$, where a and b are constants.
(c) Check your formula by differentiation. Explain which differentiation rules you are using.

84. **(a)** Use a computer algebra system to find

$$\int \frac{x-2}{x-1}\,dx, \quad \int \frac{x-3}{x-1}\,dx, \quad \text{and} \quad \int \frac{x-1}{x-2}\,dx.$$

(b) If a and b are constants, use your answers to part (a) to conjecture a formula for

$$\int \frac{x-a}{x-b}\,dx.$$

(c) Check your formula by differentiation. Explain which rules of differentiation you are using.

85. **(a)** Use a computer algebra system to find

$$\int \frac{1}{(x-1)(x-3)}\,dx, \quad \int \frac{1}{(x-1)(x-4)}\,dx$$

and

$$\int \frac{1}{(x-1)(x+3)}\,dx.$$

(b) If a and b are constants, use your answers to part (a) to conjecture a formula for

$$\int \frac{1}{(x-a)(x-b)}\,dx.$$

(c) Check your formula by differentiation. Explain which rules of differentiation you are using.

CHECK YOUR UNDERSTANDING

Are the statements in Problems 1–26 true or false? Give an explanation for your answer.

1. A function $f(x)$ has at most one derivative.

2. An antiderivative of $3\sqrt{x+1}$ is $2(x+1)^{3/2}$.

3. An antiderivative of $3x^2$ is $x^3+\pi$.

4. An antiderivative of $1/x$ is $\ln|x|+\ln 2$.

5. An antiderivative of e^{-x^2} is $-e^{-x^2}/2x$.

6. $\int f(x)dx = (1/x)\int xf(x)dx$.

7. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)=F(x)+2$, then $G(x)$ is an antiderivative of $f(x)$.

8. If $F(x)$ is an antiderivative of $f(x)$, then $y=F(x)$ is a solution to the differential equation $dy/dx=f(x)$.

9. If $y=F(x)$ is a solution to the differential equation $dy/dx=f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

10. If an object has constant nonzero acceleration, then the position of the object as a function of time is a quadratic polynomial.

11. If $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$ for $-\infty < x < \infty$ and $F(5) > G(5)$, then $F(10) > G(10)$.

12. In an initial value problem for the differential equation $dy/dx=f(x)$, the value of y at $x=0$ is always specified.

13. If $f(x)$ is positive for all x, then there is a solution of the differential equation $dy/dx=f(x)$ where $y(x)$ is positive for all x.

14. If $f(x)>0$ for all x then every solution of the differential equation $dy/dx=f(x)$ is an increasing function.

15. If two solutions of a differential equation $dy/dx=f(x)$ have different values at $x=3$ then they have different values at every x.

16. If the function $y=f(x)$ is a solution of the differential equation $dy/dx=\sin x/x$, then the function $y=f(x)+5$ is also a solution.

17. There is only one solution $y(t)$ to the initial value problem $dy/dt=3t^2$, $y(1)=\pi$.

18. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then $F(x)\cdot G(x)$ is an antiderivative of $f(x)\cdot g(x)$.

19. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval then $F(x)-G(x)$ is a constant function.

20. Every continuous function has an antiderivative.

21. A ball thrown downward at 10 feet per second from the top of a 100 foot building hits the ground in less than 3 seconds.

22. $\int_0^x \sin(t^2)dt$ is an antiderivative of $\sin(x^2)$.

23. If $F(x)=\int_0^x f(t)dt$, then $F(5)-F(3)=\int_3^5 f(t)dt$.

24. If $F(x)=\int_0^x f(t)dt$, then $F(x)$ must be increasing.

25. If $F(x)=\int_0^x f(t)dt$ and $G(x)=\int_2^x f(t)dt$, then $F(x)=G(x)+C$.

26. If $F(x)=\int_0^x f(t)dt$ and $G(x)=\int_0^x g(t)dt$, then $F(x)+G(x)=\int_0^x (f(t)+g(t))dt$.

PROJECTS FOR CHAPTER SIX

1. Distribution of Resources

Whether a resource is distributed evenly among members of a population is often an important political or economic question. How can we measure this? How can we decide if the distribution of wealth in this country is becoming more or less equitable over time? How can we measure which country has the most equitable income distribution? This problem describes a way of making such measurements. Suppose the resource is distributed evenly. Then any 20% of the population will have 20% of the resource. Similarly, any 30% will have 30% of the resource and so on. If, however, the resource is not distributed evenly, the poorest $p\%$ of the population (in terms of this resource) will not have $p\%$ of the goods. Suppose $F(x)$ represents the fraction of the resources owned by the poorest fraction x of the population. Thus $F(0.4) = 0.1$ means that the poorest 40% of the population owns 10% of the resource.

(a) What would F be if the resource were distributed evenly?
(b) What must be true of any such F? What must $F(0)$ and $F(1)$ equal? Is F increasing or decreasing? Is the graph of F concave up or concave down?
(c) Gini's index of inequality, G, is one way to measure how evenly the resource is distributed. It is defined by

$$G = 2\int_0^1 [x - F(x)]\, dx.$$

Show graphically what G represents.

2. Yield from an Apple Orchard

Figure 6.39 is a graph of the annual yield, $y(t)$, in bushels per year, from an orchard t years after planting. The trees take about 10 years to get established, but for the next 20 years they give a substantial yield. After about 30 years, however, age and disease start to take their toll, and the annual yield falls off.[1]

(a) Represent on a sketch of Figure 6.39 the total yield, $F(M)$, up to M years, with $0 \le M \le 60$. Write an expression for $F(M)$ in terms of $y(t)$.
(b) Sketch a graph of $F(M)$ against M for $0 \le M \le 60$.
(c) Write an expression for the average annual yield, $a(M)$, up to M years.
(d) When should the orchard be cut down and replanted? Assume that we want to maximize average revenue per year, and that fruit prices remain constant, so that this is achieved by maximizing average annual yield. Use the graph of $y(t)$ to estimate the time at which the average annual yield is a maximum. Explain your answer geometrically and symbolically.

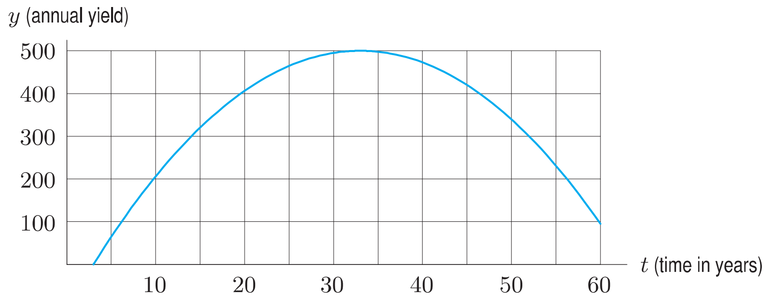


Figure 6.39

3. Slope Fields

Suppose we want to sketch the antiderivative, F, of the function f. To get an accurate graph of F, we must be careful about making F have the right slope at every point. The slope of F at any point (x, y) on its graph should be $f(x)$, since $F'(x) = f(x)$. We arrange this as follows: at the point (x, y) in the plane, draw a small line segment with slope $f(x)$. Do this at many points. We call such a diagram a *slope field*. If $f(x) = x$, we get the slope field in Figure 6.40.

[1] From Peter D. Taylor, *Calculus: The Analysis of Functions*, (Toronto: Wall & Emerson, Inc., 1992).

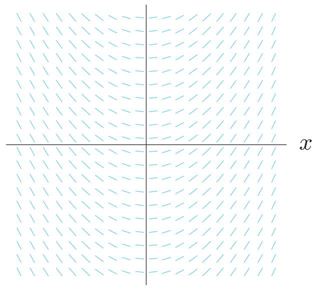


Figure 6.40: Slope field of $f(x) = x$

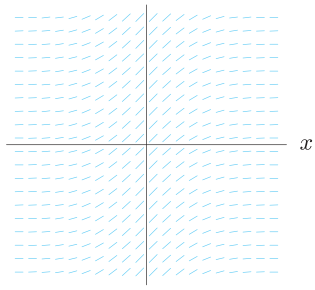


Figure 6.41: Slope field of $f(x) = e^{-x^2}$

Notice how the lines in Figure 6.40 seem to be arranged in a parabolic pattern. This is because the general antiderivative of x is $x^2/2 + C$, so the lines are all the tangent lines to the family of parabolas $y = x^2/2 + C$. This suggests a way of finding antiderivatives graphically even if we can't write down a formula for them: plot the slopes, and see if they suggest the graph of an antiderivative. For example, if you do this with $f(x) = e^{-x^2}$, which is one of the functions that does not have an elementary antiderivative, you get Figure 6.41.

You can see the ghost of the graph of a function lurking behind the slopes in Figure 6.41; in fact there is a whole stack of them. If you move across the plane in the direction suggested by the slope field at every point, you will trace out a curve. The slope field is tangent to the curve everywhere, so this is the graph of an antiderivative of e^{-x^2}.

(a) (i) Sketch a graph of $f(t) = \dfrac{\sin t}{t}$.

(ii) What does your graph tell you about the behavior of

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

for $x > 0$? Is $\text{Si}(x)$ always increasing or always decreasing? Does $\text{Si}(x)$ cross the x-axis for $x > 0$?

(iii) By drawing the slope field for $f(t) = \dfrac{\sin t}{t}$, decide whether $\lim\limits_{x\to\infty} \text{Si}(x)$ exists.

(b) (i) Use your calculator or computer to sketch a graph of $y = x^{\sin x}$ for $0 < x \le 20$.

(ii) Using your answer to part (i), sketch by hand a graph of the function F, where

$$F(x) = \int_0^x t^{\sin t}\,dt.$$

(iii) Use a slope field program to check your answer to part (ii).

(c) Let $F(x)$ be the antiderivative of $\sin(x^2)$ satisfying $F(0) = 0$.

(i) Describe any general features of the graph of F that you can deduce by looking at the graph of $\sin(x^2)$ in Figure 6.42.

(ii) By drawing a slope field (using a calculator or computer), sketch a graph of F. Does F ever cross the x-axis in the region $x > 0$? Does $\lim\limits_{x\to\infty} F(x)$ exist?

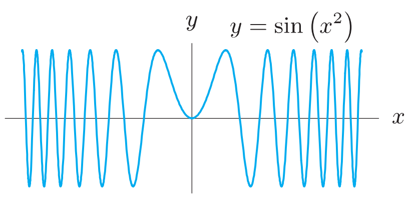


Figure 6.42