Chapter Seven

INTEGRATION

Contents

7.1 INTEGRATION BY SUBSTITUTION

In Chapter 3, we learned rules to differentiate any function obtained by combining constants, powers of x, $\sin x$, $\cos x$, e^x, and $\ln x$, using addition, multiplication, division, or composition of functions. Such functions are called *elementary*.

In the next few sections, we introduce two methods of antidifferentiation: substitution and integration by parts, which reverse the chain and product rules, respectively. However, there is a great difference between looking for derivatives and looking for antiderivatives. Every elementary function has elementary derivatives, but many elementary functions do not have elementary antiderivatives. Some examples are $\sqrt{x^3+1}$, $(\sin x)/x$, and e^{-x^2}. These are not exotic functions, but ordinary functions that arise naturally.

The Guess-and-Check Method

A good strategy for finding simple antiderivatives is to *guess* an answer (using knowledge of differentiation rules) and then *check* the answer by differentiating it. If we get the expected result, then we're done; otherwise, we revise the guess and check again.

The method of guess-and-check is useful in reversing the chain rule. According to the chain rule,

$$\frac{d}{dx}(f(g(x))) = \underbrace{f'}_{\text{Derivative of outside}}(\overbrace{g(x)}^{\text{Inside}}) \cdot \underbrace{g'(x)}_{\text{Derivative of inside}}.$$

Thus, any function which is the result of applying the chain rule is the product of two factors: the "derivative of the outside" and the "derivative of the inside." If a function has this form, its antiderivative is $f(g(x))$.

Example 1 Find $\int 3x^2\cos(x^3)\,dx$.

Solution The function $3x^2\cos(x^3)$ looks like the result of applying the chain rule: there is an "inside" function x^3 and its derivative $3x^2$ appears as a factor. Since the outside function is a cosine which has a sine as an antiderivative, we guess $\sin(x^3)$ for the antiderivative. Differentiating to check gives

$$\frac{d}{dx}(\sin(x^3)) = \cos(x^3)\cdot(3x^2).$$

Since this is what we began with, we know that

$$\int 3x^2\cos(x^3)\,dx = \sin(x^3) + C.$$

The basic idea of this method is to try to find an inside function whose derivative appears as a factor. This works even when the derivative is missing a constant factor, as in the next example.

Example 2 Find $\int te^{(t^2+1)}\,dt$.

Solution It looks like t^2+1 is an inside function. So we guess $e^{(t^2+1)}$ for the antiderivative, since taking the derivative of an exponential results in the reappearance of the exponential together with other terms from the chain rule. Now we check:

$$\frac{d}{dt}\left(e^{(t^2+1)}\right) = \left(e^{(t^2+1)}\right)\cdot 2t.$$

The original guess was too large by a factor of 2. We change the guess to $\frac{1}{2}e^{(t^2+1)}$ and check again:

$$\frac{d}{dt}\left(\frac{1}{2}e^{(t^2+1)}\right) = \frac{1}{2}e^{(t^2+1)} \cdot 2t = e^{(t^2+1)} \cdot t.$$

Thus, we know that

$$\int te^{(t^2+1)}\,dt = \frac{1}{2}e^{(t^2+1)} + C.$$

Example 3 Find $\int x^3\sqrt{x^4+5}\,dx$.

Solution Here the inside function is x^4+5, and its derivative appears as a factor, with the exception of a missing 4. Thus, the integrand we have is more or less of the form

$$g'(x)\sqrt{g(x)},$$

with $g(x) = x^4+5$. Since $x^{3/2}/(3/2)$ is an antiderivative of the outside function $\sqrt{x}$, we might guess that an antiderivative is

$$\frac{(g(x))^{3/2}}{3/2} = \frac{(x^4+5)^{3/2}}{3/2}.$$

Let's check and see:

$$\frac{d}{dx}\left(\frac{(x^4+5)^{3/2}}{3/2}\right) = \frac{3}{2}\frac{(x^4+5)^{1/2}}{3/2} \cdot 4x^3 = 4x^3(x^4+5)^{1/2},$$

so $\dfrac{(x^4+5)^{3/2}}{3/2}$ is too big by a factor of 4. The correct antiderivative is

$$\frac{1}{4}\frac{(x^4+5)^{3/2}}{3/2} = \frac{1}{6}(x^4+5)^{3/2}.$$

Thus

$$\int x^3\sqrt{x^4+5}\,dx = \frac{1}{6}(x^4+5)^{3/2} + C.$$

As a final check:

$$\frac{d}{dx}\left(\frac{1}{6}(x^4+5)^{3/2}\right) = \frac{1}{6}\cdot\frac{3}{2}(x^4+5)^{1/2}\cdot 4x^3 = x^3(x^4+5)^{1/2}.$$

As we have seen in the preceding examples, antidifferentiating a function often involves "correcting for" constant factors: if differentiation produces an extra factor of 4, antidifferentiation will require a factor of $\frac{1}{4}$.

The Method of Substitution

When the integrand is complicated, it helps to formalize this guess-and-check method as follows:

To Make a Substitution

Let w be the "inside function" and $dw = w'(x)\,dx = \dfrac{dw}{dx}dx$.

Let's redo the first example using a substitution.

Example 4 Find $\int 3x^2\cos(x^3)\,dx$.

Solution

As before, we look for an inside function whose derivative appears—in this case x^3. We let $w = x^3$. Then $dw = w'(x)\,dx = 3x^2\,dx$. The original integrand can now be completely rewritten in terms of the new variable w:

$$\int 3x^2 \cos(x^3)\,dx = \int \cos\underbrace{(x^3)}_{w} \cdot \underbrace{3x^2\,dx}_{dw} = \int \cos w\,dw = \sin w + C = \sin(x^3) + C.$$

By changing the variable to w, we can simplify the integrand. We now have $\cos w$, which can be antidifferentiated more easily. The final step, after antidifferentiating, is to convert back to the original variable, x.

Why Does Substitution Work?

The substitution method makes it look as if we can treat dw and dx as separate entities, even canceling them in the equation $dw = (dw/dx)dx$. Let's see why this works. Suppose we have an integral of the form $\int f(g(x))g'(x)\,dx$, where $g(x)$ is the inside function and $f(x)$ is the outside function. If F is an antiderivative of f, then $F' = f$, and by the chain rule $\frac{d}{dx}(F(g(x))) = f(g(x))g'(x)$. Therefore,

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Now write $w = g(x)$ and $dw/dx = g'(x)$ on both sides of this equation:

$$\int f(w)\frac{dw}{dx}\,dx = F(w) + C.$$

On the other hand, knowing that $F' = f$ tells us that

$$\int f(w)\,dw = F(w) + C.$$

Thus, the following two integrals are equal:

$$\int f(w)\frac{dw}{dx}\,dx = \int f(w)\,dw.$$

Substituting w for the inside function and writing $dw = w'(x)dx$ leaves the indefinite integral unchanged.

Let's revisit the second example that we did by guess-and-check.

Example 5 Find $\displaystyle\int te^{(t^2+1)}\,dt$.

Solution

Here the inside function is $t^2 + 1$, with derivative $2t$. Since there is a factor of t in the integrand, we try

$$w = t^2 + 1.$$

Then

$$dw = w'(t)\,dt = 2t\,dt.$$

Notice, however, that the original integrand has only $t\,dt$, not $2t\,dt$. We therefore write

$$\frac{1}{2}\,dw = t\,dt$$

and then substitute:

$$\int te^{(t^2+1)}\,dt = \int e^{\overbrace{(t^2+1)}^{w}} \cdot \underbrace{t\,dt}_{\frac{1}{2}dw} = \int e^w\,\frac{1}{2}\,dw = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{(t^2+1)} + C.$$

This gives the same answer as we found using guess-and-check.

Why didn't we put $\frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + \frac{1}{2}C$ in the preceding example? Since the constant C is arbitrary, it does not really matter whether we add C or $\frac{1}{2}C$. The convention is always to add C to whatever antiderivative we have calculated.

Now let's redo the third example that we solved previously by guess-and-check.

Example 6 Find $\displaystyle\int x^3\sqrt{x^4+5}\,dx$.

Solution The inside function is x^4+5, with derivative $4x^3$. The integrand has a factor of x^3, and since the only thing missing is a constant factor, we try

$$w = x^4 + 5.$$

Then

$$dw = w'(x)\,dx = 4x^3\,dx,$$

giving

$$\frac{1}{4}dw = x^3\,dx.$$

Thus,

$$\int x^3\sqrt{x^4+5}\,dx = \int \sqrt{w}\,\frac{1}{4}dw = \frac{1}{4}\int w^{1/2}\,dw = \frac{1}{4}\cdot\frac{w^{3/2}}{3/2} + C = \frac{1}{6}(x^4+5)^{3/2} + C.$$

Once again, we get the same result as with guess-and-check.

Warning

We saw in the preceding examples that we can apply the substitution method when a *constant* factor is missing from the derivative of the inside function. However, we may not be able to use substitution if anything other than a constant factor is missing. For example, setting $w = x^4+5$ to find

$$\int x^2\sqrt{x^4+5}\,dx$$

does us no good because $x^2\,dx$ is not a constant multiple of $dw = 4x^3\,dx$. Substitution works if the integrand contains the derivative of the inside function, *to within a constant factor*.

Some people prefer the substitution method over guess-and-check since it is more systematic, but both methods achieve the same result. For simple problems, guess-and-check can be faster.

Example 7 Find $\displaystyle\int e^{\cos\theta}\sin\theta\,d\theta$.

Solution We let $w = \cos\theta$ since its derivative is $-\sin\theta$ and there is a factor of $\sin\theta$ in the integrand. This gives

$$dw = w'(\theta)\,d\theta = -\sin\theta\,d\theta,$$

so

$$-dw = \sin\theta\,d\theta.$$

Thus

$$\int e^{\cos\theta}\sin\theta\,d\theta = \int e^w\,(-dw) = (-1)\int e^w\,dw = -e^w + C = -e^{\cos\theta} + C.$$

Example 8 Find $\displaystyle\int \frac{e^t}{1+e^t}\,dt$.

Solution Observing that the derivative of $1+e^t$ is e^t, we see $w = 1+e^t$ is a good choice. Then $dw = e^t\,dt$, so that

$$\int \frac{e^t}{1+e^t}\,dt = \int \frac{1}{1+e^t}\,e^t\,dt = \int \frac{1}{w}\,dw = \ln|w| + C$$
$$= \ln|1+e^t| + C$$
$$= \ln(1+e^t) + C. \qquad \text{(Since } (1+e^t) \text{ is always positive.)}$$

Since the numerator is $e^t\,dt$, we might also have tried $w = e^t$. This substitution leads to the integral $\int (1/(1+w))dw$, which is better than the original integral but requires another substitution, $u = 1+w$, to finish. There are often several different ways of doing an integral by substitution.

Notice the pattern in the previous example: having a function in the denominator and its derivative in the numerator leads to a natural logarithm. The next example follows the same pattern.

Example 9 Find $\displaystyle\int \tan\theta\,d\theta$.

Solution Recall that $\tan\theta = (\sin\theta)/(\cos\theta)$. If $w = \cos\theta$, then $dw = -\sin\theta\,d\theta$, so

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta = \int \frac{-dw}{w} = -\ln|w| + C = -\ln|\cos\theta| + C.$$

Definite Integrals by Substitution

Example 10 Compute $\displaystyle\int_0^2 xe^{x^2}\,dx$.

Solution To evaluate this definite integral using the Fundamental Theorem of Calculus, we first need to find an antiderivative of $f(x) = xe^{x^2}$. The inside function is x^2, so we let $w = x^2$. Then $dw = 2x\,dx$, so $\frac{1}{2}\,dw = x\,dx$. Thus,

$$\int xe^{x^2}\,dx = \int e^w \frac{1}{2}\,dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{x^2} + C.$$

Now we find the definite integral

$$\int_0^2 xe^{x^2}\,dx = \frac{1}{2}e^{x^2}\Big|_0^2 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1).$$

There is another way to look at the same problem. After we established that

$$\int xe^{x^2}\,dx = \frac{1}{2}e^w + C,$$

our next two steps were to replace w by x^2, and then x by 2 and 0. We could have directly replaced the original limits of integration, $x = 0$ and $x = 2$, by the corresponding w limits. Since $w = x^2$, the w limits are $w = 0^2 = 0$ (when $x = 0$) and $w = 2^2 = 4$ (when $x = 2$), so we get

$$\int_{x=0}^{x=2} xe^{x^2}\,dx = \frac{1}{2}\int_{w=0}^{w=4} e^w\,dw = \frac{1}{2}e^w\Big|_0^4 = \frac{1}{2}\left(e^4 - e^0\right) = \frac{1}{2}(e^4 - 1).$$

As we would expect, both methods give the same answer.

To Use Substitution to Find Definite Integrals

Either

- Compute the indefinite integral, expressing an antiderivative in terms of the original variable, and then evaluate the result at the original limits,

or

- Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative back to the original variable.

Example 11 Evaluate $\displaystyle\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\,d\theta$.

Solution To use substitution, we must decide what w should be. There are two possible inside functions, $\tan\theta$ and $\cos\theta$. Now

$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{\cos^2\theta} \quad \text{and} \quad \frac{d}{d\theta}(\cos\theta) = -\sin\theta,$$

and since the integral contains a factor of $1/\cos^2\theta$ but not of $\sin\theta$, we try $w = \tan\theta$. Then $dw = (1/\cos^2\theta)d\theta$. When $\theta = 0$, $w = \tan 0 = 0$, and when $\theta = \pi/4$, $w = \tan(\pi/4) = 1$, so

$$\int_0^{\pi/4} \frac{\tan^3\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4} (\tan\theta)^3 \cdot \frac{1}{\cos^2\theta}\,d\theta = \int_0^1 w^3\,dw = \frac{1}{4}w^4\bigg|_0^1 = \frac{1}{4}.$$

Example 12 Evaluate $\displaystyle\int_1^3 \frac{dx}{5-x}$.

Solution Let $w = 5 - x$, so $dw = -dx$. When $x = 1$, $w = 4$, and when $x = 3$, $w = 2$, so

$$\int_1^3 \frac{dx}{5-x} = \int_4^2 \frac{-dw}{w} = -\ln|w|\bigg|_4^2 = -(\ln 2 - \ln 4) = \ln\left(\frac{4}{2}\right) = \ln 2 \approx 0.69.$$

Notice that we write the limit $w = 4$ at the bottom, even though it is larger than $w = 2$, because $w = 4$ corresponds to the lower limit $x = 1$.

More Complex Substitutions

In the examples of substitution presented so far, we guessed an expression for w and hoped to find dw (or some constant multiple of it) in the integrand. What if we are not so lucky? It turns out that it often works to let w be some messy expression contained inside, say, a cosine or under a root, even if we cannot see immediately how such a substitution helps.

Example 13 Find $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$.

Solution This time, the derivative of the inside function is nowhere to be seen. Nevertheless, we try $w = 1 + \sqrt{x}$. Then $w - 1 = \sqrt{x}$, so $(w-1)^2 = x$. Therefore $2(w-1)\,dw = dx$. We have

$$\begin{aligned}
\int \sqrt{1+\sqrt{x}}\,dx &= \int \sqrt{w}\,2(w-1)\,dw = 2\int w^{1/2}(w-1)\,dw \\
&= 2\int (w^{3/2} - w^{1/2})\,dw = 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C \\
&= 2\left(\frac{2}{5}(1+\sqrt{x})^{5/2} - \frac{2}{3}(1+\sqrt{x})^{3/2}\right) + C.
\end{aligned}$$

Notice that the substitution in the preceding example again converts the inside of the messiest function into something simple. In addition, since the derivative of the inside function is not waiting for us, we have to solve for x so that we can get dx entirely in terms of w and dw.

Example 14 Find $\int (x+7)\sqrt[3]{3-2x}\,dx$.

Solution Here, instead of the derivative of the inside function (which is -2), we have the factor $(x+7)$. However, substituting $w = 3-2x$ turns out to help anyway. Then $dw = -2\,dx$, so $(-1/2)\,dw = dx$. Now we must convert everything to w, including $x+7$. If $w = 3-2x$, then $2x = 3-w$, so $x = 3/2 - w/2$, and therefore we can write $x+7$ in terms of w. Thus

$$\begin{aligned}
\int (x+7)\sqrt[3]{3-2x}\,dx &= \int \left(\frac{3}{2} - \frac{w}{2} + 7\right)\sqrt[3]{w}\left(-\frac{1}{2}\right)dw \\
&= -\frac{1}{2}\int \left(\frac{17}{2} - \frac{w}{2}\right) w^{1/3}\,dw \\
&= -\frac{1}{4}\int (17-w)w^{1/3}\,dw \\
&= -\frac{1}{4}\int (17w^{1/3} - w^{4/3})\,dw \\
&= -\frac{1}{4}\left(17\frac{w^{4/3}}{4/3} - \frac{w^{7/3}}{7/3}\right) + C \\
&= -\frac{1}{4}\left(\frac{51}{4}(3-2x)^{4/3} - \frac{3}{7}(3-2x)^{7/3}\right) + C.
\end{aligned}$$

Looking back over the solution, the reason this substitution works is that it converts $\sqrt[3]{3-2x}$, the messiest part of the integrand, to $\sqrt[3]{w}$, which can be combined with the other term and then integrated.

Exercises and Problems for Section 7.1

Exercises

1. Use substitution to express each of the following integrals as a multiple of $\int_a^b (1/w)\,dw$ for some a and b. Then evaluate the integrals.

(a) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$ **(b)** $\displaystyle\int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$

2. **(a)** Find the derivatives of $\sin(x^2+1)$ and $\sin(x^3+1)$.
(b) Use your answer to part (a) to find antiderivatives of:
(i) $x\cos(x^2+1)$ (ii) $x^2\cos(x^3+1)$
(c) Find the general antiderivatives of:
(i) $x\sin(x^2+1)$ (ii) $x^2\sin(x^3+1)$

Find the integrals in Exercises 3–46. Check your answers by differentiation.

3. $\int te^{t^2}\,dt$

4. $\int e^{3x}\,dx$

5. $\int e^{-x}\,dx$

6. $\int 25e^{-0.2t}\,dt$

7. $\int t\cos(t^2)\,dt$

8. $\int \sin(2x)\,dx$

9. $\int \sin(3-t)\,dt$

10. $\int xe^{-x^2}\,dx$

11. $\int (r+1)^3\,dr$

12. $\int y(y^2+5)^8\,dy$

13. $\int t^2(t^3-3)^{10}\,dt$

14. $\int x^2(1+2x^3)^2\,dx$

15. $\int x(x^2+3)^2\,dx$

16. $\int x(x^2-4)^{7/2}\,dx$

17. $\int y^2(1+y)^2\,dy$

18. $\int (2t-7)^{73}\,dt$

19. $\displaystyle\int \frac{dy}{y+5}$

20. $\displaystyle\int \frac{1}{\sqrt{4-x}}\,dx$

21. $\int (x^2+3)^2\,dx$

22. $\int x^2e^{x^3+1}\,dx$

23. $\int \sin\theta(\cos\theta+5)^7\,d\theta$

24. $\int \sqrt{\cos 3t}\sin 3t\,dt$

25. $\int \sin^6\theta\cos\theta\,d\theta$

26. $\int \sin^3\alpha\cos\alpha\,d\alpha$

27. $\int \sin^6(5\theta)\cos(5\theta)\,d\theta$

28. $\int \tan(2x)\,dx$

29. $\int \frac{(\ln z)^2}{z}\,dz$

30. $\int \frac{e^t+1}{e^t+t}\,dt$

31. $\int \frac{y}{y^2+4}\,dy$

32. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

33. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy$

34. $\int \frac{1+e^x}{\sqrt{x+e^x}}\,dx$

35. $\int \frac{e^x}{2+e^x}\,dx$

36. $\int \frac{x+1}{x^2+2x+19}\,dx$

37. $\int \frac{t}{1+3t^2}\,dt$

38. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

39. $\int \frac{(t+1)^2}{t^2}\,dt$

40. $\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx$

41. $\int \cosh x\,dx$

42. $\int \sinh 3t\,dt$

43. $\int (\sinh z)e^{\cosh z}\,dz$

44. $\int \cosh(2w+1)\,dw$

45. $\int x\cosh x^2\,dx$

46. $\int \cosh^2 x\sinh x\,dx$

For the functions in Exercises 47–54, find the general antiderivative. Check your answers by differentiation.

47. $p(t)=\pi t^3+4t$

48. $f(x)=\sin 3x$

49. $f(x)=2x\cos(x^2)$

50. $r(t)=12t^2\cos(t^3)$

51. $f(x)=\sin(2-5x)$

52. $f(x)=e^{\sin x}\cos x$

53. $f(x)=\frac{x}{x^2+1}$

54. $f(x)=\frac{1}{3\cos^2(2x)}$

For Exercises 55–62, use the Fundamental Theorem to calculate the definite integrals.

55. $\int_0^{\pi}\cos(x+\pi)\,dx$

56. $\int_0^{1/2}\cos(\pi x)\,dx$

57. $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$

58. $\int_1^2 2xe^{x^2}\,dx$

59. $\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx$

60. $\int_{-1}^{e-2}\frac{1}{t+2}dt$

61. $\int_1^4 \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

62. $\int_0^2 \frac{x}{(1+x^2)^2}\,dx$

For Exercises 63–68, evaluate the definite integrals. Whenever possible, use the Fundamental Theorem of Calculus, perhaps after a substitution. Otherwise, use numerical methods.

63. $\int_{-1}^3 (x^3+5x)\,dx$

64. $\int_{-1}^1 \frac{1}{1+y^2}\,dy$

65. $\int_1^3 \frac{1}{x}\,dx$

66. $\int_1^3 \frac{dt}{(t+7)^2}$

67. $\int_{-1}^2 \sqrt{x+2}\,dx$

68. $\int_1^2 \frac{\sin t}{t}\,dt$

Find the integrals in Exercises 69–76.

69. $\int y\sqrt{y+1}\,dy$

70. $\int z(z+1)^{1/3}\,dz$

71. $\int \frac{t^2+t}{\sqrt{t+1}}\,dt$

72. $\int \frac{dx}{2+2\sqrt{x}}$

73. $\int x^2\sqrt{x-2}\,dx$

74. $\int (z+2)\sqrt{1-z}\,dz$

75. $\int \frac{t}{\sqrt{t+1}}\,dt$

76. $\int \frac{3x-2}{\sqrt{2x+1}}\,dx$

Problems

In Problems 77–80, show the two integrals are equal using a substitution.

77. $\int_0^{\pi/3} 3\sin^2(3x)\,dx=\int_0^{\pi}\sin^2(y)\,dy$

78. $\int_1^2 2\ln(s^2+1)\,ds=\int_1^4 \frac{\ln(t+1)}{\sqrt{t}}\,dt$

79. $\int_1^e (\ln w)^3\,dw=\int_0^1 z^3e^z\,dz$

80. $\int_0^{\pi}(\pi-x)\cos x\,dx=\int_0^{\pi}x\cos(\pi-x)\,dx$

81. Using the substitution $w=x^2$, find a function $g(w)$ such that $\int_{\sqrt{a}}^{\sqrt{b}} dx=\int_a^b g(w)\,dw$ for all $0<a<b$.

82. Using the substitution $w=e^x$, find a function $g(w)$ such that $\int_a^b e^{-x}dx=\int_{e^a}^{e^b} g(w)\,dw$ for all $a<b$.

In Problems 83–87, explain why the two antiderivatives are really, despite their apparent dissimilarity, different expressions of the same problem. You do not need to evaluate the integrals.

83. $\displaystyle\int \frac{e^x\,dx}{1+e^{2x}}$ and $\displaystyle\int \frac{\cos x\,dx}{1+\sin^2 x}$

84. $\displaystyle\int \frac{\ln x}{x}\,dx$ and $\displaystyle\int x\,dx$

85. $\displaystyle\int e^{\sin x}\cos x\,dx$ and $\displaystyle\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx$

86. $\displaystyle\int (\sin x)^3\cos x\,dx$ and $\displaystyle\int (x^3+1)^3x^2\,dx$

87. $\displaystyle\int \sqrt{x+1}\,dx$ and $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

88. Integrate:

(a) $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$ **(b)** $\displaystyle\int \frac{1}{\sqrt{x+1}}\,dx$ **(c)** $\displaystyle\int \frac{1}{\sqrt{x}+1}\,dx$

89. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

(a) $\displaystyle\int x\sin(x^2)\,dx$ **(b)** $\displaystyle\int x^2\sin x\,dx$

(c) $\displaystyle\int \frac{x^2}{1+x^2}\,dx$ **(d)** $\displaystyle\int \frac{x}{(1+x^2)^2}\,dx$

(e) $\displaystyle\int x^3e^{x^2}\,dx$ **(f)** $\displaystyle\int \frac{\sin x}{2+\cos x}\,dx$

In Problems 90–96, find the exact area.

90. Under $f(x)=xe^{x^2}$ between $x=0$ and $x=2$.

91. Under $f(x)=1/(x+1)$ between $x=0$ and $x=2$.

92. Under $f(x)=\sinh(x/2)$ between $x=0$ and $x=2$.

93. Under $f(\theta)=(e^{\theta+1})^3$ for $0\le\theta\le 2$.

94. Between e^t and e^{t+1} for $0\le t\le 2$.

95. Between $y=e^x$, $y=3$, and the y-axis.

96. Under one arch of the curve $V(t)=V_0\sin(\omega t)$, where $V_0>0$ and $\omega>0$.

97. Find the exact average value of $f(x)=1/(x+1)$ on the interval $x=0$ to $x=2$. Sketch a graph showing the function and the average value.

98. Let $g(x)=f(2x)$. Show that the average value of f on the interval $[0,2b]$ is the same as the average value of g on the interval $[0,b]$.

99. Suppose $\int_0^2 g(t)\,dt=5$. Calculate the following:

(a) $\int_0^4 g(t/2)\,dt$ **(b)** $\int_0^2 g(2-t)\,dt$

100. Suppose $\int_0^1 f(t)\,dt=3$. Calculate the following:

(a) $\int_0^{0.5} f(2t)\,dt$ **(b)** $\int_0^1 f(1-t)\,dt$

(c) $\int_1^{1.5} f(3-2t)\,dt$

101. **(a)** Calculate exactly: $\int_{-\pi}^{\pi}\cos^2\theta\sin\theta\,d\theta$.

(b) Calculate the exact area under the curve $y=\cos^2\theta\sin\theta$ between $\theta=0$ and $\theta=\pi$.

102. Find $\int 4x(x^2+1)\,dx$ using two methods:

(a) Do the multiplication first, and then antidifferentiate.

(b) Use the substitution $w=x^2+1$.

(c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

103. **(a)** Find $\int\sin\theta\cos\theta\,d\theta$.

(b) You probably solved part (a) by making the substitution $w=\sin\theta$ or $w=\cos\theta$. (If not, go back and do it that way.) Now find $\int\sin\theta\cos\theta\,d\theta$ by making the *other* substitution.

(c) There is yet another way of finding this integral which involves the trigonometric identities

$$\sin(2\theta)=2\sin\theta\cos\theta$$
$$\cos(2\theta)=\cos^2\theta-\sin^2\theta.$$

Find $\int\sin\theta\cos\theta\,d\theta$ using one of these identities and then the substitution $w=2\theta$.

(d) You should now have three different expressions for the indefinite integral $\int\sin\theta\cos\theta\,d\theta$. Are they really different? Are they all correct? Explain.

104. Find the solution of the initial value problem

$$y'=\tan x+1,\quad y(0)=1.$$

105. Let $I_{m,n}=\int_0^1 x^m(1-x)^n dx$ for constant m,n. Show that $I_{m,n}=I_{n,m}$.

106. Let $f(t)$ be the velocity in meters/second of a car at time t in seconds. Give an integral for the change of position of the car

(a) For the time interval $0\le t\le 60$.

(b) In terms of T in minutes, for the same time interval.

107. Let $f(t)$ be the rate of flow, in cubic meters per hour, of a flooding river at time t in hours. Give an integral for the total flow of the river

(a) Over the 3-day period, $0\le t\le 72$.

(b) In terms of time T in days over the same 3-day period.

108. With t in years since 2000, the population, P, of the world in billions can be modeled by $P=6.1e^{0.012t}$.

(a) What does this model predict for the world population in 2010? In 2020?

(b) Use the Fundamental Theorem to predict the average population of the world between 2000 and 2010.

109. Oil is leaking out of a ruptured tanker at the rate of $r(t)=50e^{-0.02t}$ thousand liters per minute.

(a) At what rate, in liters per minute, is oil leaking out at $t=0$? At $t=60$?

(b) How many liters leak out during the first hour?

110. Throughout much of the 20^{th} century, the yearly consumption of electricity in the US increased exponentially at a continuous rate of 7% per year. Assume this trend continues and that the electrical energy consumed in 1900 was 1.4 million megawatt-hours.

(a) Write an expression for yearly electricity consumption as a function of time, t, in years since 1900.
(b) Find the average yearly electrical consumption throughout the 20^{th} century.
(c) During what year was electrical consumption closest to the average for the century?
(d) Without doing the calculation for part (c), how could you have predicted which half of the century the answer would be in?

111. An electric current, $I(t)$, flowing out of a capacitor, decays according to $I(t) = I_0e^{-t}$, where t is time. Find the charge, $Q(t)$, remaining in the capacitor at time t. The initial charge is Q_0 and $Q(t)$ is related to $I(t)$ by

$$Q'(t) = -I(t).$$

112. If we assume that wind resistance is proportional to velocity, then the downward velocity, v, of a body of mass m falling vertically is given by

$$v = \frac{mg}{k}\left(1 - e^{-kt/m}\right),$$

where g is the acceleration due to gravity and k is a constant. Find the height, h, above the surface of the earth as a function of time. Assume the body starts at height h_0.

113. If we assume that wind resistance is proportional to the square of velocity, then the downward velocity, v, of a falling body is given by

$$v = \sqrt{\frac{g}{k}}\left(\frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\right).$$

Use the substitution $w = e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}$ to find the height, h, of the body above the surface of the earth as a function of time. Assume the body starts at a height h_0.

114. **(a)** Between 1995 and 2005, ACME Widgets sold at a continuous rate of $R = R_0e^{0.15t}$ widgets per year, where t is time in years since January 1, 1995. Suppose they were selling widgets at a rate of 1000 per year on January 1, 1995. How many widgets did they sell between 1995 and 2005? How many did they sell if the rate on January 1, 1995 was 150,000,000 widgets per year?
(b) In the first case (1000 widgets per year on January 1, 1995), how long did it take for half the widgets in the ten year period to be sold? In the second case (150,000,000 widgets per year on January 1, 1995), when had half the widgets in the ten year period been sold?
(c) In 2005, ACME advertised that half the widgets it had sold in the previous ten years were still in use. Based on your answer to part (b), how long must a widget last in order to justify this claim?

115. The rate at which water is flowing into a tank is $r(t)$ gallons/minute, with t in minutes.

(a) Write an expression approximating the amount of water entering the tank during the interval from time t to time $t + \Delta t$, where Δt is small.
(b) Write a Riemann sum approximating the total amount of water entering the tank between $t = 0$ and $t = 5$. Write an exact expression for this amount.
(c) By how much has the amount of water in the tank changed between $t = 0$ and $t = 5$ if $r(t) = 20e^{0.02t}$?
(d) If $r(t)$ is as in part (c), and if the tank contains 3000 gallons initially, find a formula for $Q(t)$, the amount of water in the tank at time t.

7.2 INTEGRATION BY PARTS

The method of substitution reverses the chain rule. Now we introduce *integration by parts*, which is based on the product rule.

Example 1 Find $\int xe^x\,dx$.

Solution We are looking for a function whose derivative is xe^x. The product rule might lead us to guess xe^x, because we know that the derivative has two terms, one of which is xe^x:

$$\frac{d}{dx}(xe^x) = \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) = e^x + xe^x.$$

Of course, our guess is wrong because of the extra e^x. But we can adjust our guess by subtracting e^x; this leads us to try $xe^x - e^x$. Let's check it:

$$\frac{d}{dx}(xe^x - e^x) = \frac{d}{dx}(xe^x) - \frac{d}{dx}(e^x) = e^x + xe^x - e^x = xe^x.$$

It works, so $\int xe^x\,dx = xe^x - e^x + C$.

Example 2 Find $\int \theta \cos\theta \, d\theta$.

Solution We guess the antiderivative is $\theta \sin\theta$ and use the product rule to check:

$$\frac{d}{d\theta}(\theta\sin\theta) = \frac{d(\theta)}{d\theta}\sin\theta + \theta\frac{d}{d\theta}(\sin\theta) = \sin\theta + \theta\cos\theta.$$

To correct for the extra $\sin\theta$ term, we must subtract from our original guess something whose derivative is $\sin\theta$. Since $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$, we try:

$$\frac{d}{d\theta}(\theta\sin\theta + \cos\theta) = \frac{d}{d\theta}(\theta\sin\theta) + \frac{d}{d\theta}(\cos\theta) = \sin\theta + \theta\cos\theta - \sin\theta = \theta\cos\theta.$$

Thus, $\int \theta\cos\theta \, d\theta = \theta\sin\theta + \cos\theta + C$.

The General Formula for Integration by Parts

We can formalize the process illustrated in the last two examples in the following way. We begin with the product rule:

$$\frac{d}{dx}(uv) = u'v + uv'$$

where u and v are functions of x with derivatives u' and v', respectively. We rewrite this as:

$$uv' = \frac{d}{dx}(uv) - u'v$$

and then integrate both sides:

$$\int uv' \, dx = \int \frac{d}{dx}(uv)\, dx - \int u'v \, dx.$$

Since an antiderivative of $\frac{d}{dx}(uv)$ is just uv, we get the following formula:

Integration by Parts

$$\int uv' \, dx = uv - \int u'v \, dx.$$

This formula is useful when the integrand can be viewed as a product and when the integral on the right-hand side is simpler than that on the left. In effect, we were using integration by parts in the previous two examples. In Example 1, we let $xe^x = (x)\cdot(e^x) = uv'$, and choose $u = x$ and $v' = e^x$. Thus, $u' = 1$ and $v = e^x$, so

$$\int \underbrace{(x)}_{u}\underbrace{(e^x)}_{v'} \, dx = \underbrace{(x)}_{u}\underbrace{(e^x)}_{v} - \int \underbrace{(1)}_{u'}\underbrace{(e^x)}_{v} \, dx = xe^x - e^x + C.$$

So uv represents our first guess, and $\int u'v \, dx$ represents the correction to our guess.

Notice what would have happened if we took $v = e^x + C_1$. Then

$$\begin{aligned}\int xe^x \, dx &= x(e^x + C_1) - \int (e^x + C_1)\, dx \\ &= xe^x + C_1 x - e^x - C_1 x + C \\ &= xe^x - e^x + C,\end{aligned}$$

as before. Thus, it is not necessary to include an arbitrary constant in the antiderivative for v; any antiderivative will do.

What would have happened if we had picked u and v' the other way around? If $u = e^x$ and $v' = x$, then $u' = e^x$ and $v = x^2/2$. The formula for integration by parts then gives

$$\int xe^x \, dx = \frac{x^2}{2}e^x - \int \frac{x^2}{2} \cdot e^x \, dx,$$

which is true but not helpful since the integral on the right is worse than the one on the left. To use this method, we must choose u and v' to make the integral on the right easier to find than the integral on the left.

How to Choose u and v'

- Whatever you let v' be, you need to be able to find v.
- It helps if u' is simpler than u (or at least no more complicated than u).
- It helps if v is simpler than v' (or at least no more complicated than v').

If we pick $v' = x$ in Example 1, then $v = x^2/2$, which is certainly "worse" than v'.

There are some examples which don't look like good candidates for integration by parts because they don't appear to involve products, but for which the method works well. Such examples often involve $\ln x$ or the inverse trigonometric functions. Here is one:

Example 3 Find $\displaystyle\int_2^3 \ln x \, dx$.

Solution This does not look like a product unless we write $\ln x = (1)(\ln x)$. Then we might say $u = 1$ so $u' = 0$, which certainly makes things simpler. But if $v' = \ln x$, what is v? If we knew, we would not need integration by parts. Let's try the other way: if $u = \ln x$, $u' = 1/x$ and if $v' = 1$, $v = x$, so

$$\int_2^3 \underbrace{(\ln x)}_{u}\underbrace{(1)}_{v'} \, dx = \underbrace{(\ln x)}_{u}\underbrace{(x)}_{v}\Bigg|_2^3 - \int_2^3 \underbrace{\left(\frac{1}{x}\right)}_{u'} \cdot \underbrace{(x)}_{v} \, dx$$

$$= x\ln x\Big|_2^3 - \int_2^3 1 \, dx = (x \ln x - x)\Big|_2^3$$

$$= 3\ln 3 - 3 - 2\ln 2 + 2 = 3\ln 3 - 2\ln 2 - 1.$$

Notice that when doing a definite integral by parts, we must remember to put the limits of integration (here 2 and 3) on the uv term (in this case $x \ln x$) as well as on the integral $\int u'v \, dx$.

Example 4 Find $\displaystyle\int x^6 \ln x \, dx$.

Solution View $x^6 \ln x$ as uv' where $u = \ln x$ and $v' = x^6$. Then $v = \frac{1}{7}x^7$ and $u' = 1/x$, so integration by parts gives us:

$$\int x^6 \ln x \, dx = \int (\ln x) x^6 \, dx = (\ln x)\left(\frac{1}{7}x^7\right) - \int \frac{1}{7}x^7 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{7}x^7 \ln x - \frac{1}{7}\int x^6 \, dx$$

$$= \frac{1}{7}x^7 \ln x - \frac{1}{49}x^7 + C.$$

In Example 4 we did not choose $v' = \ln x$, because it is not immediately clear what v would be. In fact, we used integration by parts in Example 3 to find the antiderivative of $\ln x$. Also, using $u = \ln x$, as we have done, gives $u' = 1/x$, which can be considered simpler than $u = \ln x$. This shows that u does not have to be the first factor in the integrand (here x^6).

Example 5 Find $\int x^2 \sin 4x\, dx$.

Solution If we let $v' = \sin 4x$, then $v = -\frac{1}{4}\cos 4x$, which is no worse than v'. Also letting $u = x^2$, we get $u' = 2x$, which is simpler than $u = x^2$. Using integration by parts:

$$\begin{aligned}\int x^2 \sin 4x\, dx &= x^2\left(-\frac{1}{4}\cos 4x\right) - \int 2x\left(-\frac{1}{4}\cos 4x\right) dx \\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x\, dx.\end{aligned}$$

The trouble is we still have to grapple with $\int x\cos 4x\, dx$. This can be done by using integration by parts again with a new u and v, namely $u = x$ and $v' = \cos 4x$:

$$\begin{aligned}\int x\cos 4x\, dx &= x\left(\frac{1}{4}\sin 4x\right) - \int 1\cdot\frac{1}{4}\sin 4x\, dx \\ &= \frac{1}{4}x\sin 4x - \frac{1}{4}\cdot\left(-\frac{1}{4}\cos 4x\right) + C \\ &= \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.\end{aligned}$$

Thus,

$$\begin{aligned}\int x^2\sin 4x\, dx &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\int x\cos 4x\, dx \\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{2}\left(\frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C\right) \\ &= -\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + C.\end{aligned}$$

Notice that, in this example, each time we used integration by parts, the exponent of x went down by 1. In addition, when the arbitrary constant C is multiplied by $\frac{1}{2}$, it is still represented by C.

Example 6 Find $\int \cos^2\theta\, d\theta$.

Solution Using integration by parts with $u = \cos\theta$, $v' = \cos\theta$ gives $u' = -\sin\theta$, $v = \sin\theta$, so we get

$$\int \cos^2\theta\, d\theta = \cos\theta\sin\theta + \int \sin^2\theta\, d\theta.$$

Substituting $\sin^2\theta = 1 - \cos^2\theta$ leads to

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \cos\theta\sin\theta + \int(1-\cos^2\theta)\, d\theta \\ &= \cos\theta\sin\theta + \int 1\, d\theta - \int\cos^2\theta\, d\theta.\end{aligned}$$

Looking at the right side, we see that the original integral has reappeared. If we move it to the left, we get

$$2\int \cos^2\theta\,d\theta = \cos\theta\sin\theta + \int 1\,d\theta = \cos\theta\sin\theta + \theta + C.$$

Dividing by 2 gives

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta + C.$$

Problem 48 asks you to do this integral by another method.

The previous example illustrates a useful technique: Use integration by parts to transform the integral into an expression containing another copy of the same integral, possibly multiplied by a coefficient, then solve for the original integral. It may be necessary to apply integration by parts twice, as the next example shows.

Example 7 Use integration by parts twice to find $\int e^{2x}\sin(3x)\,dx$.

Solution Using integration by parts with $u = e^{2x}$ and $v' = \sin(3x)$ gives $u' = 2e^{2x}$, $v = -\frac{1}{3}\cos(3x)$, so we get

$$\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\,dx.$$

On the right side we have an integral similar to the original one, with the sine replaced by a cosine. Using integration by parts on that integral in the same way gives

$$\int e^{2x}\cos(3x)\,dx = \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\,dx.$$

Substituting this into the expression we obtained for the original integral gives

$$\begin{aligned}\int e^{2x}\sin(3x)\,dx &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\,dx\right)\\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}\int e^{2x}\sin(3x)\,dx.\end{aligned}$$

The right side now has a copy of the original integral, multiplied by $-4/9$. Moving it to the left, we get

$$\left(1+\frac{4}{9}\right)\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x).$$

Dividing through by the coefficient on the left, $(1 + 4/9) = 13/9$, we get

$$\begin{aligned}\int e^{2x}\sin(3x)\,dx &= \frac{9}{13}\left(-\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)\right)\\ &= \frac{1}{13}e^{2x}\left(2\sin(3x) - 3\cos(3x)\right) + C.\end{aligned}$$

Exercises and Problems for Section 7.2

Exercises

1. Use integration by parts to express $\int x^2 e^x \, dx$ in terms of

(a) $\int x^3 e^x \, dx$ **(b)** $\int x e^x \, dx$

2. Write $\arctan x = 1 \cdot \arctan x$ to find $\int \arctan x \, dx$.

Find the integrals in Exercises 3–30.

3. $\displaystyle\int t \sin t \, dt$

4. $\displaystyle\int t^2 \sin t \, dt$

5. $\displaystyle\int t e^{5t} \, dt$

6. $\displaystyle\int t^2 e^{5t} \, dt$

7. $\displaystyle\int p e^{-0.1p} \, dp$

8. $\displaystyle\int (z+1) e^{2z} \, dz$

9. $\displaystyle\int y \ln y \, dy$

10. $\displaystyle\int x^3 \ln x \, dx$

11. $\displaystyle\int q^5 \ln 5q \, dq$

12. $\displaystyle\int \theta^2 \cos 3\theta \, d\theta$

13. $\displaystyle\int \sin^2 \theta \, d\theta$

14. $\displaystyle\int \cos^2(3\alpha + 1) \, d\alpha$

15. $\displaystyle\int (\ln t)^2 \, dt$

16. $\displaystyle\int y\sqrt{y+3} \, dy$

17. $\displaystyle\int (t+2)\sqrt{2+3t} \, dt$

18. $\displaystyle\int (\theta+1)\sin(\theta+1) \, d\theta$

19. $\displaystyle\int \frac{z}{e^z} \, dz$

20. $\displaystyle\int \frac{\ln x}{x^2} \, dx$

21. $\displaystyle\int \frac{y}{\sqrt{5-y}} \, dy$

22. $\displaystyle\int \frac{t+7}{\sqrt{5-t}} \, dt$

23. $\displaystyle\int x(\ln x)^4 \, dx$

24. $\displaystyle\int \arcsin w \, dw$

25. $\displaystyle\int \arctan 7z \, dz$

26. $\displaystyle\int x \arctan x^2 \, dx$

27. $\displaystyle\int x^3 e^{x^2} \, dx$

28. $\displaystyle\int x^5 \cos x^3 \, dx$

29. $\displaystyle\int x \sinh x \, dx$

30. $\displaystyle\int (x-1)\cosh x \, dx$

Evaluate the integrals in Exercises 31–38 both exactly [e.g. $\ln(3\pi)$] and numerically [e.g. $\ln(3\pi) \approx 2.243$].

31. $\displaystyle\int_1^5 \ln t \, dt$

32. $\displaystyle\int_3^5 x \cos x \, dx$

33. $\displaystyle\int_0^{10} z e^{-z} \, dz$

34. $\displaystyle\int_1^3 t \ln t \, dt$

35. $\displaystyle\int_0^1 \arctan y \, dy$

36. $\displaystyle\int_0^5 \ln(1+t) \, dt$

37. $\displaystyle\int_0^1 \arcsin z \, dz$

38. $\displaystyle\int_0^1 u \arcsin u^2 \, du$

39. For each of the following integrals, indicate whether integration by substitution or integration by parts is more appropriate. Do not evaluate the integrals.

(a) $\displaystyle\int x \sin x \, dx$ **(b)** $\displaystyle\int \frac{x^2}{1+x^3} \, dx$

(c) $\displaystyle\int x e^{x^2} \, dx$ **(d)** $\displaystyle\int x^2 \cos(x^3) \, dx$

(e) $\displaystyle\int \frac{1}{\sqrt{3x+1}} \, dx$ **(f)** $\displaystyle\int x^2 \sin x \, dx$

(g) $\displaystyle\int \ln x \, dx$

40. Find $\int_1^2 \ln x \, dx$ numerically. Find $\int_1^2 \ln x \, dx$ using antiderivatives. Check that your answers agree.

Problems

In Problems 41–46, find the exact area.

41. Under $y = te^{-t}$ for $0 \le t \le 2$.

42. Under $f(z) = \arctan z$ for $0 \le z \le 2$.

43. Under $f(y) = \arcsin y$ for $0 \le y \le 1$.

44. Between $y = \ln x$ and $y = \ln(x^2)$ for $1 \le x \le 2$.

45. Between $f(t) = \ln(t^2 - 1)$ and $g(t) = \ln(t-1)$ for $2 \le t \le 3$.

46. Under the first arch of $f(x) = x \sin x$.

47. In Exercise 13, you evaluated $\int \sin^2 \theta \, d\theta$ using integration by parts. (If you did not do it by parts, do so now!) Redo this integral using the identity $\sin^2 \theta = (1 - \cos 2\theta)/2$. Explain any differences in the form of the answer obtained by the two methods.

48. Compute $\int \cos^2 \theta \, d\theta$ in two different ways and explain any differences in the form of your answers. (The identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ may be useful.)

49. Use integration by parts twice to find $\int e^x \sin x \, dx$.

50. Use integration by parts twice to find $\int e^\theta \cos \theta \, d\theta$.

51. Use the results from Problems 49 and 50 and integration by parts to find $\int x e^x \sin x \, dx$.

52. Use the results from Problems 49 and 50 and integration by parts to find $\int \theta e^{\theta} \cos\theta \, d\theta$.

In Problems 53–56, derive the given formulas.

53. $$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

54. $$\int x^n \cos ax \, dx = \frac{1}{a} x^n \sin ax - \frac{n}{a} \int x^{n-1} \sin ax \, dx$$

55. $$\int x^n \sin ax \, dx = -\frac{1}{a} x^n \cos ax + \frac{n}{a} \int x^{n-1} \cos ax \, dx$$

56. $$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

57. Integrating $e^{ax} \sin bx$ by parts twice yields a result of the form
$$\int e^{ax} \sin bx \, dx = e^{ax}(A \sin bx + B \cos bx) + C.$$

(a) Find the constants A and B in terms of a and b. [Hint: Don't actually perform the integration by parts.]
(b) Evaluate $\int e^{ax} \cos bx \, dx$ by modifying the method in part (a). [Again, it is not necessary to perform the integration, as the result has the same form as that in part (a).]

58. Estimate $\int_0^{10} f(x) g'(x) \, dx$ if $f(x) = x^2$ and g has the values in the following table.

x	0	2	4	6	8	10
$g(x)$	2.3	3.1	4.1	5.5	5.9	6.1

59. Let f be twice differentiable with $f(0) = 6$, $f(1) = 5$, and $f'(1) = 2$. Evaluate the integral $\int_0^1 x f''(x) dx$.

60. Let $F(a)$ be the area under the graph of $y = x^2 e^{-x}$ between $x = 0$ and $x = a$, for $a > 0$.

(a) Find a formula for $F(a)$.
(b) Is F an increasing or decreasing function?
(c) Is F concave up or concave down for $0 < a < 2$?

61. The concentration, C, in ng/ml, of a drug in the blood as a function of the time, t, in hours since the drug was administered is given by $C = 15te^{-0.2t}$. The area under the concentration curve is a measure of the overall effect of the drug on the body, called the bioavailability. Find the bioavailability of the drug between $t = 0$ and $t = 3$.

62. The voltage, V, in an electric circuit is given as a function of time, t, by
$$V = V_0 \cos(\omega t + \phi).$$
Each of the positive constants, V_0, ω, ϕ is increased (while the other two are held constant). What is the effect of each increase on the following quantities:

(a) The maximum value of V?
(b) The maximum value of dV/dt?
(c) The average value of V^2 over one period of V?

63. During a surge in the demand for electricity, the rate, r, at which energy is used can be approximated by
$$r = te^{-at},$$
where t is the time in hours and a is a positive constant.

(a) Find the total energy, E, used in the first T hours. Give your answer as a function of a.
(b) What happens to E as $T \to \infty$?

64. Use integration by parts on $\int_0^x f''(t)(x - t) \, dt$ with $u(t) = x - t$, $v'(t) = f''(t)$ to show that
$$f(x) - f(0) = f'(0)x + \int_0^x f''(t)(x - t) \, dt.$$

65. In describing the behavior of an electron, we use wave functions $\Psi_1, \Psi_2, \Psi_3, \ldots$ of the form
$$\Psi_n(x) = C_n \sin(n\pi x) \qquad \text{for } n = 1, 2, 3, \ldots$$
where x is the distance from a fixed point and C_n is a positive constant.

(a) Find C_1 so that Ψ_1 satisfies
$$\int_0^1 (\Psi_1(x))^2 \, dx = 1.$$
This is called normalizing the wave function Ψ_1.
(b) For any integer n, find C_n so that Ψ_n is normalized.

7.3 TABLES OF INTEGRALS

Since so few functions have elementary antiderivatives, they have been compiled in a list called a table of integrals.[1] A short table of indefinite integrals is given inside the back cover of this book. The key to using these tables is being able to recognize the general class of function that you are trying to integrate, so you can know in what section of the table to look.

Warning: This section involves long division of polynomials and completing the square. You may want to review these topics!

[1]See, for example, *CRC Standard Mathematical Tables* (Boca Raton, Fl: CRC Press). Many computer programs and calculators can compute antiderivatives as well.

Using the Table of Integrals

Part I of the table inside the back cover gives the antiderivatives of the basic functions x^n, a^x, $\ln x$, $\sin x$, $\cos x$, and $\tan x$. (The antiderivative for $\ln x$ is found using integration by parts and is a special case of the more general formula III-13.) Most of these are already familiar.

Part II of the table contains antiderivatives of functions involving products of e^x, $\sin x$, and $\cos x$. All of these antiderivatives were obtained using integration by parts.

Example 1 Find $\displaystyle\int \sin 7z \sin 3z \, dz$.

Solution Since the integrand is the product of two sines, we should use II-10 in the table,

$$\int \sin 7z \sin 3z \, dz = -\frac{1}{40}(7\cos 7z \sin 3z - 3\cos 3z \sin 7z) + C.$$

Part III of the table contains antiderivatives for products of a polynomial and e^x, $\sin x$, or $\cos x$. It also has an antiderivative for $x^n \ln x$, which can easily be used to find the antiderivatives of the product of a general polynomial and $\ln x$. Each *reduction formula* is used repeatedly to reduce the degree of the polynomial until a zero degree polynomial is obtained.

Example 2 Find $\displaystyle\int (x^5 + 2x^3 - 8)e^{3x} \, dx$.

Solution Since $p(x) = x^5 + 2x^3 - 8$ is a polynomial multiplied by e^{3x}, this is of the form in III-14. Now $p'(x) = 5x^4 + 6x^2$ and $p''(x) = 20x^3 + 12x$, and so on, giving

$$\begin{aligned}\int (x^5 + 2x^3 - 8)e^{3x} \, dx \;=\; e^{3x} \bigg[&\frac{1}{3}(x^5 + 2x^3 - 8) - \frac{1}{9}(5x^4 + 6x^2) + \frac{1}{27}(20x^3 + 12x) \\ &- \frac{1}{81}(60x^2 + 12) + \frac{1}{243}(120x) - \frac{1}{729} \cdot 120\bigg] + C.\end{aligned}$$

Here we have the successive derivatives of the original polynomial $x^5 + 2x^3 - 8$, occurring with alternating signs and multiplied by successive powers of 1/3.

Part IV of the table contains reduction formulas for the antiderivatives of $\cos^n x$ and $\sin^n x$, which can be obtained by integration by parts. When n is a positive integer, formulas IV-17 and IV-18 can be used repeatedly to reduce the power n until it is 0 or 1.

Example 3 Find $\displaystyle\int \sin^6 \theta \, d\theta$.

Solution Use IV-17 repeatedly:

$$\int \sin^6 \theta \, d\theta = -\frac{1}{6}\sin^5 \theta \cos \theta + \frac{5}{6}\int \sin^4 \theta \, d\theta$$

$$\int \sin^4 \theta \, d\theta = -\frac{1}{4}\sin^3 \theta \cos \theta + \frac{3}{4}\int \sin^2 \theta \, d\theta$$

$$\int \sin^2 \theta \, d\theta = -\frac{1}{2}\sin \theta \cos \theta + \frac{1}{2}\int 1 \, d\theta.$$

Calculate $\int \sin^2 \theta \, d\theta$ first, and use this to find $\int \sin^4 \theta \, d\theta$; then calculate $\int \sin^6 \theta \, d\theta$. Putting this all together, we get

$$\int \sin^6 \theta \, d\theta = -\frac{1}{6}\sin^5 \theta \cos \theta - \frac{5}{24}\sin^3 \theta \cos \theta - \frac{15}{48}\sin \theta \cos \theta + \frac{15}{48}\theta + C.$$

The last item in **Part IV** of the table is not a formula: it is advice on how to antidifferentiate products of integer powers of $\sin x$ and $\cos x$. There are various techniques to choose from, depending on the nature (odd or even, positive or negative) of the exponents.

Example 4 Find $\int \cos^3 t \sin^4 t \, dt$.

Solution Here the exponent of $\cos t$ is odd, so IV-23 recommends making the substitution $w = \sin t$. Then $dw = \cos t \, dt$. To make this work, we'll have to separate off one of the cosines to be part of dw. Also, the remaining even power of $\cos t$ can be rewritten in terms of $\sin t$ by using $\cos^2 t = 1 - \sin^2 t = 1 - w^2$, so that

$$\begin{aligned}\int \cos^3 t \sin^4 t \, dt &= \int \cos^2 t \sin^4 t \cos t \, dt \\ &= \int (1 - w^2) w^4 \, dw = \int (w^4 - w^6) \, dw \\ &= \frac{1}{5} w^5 - \frac{1}{7} w^7 + C = \frac{1}{5} \sin^5 t - \frac{1}{7} \sin^7 t + C.\end{aligned}$$

Example 5 Find $\int \cos^2 x \sin^4 x \, dx$.

Solution In this example, both exponents are even. The advice given in IV-23 is to convert to all sines or all cosines. We'll convert to all sines by substituting $\cos^2 x = 1 - \sin^2 x$, and then we'll multiply out the integrand:

$$\int \cos^2 x \sin^4 x \, dx = \int (1 - \sin^2 x) \sin^4 x \, dx = \int \sin^4 x \, dx - \int \sin^6 x \, dx.$$

In Example 3 we found $\int \sin^4 x \, dx$ and $\int \sin^6 x \, dx$. Put them together to get

$$\begin{aligned}\int \cos^2 x \sin^4 x \, dx &= -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x \\ &\quad - \left(-\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{15}{48} \sin x \cos x + \frac{15}{48} x \right) + C \\ &= \frac{1}{6} \sin^5 x \cos x - \frac{1}{24} \sin^3 x \cos x - \frac{3}{48} \sin x \cos x + \frac{3}{48} x + C.\end{aligned}$$

The last two parts of the table are concerned with quadratic functions: **Part V** has expressions with quadratic denominators; **Part VI** contains square roots of quadratics. The quadratics that appear in these formulas are of the form $x^2 \pm a^2$ or $a^2 - x^2$, or in factored form $(x - a)(x - b)$, where a and b are different constants. Quadratics can be converted to these forms by factoring or completing the square.

Preparing to Use the Table: Transforming the Integrand

To use the integral table, we often need to manipulate or reshape integrands to fit entries in the table. The manipulations that tend to be useful are factoring, long division, completing the square, and substitution.

Using Factoring

Example 6 Find $\int \frac{3x + 7}{x^2 + 6x + 8} \, dx$.

Solution In this case we factor the denominator to get it into a form in the table:

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

Now in V-27 we let $a = -2$, $b = -4$, $c = 3$, and $d = 7$, to obtain

$$\int \frac{3x+7}{x^2+6x+8}\,dx = \frac{1}{2}(\ln|x+2| - (-5)\ln|x+4|) + C.$$

Long Division

Example 7 Find $\displaystyle\int \frac{x^2}{x^2+4}\,dx$.

Solution A good rule of thumb when integrating a rational function whose numerator has a degree greater than or equal to that of the denominator is to start by doing *long division*. This results in a polynomial plus a simpler rational function as a remainder. Performing long division here, we obtain:

$$\frac{x^2}{x^2+4} = 1 - \frac{4}{x^2+4}.$$

Then, by V-24 with $a = 2$, we obtain:

$$\int \frac{x^2}{x^2+4}\,dx = \int 1\,dx - 4\int \frac{1}{x^2+4}\,dx = x - 4\cdot\frac{1}{2}\arctan\frac{x}{2} + C = x - 2\arctan\frac{x}{2} + C.$$

Completing the Square to Rewrite the Quadratic in the Form $w^2 + a^2$

Example 8 Find $\displaystyle\int \frac{1}{x^2+6x+14}\,dx$.

Solution By completing the square, we can get this integrand into a form in the table:

$$\begin{aligned} x^2+6x+14 &= (x^2+6x+9) - 9 + 14 \\ &= (x+3)^2 + 5. \end{aligned}$$

Let $w = x + 3$. Then $dw = dx$ and so the substitution gives

$$\int \frac{1}{x^2+6x+14}\,dx = \int \frac{1}{w^2+5}\,dw = \frac{1}{\sqrt{5}}\arctan\frac{w}{\sqrt{5}} + C = \frac{1}{\sqrt{5}}\arctan\frac{x+3}{\sqrt{5}} + C,$$

where the antidifferentiation uses V-24 with $a^2 = 5$.

Substitution

Example 9 Find $\displaystyle\int e^t \sin(5t+7)\,dt$.

Solution This looks similar to II-8. To make the correspondence more complete, let's try the substitution $w = 5t + 7$. Then $dw = 5\,dt$, so $dt = \frac{1}{5}\,dw$. Also, $t = (w-7)/5$. Then the integral becomes

$$\begin{aligned} \int e^t \sin(5t+7)\,dt &= \int e^{(w-7)/5}\sin w\,\frac{dw}{5} \\ &= \frac{e^{-7/5}}{5}\int e^{w/5}\sin w\,dw. \qquad \text{(Since } e^{(w-7)/5} = e^{w/5}e^{-7/5} \text{ and } e^{-7/5} \text{ is a constant)} \end{aligned}$$

Now we can use II-8 with $a = \frac{1}{5}$ and $b = 1$ to write

$$\int e^{w/5}\sin w\,dw = \frac{1}{(\frac{1}{5})^2 + 1^2}e^{w/5}\left(\frac{\sin w}{5} - \cos w\right) + C,$$

so

$$\int e^t \sin(5t+7)\,dt = \frac{e^{-7/5}}{5}\left(\frac{25}{26}e^{(5t+7)/5}\left(\frac{\sin(5t+7)}{5} - \cos(5t+7)\right)\right) + C$$
$$= \frac{5e^t}{26}\left(\frac{\sin(5t+7)}{5} - \cos(5t+7)\right) + C.$$

Exercises and Problems for Section 7.3

Exercises

For Exercises 1–40, antidifferentiate using the table of integrals. You may need to transform the integrand first.

1. $\int e^{-3\theta}\cos\theta\,d\theta$

2. $\int x^5 \ln x\,dx$

3. $\int x^3 \sin 5x\,dx.$

4. $\int (x^2+3)\ln x\,dx.$

5. $\int (x^3+5)^2\,dx.$

6. $\int \sin w \cos^4 w\,dw$

7. $\int \sin^4 x\,dx$

8. $\int x^3 e^{2x}\,dx$

9. $\int \frac{1}{3+y^2}\,dy$

10. $\int \frac{dx}{9x^2+16}$

11. $\int \frac{dx}{\sqrt{25-16x^2}}$

12. $\int \frac{dx}{\sqrt{9x^2+25}}$

13. $\int \sin 3\theta \cos 5\theta\,d\theta$

14. $\int \sin 3\theta \sin 5\theta\,d\theta$

15. $\int x^2 e^{3x}\,dx$

16. $\int x^2 e^{x^3}\,dx$

17. $\int x^4 e^{3x}\,dx$

18. $\int u^5 \ln(5u)\,du$

19. $\int \frac{1}{\cos^3 x}\,dx$

20. $\int \frac{t^2+1}{t^2-1}\,dt$

21. $\int x^3 \sin x^2\,dx$

22. $\int \cos 2y \cos 7y\,dy$

23. $\int y^2 \sin 2y\,dy$

24. $\int e^{5x}\sin 3x\,dx$

25. $\int \frac{1}{\cos^5 x}\,dx.$

26. $\int \frac{1}{\sin^2 2\theta}\,d\theta$

27. $\int \frac{1}{\sin^3 3\theta}\,d\theta$

28. $\int \frac{1}{\cos^4 7x}\,dx$

29. $\int \frac{1}{x^2+4x+3}\,dx$

30. $\int \frac{1}{x^2+4x+4}\,dx$

31. $\int \frac{dz}{z(z-3)}$

32. $\int \frac{dy}{4-y^2}$

33. $\int \frac{1}{1+(z+2)^2}\,dz$

34. $\int \frac{1}{y^2+4y+5}\,dy$

35. $\int \tan^4 x\,dx$

36. $\int \sin^3 x\,dx$

37. $\int \sin^3 3\theta \cos^2 3\theta\,d\theta$

38. $\int z e^{2z^2}\cos(2z^2)\,dz$

39. $\int \sinh^3 x \cosh^2 x\,dx$

40. $\int \sinh^2 x \cosh^3 x\,dx$

For Problems 41–50, evaluate the definite integrals. Whenever possible, use the Fundamental Theorem of Calculus, perhaps after a substitution. Otherwise, use numerical methods.

41. $\int_0^1 \sqrt{3-x^2}\,dx$

42. $\int_{-\pi}^{\pi} \sin 5x \cos 6x\,dx$

43. $\int_1^2 (x-2x^3)\ln x\,dx$

44. $\int_0^{\pi/12} \sin(3\alpha)\,d\alpha$

45. $\int_0^1 \frac{1}{x^2+2x+1}\,dx$

46. $\int_0^1 \frac{dx}{x^2+2x+5}$

47. $\int_{\pi/4}^{\pi/3} \frac{dx}{\sin^3 x}$

48. $\int_{-3}^{-1} \frac{dx}{\sqrt{x^2+6x+10}}$

49. $\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx$

50. $\int_0^1 \frac{(x+2)}{(x+2)^2+1}\,dx$

Problems

51. Show that for all integers m and n, with $m \neq \pm n$, $\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = 0$.

52. Show that for all integers m and n, with $m \neq \pm n$, $\int_{-\pi}^{\pi} \cos m\theta \cos n\theta \, d\theta = 0$.

53. The voltage, V, in an electrical outlet is given as a function of time, t, by the function $V = V_0 \cos(120\pi t)$, where V is in volts and t is in seconds, and V_0 is a positive constant representing the maximum voltage.

(a) What is the average value of the voltage over 1 second?

(b) Engineers do not use the average voltage. They use the root mean square voltage defined by $\overline{V} = \sqrt{\text{average of } (V^2)}$. Find $\overline{V}$ in terms of V_0. (Take the average over 1 second.)

(c) The standard voltage in an American house is 110 volts, meaning that $\overline{V} = 110$. What is V_0?

54. For some constants A and B, the rate of production, $R(t)$, of oil in a new oil well is modeled by:

$$R(t) = A + Be^{-t} \sin(2\pi t)$$

where t is the time in years, A is the equilibrium rate, and B is the "variability" coefficient.

(a) Find the total amount of oil produced in the first N years of operation. (Take N to be an integer.)

(b) Find the average amount of oil produced per year over the first N years (where N is an integer).

(c) From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$.

(d) Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations.

(e) Do you think it is reasonable to expect this model to hold over a very long period? Why or why not?

7.4 ALGEBRAIC IDENTITIES AND TRIGONOMETRIC SUBSTITUTIONS

Although not all functions have elementary antiderivatives, many do. In this section we introduce two powerful methods of integration which show that large classes of functions have elementary antiderivatives. The first is the method of partial fractions, which depends on an algebraic identity, and allows us to integrate rational functions. The second is the method of trigonometric substitutions, which allows us to handle expressions involving the square root of a quadratic polynomial. Some of the formulas in the table of integrals can be derived using the techniques of this section.

Method of Partial Fractions

The integral of some rational functions can be obtained by splitting the integrand into *partial fractions*. For example, to find

$$\int \frac{1}{(x-2)(x-5)} \, dx,$$

the integrand is split into partial fractions with denominators $(x-2)$ and $(x-5)$. We write

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5},$$

where A and B are constants that need to be found. Multiplying by $(x-2)(x-5)$ gives the identity

$$1 = A(x-5) + B(x-2)$$

so

$$1 = (A+B)x - 5A - 2B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficients of x on both sides must be equal. So

$$\begin{aligned} -5A - 2B &= 1 \\ A + B &= 0. \end{aligned}$$

Solving these equations gives $A = -1/3$, $B = 1/3$. Thus,

$$\frac{1}{(x-2)(x-5)} = \frac{-1/3}{x-2} + \frac{1/3}{x-5}.$$

Example 1 Use partial fractions to integrate $\int \frac{1}{(x-2)(x-5)}\,dx$.

Solution We split the integrand into partial fractions, each of which can be integrated:

$$\int \frac{1}{(x-2)(x-5)}\,dx = \int \left(\frac{-1/3}{x-2} + \frac{1/3}{x-5}\right) dx = -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C.$$

You can check that using formula V-26 in the integral table gives the same result.

This method can be used to derive formulas V-26 and V-27 in the integral table. A similar method works whenever the denominator of the integrand factors into distinct linear factors and the numerator has degree less than the denominator.

Example 2 Find $\int \frac{x+2}{x^2+x}\,dx$.

Solution We factor the denominator and split the integrand into partial fractions:

$$\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}.$$

Multiplying by $x(x+1)$ gives the identity

$$\begin{aligned} x+2 &= A(x+1) + Bx \\ &= (A+B)x + A. \end{aligned}$$

Equating constant terms and coefficients of x gives $A = 2$ and $A + B = 1$, so $B = -1$. Then we split the integrand into two parts and integrate:

$$\int \frac{x+2}{x^2+x}\,dx = \int \left(\frac{2}{x} - \frac{1}{x+1}\right) dx = 2\ln|x| - \ln|x+1| + C.$$

The next example illustrates what to do if there is a repeated factor in the denominator.

Example 3 Calculate $\int \frac{10x-2x^2}{(x-1)^2(x+3)}\,dx$ using partial fractions of the form $\frac{A}{x-1}, \frac{B}{(x-1)^2}, \frac{C}{x+3}$.

Solution We are given that the squared factor, $(x-1)^2$, leads to partial fractions of the form:

$$\frac{10x-2x^2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

Multiplying through by $(x-1)^2(x+3)$ gives

$$\begin{aligned} 10x - 2x^2 &= A(x-1)(x+3) + B(x+3) + C(x-1)^2 \\ &= (A+C)x^2 + (2A+B-2C)x - 3A + 3B + C. \end{aligned}$$

Equating the coefficients of x^2 and x and the constant terms, we get the simultaneous equations:

$$\begin{aligned} A + C &= -2 \\ 2A + B - 2C &= 10 \\ -3A + 3B + C &= 0 \end{aligned}$$

Solving gives $A = 1, B = 2, C = -3$. Thus, we obtain three integrals which can be evaluated:

$$\begin{aligned}\int \frac{10x - 2x^2}{(x-1)^2(x+3)}\,dx &= \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{3}{x+3}\right)dx\\ &= \ln|x-1| - \frac{2}{(x-1)} - 3\ln|x+3| + K.\end{aligned}$$

For the second integral, we use the fact that $\int 2/(x-1)^2 dx = 2\int (x-1)^{-2}dx = -2(x-1)^{-1} + K$.

If there is a quadratic in the denominator which cannot be factored, we must allow a numerator of the form $Ax + B$ in the numerator, as the next example shows.

Example 4 Find $\displaystyle\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx$ using partial fractions of the form $\dfrac{Ax+B}{x^2+1}$ and $\dfrac{C}{x-2}$.

Solution We are given that the quadratic denominator, (x^2+1), which cannot be factored further, has a numerator of the form $Ax + B$, so we have

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}.$$

Multiplying by $(x^2+1)(x-2)$ gives

$$\begin{aligned}2x^2 - x - 1 &= (Ax+B)(x-2) + C(x^2+1)\\ &= (A+C)x^2 + (B-2A)x + C - 2B.\end{aligned}$$

Equating the coefficients of x^2 and x and the constant terms gives the simultaneous equations

$$\begin{aligned}A + C &= 2\\ B - 2A &= -1\\ C - 2B &= -1.\end{aligned}$$

Solving gives $A = B = C = 1$, so we rewrite the integral as follows:

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx = \int \left(\frac{x+1}{x^2+1} + \frac{1}{x-2}\right)dx.$$

This identity is useful provided we can perform the integration on the right-hand side. The first integral can be done if it is split into two; the second integral is similar to those in the previous examples. We have

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x-2}\,dx.$$

To calculate $\int (x/(x^2+1))\,dx$, substitute $w = x^2 + 1$, or guess and check. The final result is

$$\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx = \frac{1}{2}\ln|x^2+1| + \arctan x + \ln|x-2| + K.$$

Finally, the next example shows what to do if the numerator has degree larger than the denominator.

Example 5 Calculate $\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\,dx$ using long division.

Solution The degree of the numerator is greater than the degree of the denominator, so we divide first:

$$\frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10} = \frac{x(x^2 - 7x + 10) + 1}{x^2 - 7x + 10} = x + \frac{1}{x^2 - 7x + 10}.$$

The remainder, in this case $1/(x^2 - 7x + 10)$, is a rational function on which we try to use partial fractions. We have

$$\frac{1}{x^2 - 7x + 10} = \frac{1}{(x-2)(x-5)}$$

so in this case we use the result of Example 1 to obtain

$$\int \frac{x^3 - 7x^2 + 10x + 1}{x^2 - 7x + 10}\,dx = \int \left(x + \frac{1}{(x-2)(x-5)}\right) dx = \frac{x^2}{2} - \frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C.$$

Many, though not all, rational functions can be integrated by the strategy suggested by the previous examples.

Strategy for Integrating a Rational Function, $\frac{P(x)}{Q(x)}$

- If degree of $P(x) \geq$ degree of $Q(x)$, try long division and the method of partial fractions on the remainder.
- If $Q(x)$ is the product of distinct linear factors, use partial fractions of the form
$$\frac{A}{(x-c)}.$$
- If $Q(x)$ contains a repeated linear factor, $(x-c)^n$, use partial fractions of the form
$$\frac{A_1}{(x-c)} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}.$$
- If $Q(x)$ contains an unfactorable quadratic $q(x)$, try a partial fraction of the form
$$\frac{Ax+B}{q(x)}.$$

To use this method, we need to be able to integrate each partial fraction. We know how to integrate terms of the form $A/(x-c)^n$ using the power rule (if $n > 1$) and logarithms (if $n = 1$). Next we see how to integrate terms of the form $(Ax+B)/q(x)$, where $q(x)$ is an unfactorable quadratic.

Trigonometric Substitutions

Section 7.1 showed how substitutions could be used to transform complex integrands. Now we see how substitution of $\sin\theta$ or $\tan\theta$ can be used for integrands involving square roots of quadratics or unfactorable quadratics.

Sine Substitutions

Substitutions involving $\sin\theta$ make use of the Pythagorean identity, $\cos^2\theta + \sin^2\theta = 1$, to simplify an integrand involving $\sqrt{a^2 - x^2}$.

Example 6 Find $\int \frac{1}{\sqrt{1-x^2}}\,dx$ using the substitution $x=\sin\theta$.

Solution If $x=\sin\theta$, then $dx=\cos\theta\,d\theta$, and substitution gives

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta.$$

Now either $\sqrt{\cos^2\theta}=\cos\theta$ or $\sqrt{\cos^2\theta}=-\cos\theta$ depending on the values taken by θ. If we choose $-\pi/2\le\theta\le\pi/2$, then $\cos\theta\ge 0$, so $\sqrt{\cos^2\theta}=\cos\theta$. Then

$$\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{\cos\theta}{\cos\theta}\,d\theta = \int 1\,d\theta = \theta + C = \arcsin x + C.$$

The last step uses the fact that $\theta=\arcsin x$ if $x=\sin\theta$ and $-\pi/2\le\theta\le\pi/2$.

From now on, when we substitute $\sin\theta$, we assume that the interval $-\pi/2\le\theta\le\pi/2$ has been chosen. Notice that the previous example is the case $a=1$ of VI-28 in the table of integrals. The next example illustrates how to choose the substitution when $a\ne 1$.

Example 7 Use a trigonometric substitution to find $\int \frac{1}{\sqrt{4-x^2}}\,dx$.

Solution This time we choose $x=2\sin\theta$, with $-\pi/2\le\theta\le\pi/2$, so that $4-x^2$ becomes a perfect square:

$$\sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}=2\sqrt{1-\sin^2\theta}=2\sqrt{\cos^2\theta}=2\cos\theta.$$

Then $dx=2\cos\theta\,d\theta$, so substitution gives

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{2\cos\theta}2\cos\theta\,d\theta = \int 1\,d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C.$$

The general rule for choosing a sine substitution is:

> To simplify $\sqrt{a^2-x^2}$, for constant a, try $x=a\sin\theta$, with $-\pi/2\le\theta\le\pi/2$.

Notice $\sqrt{a^2-x^2}$ is only defined on the interval $[-a,a]$. Assuming that the domain of the integrand is $[-a,a]$, the substitution $x=a\sin\theta$, with $-\pi/2\le\theta\le\pi/2$, is valid for all x in the domain, because its range is $[-a,a]$ and it has an inverse $\theta=\arcsin(x/a)$ on $[-a,a]$.

Example 8 Find the area of the ellipse $4x^2+y^2=9$.

Solution Solving for y shows that $y=\sqrt{9-4x^2}$ gives the upper half of the ellipse. From Figure 7.1, we see that

$$\text{Area } = 4\int_0^{3/2}\sqrt{9-4x^2}\,dx.$$

To decide which trigonometric substitution to use, we write the integrand as

$$\sqrt{9-4x^2} = 2\sqrt{\frac{9}{4}-x^2} = 2\sqrt{\left(\frac{3}{2}\right)^2-x^2}.$$

This suggests that we should choose $x = (3/2)\sin\theta$, so that $dx = (3/2)\cos\theta\, d\theta$ and

$$\sqrt{9-4x^2} = 2\sqrt{\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\sin^2\theta} = 2\left(\frac{3}{2}\right)\sqrt{1-\sin^2\theta} = 3\cos\theta.$$

When $x = 0$, $\theta = 0$, and when $x = 3/2$, $\theta = \pi/2$, so

$$4\int_0^{3/2}\sqrt{9-4x^2}\,dx = 4\int_0^{\pi/2} 3\cos\theta\left(\frac{3}{2}\right)\cos\theta\,d\theta = 18\int_0^{\pi/2}\cos^2\theta\,d\theta.$$

Using Example 6 on page 344 or table of integrals IV-18, we find

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta + C.$$

So we have

$$\text{Area } = 4\int_0^{3/2}\sqrt{9-4x^2}\,dx = \frac{18}{2}(\cos\theta\sin\theta+\theta)\Big|_0^{\pi/2} = 9\left(0+\frac{\pi}{2}\right) = \frac{9\pi}{2}.$$

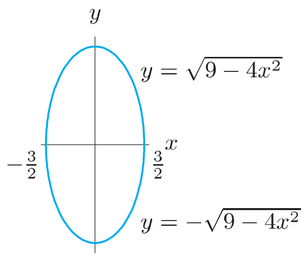


Figure 7.1: The ellipse $4x^2+y^2=9$

Tangent Substitutions

Integrals involving a^2+x^2 may be simplified by a substitution involving $\tan\theta$ and the trigonometric identities $\tan\theta = \sin\theta/\cos\theta$ and $\cos^2\theta+\sin^2\theta = 1$.

Example 9 Find $\displaystyle\int \frac{1}{x^2+9}\,dx$ using the substitution $x = 3\tan\theta$.

Solution If $x = 3\tan\theta$, then $dx = (3/\cos^2\theta)\,d\theta$, so

$$\int\frac{1}{x^2+9}\,dx = \int\left(\frac{1}{9\tan^2\theta+9}\right)\left(\frac{3}{\cos^2\theta}\right)d\theta = \frac{1}{3}\int\frac{1}{\left(\frac{\sin^2\theta}{\cos^2\theta}+1\right)\cos^2\theta}\,d\theta$$

$$= \frac{1}{3}\int\frac{1}{\sin^2\theta+\cos^2\theta}\,d\theta = \frac{1}{3}\int 1\,d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arctan\left(\frac{x}{3}\right)+C.$$

To simplify a^2+x^2 or $\sqrt{a^2+x^2}$, for constant a, try $x = a\tan\theta$, with $-\pi/2 < \theta < \pi/2$.

Note that $a^2 + x^2$ and $\sqrt{a^2 + x^2}$ are defined on $(-\infty, \infty)$. Assuming that the domain of the integrand is $(-\infty, \infty)$, the substitution $x = a\tan\theta$, with $-\pi/2 < \theta < \pi/2$, is valid for all x in the domain, because its range is $(-\infty, \infty)$ and it has an inverse $\theta = \arctan(x/a)$ on $(-\infty, \infty)$.

Example 10 Use a tangent substitution to show that the following two integrals are equal:

$$\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \frac{1}{\cos^3\theta}\,d\theta.$$

What area do these integrals represent?

Solution We put $x = \tan\theta$, with $-\pi/2 < \theta < \pi/2$, so that $dx = (1/\cos^2\theta)\,d\theta$, and

$$\sqrt{1+x^2} = \sqrt{1 + \frac{\sin^2\theta}{\cos^2\theta}} = \sqrt{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{1}{\cos\theta}.$$

When $x = 0$, $\theta = 0$, and when $x = 1$, $\theta = \pi/4$, so

$$\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\pi/4} \left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\cos^2\theta}\right) d\theta = \int_0^{\pi/4} \frac{1}{\cos^3\theta}\,d\theta.$$

The left-hand integral represents the area under the hyperbola $y^2 - x^2 = 1$ in Figure 7.2.

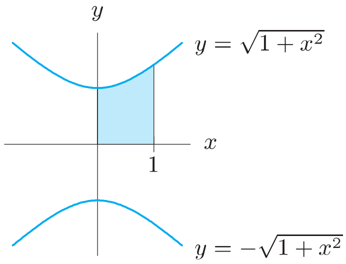


Figure 7.2: The hyperbola $y^2 - x^2 = 1$

Completing the Square to Use a Trigonometric Substitution

To make a trigonometric substitution, we may first need to complete the square.

Example 11 Find $\displaystyle\int \frac{3}{\sqrt{2x - x^2}}\,dx.$

Solution To use a sine or tangent substitution, the expression under the square root sign should be in the form $a^2 + x^2$ or $a^2 - x^2$. Completing the square, we get

$$2x - x^2 = 1 - (x-1)^2.$$

This suggests we substitute $x - 1 = \sin\theta$, or $x = \sin\theta + 1$. Then $dx = \cos\theta\,d\theta$, and

$$\int \frac{3}{\sqrt{2x-x^2}}\,dx = \int \frac{3}{\sqrt{1-(x-1)^2}}\,dx = \int \frac{3}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$
$$= \int \frac{3}{\cos\theta}\cos\theta\,d\theta = \int 3\,d\theta = 3\theta + C.$$

Since $x - 1 = \sin\theta$, we have $\theta = \arcsin(x-1)$, so

$$\int \frac{3}{\sqrt{2x-x^2}}\,dx = 3\arcsin(x-1) + C.$$

Example 12 Find $\displaystyle\int \frac{1}{x^2+x+1}\,dx$.

Solution Completing the square, we get

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2.$$

This suggests we substitute $x+1/2=(\sqrt{3}/2)\tan\theta$, or $x=-1/2+(\sqrt{3}/2)\tan\theta$. Then $dx=(\sqrt{3}/2)(1/\cos^2\theta)\,d\theta$, so

$$\begin{aligned}\int \frac{1}{x^2+x+1}\,dx &= \int \left(\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)\left(\frac{\sqrt{3}}{2}\frac{1}{\cos^2\theta}\right)d\theta\\ &= \frac{\sqrt{3}}{2}\int\left(\frac{1}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}\right)\left(\frac{1}{\cos^2\theta}\right)d\theta=\frac{2}{\sqrt{3}}\int\frac{1}{(\tan^2\theta+1)\cos^2\theta}\,d\theta\\ &= \frac{2}{\sqrt{3}}\int\frac{1}{\sin^2\theta+\cos^2\theta}\,d\theta=\frac{2}{\sqrt{3}}\int 1\,d\theta=\frac{2}{\sqrt{3}}\theta+C.\end{aligned}$$

Since $x+1/2=(\sqrt{3}/2)\tan\theta$, we have $\theta=\arctan((2/\sqrt{3})x+1/\sqrt{3})$, so

$$\int\frac{1}{x^2+x+1}\,dx=\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)+C.$$

Exercises and Problems for Section 7.4

Exercises

Split the functions in Exercises 1–7 into partial fractions.

1. $\dfrac{20}{25-x^2}$

2. $\dfrac{x+1}{6x+x^2}$

3. $\dfrac{8}{y^3-4y}$

4. $\dfrac{2(1+s)}{s(s^2+3s+2)}$

5. $\dfrac{2}{s^4-1}$

6. $\dfrac{2y}{y^3-y^2+y-1}$

7. $\dfrac{1}{w^4-w^3}$

In Exercises 8–14, find the antiderivative of the function in the given exercise.

8. Exercise 1

9. Exercise 2

10. Exercise 3

11. Exercise 4

12. Exercise 5

13. Exercise 6

14. Exercise 7

In Exercises 15–19, evaluate the integral.

15. $\displaystyle\int\frac{3x^2-8x+1}{x^3-4x^2+x+6}\,dx$; use $\dfrac{A}{x-2}+\dfrac{B}{x+1}+\dfrac{C}{x-3}$.

16. $\displaystyle\int\frac{dx}{x^3-x^2}$; use $\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$.

17. $\displaystyle\int\frac{10x+2}{x^3-5x^2+x-5}\,dx$; use $\dfrac{A}{x-5}+\dfrac{Bx+C}{x^2+1}$.

18. $\displaystyle\int\frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x}\,dx$;
use division and $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x+11}$.

19. $\displaystyle\int\frac{x^4+3x^3+2x^2+1}{x^2+3x+2}\,dx$; use division.

In Exercises 20–22, use the substitution to find the integral.

20. $\displaystyle\int\frac{1}{\sqrt{9-4x^2}}dx,\quad x=\frac{3}{2}\sin t$

21. $\displaystyle\int\frac{1}{x^2+4x+5}dx,\quad x=\tan t-2$

22. $\displaystyle\int\frac{1}{\sqrt{4x-3-x^2}}dx,\quad x=\sin t+2$

23. Which of the following integrals are best done by a trigonometric substitution, and what substitution?

(a) $\int \sqrt{9-x^2}\, dx$ **(b)** $\int x\sqrt{9-x^2}\, dx$

24. Give a substitution (not necessarily trigonometric) which could be used to compute the following integrals:

(a) $\int \frac{x}{\sqrt{x^2+10}}\, dx$ **(b)** $\int \frac{1}{\sqrt{x^2+10}}\, dx$

Problems

25. **(a)** Evaluate $\int \frac{3x+6}{x^2+3x}\, dx$ by partial fractions.

(b) Show that your answer to part (a) agrees with the answer you get by using the integral tables.

26. Calculate $\int \frac{x}{(x-a)(x-b)}\, dx$ for

(a) $a \neq b$ **(b)** $a = b$

Complete the square and give a substitution (not necessarily trigonometric) which could be used to compute the integrals in Problems 27–34.

27. $\int \frac{1}{x^2+2x+2}\, dx$

28. $\int \frac{1}{x^2+6x+25} dx$

29. $\int \frac{dy}{y^2+3y+3}$

30. $\int \frac{x+1}{x^2+2x+2}\, dx$

31. $\int \frac{4}{\sqrt{2z-z^2}}\, dz$

32. $\int \frac{z-1}{\sqrt{2z-z^2}}\, dz$

33. $\int (t+2)\sin(t^2+4t+7)\, dt$

34. $\int (2-\theta)\cos(\theta^2-4\theta)d\theta$

Calculate the integrals in Problems 35–52.

35. $\int \frac{1}{(x-5)(x-3)}\, dx$

36. $\int \frac{1}{(x+2)(x+3)}\, dx$

37. $\int \frac{1}{(x+7)(x-2)}\, dx$

38. $\int \frac{x}{x^2-3x+2} dx$

39. $\int \frac{dz}{z^2+z}$

40. $\int \frac{dx}{x^2+5x+4}$

41. $\int \frac{dP}{3P-3P^2}$

42. $\int \frac{3x+1}{x^2-3x+2} dx$

43. $\int \frac{y+2}{2y^2+3y+1}\, dy$

44. $\int \frac{x+1}{x^3+x}\, dx$

45. $\int \frac{x-2}{x^2+x^4}\, dx$

46. $\int \frac{x^2}{\sqrt{9-x^2}}\, dx$

47. $\int \frac{y^2}{25+y^2}\, dy$

48. $\int \frac{dt}{t^2\sqrt{1+t^2}}$

49. $\int \frac{dz}{(4-z^2)^{3/2}}$

50. $\int \frac{10}{(s+2)(s^2+1)}\, ds$

51. $\int \frac{1}{x^2+4x+13} dx$

52. $\int \frac{e^x\, dx}{(e^x-1)(e^x+2)}$

Find the exact area of the regions in Problems 53–58.

53. Bounded by $y = 3x/((x-1)(x-4))$, $y = 0$, $x = 2$, $x = 3$.

54. Bounded by $y = (3x^2+x)/((x^2+1)(x+1))$, $y = 0$, $x = 0$, $x = 1$.

55. Bounded by $y = x^2/\sqrt{1-x^2}$, $y = 0$, $x = 0$, $x = 1/2$.

56. Bounded by $y = x^3/\sqrt{4-x^2}$, $y = 0$, $x = 0$, $x = \sqrt{2}$.

57. Bounded by $y = 1/\sqrt{x^2+9}$, $y = 0$, $x = 0$, $x = 3$.

58. Bounded by $y = 1/(x\sqrt{x^2+9})$, $y = 0$, $x = \sqrt{3}$, $x = 3$.

Calculate the integrals in Problems 59–61 by partial fractions and using the indicated substitution. Show that the results you get are the same.

59. $\int \frac{dx}{1-x^2}$; substitution $x = \sin\theta$.

60. $\int \frac{2x}{x^2-1}\, dx$; substitution $w = x^2 - 1$.

61. $\int \frac{3x^2+1}{x^3+x}\, dx$; substitution $w = x^3 + x$.

62. **(a)** Show $\int \frac{1}{\sin^2\theta}\, d\theta = -\frac{1}{\tan\theta} + C$.

(b) Calculate $\int \frac{dy}{y^2\sqrt{5-y^2}}$.

Solve Problems 63–65 without using integral tables.

63. Calculate the integral $\int \frac{1}{(x-a)(x-b)}\, dx$ for

(a) $a \neq b$ **(b)** $a = b$

64. Calculate the integral $\int \frac{x}{(x-a)(x-b)}\, dx$ for

(a) $a \neq b$ **(b)** $a = b$

65. Calculate the integral $\int \frac{1}{x^2-a}\, dx$ for

(a) $a > 0$ **(b)** $a = 0$ **(c)** $a < 0$

66. A rumor is spread in a school. For $0 < a < 1$ and $b > 0$, the time t at which a fraction p of the school population has heard the rumor is given by

$$t(p) = \int_a^p \frac{b}{x(1-x)}\, dx.$$

(a) Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so it has only one $\ln$ term.
(b) At time $t = 0$ one percent of the school population ($p = 0.01$) has heard the rumor. What is a?
(c) At time $t = 1$ half the school population ($p = 0.5$) has heard the rumor. What is b?
(d) At what time has 90% of the school population ($p = 0.9$) heard the rumor?

67. The Law of Mass Action tells us that the time, T, taken by a chemical to create a quantity x_0 of the product (in molecules) is given by

$$T = \int_0^{x_0} \frac{k\, dx}{(a-x)(b-x)}$$

where a and b are initial quantities of the two ingredients used to make the product, and k is a positive constant. Suppose $0 < a < b$.

(a) Find the time taken to make a quantity $x_0 = a/2$ of the product.
(b) What happens to T as $x_0 \to a$?

68. The moment generating function, $m(t)$, which gives useful information about the normal distribution of statistics, is defined by

$$m(t) = \int_{-\infty}^{\infty} e^{tx} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx.$$

Find a formula for $m(t)$. [Hint: Complete the square and use the fact that $\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$.]

7.5 APPROXIMATING DEFINITE INTEGRALS

The methods of the last few sections allow us to get exact answers for definite integrals in a variety of cases. However, many functions do not have elementary antiderivatives. To evaluate the definite integrals of such functions, we cannot use the Fundamental Theorem; we must use numerical methods.

We already know how to approximate a definite integral numerically using left- and right-hand Riemann sums. In the next two sections we introduce better methods for approximating definite integrals—better in the sense that they give more accurate results with less work than that required to find the left- and right-hand sums.

The Midpoint Rule

In the left- and right-hand Riemann sums, the heights of the rectangles are found using the left-hand or right-hand endpoints, respectively, of the subintervals. For the *midpoint rule*, we use the midpoint of each of the subintervals.

For example, in approximating $\int_1^2 f(x)\, dx$ by a Riemann sum with two subdivisions, we first divide the interval $1 \leq x \leq 2$ into two pieces. The midpoint of the first subinterval is 1.25 and the midpoint of the second is 1.75. The heights of the two rectangles are $f(1.25)$ and $f(1.75)$, respectively. (See Figure 7.3.) The Riemann sum is

$$f(1.25)0.5 + f(1.75)0.5.$$

Figure 7.3 shows that evaluating f at the midpoint of each subdivision usually gives a better approximation to the area under the curve than evaluating f at either end. For this particular f, it appears that each rectangle is partly above and partly below the graph on each subinterval. Furthermore, the area under the curve which is not under the rectangle appears to be nearly equal to the area under the rectangle which is above the curve. In fact, this new midpoint Riemann sum is generally a better approximation to the definite integral than the left- or right-hand sum with the same number of subdivisions, n.

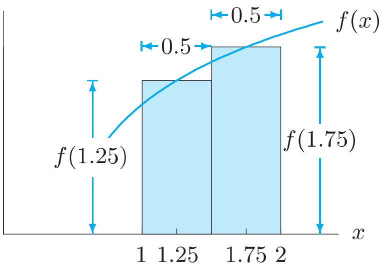


Figure 7.3: Midpoint rule with two subdivisions

So far, we have three ways of estimating an integral using a Riemann sum:

1. The **left rule** uses the left endpoint of each subinterval.
2. The **right rule** uses the right endpoint of each subinterval.
3. The **midpoint rule** uses the midpoint of each subinterval.

We write $\text{LEFT}(n)$, $\text{RIGHT}(n)$, and $\text{MID}(n)$ to denote the results obtained by using these rules with n subdivisions.

Example 1 For $\displaystyle\int_1^2 \frac{1}{x}\,dx$, compute $\text{LEFT}(2)$, $\text{RIGHT}(2)$ and $\text{MID}(2)$, and compare your answers with the exact value of the integral.

Solution For $n = 2$ subdivisions of the interval $[1, 2]$, we use $\Delta x = 0.5$. Then

$$\text{LEFT}(2) = f(1)(0.5) + f(1.5)(0.5) = \frac{1}{1}(0.5) + \frac{1}{1.5}(0.5) = 0.8333\ldots$$

$$\text{RIGHT}(2) = f(1.5)(0.5) + f(2)(0.5) = \frac{1}{1.5}(0.5) + \frac{1}{2}(0.5) = 0.5833\ldots$$

$$\text{MID}(2) = f(1.25)(0.5) + f(1.75)(0.5) = \frac{1}{1.25}(0.5) + \frac{1}{1.75}(0.5) = 0.6857\ldots$$

All three Riemann sums in this example are approximating

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931\ldots.$$

With only two subdivisions, the left and right rules give quite poor approximations but the midpoint rule is already fairly close to the exact answer.

Figure 7.4(a) illustrates why the left and right rules are so inaccurate. Since $f(x) = 1/x$ is decreasing from 1 to 2, the left rule overestimates on each subdivision while the right rule underestimates. However, the midpoint rule approximates with rectangles on each subdivision that are each partly above and partly below the graph, so the errors tend to balance out. (See Figure 7.4(b).)

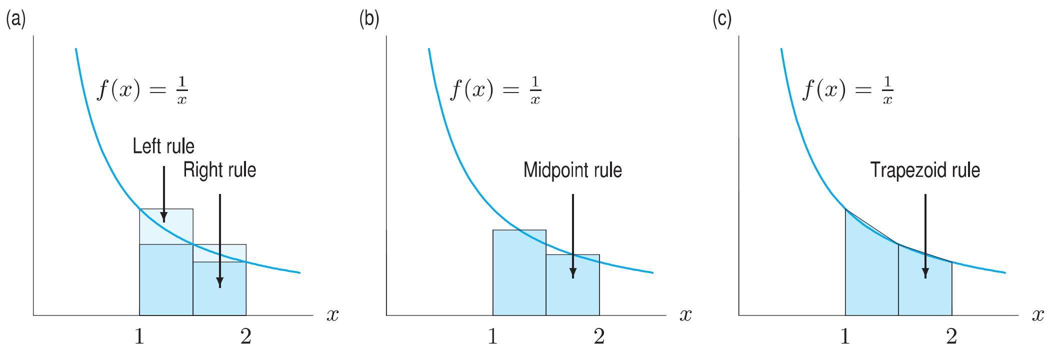


Figure 7.4: Left, right, midpoint, and trapezoid approximations to $\int_1^2 \frac{1}{x}\,dx$

The Trapezoid Rule

We have just seen how the midpoint rule can have the effect of balancing out the errors of the left and right rules. There is another way of balancing these errors: we average the results from the left and right rules. This approximation is called the *trapezoid rule*:

$$\text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}.$$

The trapezoid rule averages the values of f at the left and right endpoints of each subinterval and multiplies by Δx. This is the same as approximating the area under the graph of f in each subinterval by a trapezoid (see Figure 7.5).

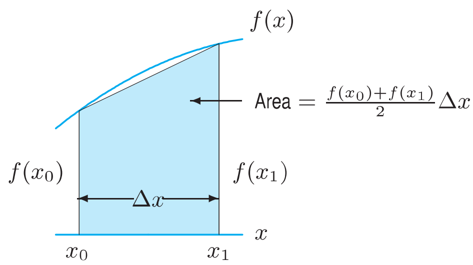


Figure 7.5: Area used in the trapezoid rule

Example 2 For $\int_1^2 \frac{1}{x}\,dx$, compare the trapezoid rule with two subdivisions with the left, right, and midpoint rules.

Solution In the previous example we got $\text{LEFT}(2) = 0.8333\ldots$ and $\text{RIGHT}(2) = 0.5833\ldots$. The trapezoid rule is the average of these, so $\text{TRAP}(2) = 0.7083\ldots$. (See Figure 7.4(c).) The exact value of the integral is $0.6931\ldots$, so the trapezoid rule is better than the left or right rules. The midpoint rule is still the best, however, since $\text{MID}(2) = 0.6857\ldots$.

Is the Approximation an Over- or Underestimate?

It is useful to know when a rule is producing an overestimate and when it is producing an underestimate. In Chapter 5 we saw that the following relationship holds.

If f is increasing on $[a, b]$, then

$$\text{LEFT}(n) \le \int_a^b f(x)\,dx \le \text{RIGHT}(n).$$

If f is decreasing on $[a, b]$, then

$$\text{RIGHT}(n) \le \int_a^b f(x)\,dx \le \text{LEFT}(n).$$

The Trapezoid Rule

If the graph of the function is concave down on $[a, b]$, then each trapezoid lies below the graph and the trapezoid rule underestimates. If the graph is concave up on $[a, b]$, the trapezoid rule overestimates. (See Figure 7.6.)

Figure 7.6: Error in the trapezoid rule

The Midpoint Rule

To understand the relationship between the midpoint rule and concavity, take a rectangle whose top intersects the curve at the midpoint of a subinterval. Draw a tangent to the curve at the midpoint; this gives a trapezoid. See Figure 7.7. (This is *not* the same trapezoid as in the trapezoid rule.) The midpoint rectangle and the new trapezoid have the same area, because the shaded triangles in Figure 7.7 are congruent. Hence, if the graph of the function is concave down, the midpoint rule overestimates; if the graph is concave up, the midpoint rule underestimates. (See Figure 7.8.)

If the graph of f is concave down on $[a, b]$, then

$$\text{TRAP}(n) \leq \int_a^b f(x)\, dx \leq \text{MID}(n).$$

If the graph of f is concave up on $[a, b]$, then

$$\text{MID}(n) \leq \int_a^b f(x)\, dx \leq \text{TRAP}(n).$$

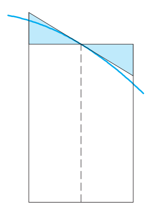


Figure 7.7: Midpoint rectangle and trapezoid with same area

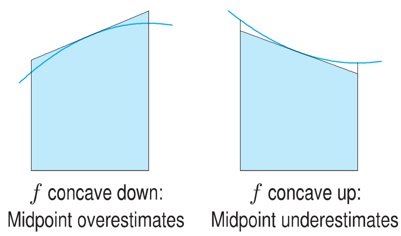


Figure 7.8: Error in the midpoint rule

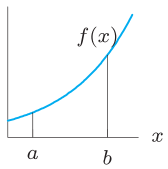
Exercises and Problems for Section 7.5

Exercises

In Exercises 1–6, sketch the area given by the following approximations to $\int_a^b f(x)dx$. Identify each approximation as an overestimate or an underestimate.

(a) LEFT(2) **(b)** RIGHT(2)
(c) TRAP(2) **(d)** MID(2)

1.



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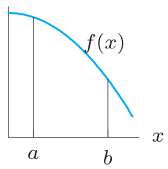

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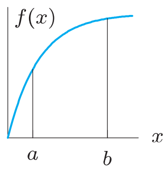

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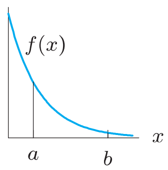

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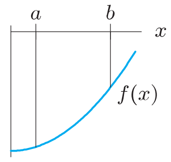
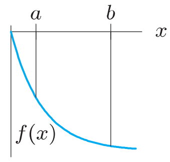


6.



7. Calculate the following approximations to $\int_0^6 x^2 dx$.

(a) LEFT(2) **(b)** RIGHT(2)
(c) TRAP(2) **(d)** MID(2)

8. **(a)** Find LEFT(2) and RIGHT(2) for $\int_0^4 (x^2 + 1)\,dx$.
(b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

9. **(a)** Find MID(2) and TRAP(2) for $\int_0^4 (x^2 + 1)\,dx$.
(b) Illustrate your answers to part (a) graphically. Is each approximation an underestimate or overestimate?

10. Calculate the following approximations to $\int_0^\pi \sin\theta\,d\theta$.

(a) LEFT(2) **(b)** RIGHT(2)
(c) TRAP(2) **(d)** MID(2)

Problems

11. **(a)** Estimate $\int_0^1 1/(1 + x^2)\,dx$ by subdividing the interval into eight parts using:
(i) the left Riemann sum
(ii) the right Riemann sum
(iii) the trapezoidal rule
(b) Since the exact value of the integral is $\pi/4$, you can estimate the value of π using part(a). Explain why your first estimate is too large and your second estimate too small.

12. Using the table, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using LEFT, RIGHT, and TRAP.

Time, t	0	1	2	3	4	5	6
Velocity, v	3	4	5	4	7	8	11

13. Using Figure 7.9, order the following approximations to the integral $\int_0^3 f(x)dx$ and its exact value from smallest to largest:
LEFT(n), RIGHT(n), MID(n), TRAP(n), Exact value.

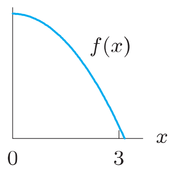


Figure 7.9

14. The results from the left, right, trapezoid, and midpoint rules used to approximate $\int_0^1 g(t)\,dt$, with the same number of subdivisions for each rule, are as follows:
0.601, 0.632, 0.633, 0.664.

(a) Using Figure 7.10, match each rule with its approximation.
(b) Between which two consecutive approximations does the true value of the integral lie?

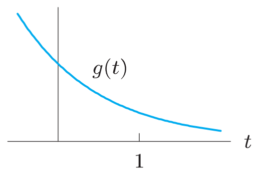
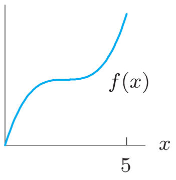


Figure 7.10

For the functions in Problems 15–18, pick which approximation—left, right, trapezoid, or midpoint—is guaranteed to give an overestimate for $\int_0^5 f(x)\,dx$, and which is guaranteed to give an underestimate. (There may be more than one.)

15.



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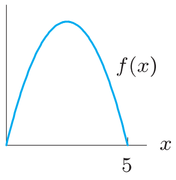

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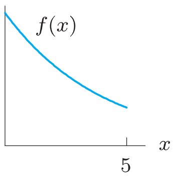

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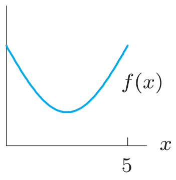

19. Using a fixed number of subdivisions, we approximate the integrals of f and g on the interval in Figure 7.11.

(a) For which function, f or g, is LEFT more accurate? RIGHT? Explain.

(b) For which function, f or g, is TRAP more accurate? MID? Explain.

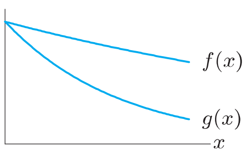

Figure 7.11

20. (a) Values for $f(x)$ are in the table. Which of the four approximation methods in this section is most likely to give the best estimate of $\int_0^{12} f(x)\,dx$? Estimate the integral using this method.

(b) Assume $f(x)$ is continuous with no critical points or points of inflection on the interval $0 \leq x \leq 12$. Is the estimate found in part (a) an over- or underestimate? Explain.

x	0	3	6	9	12
$f(x)$	100	97	90	78	55

21. (a) Find the exact value of $\int_0^{2\pi} \sin\theta\,d\theta$.

(b) Explain, using pictures, why the MID(1) and MID(2) approximations to this integral give the exact value.

(c) Does MID(3) give the exact value of this integral? How about MID(n)? Explain.

22. (a) Show geometrically why $\int_0^1 \sqrt{2-x^2}\,dx = \frac{\pi}{4} + \frac{1}{2}$. [Hint: Break up the area under $y = \sqrt{2-x^2}$ from $x = 0$ to $x = 1$ into two pieces: a sector of a circle and a right triangle.]

(b) Approximate $\int_0^1 \sqrt{2-x^2}\,dx$ for $n = 5$ using the left, right, trapezoid, and midpoint rules. Compute the error in each case using the answer to part (a), and compare the errors.

23. The width, in feet, at various points along the fairway of a hole on a golf course is given in Figure 7.12. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway.

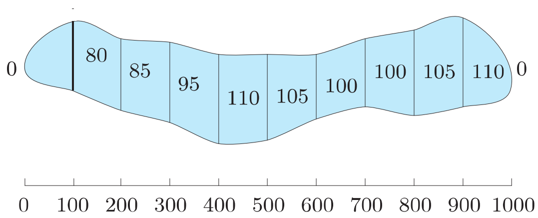

Figure 7.12

Problems 24–28 involve approximating $\int_a^b f(x)\,dx$.

24. Show RIGHT(n) = LEFT(n) + $f(b)\Delta x - f(a)\Delta x$.

25. Show TRAP(n) = LEFT(n) + $\frac{1}{2}(f(b) - f(a))\Delta x$.

26. Show LEFT($2n$) = $\frac{1}{2}$(LEFT(n) + MID(n)).

27. Check that the equations in Problems 24 and 25 hold for $\int_1^2 (1/x)\,dx$ when $n = 10$.

28. Suppose that $a = 2$, $b = 5$, $f(2) = 13$, $f(5) = 21$ and that LEFT(10) = 3.156 and MID(10) = 3.242. Use Problems 24–26 to compute RIGHT(10), TRAP(10), LEFT(20), RIGHT(20), and TRAP(20).

7.6 APPROXIMATION ERRORS AND SIMPSON'S RULE

When we compute an approximation, we are always concerned about the error, namely the difference between the exact answer and the approximation. We usually do not know the exact error; if we did, we would also know the exact answer. Often the best we can get is an upper bound on the error and some idea of how much work is involved in making the error smaller. The study of numerical approximations is really the study of errors. The errors for some methods are much smaller than those for others. The errors for the midpoint and trapezoid rules are related to each other in a way that suggests an even better method, called Simpson's rule. We work with the example $\int_1^2 (1/x)\,dx$ because we know the exact value of this integral ($\ln 2$) and we can investigate the behavior of the errors.

Error in Left and Right Rules

For any approximation, we take

$$\text{Error} = \text{Actual value} - \text{Approximate value}.$$

Let us see what happens to the error in the left and right rules as we increase n. We increase n each time by a factor of 5 starting at $n = 2$. The results are in Table 7.1. A positive error indicates that the Riemann sum is less than the exact value, $\ln 2$. Notice that the errors for the left and right rules have opposite signs but are approximately equal in magnitude. (See Figure 7.13.) The best way to try to get the errors to cancel is to average the left and right rules; this average is the trapezoid rule. If we had not already thought of the trapezoid rule, we might have been led to invent it by this observation.

There is another pattern to the errors in Table 7.1. If we compute the *ratio* of the errors in Table 7.2, we see that the error[2] in both the left and right rules decreases by a factor of about 5 as n increases by a factor of 5.

Table 7.1 *Errors for the left and right rule approximation to* $\int_1^2 \frac{1}{x}\,dx = \ln 2 \approx 0.6931471806$

n	Error in left rule	Error in right rule
2	-0.1402	0.1098
10	-0.0256	0.0244
50	-0.0050	0.0050
250	-0.0010	0.0010

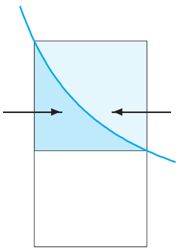

Figure 7.13: Errors in left and right sums

There is nothing special about the number 5; the same holds for any factor. To get one extra digit of accuracy in any calculation, we must make the error $1/10$ as big, so we must increase n by a factor of 10. In fact, *for the left or right rules, each extra digit of accuracy requires about 10 times the work*. The calculator used to produce these tables took about half a second to compute the left rule approximation for $n = 50$, and this yields $\ln 2$ to two digits. To get three correct digits, n would need to be around 500 and the time would be about 5 seconds. Four digits requires $n = 5000$ and 50 seconds. Ten digits requires $n = 5 \cdot 10^9$ and $5 \cdot 10^7$ seconds, which is more than a year! Clearly, the errors for the left and right rules do not decrease fast enough as n increases for practical use.

Table 7.2 *Ratio of the errors as n increases for* $\int_1^2 \frac{1}{x}\,dx$

	Ratio of errors in left rule	Ratio of errors in right rule
Error(2)/Error(10)	5.47	4.51
Error(10)/Error(50)	5.10	4.90
Error(50)/Error(250)	5.02	4.98

[2]The values in Table 7.1 are rounded to 4 decimal places; those in Table 7.2 were computed using more decimal places and then rounded.

Error in Trapezoid and Midpoint Rules

Table 7.3 shows that the trapezoid and midpoint rules produce much better approximations to $\int_1^2 (1/x)\,dx$ than the left and right rules.

Again there is a pattern to the errors. For each n, the midpoint rule is noticeably better than the trapezoid rule; the error for the midpoint rule, in absolute value, seems to be about half the error of the trapezoid rule. To see why, compare the shaded areas in Figure 7.14. Also, notice in Table 7.3 that the errors for the two rules have opposite signs; this is due to concavity.

We are interested in how the errors behave as n increases. Table 7.4 gives the ratios of the errors for each rule. For each rule, we see that as n increases by a factor of 5, the error decreases by a factor of about $25 = 5^2$. In fact, it can be shown that this squaring relationship holds for any factor, so increasing n by a factor of 10 will decrease the error by a factor of about $100 = 10^2$. Reducing the error by a factor of 100 is equivalent to adding two more decimal places of accuracy to the result.

Table 7.3 *The errors for the trapezoid and midpoint rules for* $\int_1^2 \frac{1}{x}\,dx$

n	Error in trapezoid rule	Error in midpoint rule
2	−0.0152	0.0074
10	−0.00062	0.00031
50	−0.0000250	0.0000125
250	−0.0000010	0.0000005

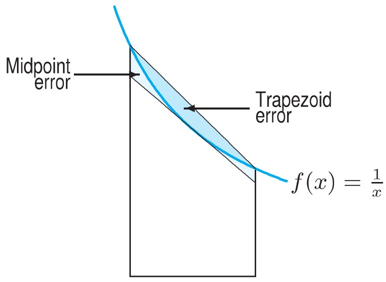

Figure 7.14: Errors in the midpoint and trapezoid rules

In other words: *In the trapezoid or midpoint rules, each extra 2 digits of accuracy requires about 10 times the work.*

This result shows the advantage of the midpoint and trapezoid rules over the left and right rules: less additional work needs to be done to get another decimal place of accuracy. The calculator used to produce these tables again took about half a second to compute the midpoint rule for $\int_1^2 \frac{1}{x}\,dx$ with $n = 50$, and this gets 4 digits correct. Thus to get 6 digits would take $n = 500$ and 5 seconds, to get 8 digits would take 50 seconds, and to get 10 digits would take 500 seconds, or about 10 minutes. That is still not great, but it is certainly better than the 1 year required by the left or right rule.

Table 7.4 *Ratios of the errors as n increases for* $\int_1^2 \frac{1}{x}\,dx$

	Ratio of errors in trapezoid rule	Ratio of errors in midpoint rule
Error(2)/Error(10)	24.33	23.84
Error(10)/Error(50)	24.97	24.95
Error(50)/Error(250)	25.00	25.00

Simpson's Rule

Still more improvement is possible. Observing that the trapezoid error has the opposite sign and about twice the magnitude of the midpoint error, we may guess that a weighted average of the two rules, with the midpoint rule weighted twice the trapezoid rule, will have a much smaller error. This approximation is called *Simpson's rule*[3]:

$$\text{SIMP}(n) = \frac{2 \cdot \text{MID}(n) + \text{TRAP}(n)}{3}.$$

[3]Some books and computer programs use slightly different terminology for Simpson's rule; what we call $n = 50$, they call $n = 100$.

Table 7.5 gives the errors for Simpson's rule. Notice how much smaller the errors are than the previous errors. Of course, it is a little unfair to compare Simpson's rule at $n = 50$, say, with the previous rules, because Simpson's rule must compute the value of f at both the midpoint and the endpoints of each subinterval and hence involves evaluating the function at twice as many points. We know by our previous analysis, however, that even if we did compute the other rules at $n = 100$ to compare with Simpson's rule at $n = 50$, the other errors would only decrease by a factor of 2 for the left and right rules and by a factor of 4 for the trapezoid and midpoint rules.

We see in Table 7.5 that as n increases by a factor of 5, the errors decrease by a factor of about 600, or about 5^4. Again this behavior holds for any factor, so increasing n by a factor of 10 decreases the error by a factor of about 10^4. In other words: *In Simpson's rule, each extra 4 digits of accuracy requires about 10 times the work.*

Table 7.5 *The errors for Simpson's rule and the ratios of the errors*

n	Error	Ratio
2	-0.0001067877	
		550.15
10	-0.0000001940	
		632.27
50	-0.0000000003	

This is a great improvement over either the midpoint or trapezoid rules, which only give two extra digits of accuracy when n is increased by a factor of 10. Simpson's rule is so efficient that we get 9 digits correct with $n = 50$ in about 1 second on our calculator. Doubling n will decrease the error by a factor of about $2^4 = 16$ and hence will give the tenth digit. The total time is 2 seconds, which is pretty good.

In general, Simpson's rule achieves a reasonable degree of accuracy when using relatively small values of n, and is a good choice for an all-purpose method for estimating definite integrals.

Analytical View of the Trapezoid and Simpson's Rules

Our approach to approximating $\int_a^b f(x)\,dx$ numerically has been empirical: try a method, see how the error behaves, and then try to improve it. We can also develop the various rules for numerical integration by making better and better approximations to the integrand, f. The left, right, and midpoint rules are all examples of approximating f by a constant (flat) function on each subinterval. The trapezoid rule is obtained by approximating f by a linear function on each subinterval. Simpson's rule can, in the same spirit, be obtained by approximating f by quadratic functions. The details are given in Problems 9 and 10 on page 371.

How the Error Depends on the Integrand

Other factors besides the size of n affect the size of the error in each of the rules. Instead of looking at how the error behaves as we increase n, let's leave n fixed and imagine trying our approximation methods on different functions. We observe that the error in the left or right rule depends on how steeply the graph of f rises or falls. A steep curve makes the triangular regions missed by the left or right rectangles tall and hence large in area. This observation suggests that the error in the left or right rules depends on the size of the derivative of f (see Figure 7.15).

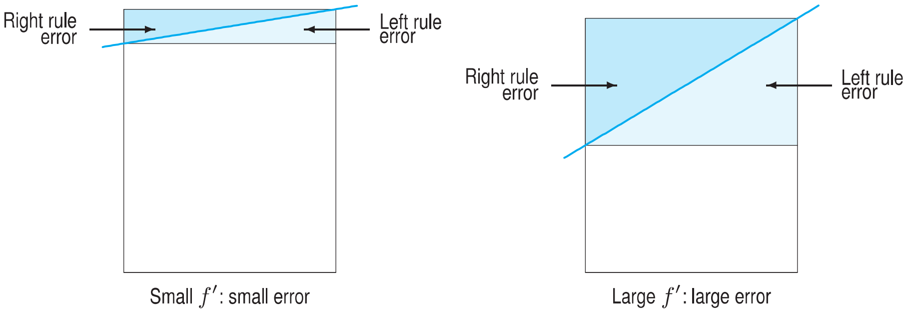

Figure 7.15: The error in the left and right rules depends on the steepness of the curve

From Figure 7.16 it appears that the errors in the trapezoid and midpoint rules depend on how much the curve is bent up or down. In other words, the concavity, and hence the size of the second derivative of f, has an effect on the errors of these two rules. Finally, it can be shown[4] that the error in Simpson's rule depends on the size of the *fourth* derivative of f, written $f^{(4)}$.

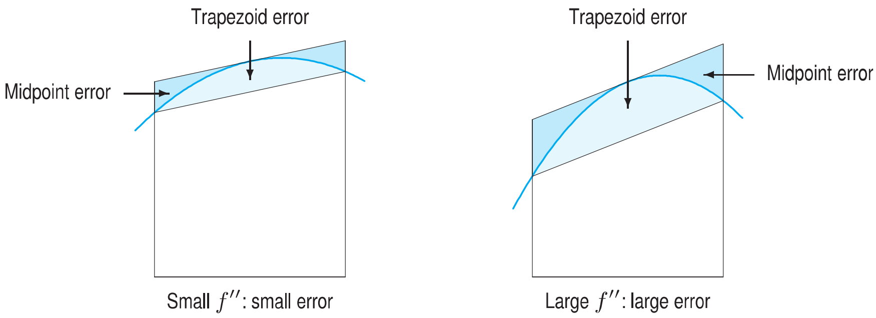

Figure 7.16: The error in the trapezoid and midpoint rules depends on how bent the curve is

Exercises and Problems for Section 7.6

Exercises

1. Estimate $\int_0^6 x^2 dx$ using SIMP(2).
2. **(a)** Using the result of Problem 9 on page 365, compute SIMP(2) for $\int_0^4 (x^2+1)\,dx$.
   (b) Use the Fundamental Theorem of Calculus to find $\int_0^4 (x^2+1)\,dx$ exactly.
   (c) What is the error in SIMP(2) for this integral?

Problems

3. In this problem you will investigate the behavior of the errors in the approximation of the integral
$$\int_1^2 \frac{1}{x}\,dx \approx 0.6931471806\ldots.$$
   (a) For $n = 2, 4, 8, 16, 32, 64, 128$ subdivisions, find the left and right approximations and the errors in each.
   (b) What are the signs of the errors in the left and right approximations? How do the errors change if n is doubled?
   (c) For the values of n in part (a), compute the midpoint and trapezoid approximations and the errors in each.
   (d) What are the signs of the errors in the midpoint and trapezoid approximations? How do the errors change if n is doubled?
   (e) For $n = 2, 4, 8, 16, 32$, compute Simpson's rule approximation and the error in each. How do these errors change as n doubles?

[4]See Kendall E. Atkinson, *An Introduction to Numerical Analysis* (New York: John Wiley and Sons, 1978).

4. **(a)** What is the exact value of $\int_0^2 (x^3 + 3x^2)\, dx$?
   (b) Find SIMP(n) for $n = 2, 4, 100$. What do you notice?
5. **(a)** What is the exact value of $\int_0^4 e^x\, dx$?
   (b) Find LEFT(2), RIGHT(2), TRAP(2), MID(2), and SIMP(2). Compute the error for each.
   (c) Repeat part (b) with $n = 4$ (instead of $n = 2$).
   (d) For each rule in part (b), as n goes from $n = 2$ to $n = 4$, does the error go down approximately as you would expect? Explain.
6. The approximation to a definite integral using $n = 10$ is 2.346; the exact value is 4.0. If the approximation was found using each of the following rules, use the same rule to estimate the integral with $n = 30$.
   (a) LEFT **(b)** TRAP **(c)** SIMP
7. A computer takes 3 seconds to compute a particular definite integral accurate to 2 decimal places. How long does it take the computer to get 10 decimal places of accuracy using each of the following rules? Give your answer in seconds and in appropriate time units (minutes, hours, days, or years).
   (a) LEFT **(b)** MID **(c)** SIMP
8. Table 7.6 gives approximations to an integral whose true value is 7.621372.
   (a) Does the integrand function appear to be increasing or decreasing? Concave up or concave down?
   (b) Fill in the errors for $n = 3$ in the middle column in Table 7.6.
   (c) Estimate the errors for $n = 30$ and fill in the right hand column in Table 7.6.

Table 7.6

	Approximation $n = 3$	Error $n = 3$	Error $n = 30$
LEFT	5.416101		
RIGHT	9.307921		
TRAP	7.362011		
MID	7.742402		
SIMP	7.615605		

Problems 9–10 show how Simpson's rule can be obtained by approximating the integrand, f, by quadratic functions.

9. Suppose that $a < b$ and that m is the midpoint $m = (a + b)/2$. Let $h = b - a$. The purpose of this problem is to show that if f is a quadratic function, then

$$\int_a^b f(x)\, dx = \frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right).$$

   (a) Show that this equation holds for the functions $f(x) = 1$, $f(x) = x$, and $f(x) = x^2$.
   (b) Use part (a) and the properties of the integral on page 284 to show that the equation holds for any quadratic function, $f(x) = Ax^2 + Bx + C$.

10. Consider the following method for approximating $\int_a^b f(x)\, dx$. Divide the interval $[a, b]$ into n equal subintervals. On each subinterval approximate f by a quadratic function that agrees with f at both endpoints and at the midpoint of the subinterval.
   (a) Explain why the integral of f on the subinterval $[x_i, x_{i+1}]$ is approximately equal to the expression

$$\frac{h}{3}\left(\frac{f(x_i)}{2} + 2f(m_i) + \frac{f(x_{i+1})}{2}\right),$$

   where m_i is the midpoint of the subinterval, $m_i = (x_i + x_{i+1})/2$. (See Problem 9.)
   (b) Show that if we add up these approximations for each subinterval, we get Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{2 \cdot \mathrm{MID}(n) + \mathrm{TRAP}(n)}{3}.$$

7.7 IMPROPER INTEGRALS

Our original discussion of the definite integral $\int_a^b f(x)\, dx$ assumed that the interval $a \le x \le b$ was of finite length and that f was continuous. Integrals that arise in applications don't necessarily have these nice properties. In this section we investigate a class of integrals, called *improper* integrals, in which one limit of integration is infinite or the integrand is unbounded. As an example, to estimate the mass of the earth's atmosphere, we might calculate an integral which sums the mass of the air up to different heights. In order to represent the fact that the atmosphere does not end at a specific height, we let the upper limit of integration get larger and larger, or tend to infinity.

We will usually consider only improper integrals with positive integrands since they are the most common.

One Type of Improper Integral: When the Limit of Integration Is Infinite

Here is an example of an improper integral:

$$\int_1^\infty \frac{1}{x^2}\, dx.$$

To evaluate this integral, we first compute the definite integral $\int_1^b (1/x^2)\, dx$:

$$\int_1^b \frac{1}{x^2}\, dx = -x^{-1}\Big|_1^b = -\frac{1}{b} + \frac{1}{1}.$$

Now take the limit as $b \to \infty$. Since

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\, dx = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1,$$

we say that the improper integral $\int_1^\infty (1/x^2)\, dx$ *converges* to 1.

If we think in terms of areas, the integral $\int_1^\infty (1/x^2)\, dx$ represents the area under $f(x) = 1/x^2$ from $x = 1$ extending infinitely far to the right. (See Figure 7.17(a).) It may seem strange that this region has finite area. What our limit computations are saying is that

$$\text{When } b = 10: \quad \int_1^{10} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^{10} = -\frac{1}{10} + 1 = 0.9$$

$$\text{When } b = 100: \quad \int_1^{100} \frac{1}{x^2}\, dx = -\frac{1}{100} + 1 = 0.99$$

$$\text{When } b = 1000: \quad \int_1^{1000} \frac{1}{x^2}\, dx = -\frac{1}{1000} + 1 = 0.999$$

and so on. In other words, as b gets larger and larger, the area between $x = 1$ and $x = b$ tends to 1. See Figure 7.17(b). Thus, it does make sense to declare that $\int_1^\infty (1/x^2)\, dx = 1$.

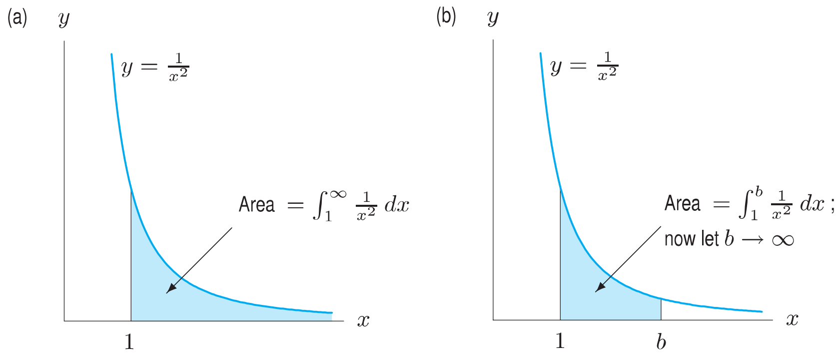

Figure 7.17: Area representation of improper integral

Of course, in another example, we might not get a finite limit as b gets larger and larger. In that case we say the improper integral *diverges*.

Suppose $f(x)$ is positive for $x \geq a$.

If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ is a finite number, we say that $\int_a^\infty f(x)\,dx$ **converges** and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

Otherwise, we say that $\int_a^\infty f(x)\,dx$ **diverges**. We define $\int_{-\infty}^b f(x)\,dx$ similarly.

Example 1 Does the improper integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ converge or diverge?

Solution We consider

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = \int_1^b x^{-1/2}\,dx = 2x^{1/2}\Big|_1^b = 2b^{1/2} - 2.$$

We see that $\int_1^b (1/\sqrt{x})\,dx$ grows without bound as $b \to \infty$. We have shown that the area under the curve in Figure 7.18 is not finite. Thus we say the integral $\int_1^\infty (1/\sqrt{x})\,dx$ *diverges*. We could also say $\int_1^\infty (1/\sqrt{x})\,dx = \infty$.

Notice that $f(x) \to 0$ as $x \to \infty$ does not guarantee convergence of $\int_a^\infty f(x)\,dx$.

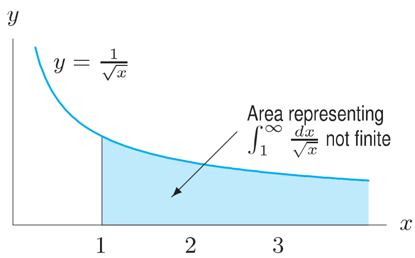

Figure 7.18: $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges

What is the difference between the functions $1/x^2$ and $1/\sqrt{x}$ that makes the area under the graph of $1/x^2$ approach 1 as $x \to \infty$, whereas the area under $1/\sqrt{x}$ grows very large? Both functions approach 0 as x grows, so as b grows larger, smaller bits of area are being added to the definite integral. The difference between the functions is subtle: the values of the function $1/\sqrt{x}$ *don't shrink fast enough* for the integral to have a finite value. Of the two functions, $1/x^2$ drops to 0 much faster than $1/\sqrt{x}$, and this feature keeps the area under $1/x^2$ from growing beyond 1.

Example 2 Find $\displaystyle\int_0^\infty e^{-5x}\,dx$.

Solution First we consider $\int_0^b e^{-5x}\,dx$:

$$\int_0^b e^{-5x}\,dx = -\frac{1}{5}e^{-5x}\Big|_0^b = -\frac{1}{5}e^{-5b} + \frac{1}{5}.$$

Since $e^{-5b} = \dfrac{1}{e^{5b}}$, this term tends to 0 as b approaches infinity, so $\int_0^\infty e^{-5x}\,dx$ converges. Its value is

$$\int_0^\infty e^{-5x}\,dx = \lim_{b\to\infty} \int_0^b e^{-5x}\,dx = \lim_{b\to\infty} \left(-\frac{1}{5}e^{-5b} + \frac{1}{5}\right) = 0 + \frac{1}{5} = \frac{1}{5}.$$

Since e^{5x} grows very rapidly, we expect that e^{-5x} will approach 0 rapidly. The fact that the area approaches $1/5$ instead of growing without bound is a consequence of the speed with which the integrand e^{-5x} approaches 0.

Example 3 Determine for which values of the exponent, p, the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

Solution For $p \neq 1$,

$$\int_1^b x^{-p}\,dx = \frac{1}{-p+1}x^{-p+1}\bigg|_1^b = \left(\frac{1}{-p+1}b^{-p+1} - \frac{1}{-p+1}\right).$$

The important question is whether the exponent of b is positive or negative. If it is negative, then as b approaches infinity, b^{-p+1} approaches 0. If the exponent is positive, then b^{-p+1} grows without bound as b approaches infinity. What happens if $p = 1$? In this case we get

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x\bigg|_1^b = \lim_{b\to\infty} \ln b - \ln 1.$$

Since $\ln b$ becomes arbitrarily large as b approaches infinity, the integral grows without bound. We conclude that $\int_1^\infty (1/x^p)\,dx$ diverges precisely when $p \leq 1$. For $p > 1$ the integral has the value

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty}\left(\frac{1}{-p+1}b^{-p+1} - \frac{1}{-p+1}\right) = -\left(\frac{1}{-p+1}\right) = \frac{1}{p-1}.$$

Application of Improper Integrals to Energy

The energy, E, required to separate two charged particles, originally a distance a apart, to a distance b, is given by the integral

$$E = \int_a^b \frac{kq_1q_2}{r^2}\,dr$$

where q_1 and q_2 are the magnitudes of the charges and k is a constant. If q_1 and q_2 are in coulombs, a and b are in meters, and E is in joules, the value of the constant k is $9\cdot 10^9$.

Example 4 A hydrogen atom consists of a proton and an electron, with opposite charges of magnitude $1.6\cdot 10^{-19}$ coulombs. Find the energy required to take a hydrogen atom apart (that is, to move the electron from its orbit to an infinite distance from the proton). Assume that the initial distance between the electron and the proton is the Bohr radius, $R_B = 5.3\cdot 10^{-11}$ meter.

Solution Since we are moving from an initial distance of R_B to a final distance of ∞, the energy is represented by the improper integral

$$E = \int_{R_B}^\infty k\frac{q_1q_2}{r^2}\,dr = kq_1q_2 \lim_{b\to\infty}\int_{R_B}^b \frac{1}{r^2}\,dr$$
$$= kq_1q_2 \lim_{b\to\infty} -\frac{1}{r}\bigg|_{R_B}^b = kq_1q_2\lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{R_B}\right) = \frac{kq_1q_2}{R_B}.$$

Substituting numerical values, we get

$$E = \frac{(9\cdot 10^9)(1.6\cdot 10^{-19})^2}{5.3\cdot 10^{-11}} \approx 4.35\cdot 10^{-18} \text{ joules}.$$

This is about the amount of energy needed to lift a speck of dust 0.000000025 inch off the ground. (In other words, not much!)

What happens if the limits of integration are $-\infty$ and ∞? In this case, we break the integral at any point and write the original integral as a sum of two new improper integrals.

We can use any (finite) number c to define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if both of the new integrals have a finite value do we add the values to get a finite value for the original integral.

It is not hard to show that the preceding definition does not depend on the choice for c.

Another Type of Improper Integral: When the Integrand Becomes Infinite

There is another way for an integral to be improper. The interval may be finite but the function may be unbounded near some points in the interval. For example, consider $\int_0^1 (1/\sqrt{x})\,dx$. Since the graph of $y = 1/\sqrt{x}$ has a vertical asymptote at $x = 0$, the region between the graph, the x-axis, and the lines $x = 0$ and $x = 1$ is unbounded. Instead of extending to infinity in the horizontal direction as in the previous improper integrals, this region extends to infinity in the vertical direction. See Figure 7.19(a). We handle this improper integral in a similar way as before: we compute $\int_a^1 (1/\sqrt{x})\,dx$ for values of a slightly larger than 0 and look at what happens as a approaches 0 from the positive side. (This is written as $a \to 0^+$.)

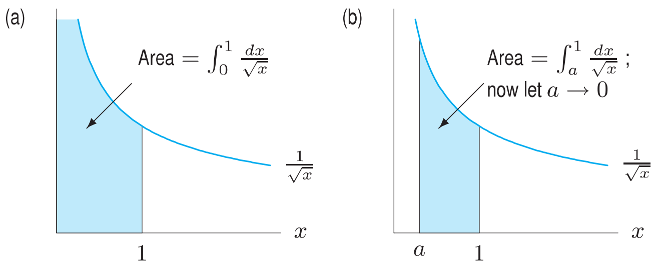

Figure 7.19: Area representation of improper integral

First we compute the integral:

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big|_a^1 = 2 - 2a^{1/2}.$$

Now we take the limit:

$$\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} (2 - 2a^{1/2}) = 2.$$

Since the limit is finite, we say the improper integral converges, and that

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Geometrically, what we have done is to calculate the finite area between $x = a$ and $x = 1$ and take the limit as a tends to 0 from the right. See Figure 7.19(b). Since the limit exists, the integral converges to 2. If the limit did not exist, we would say the improper integral diverges.

Example 5 Investigate the convergence of $\int_0^2 \frac{1}{(x-2)^2}\,dx$.

Solution This is an improper integral since the integrand tends to infinity as x approaches 2, and is undefined at $x = 2$. Since the trouble is at the right endpoint, we replace the upper limit by b, and let b tend to 2 from the left. This is written $b \to 2^-$, with the "–" signifying that 2 is approached from below. See Figure 7.20.

$$\int_0^2 \frac{1}{(x-2)^2}\,dx = \lim_{b\to 2^-}\int_0^b \frac{1}{(x-2)^2}\,dx = \lim_{b\to 2^-}(-1)(x-2)^{-1}\Big|_0^b = \lim_{b\to 2^-}\left(-\frac{1}{(b-2)} - \frac{1}{2}\right).$$

Therefore, since $\lim_{b\to 2^-}\left(-\frac{1}{b-2}\right)$ does not exist, the integral diverges.

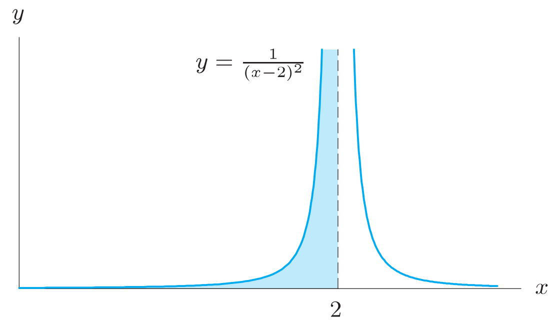

Figure 7.20: Shaded area represents $\int_0^2 \frac{1}{(x-2)^2}\,dx$

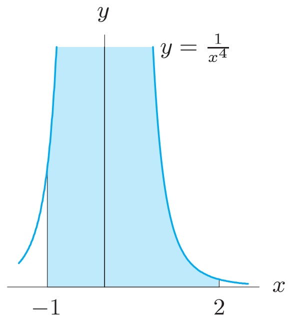

Figure 7.21: Shaded area represents $\int_{-1}^2 \frac{1}{x^4}\,dx$

> Suppose $f(x)$ is positive and continuous on $a \le x < b$ and tends to infinity as $x \to b$. If $\lim_{c\to b^-}\int_a^c f(x)\,dx$ is a finite number, we say that $\int_a^b f(x)\,dx$ **converges** and define
>
> $$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$
>
> Otherwise, we say that $\int_a^b f(x)\,dx$ **diverges**.

When $f(x)$ tends to infinity as x approaches a, we define convergence in a similar way. In addition, an integral can be improper because the integrand tends to infinity *inside* the interval of integration rather than at an endpoint. In this case, we break the given integral into two (or more) improper integrals so that the integrand tends to infinity only at endpoints.

> Suppose that $f(x)$ is positive and continuous on $[a, b]$ except at the point c. If $f(x)$ tends to infinity as $x \to c$, then we define
>
> $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$
>
> If *either* of the two new improper integrals diverges, we say the original integral diverges. Only if *both* of the new integrals have a finite value do we add the values to get a finite value for the original integral.

Example 6 Investigate the convergence of $\displaystyle\int_{-1}^{2} \frac{1}{x^4}\,dx$.

Solution See the graph in Figure 7.21. The trouble spot is $x = 0$, rather than $x = -1$ or $x = 2$. To handle this situation, we break the given improper integral into two other improper integrals each of which have $x = 0$ as one of the endpoints:

$$\int_{-1}^{2} \frac{1}{x^4}\,dx = \int_{-1}^{0} \frac{1}{x^4}\,dx + \int_{0}^{2} \frac{1}{x^4}\,dx.$$

We can now use the previous technique to evaluate the new integrals, if they converge. Since

$$\int_{0}^{2} \frac{1}{x^4}\,dx = \lim_{a\to 0^+} -\frac{1}{3}x^{-3}\Big|_a^2 = \lim_{a\to 0^+} \left(-\frac{1}{3}\right)\left(\frac{1}{8} - \frac{1}{a^3}\right)$$

the integral $\int_0^2 (1/x^4)\,dx$ diverges. Thus, the original integral diverges. A similar computation shows that $\int_{-1}^{0} (1/x^4)\,dx$ also diverges.

It is easy to miss an improper integral when the integrand tends to infinity inside the interval. For example, it is fundamentally incorrect to say that $\int_{-1}^{2} (1/x^4)\,dx = -\frac{1}{3}x^{-3}\Big|_{-1}^{2} = -\frac{1}{24} - \frac{1}{3} = -\frac{3}{8}$.

Example 7 Find $\displaystyle\int_{0}^{6} \frac{1}{(x-4)^{2/3}}\,dx$.

Solution Figure 7.22 shows that the trouble spot is at $x = 4$, so we break the integral at $x = 4$ and consider the separate parts.

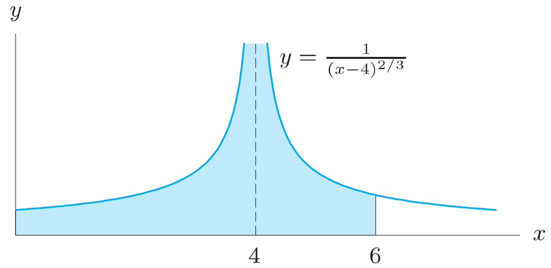

Figure 7.22: Shaded area represents $\int_0^6 \frac{1}{(x-4)^{2/3}}\,dx$

We have

$$\int_{0}^{4} \frac{1}{(x-4)^{2/3}}\,dx = \lim_{b\to 4^-} 3(x-4)^{1/3}\Big|_0^b = \lim_{b\to 4^-} \left(3(b-4)^{1/3} - 3(-4)^{1/3}\right) = 3(4)^{1/3}.$$

Similarly,

$$\int_{4}^{6} \frac{1}{(x-4)^{2/3}}\,dx = \lim_{a\to 4^+} 3(x-4)^{1/3}\Big|_a^6 = \lim_{a\to 4^+} \left(3\cdot 2^{1/3} - 3(a-4)^{1/3}\right) = 3(2)^{1/3}.$$

Since both of these integrals converge, the original integral converges:

$$\int_{0}^{6} \frac{1}{(x-4)^{2/3}}\,dx = 3(4)^{1/3} + 3(2)^{1/3} \approx 8.54.$$

Finally, there is a question of what to do when an integral is improper at both endpoints. In this case, we just break the integral at any interior point of the interval. The original integral diverges if either or both of the new integrals diverge.

Example 8 Investigate the convergence of $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$.

Solution This integral is improper both because the upper limit is ∞ and because the function is undefined at $x = 0$. We break the integral into two parts at, say, $x = 1$. We know by Example 3 that $\int_1^\infty (1/x^2)\,dx$ has a finite value. However, the other part, $\int_0^1 (1/x^2)\,dx$, diverges since:

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+} -x^{-1}\Big|_a^1 = \lim_{a\to 0^+}\left(\frac{1}{a} - 1\right).$$

Therefore $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$ diverges as well.

Exercises and Problems for Section 7.7

Exercises

1. Shade the area represented by:
(a) $\int_1^\infty (1/x^2)\,dx$ **(b)** $\int_0^1 (1/\sqrt{x})\,dx$

2. Evaluate the improper integral $\int_0^\infty e^{-0.4x}\,dx$ and sketch the area it represents.

3. **(a)** Use a calculator or computer to estimate $\int_0^b xe^{-x}\,dx$ for $b = 5, 10, 20$.
(b) Use your answers to part (a) to estimate the value of $\int_0^\infty xe^{-x}\,dx$, assuming it is finite.

4. **(a)** Sketch the the area represented by the improper integral $\int_{-\infty}^\infty e^{-x^2}\,dx$.
(b) Use a calculator or computer to estimate $\int_{-a}^a e^{-x^2}\,dx$ for $a = 1,2,3,4,5$.
(c) Use the answers to part (b) to estimate the value of $\int_{-\infty}^\infty e^{-x^2}\,dx$, assuming it is finite.

Calculate the integrals in Exercises 5–35, if they converge. You may calculate the limits by appealing to the dominance of one function over another, or by L'Hopital's rule.

5. $\displaystyle\int_1^\infty \frac{1}{5x+2}\,dx$

6. $\displaystyle\int_1^\infty \frac{1}{(x+2)^2}\,dx$

7. $\displaystyle\int_0^1 \ln x\,dx$

8. $\displaystyle\int_0^\infty e^{-\sqrt{x}}\,dx$

9. $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

10. $\displaystyle\int_1^\infty e^{-2x}\,dx$

11. $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx$

12. $\displaystyle\int_1^\infty \frac{x}{4+x^2}\,dx$

13. $\displaystyle\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx$

14. $\displaystyle\int_{-\infty}^\infty \frac{dz}{z^2+25}$

15. $\displaystyle\int_0^4 \frac{dx}{\sqrt{16-x^2}}$

16. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx$

17. $\displaystyle\int_0^1 \frac{1}{v}\,dv$

18. $\displaystyle\int_0^1 \frac{x^4+1}{x}\,dx$

19. $\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx$

20. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+1}}\,dx$

21. $\displaystyle\int_0^4 \frac{-1}{u^2-16}\,du$

22. $\displaystyle\int_1^\infty \frac{y}{y^4+1}\,dy$

23. $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$

24. $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$

25. $\displaystyle\int_{16}^{20} \frac{1}{y^2-16}\,dy$

26. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

27. $\displaystyle\int_0^\pi \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx$

28. $\displaystyle\int_3^\infty \frac{dx}{x(\ln x)^2}$

29. $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

30. $\displaystyle\int_4^\infty \frac{dx}{(x-1)^2}$

31. $\displaystyle\int_4^\infty \frac{dx}{x^2-1}$

32. $\displaystyle\int_7^\infty \frac{dy}{\sqrt{y-5}}$

33. $\displaystyle\int_{-3}^3 \frac{y\,dy}{\sqrt{9-y^2}}$

34. $\displaystyle\int_3^6 \frac{d\theta}{(4-\theta)^2}$

35. $\displaystyle\int_1^\infty \frac{d}{dx}\left(\frac{\ln x}{x}\right)dx$

36. Use the fact that $\dfrac{d}{dx}\dfrac{\sin x}{x} = \dfrac{x\cos x - \sin x}{x^2}$ to evaluate, if it exists:

$$\int_0^\pi \frac{x\cos x - \sin x}{x^2}\,dx.$$

Problems

37. In statistics we encounter $P(x)$, a function defined by

$$P(x) = \frac{1}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

Use a calculator or computer to evaluate

(a) $P(1)$ **(b)** $P(\infty)$

38. Find the area under the curve $y = xe^{-x}$ for $x \geq 0$.

39. Find the area under the curve $y = 1/\cos^2 t$ between $t = 0$ and $t = \pi/2$.

For what values of p do the integrals in Problems 40–41 converge or diverge? What is the value of the integral when it converges?

40. $\displaystyle\int_e^\infty x^p \ln x\, dx$ **41.** $\displaystyle\int_0^e x^p \ln x\, dx$

42. For $\alpha > 0$, calculate

(a) $\displaystyle\int_0^\infty \frac{e^{-y/\alpha}}{\alpha}\, dy$ **(b)** $\displaystyle\int_0^\infty \frac{ye^{-y/\alpha}}{\alpha}\, dy$

(c) $\displaystyle\int_0^\infty \frac{y^2 e^{-y/\alpha}}{\alpha}\, dy$

43. Given that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$, calculate the exact value of

$$\int_{-\infty}^{\infty} e^{-(x-a)^2/b}\, dx.$$

44. Assuming $g(x)$ is a differentiable function whose values are bounded for all x, derive Stein's identity, which is used in statistics:

$$\int_{-\infty}^{\infty} g'(x)e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} xg(x)e^{-x^2/2}\, dx.$$

The k^{th} moment, m_k of the normal distribution is defined by

$$m_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^k e^{-x^2/2} dx.$$

In Problems 45–48, use the fact that $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ to calculate the moments. Assume all the integrals converge.

45. m_1 **46.** m_2 **47.** m_3 **48.** m_4

49. The gamma function is defined for all $x > 0$ by the rule

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) Find $\Gamma(1)$ and $\Gamma(2)$.
(b) Integrate by parts with respect to t to show that, for positive n,
$$\Gamma(n+1) = n\Gamma(n).$$
(c) Find a simple expression for $\Gamma(n)$ for positive integers n .

50. The rate, r, at which people get sick during an epidemic of the flu can be approximated by $r = 1000te^{-0.5t}$, where r is measured in people/day and t is measured in days since the start of the epidemic.

(a) Sketch a graph of r as a function of t.
(b) When are people getting sick fastest?
(c) How many people get sick altogether?

51. Find the energy required to separate opposite electric charges of magnitude 1 coulomb. The charges are initially 1 meter apart and one is moved infinitely far from the other. (The definition of energy is on page 374.)

7.8 COMPARISON OF IMPROPER INTEGRALS

Making Comparisons

Sometimes it is difficult to find the exact value of an improper integral by antidifferentiation, but it may be possible to determine whether an integral converges or diverges. The key is to *compare* the given integral to one whose behavior we already know. Let's look at an example.

Example 1 Determine whether $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+5}}\, dx$ converges.

Solution First, let's see what this integrand does as $x \to \infty$. For large x, the 5 becomes insignificant compared with the x^3, so

$$\frac{1}{\sqrt{x^3+5}} \approx \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}.$$

Since

$$\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} -2x^{-1/2}\Big|_1^b = \lim_{b\to\infty}\left(2-2b^{-1/2}\right) = 2,$$

the integral $\int_1^\infty (1/x^{3/2})\,dx$ converges. So we expect our integral to converge as well.

In order to confirm this, we observe that for $0 \le x^3 \le x^3+5$, we have

$$\frac{1}{\sqrt{x^3+5}} \le \frac{1}{\sqrt{x^3}}.$$

and so for $b \ge 1$,

$$\int_1^b \frac{1}{\sqrt{x^3+5}}\,dx \le \int_1^b \frac{1}{\sqrt{x^3}}\,dx.$$

(See Figure 7.23.) Since $\int_1^b (1/\sqrt{x^3+5})\,dx$ increases as b approaches infinity but is always smaller than $\int_1^b (1/x^{3/2})\,dx < \int_1^\infty (1/x^{3/2})\,dx = 2$, we know $\int_1^\infty (1/\sqrt{x^3+5})\,dx$ must have a finite value less than 2. Thus,

$$\int_1^\infty \frac{dx}{\sqrt{x^3+5}} \text{ converges to a value less than 2.}$$

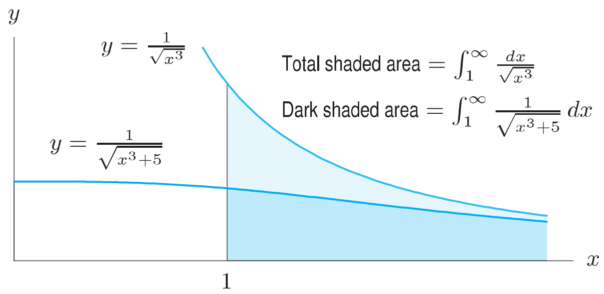

Figure 7.23: Graph showing $\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \le \int_1^\infty \frac{dx}{\sqrt{x^3}}$

A little more work is required to estimate the value of a convergent improper integral.

Example 2 Estimate the value of $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx$ with an error of less than 0.01 using the approximation

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \approx \int_1^b \frac{1}{\sqrt{x^3+5}}\,dx.$$

Solution We must figure out how large a value of b to take. Since

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx = \int_1^b \frac{1}{\sqrt{x^3+5}}\,dx + \int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx,$$

we find b such that the tail of the integral satisfies the inequality

$$\left|\int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx\right| < 0.01.$$

From the solution of Example 1, we have

$$0 < \int_b^\infty \frac{1}{\sqrt{x^3+5}}\,dx < \int_b^\infty \frac{1}{\sqrt{x^3}}\,dx = \frac{2}{\sqrt{b}}.$$

We choose b such that $2/\sqrt{b} < 0.01$, which means that $b > 40{,}000$. Then, picking $b = 50{,}000$, we have

$$\int_1^\infty \frac{1}{\sqrt{x^3+5}}\,dx \approx \int_1^{50{,}000} \frac{1}{\sqrt{x^3+5}}\,dx = 1.699,$$

with an error of less than 0.01.

Notice that we first looked at the behavior of the integrand as $x \to \infty$. This is useful because the convergence or divergence of the integral is determined by what happens as $x \to \infty$.

The Comparison Test for $\int_a^\infty f(x)\,dx$

Assume $f(x)$ is positive. Making a comparison involves two stages:

1. Guess, by looking at the behavior of the integrand for large x, whether the integral converges or not. (This is the "behaves like" principle.)
2. Confirm the guess by comparison:
   - If $0 \le f(x) \le g(x)$ and $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
   - If $0 \le g(x) \le f(x)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Example 3 Decide whether $\displaystyle\int_4^\infty \frac{dt}{(\ln t)-1}$ converges or diverges.

Solution Since $\ln t$ grows without bound as $t \to \infty$, the -1 is eventually going to be insignificant in comparison to $\ln t$. Thus, as far as convergence is concerned,

$$\int_4^\infty \frac{1}{(\ln t)-1}\,dt \quad \text{behaves like} \quad \int_4^\infty \frac{1}{\ln t}\,dt.$$

Does $\int_4^\infty (1/\ln t)\,dt$ converge or diverge? Since $\ln t$ grows very slowly, $1/\ln t$ goes to zero very slowly, and so the integral probably does not converge. We know that $(\ln t) - 1 < \ln t < t$ for all positive t. So, provided $t > e$, we take reciprocals:

$$\frac{1}{(\ln t)-1} > \frac{1}{\ln t} > \frac{1}{t}.$$

Since $\int_4^\infty (1/t)\,dt$ diverges, we conclude that

$$\int_4^\infty \frac{1}{(\ln t)-1}\,dt \ \text{diverges.}$$

How Do We Know What To Compare With?

In Examples 1 and 3, we investigated the convergence of an integral by comparing it with an easier integral. How did we pick the easier integral? This is a matter of trial and error, guided by any information we get by looking at the original integrand as $x \to \infty$. We want the comparison integrand to be easy and, in particular, to have a simple antiderivative.

Useful Integrals for Comparison

- $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \le 1$.
- $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$ and diverges for $p \ge 1$.
- $\displaystyle\int_0^\infty e^{-ax}\,dx$ converges for $a > 0$.

Of course, we can use any function for comparison, provided we can determine its behavior.

Example 4 Investigate the convergence of $\displaystyle\int_1^\infty \frac{(\sin x)+3}{\sqrt{x}}\,dx$.

Solution Since it looks difficult to find an antiderivative of this function, we try comparison. What happens to this integrand as $x \to \infty$? Since $\sin x$ oscillates between -1 and 1,

$$\frac{2}{\sqrt{x}} = \frac{-1+3}{\sqrt{x}} \le \frac{(\sin x)+3}{\sqrt{x}} \le \frac{1+3}{\sqrt{x}} = \frac{4}{\sqrt{x}},$$

the integrand oscillates between $2/\sqrt{x}$ and $4/\sqrt{x}$. (See Figure 7.24.)

What do $\int_1^\infty (2/\sqrt{x})\,dx$ and $\int_1^\infty (4/\sqrt{x})\,dx$ do? As far as convergence is concerned, they certainly do the same thing, and whatever that is, the original integral does it too. It is important to notice that $\sqrt{x}$ grows very slowly. This means that $1/\sqrt{x}$ gets small slowly, which means that convergence is unlikely. Since $\sqrt{x} = x^{1/2}$, the result in the preceding box (with $p = \frac{1}{2}$) tells us that $\int_1^\infty (1/\sqrt{x})\,dx$ diverges. So the comparison test tells us that the original integral diverges.

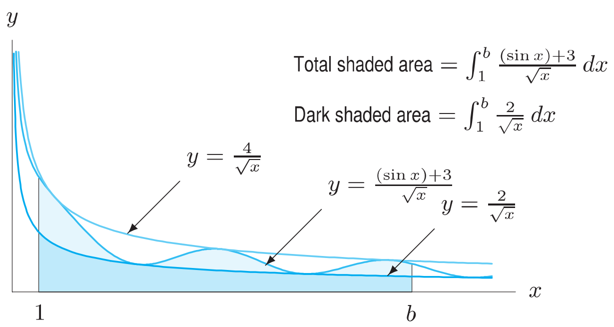

Figure 7.24: Graph showing $\int_1^b \frac{2}{\sqrt{x}}\,dx \le \int_1^b \frac{(\sin x)+3}{\sqrt{x}}\,dx$, for $b \ge 1$

Notice that there are two possible comparisons we could have made in Example 4:

$$\frac{2}{\sqrt{x}} \le \frac{(\sin x)+3}{\sqrt{x}} \quad \text{or} \quad \frac{(\sin x)+3}{\sqrt{x}} \le \frac{4}{\sqrt{x}}.$$

Since both $\int_1^\infty (2/\sqrt{x})\,dx$ and $\int_1^\infty (4/\sqrt{x})\,dx$ diverge, only the first comparison is useful. Knowing that an integral is *smaller* than a divergent integral is of no help whatsoever!

The next example shows what to do if the comparison does not hold throughout the interval of integration.

Example 5 Show $\displaystyle\int_1^\infty e^{-x^2/2}\,dx$ converges to a finite value.

Solution We know that $e^{-x^2/2}$ goes very rapidly to zero as $x \to \infty$, so we expect this integral to converge. Hence we look for some larger integrand which has a convergent integral. One possibility is $\int_1^\infty e^{-x}\,dx$, because e^{-x} has an elementary antiderivative and $\int_1^\infty e^{-x}\,dx$ converges. What is the relationship between $e^{-x^2/2}$ and e^{-x}? We know that for $x \geq 2$,

$$x \leq \frac{x^2}{2} \quad \text{so} \quad -\frac{x^2}{2} \leq -x,$$

and so, for $x \geq 2$

$$e^{-x^2/2} \leq e^{-x}.$$

Since this inequality holds only for $x \geq 2$, we split the interval of integration into two pieces:

$$\int_1^\infty e^{-x^2/2}\,dx = \int_1^2 e^{-x^2/2}\,dx + \int_2^\infty e^{-x^2/2}\,dx.$$

Now $\int_1^2 e^{-x^2/2}\,dx$ is finite (it is not improper) and $\int_2^\infty e^{-x^2/2}\,dx$ is finite by comparison with $\int_2^\infty e^{-x}\,dx$. Therefore, $\int_1^\infty e^{-x^2/2}\,dx$ is the sum of two finite pieces and therefore must be finite.

The previous example illustrates the following general principle:

> If f is continuous on $[a, b]$,
>
> $$\int_a^\infty f(x)\,dx \text{ and } \int_b^\infty f(x)\,dx$$
>
> either both converge or both diverge.

In particular, when the comparison test is applied to $\int_a^\infty f(x)\,dx$, the inequalities for $f(x)$ and $g(x)$ do not need to hold for all $x \geq a$ but only for x greater than some value, say c.

Exercises and Problems for Section 7.8

Exercises

In Exercises 1–9, use the box on page 382 and the behavior of rational and exponential functions as $x \to \infty$ to predict whether the integrals converge or diverge.

1. $\displaystyle\int_1^\infty \frac{x^2}{x^4+1}\,dx$

2. $\displaystyle\int_2^\infty \frac{x^3}{x^4-1}\,dx$

3. $\displaystyle\int_1^\infty \frac{x^2+1}{x^3+3x+2}\,dx$

4. $\displaystyle\int_1^\infty \frac{1}{x^2+5x+1}\,dx$

5. $\displaystyle\int_1^\infty \frac{x}{x^2+2x+4}\,dx$

6. $\displaystyle\int_1^\infty \frac{x^2-6x+1}{x^2+4}\,dx$

7. $\displaystyle\int_1^\infty \frac{5x+2}{x^4+8x^2+4}\,dx$

8. $\displaystyle\int_1^\infty \frac{1}{e^{5t}+2}\,dt$

9. $\displaystyle\int_1^\infty \frac{x^2+4}{x^4+3x^2+11}\,dx$

In Exercises 10–25 decide if the improper integral converges or diverges. Explain your reasoning.

10. $\displaystyle\int_{50}^\infty \frac{dz}{z^3}$

11. $\displaystyle\int_1^\infty \frac{dx}{1+x}$

12. $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

13. $\displaystyle\int_5^8 \frac{6}{\sqrt{t-5}}\,dt$

14. $\displaystyle\int_0^1 \frac{1}{x^{19/20}}\,dx$

15. $\displaystyle\int_{-1}^5 \frac{dt}{(t+1)^2}$

16. $\displaystyle\int_{-\infty}^\infty \frac{du}{1+u^2}$

17. $\displaystyle\int_1^\infty \frac{du}{u+u^2}$

18. $\displaystyle\int_1^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$

19. $\displaystyle\int_2^\infty \frac{d\theta}{\sqrt{\theta^3+1}}$

20. $\displaystyle\int_0^1 \frac{d\theta}{\sqrt{\theta^3+\theta}}$

21. $\displaystyle\int_0^\infty \frac{dy}{1+e^y}$

22. $\displaystyle\int_1^\infty \frac{2+\cos\phi}{\phi^2}\,d\phi$

23. $\displaystyle\int_0^\infty \frac{dz}{e^z+2^z}$

24. $\displaystyle\int_0^\pi \frac{2-\sin\phi}{\phi^2}\,d\phi$

25. $\displaystyle\int_4^\infty \frac{3+\sin\alpha}{\alpha}\,d\alpha$

Estimate the values of the integrals in Exercises 26–27 to two decimal places by integrating the functions on your calculator or computer for large values of the upper limit of integration.

26. $\displaystyle\int_1^\infty e^{-x^2}\,dx$

27. $\displaystyle\int_0^\infty e^{-x^2}\cos^2 x\,dx$

Problems

28. The graphs of $y = 1/x, y = 1/x^2$ and the functions $f(x)$, $g(x)$, $h(x)$, and $k(x)$ are shown in Figure 7.25.

(a) Is the area between $y = 1/x$ and $y = 1/x^2$ on the interval from $x = 1$ to ∞ finite or infinite? Explain.

(b) Using the graph, decide whether the integral of each of the functions $f(x)$, $g(x)$, $h(x)$ and $k(x)$ on the interval from $x = 1$ to ∞ converges, diverges, or whether it is impossible to tell.

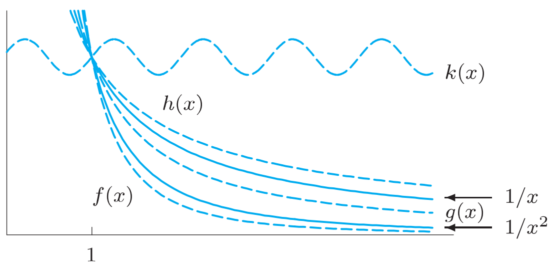

Figure 7.25

29. Suppose $\int_a^\infty f(x)\,dx$ converges. What does Figure 7.26 suggest about the convergence of $\int_a^\infty g(x)\,dx$?

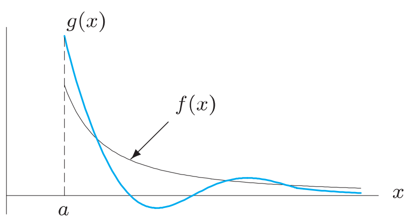

Figure 7.26

For what values of p do the integrals in Problems 30–31 converge or diverge?

30. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

31. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$

32. **(a)** Find an upper bound for

$$\int_3^\infty e^{-x^2}\,dx.$$

[Hint: $e^{-x^2} \le e^{-3x}$ for $x \ge 3$.]

(b) For any positive n, generalize the result of part (a) to find an upper bound for

$$\int_n^\infty e^{-x^2}\,dx$$

by noting that $nx \le x^2$ for $x \ge n$.

33. In Planck's Radiation Law, we encounter the integral

$$\int_1^\infty \frac{dx}{x^5(e^{1/x}-1)}.$$

(a) Explain why a graph of the tangent line to e^t at $t = 0$ tells us that for all t

$$1+t \le e^t.$$

(b) Substituting $t = 1/x$, show that for all $x \ne 0$

$$e^{1/x}-1 > \frac{1}{x}.$$

(c) Use the comparison test to show that the original integral converges.

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Integration techniques**
  Substitution, parts, partial fractions, trigonometric substitution, using tables.
- **Numerical approximations**
  Riemann sums (left, right, midpoint), trapezoid rule, Simpson's rule, approximation errors.
- **Improper integrals**
  Convergence/divergence, comparison test for integrals.

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER SEVEN

Exercises

For Exercises 1–4, find an antiderivative.

1. $q(t) = (t+1)^2$

2. $p(\theta) = 2\sin(2\theta)$

3. $f(x) = 5^x$

4. $r(t) = e^t + 5e^{5t}$

For Exercises 5–113, evaluate the following integrals. Assume a, b, c, and k are constants. Exercises 7–72 can be done without an integral table, as can some of the later problems.

5. $\int (3w+7)\,dw$

6. $\int e^{2r}\,dr$

7. $\int \sin t\,dt$

8. $\int \cos 2t\,dt$

9. $\int e^{5z}\,dz$

10. $\int \cos(x+1)\,dx$

11. $\int \sin 2\theta\,d\theta$

12. $\int (x^3-1)^4 x^2\,dx$

13. $\int \left(x^{3/2} + x^{2/3}\right)\,dx$

14. $\int (e^x + 3^x)\,dx$

15. $\int \frac{1}{e^z}\,dz$

16. $\int \left(\frac{4}{x^2} - \frac{3}{x^3}\right)\,dx$

17. $\int \frac{x^3+x+1}{x^2}dx$

18. $\int \frac{(1+\ln x)^2}{x}\,dx$

19. $\int te^{t^2}\,dt$

20. $\int x\cos x\,dx$

21. $\int x^2 e^{2x}\,dx$

22. $\int x\sqrt{1-x}\,dx$

23. $\int x\ln x\,dx$

24. $\int y\sin y\,dy$

25. $\int (\ln x)^2\,dx$

26. $\int \ln(x^2)\,dx$

27. $\int e^{0.5-0.3t}\,dt$

28. $\int \sin^2\theta\cos\theta\,d\theta$

29. $\int x\sqrt{4-x^2}\,dx$

30. $\int \frac{(u+1)^3}{u^2}\,du$

31. $\int \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy$

32. $\int \frac{1}{\cos^2 z}\,dz$

33. $\int \cos^2\theta\,d\theta$

34. $\int t^{10}(t-10)\,dt$

35. $\int \tan(2x-6)\,dx$

36. $\int \frac{(\ln x)^2}{x}\,dx$

37. $\int \frac{(t+2)^2}{t^3}\,dt$

38. $\int \left(x^2 + 2x + \frac{1}{x}\right)\,dx$

39. $\int \frac{t+1}{t^2}\,dt$

40. $\int te^{t^2+1}\,dt$

41. $\int \tan\theta\,d\theta$

42. $\int \sin(5\theta)\cos(5\theta)\,d\theta$

43. $\int \frac{x}{x^2+1}\,dx$

44. $\int \frac{dz}{1+z^2}$

45. $\int \frac{dz}{1+4z^2}$

46. $\int \cos^3 2\theta\sin 2\theta\,d\theta$

47. $\int \sin 5\theta\cos^3 5\theta\,d\theta$

48. $\int \sin^3 z\cos^3 z\,dz$

49. $\int t(t-10)^{10}\,dt$

50. $\int \cos\theta\sqrt{1+\sin\theta}\,d\theta$

51. $\int xe^x\,dx$

52. $\int t^3 e^t\,dt$

53. $\int_1^3 x(x^2+1)^{70}\,dx$

54. $\int (3z+5)^3\,dz$

55. $\int \frac{du}{9+u^2}$

56. $\int \frac{\cos w}{1+\sin^2 w}\,dw$

57. $\int \frac{1}{x}\tan(\ln x)\,dx$

58. $\int \frac{1}{x}\sin(\ln x)\,dx$

59. $\int \frac{dx}{\sqrt{1-4x^2}}$

60. $\int \frac{w\,dw}{\sqrt{16-w^2}}$

61. $\displaystyle\int \frac{e^{2y}+1}{e^{2y}}\,dy$

62. $\displaystyle\int \frac{\sin w\,dw}{\sqrt{1-\cos w}}$

63. $\displaystyle\int \frac{dx}{x\ln x}$

64. $\displaystyle\int \frac{du}{3u+8}$

65. $\displaystyle\int \frac{x\cos\sqrt{x^2+1}}{\sqrt{x^2+1}}\,dx$

66. $\displaystyle\int \frac{t^3}{\sqrt{1+t^2}}\,dt$

67. $\displaystyle\int ue^{ku}\,du$

68. $\displaystyle\int (w+5)^4 w\,dw$

69. $\displaystyle\int e^{\sqrt{2}x+3}\,dx$

70. $\displaystyle\int r(\ln r)^2\,dr$

71. $\displaystyle\int (e^x+x)^2 dx$

72. $\displaystyle\int u^2\ln u\,du$

73. $\displaystyle\int \frac{5x+6}{x^2+4}\,dx$

74. $\displaystyle\int \frac{1}{\sin^3(2x)}\,dx$

75. $\displaystyle\int \frac{dr}{r^2-100}$

76. $\displaystyle\int y^2\sin(cy)\,dy$

77. $\displaystyle\int e^{-ct}\sin kt\,dt$

78. $\displaystyle\int e^{5x}\cos(3x)\,dx$

79. $\displaystyle\int \left(x^{\sqrt{k}}+\sqrt{k}^{x}\right)\,dx$

80. $\displaystyle\int \sqrt{3+12x^2}\,dx.$

81. $\displaystyle\int \frac{1}{\sqrt{x^2-3x+2}}\,dx$

82. $\displaystyle\int \frac{x^3}{x^2+3x+2}\,dx$

83. $\displaystyle\int \frac{x^2+1}{x^2-3x+2}\,dx$

84. $\displaystyle\int \frac{dx}{ax^2+bx}$

85. $\displaystyle\int \frac{ax+b}{ax^2+2bx+c}\,dx$

86. $\displaystyle\int \left(\frac{x}{3}+\frac{3}{x}\right)^2\,dx$

87. $\displaystyle\int \frac{2^t}{2^t+1}\,dt$

88. $\displaystyle\int 10^{1-x}\,dx$

89. $\displaystyle\int (x^2+5)^3\,dx$

90. $\displaystyle\int v\arcsin v\,dv$

91. $\displaystyle\int \sqrt{4-x^2}\,dx$

92. $\displaystyle\int \frac{z^3}{z-5}\,dz$

93. $\displaystyle\int \frac{\sin w\cos w}{1+\cos^2 w}\,dw$

94. $\displaystyle\int \frac{1}{\tan(3\theta)}\,d\theta$

95. $\displaystyle\int \frac{x}{\cos^2 x}dx$

96. $\displaystyle\int \frac{x+1}{\sqrt{x}}\,dx$

97. $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

98. $\displaystyle\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}}\,dx$

99. $\displaystyle\int \frac{e^{2y}}{e^{2y}+1}\,dy$

100. $\displaystyle\int \frac{z}{(z^2-5)^3}\,dz$

101. $\displaystyle\int \frac{z}{(z-5)^3}dz$

102. $\displaystyle\int \frac{(1+\tan x)^3}{\cos^2 x}\,dx$

103. $\displaystyle\int \frac{(2x-1)e^{x^2}}{e^x}dx$

104. $\displaystyle\int (2x+1)e^{x^2}e^x\,dx$

105. $\displaystyle\int \sqrt{y^2-2y+1}(y-1)\,dy$

106. $\displaystyle\int \sin x(\sqrt{2+3\cos x})dx$

107. $\displaystyle\int (x^2-3x+2)e^{-4x}\,dx$

108. $\displaystyle\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta$

109. $\displaystyle\int \cos(2\sin x)\cos x\,dx$

110. $\displaystyle\int (x+\sin x)^3(1+\cos x)\,dx$

111. $\displaystyle\int \left(2x^3+3x+4\right)\cos(2x)\,dx$

112. $\displaystyle\int \sinh^2 x\cosh x\,dx$

113. $\displaystyle\int (x+1)\sinh(x^2+2x)\,dx$

For Exercises 114–127, evaluate the definite integrals using the Fundamental Theorem of Calculus and check your answers numerically.

114. $\displaystyle\int_0^1 x(1+x^2)^{20}\,dx$

115. $\displaystyle\int_4^1 x\sqrt{x^2+4}\,dx$

116. $\displaystyle\int_0^\pi \sin\theta(\cos\theta+5)^7\,d\theta$

117. $\displaystyle\int_0^1 \frac{x}{1+5x^2}\,dx$

118. $\displaystyle\int_1^2 \frac{x^2+1}{x}\,dx$

119. $\displaystyle\int_1^3 \ln(x^3)\,dx$

120. $\displaystyle\int_1^e (\ln x)^2\,dx$

121. $\displaystyle\int_{-\pi}^{\pi} e^{2x}\sin 2x\,dx$

122. $\displaystyle\int_0^{10} ze^{-z}\,dz$

123. $\displaystyle\int_{-\pi/3}^{\pi/4} \sin^3\theta\cos\theta\,d\theta$

124. $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

125. $\displaystyle\int_0^1 \frac{dx}{x^2+1}$

126. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos^2\theta\sin^5\theta\,d\theta$

127. $\displaystyle\int_{-2}^{0} \frac{2x+4}{x^2+4x+5}\,dx$

128. Use partial fractions on $\dfrac{1}{x^2-1}$ to find $\displaystyle\int \frac{1}{x^2-1}dx.$

129. **(a)** Use partial fractions to find $\int \frac{1}{x^2 - x} dx$.

(b) Show that your answer to part (a) agrees with the answer you get by using the integral tables.

130. Use partial fractions to find $\int \frac{1}{x(L-x)} dx$, where L is constant.

Evaluate the integrals in Exercises 131–142 using partial fractions or a trigonometric substitution (a and b are positive constants).

131. $\int \frac{1}{(x-2)(x+2)} dx$

132. $\int \frac{1}{\sqrt{25 - x^2}} dx$,

133. $\int \frac{1}{x(x+5)} dx$

134. $\int \frac{1}{\sqrt{1 - 9x^2}} dx$

135. $\int \frac{2x+3}{x(x+2)(x-1)} dx$

136. $\int \frac{3x+1}{x(x^2-1)} dx$

137. $\int \frac{1+x^2}{x(1+x)^2} dx$

138. $\int \frac{1}{x^2+2x+2} dx$

139. $\int \frac{1}{x^2+4x+5} dx$

140. $\int \frac{1}{\sqrt{a^2 - (bx)^2}} dx$

141. $\int \frac{\cos x}{\sin^3 x + \sin x} dx$

142. $\int \frac{e^x}{e^{2x} - 1} dx$

For Exercises 143–156 decide if the integral converges or diverges. If the integral converges, find its value or give a bound on its value.

143. $\int_4^\infty \frac{dt}{t^{3/2}}$

144. $\int_{10}^\infty \frac{dx}{x \ln x}$

145. $\int_0^\infty we^{-w}\, dw$

146. $\int_{-1}^1 \frac{1}{x^4} dx$

147. $\int_{-\pi/4}^{\pi/4} \tan\theta \, d\theta$

148. $\int_2^\infty \frac{1}{4+z^2} dz$

149. $\int_{10}^\infty \frac{1}{z^2-4} dz$

150. $\int_{-5}^{10} \frac{dt}{\sqrt{t+5}}$

151. $\int_0^{\pi/2} \frac{1}{\sin\phi} d\phi$

152. $\int_0^{\pi/4} \tan 2\theta \, d\theta$

153. $\int_1^\infty \frac{x}{x+1} dx$

154. $\int_0^\infty \frac{\sin^2\theta}{\theta^2+1} d\theta$

155. $\int_0^\pi \tan^2\theta d\theta$

156. $\int_0^1 (\sin x)^{-3/2} dx$

Problems

In Problems 157–159, find the exact area.

157. Under $y = (e^x)^2$ for $0 \le x \le 1$.

158. Between $y = (e^x)^3$ and $y = (e^x)^2$ for $0 \le x \le 3$.

159. Between $y = e^x$ and $y = 5e^{-x}$ and the y-axis.

160. The curves $y = \sin x$ and $y = \cos x$ cross each other infinitely often. What is the area of the region bounded by these two curves between two consecutive crossings?

161. Evaluate $\int_0^2 \sqrt{4 - x^2}\, dx$ using its geometric interpretation.

In Problems 162–165 explain why the following pairs of antiderivatives are really, despite their apparent dissimilarity, different expressions of the same problem. You do not need to evaluate the integrals.

162. $\int \frac{1}{\sqrt{1-x^2}} dx$ and $\int \frac{x\, dx}{\sqrt{1-x^4}}$

163. $\int \frac{dx}{x^2+4x+4}$ and $\int \frac{x}{(x^2+1)^2} dx$

164. $\int \frac{x}{1-x^2} dx$ and $\int \frac{1}{x \ln x} dx$

165. $\int \frac{x}{x+1} dx$ and $\int \frac{1}{x+1} dx$

In Problems 166–167, show the two integrals are equal using a substitution.

166. $\int_0^2 e^{-w^2} dw = \int_0^1 2e^{-4x^2} dx$

167. $\int_0^3 \frac{\sin t}{t} dt = \int_0^1 \frac{\sin 3t}{t} dt$

168. A function is defined by $f(t) = t^2$ for $0 \le t \le 1$ and $f(t) = 2 - t$ for $1 < t \le 2$. Compute $\int_0^2 f(t)\, dt$.

169. **(a)** Find $\int (x+5)^2 dx$ in two ways:

(i) By multiplying out

(ii) By substituting $w = x + 5$

(b) Are the results the same? Explain.

170. Suppose $\int_{-1}^1 h(z)\, dz = 7$, and that $h(z)$ is even. Calculate the following:

(a) $\int_0^1 h(z)\, dz$

(b) $\int_{-4}^{-2} 5h(z+3)\, dz$

171. Find the average (vertical) height of the shaded area in Figure 7.27.

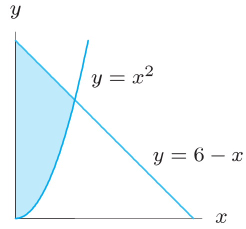

Figure 7.27

172. Find the average (horizontal) width of the shaded area in Figure 7.27.

173. **(a)** Find the average value of the following functions over one cycle:
(i) $f(t) = \cos t$
(ii) $g(t) = |\cos t|$
(iii) $k(t) = (\cos t)^2$

(b) Write the averages you have just found in ascending order. Using words and graphs, explain why the averages come out in the order they do.

174. What, if anything, is wrong with the following calculation?

$$\int_{-2}^{2} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-2}^{2} = -\frac{1}{2} - \left(-\frac{1}{-2}\right) = -1.$$

175. Let $E(x) = \int \frac{e^x}{e^x+e^{-x}}\,dx$ and $F(x) = \int \frac{e^{-x}}{e^x+e^{-x}}\,dx$.

(a) Calculate $E(x) + F(x)$.
(b) Calculate $E(x) - F(x)$.
(c) Use your results from parts (a) and (b) to calculate $E(x)$ and $F(x)$.

176. Using Figure 7.28, put the following approximations to the integral $\int_a^b f(x)\,dx$ and its exact value in order from smallest to largest: LEFT(5), LEFT(10), RIGHT(5), RIGHT(10), MID(10), TRAP(10), Exact value

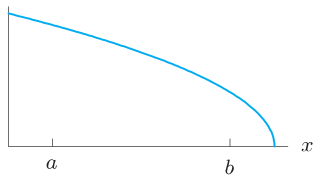

Figure 7.28

177. You estimate $\int_0^{0.5} f(x)dx$ by the trapezoid and midpoint rules with 100 steps. Which of the two estimates is an overestimate, and which is an underestimate, of the true value of the integral if

(a) $f(x) = 1 + e^{-x}$ **(b)** $f(x) = e^{-x^2}$
(c) $f(x)$ is a line

178. **(a)** Using the left rectangle rule, a computer takes two seconds to compute a particular definite integral accurate to 4 digits to the right of the decimal point. How long (in years) does it take to get 8 digits correct using the left rectangle rule? How about 12 digits? 20 digits?

(b) Repeat part (a) but this time assume that the trapezoidal rule is being used throughout.

179. Given that $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, find $\displaystyle\int_0^{\infty} x^2 e^{-x^2}\,dx$.

180. A population, P, is said to be growing logistically if the time, T, taken for it to increase from P_1 to P_2 is given by

$$T = \int_{P_1}^{P_2} \frac{k\,dP}{P(L-P)},$$

where k and L are positive constants and $P_1 < P_2 < L$.

(a) Calculate the time taken for the population to grow from $P_1 = L/4$ to $P_2 = L/2$.
(b) What happens to T as $P_2 \to L$?

181. In 2005, the average per capita income in the US was \$34,586 and increasing at a rate of $r(t) = 1556.37e^{0.045t}$ dollars per year, where t is the number of years since 2005.

(a) Estimate the average per capita income in 2015.
(b) Find a formula for the average per capita income as a function of time after 2005.

182. A patient is given an injection of Imitrex, a migraine medicine, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started.

(a) By letting $t \to \infty$, estimate the total quantity of Imitrex injected.
(b) What fraction of this dose has the patient received at the end of 5 seconds?

183. In 1990 humans generated $1.4 \cdot 10^{20}$ joules of energy from petroleum. At the time, it was estimated that all of the earth's petroleum would generate approximately 10^{22} joules. Assuming the use of energy generated by petroleum increases by 2% each year, how long will it be before all of our petroleum resources are used up?

184. An organism has a development time of T days at a temperature $H = f(t)$°C. The total the number of degree-days S required for development to maturity is a constant defined by

$$S = \int_0^T (f(t) - H_{\text{min}})dt.$$

(a) Evaluate this integral for $T = 18$, $f(t) = 30$°C, and $H_{\text{min}} = 10$°C. What are the units of S?
(b) Illustrate this definite integral on a graph. Label the features corresponding to T, $f(t)$, H_{min}, and S.
(c) Now suppose $H = g(t) = 20 + 10\cos(2\pi t/6)$°C. Assuming that S remains constant, write a definite integral which determines the new development

time, T_2. Sketch a graph illustrating this new integral. Judging from the graph, how does T_2 compare to T? Find T_2.

185. For a positive integer n, let $\Psi_n(x) = C_n \sin(n\pi x)$ be the wave function used in describing the behavior of an electron. If n and m are different positive integers, find

$$\int_0^1 \Psi_n(x) \cdot \Psi_m(x)\, dx.$$

CAS Challenge Problems

186. **(a)** Use a computer algebra system to find $\int \frac{\ln x}{x}\, dx$, $\int \frac{(\ln x)^2}{x}\, dx$, and $\int \frac{(\ln x)^3}{x}\, dx$.

(b) Guess a formula for $\int \frac{(\ln x)^n}{x}\, dx$ that works for any positive integer n.

(c) Use a substitution to check your formula.

187. **(a)** Using a computer algebra system, find $\int (\ln x)^n\, dx$ for $n = 1, 2, 3, 4$.

(b) There is a formula relating $\int (\ln x)^n\, dx$ to $\int (\ln x)^{n-1}\, dx$ for any positive integer n. Guess this formula using your answer to part (a). Check your guess using integration by parts.

In Problems 188–190:

(a) Use a computer algebra system to find the indefinite integral of the given function.

(b) Use the computer algebra system again to differentiate the result of part (a). Do not simplify.

(c) Use algebra to show that the result of part (b) is the same as the original function. Show all the steps in your calculation.

188. $\sin^3 x$

189. $\sin x \cos x \cos(2x)$

190. $\frac{x^4}{(1+x^2)^2}$

CHECK YOUR UNDERSTANDING

In Problems 1–16, decide whether the statements are true or false. Give an explanation for your answer.

1. $\int f'(x)\cos(f(x))\, dx = \sin(f(x)) + C$
2. $\int (1/f(x))\, dx = \ln|f(x)| + C$
3. $\int t\sin(5-t^2)\, dt$ can be evaluated using substitution.
4. $\int \sin^7\theta\cos^6\theta\, d\theta$ can be written as a polynomial with $\cos\theta$ as the variable.
5. $\int 1/(x^2+4x+5)\, dx$ involves a natural logarithm.
6. $\int 1/(x^2+4x-5)\, dx$ involves an arctangent.
7. $\int x^{-1}((\ln x)^2 + (\ln x)^3)\, dx$ is a polynomial with $\ln x$ as the variable.
8. $\int t\sin(5-t)\, dt$ can be evaluated by parts.
9. The midpoint rule approximation to $\int_0^1 (y^2-1)\, dy$ is always smaller than the exact value of the integral.
10. The trapezoid rule approximation is never exact.
11. If f is continuous for all x and $\int_0^\infty f(x)dx$ converges, then so does $\int_a^\infty f(x)dx$ for all positive a.
12. If $f(x)$ is a positive periodic function, then $\int_0^\infty f(x)dx$ diverges.
13. If $f(x)$ is continuous and positive for $x > 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty f(x)dx$ converges.
14. If $f(x)$ is continuous and positive for $x > 0$ and if $\lim_{x\to\infty} f(x) = \infty$, then $\int_0^\infty (1/f(x))dx$ converges.
15. If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both converge, then $\int_0^\infty (f(x)+g(x))dx$ converges.
16. If $\int_0^\infty f(x)dx$ and $\int_0^\infty g(x)dx$ both diverge, then $\int_0^\infty (f(x)+g(x))dx$ diverges.

Suppose that f is continuous for all real numbers and that $\int_0^\infty f(x)\, dx$ converges. Let a be any positive number. Decide which of the statements in Problems 17–20 are true and which are false. Give an explanation for your answer.

17. $\int_0^\infty af(x)\, dx$ converges.
18. $\int_0^\infty f(ax)\, dx$ converges.
19. $\int_0^\infty f(a+x)\, dx$ converges.
20. $\int_0^\infty (a+f(x))\, dx$ converges.

The left and right Riemann sums of a function f on the interval $[2, 6]$ are denoted by LEFT(n) and RIGHT(n), respectively, when the interval is divided into n equal parts. In Problems 21–31, decide whether the statements are true for all continuous functions, f. Give an explanation for your answer.

21. If $n = 10$, then the subdivision size is $\Delta x = 1/10$.
22. If we double the value of n, we make Δx half as large.
23. LEFT(10) $\leq$RIGHT(10)
24. As n approaches infinity, LEFT(n) approaches 0.
25. LEFT(n) $-$ RIGHT(n) $= (f(2) - f(6))\Delta x$.

26. Doubling n decreases the difference LEFT(n) − RIGHT(n) by exactly the factor $1/2$.

27. If LEFT(n) = RIGHT(n) for all n, then f is a constant function.

28. The trapezoid estimate TRAP(n) = (LEFT(n) + RIGHT$(n))/2$ is always closer to $\int_2^6 f(x)dx$ than LEFT(n) or RIGHT(n).

29. $\int_2^6 f(x)dx$ lies between LEFT(n) and RIGHT(n).

30. If LEFT$(2) < \int_a^b f(x)\,dx$, then LEFT$(4) < \int_a^b f(x)\,dx$.

31. If $0 < f' < g'$ everywhere, then the error in approximating $\int_a^b f(x)\,dx$ by LEFT(n) is less than the error in approximating $\int_a^b g(x)\,dx$ by LEFT(n).

PROJECTS FOR CHAPTER SEVEN

1. Taylor Polynomial Inequalities

(a) Use the fact that $e^x \geq 1 + x$ for all values of x and the formula

$$e^x = 1 + \int_0^x e^t\,dt$$

to show that

$$e^x \geq 1 + x + \frac{x^2}{2}$$

for all positive values of x. Generalize this idea to get inequalities involving higher-degree polynomials.

(b) Use the fact that $\cos x \leq 1$ for all x and repeated integration to show that

$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}.$$