Chapter Eight

USING THE DEFINITE INTEGRAL

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8.1 AREAS AND VOLUMES

In Chapter 5, we calculated areas using definite integrals. We obtained the integral by slicing up the region, constructing a Riemann sum, and then taking a limit.

In this section, we calculate volumes using definite integrals. To obtain the integral, we again slice up the region and construct a Riemann sum. The first two examples show how slicing can be used to find both areas and volumes.

Example 1 Use horizontal slices to set up a definite integral to calculate the area of the isosceles triangle in Figure 8.1.

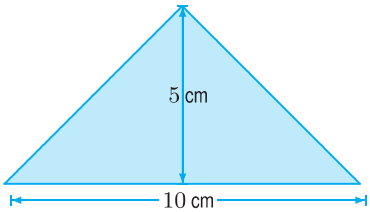

Figure 8.1: Isosceles triangle

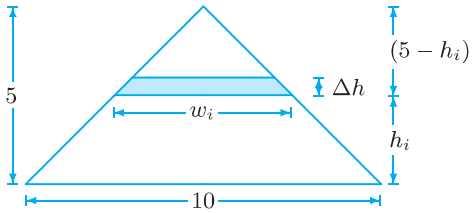

Figure 8.2: Horizontal slicing of isosceles triangle

Solution Notice that we can find the area of a triangle without using an integral; we will use this to check the result from integration:

$$\text{Area} = \frac{1}{2}\,\text{Base} \cdot \text{Height} = 25\ \text{cm}^2.$$

To calculate the area using horizontal slices, see Figure 8.2. A typical strip is approximately a rectangle of length w_i and width Δh, so

$$\text{Area of strip} \approx w_i \Delta h\ \text{cm}^2.$$

To get w_i in terms of h_i, the height above the base, use the similar triangles in Figure 8.2:

$$\frac{w_i}{10} = \frac{5 - h_i}{5}$$
$$w_i = 2(5 - h_i) = 10 - 2h_i.$$

Summing the areas of the strips gives the Riemann sum approximation:

$$\text{Area of triangle} \approx \sum_{i=1}^{n} w_i \Delta h = \sum_{i=1}^{n} (10 - 2h_i)\Delta h\ \text{cm}^2.$$

Taking the limit as $n \to \infty$ and $\Delta h \to 0$ gives the integral:

$$\text{Area of triangle} = \lim_{n\to\infty} \sum_{i=1}^{n} (10 - 2h_i)\Delta h = \int_0^5 (10 - 2h)\,dh\ \text{cm}^2.$$

Evaluating the integral gives"

$$\text{Area of triangle} = \int_0^5 (10 - 2h)\,dh = (10h - h^2)\Big|_0^5 = 25\ \text{cm}^2.$$

Notice that the limits in the definite integral are the limits for the variable h. Once we decided to slice the triangle horizontally, we knew that a typical slice has thickness Δh, so h is the variable in our definite integral, and the limits must be values of h.

Finding Volumes by Slicing

To calculate the volume of a solid using Riemann sums, we chop the solid into slices whose volumes we can estimate.

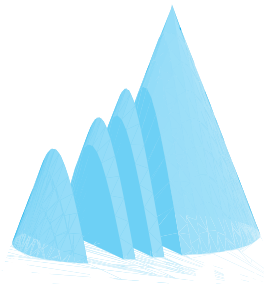

Figure 8.3: Cone cut into vertical slices

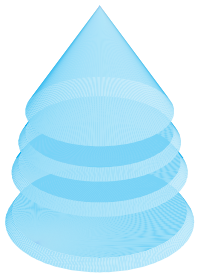

Figure 8.4: Cone cut into horizontal slices

Let's see how we might slice a cone standing with the vertex uppermost. We could divide the cone vertically into arch-shaped slices; see Figure 8.3. We could also divide the cone horizontally, giving coin-shaped slices; see Figure 8.4.

To calculate the volume of the cone, we choose the circular slices because it is easier to estimate the volumes of the coin-shaped slices.

Example 2 Use horizontal slicing to find the volume of the cone in Figure 8.5.

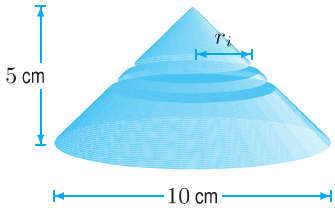

Figure 8.5: Cone

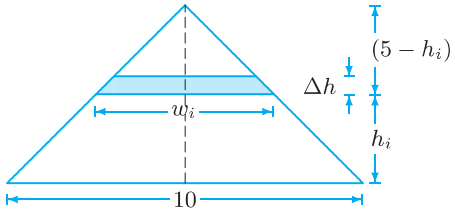

Figure 8.6: Vertical cross-section of cone in Figure 8.5

Solution Each slice is a circular disk of thickness Δh. See Figure 8.5. The disk at height h_i above the base has radius $r_i = \frac{1}{2}w_i$. From Figure 8.6 and the previous example, we have

$$w_i = 10 - 2h_i \quad \text{so} \quad r_i = 5 - h_i.$$

Each slice is approximately a cylinder of radius r_i and thickness Δh, so

$$\text{Volume of slice} \approx \pi r_i^2 \Delta h = \pi(5 - h_i)^2 \Delta h \text{ cm}^3.$$

Summing over all slices, we have

$$\text{Volume of cone} \approx \sum_{i=1}^{n} \pi(5 - h_i)^2 \Delta h \text{ cm}^3.$$

Taking the limit as $n \to \infty$, so $\Delta h \to 0$, gives

$$\text{Volume of cone} = \lim_{n\to\infty} \sum_{i=1}^{n} \pi(5 - h_i)^2 \Delta h = \int_0^5 \pi(5 - h)^2 \, dh \text{ cm}^3.$$

The integral can be evaluated using the substitution $u = 5 - h$ or by multiplying out $(5 - h)^2$. Using the substitution, we have

$$\text{Volume of cone} = \int_0^5 \pi(5 - h)^2 dh = -\frac{\pi}{3}(5 - h)^3 \Big|_0^5 = \frac{125}{3}\pi \text{ cm}^3.$$

Note that the sum represented by the $\sum$ sign is over all the strips. In the future, we will not write down the points, $h_0, h_1, \ldots h_n$, that are needed to express the Riemann sum precisely. This is unnecessary if all we want is the final expression for the definite integral. We now calculate the area of a semicircle and the volume of a hemisphere by slicing.

Example 3 Use horizontal slices to set up a definite integral representing the area of the semicircle of radius 7 cm in Figure 8.7.

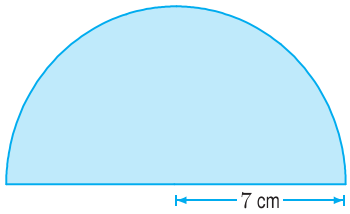

Figure 8.7: Semicircle

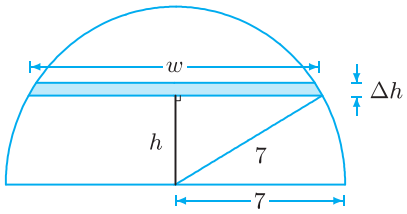

Figure 8.8: Horizontal slices of semicircle

Solution As in Example 1, to calculate the area using horizontal slices, we divide the region into strips; see Figure 8.8. A typical strip at height h above the base has width w and thickness Δh, so

$$\text{Area of strip} \approx w\Delta h \text{ cm}^2.$$

To get w in terms of h, we use the Pythagorean Theorem in Figure 8.8, giving

$$h^2 + \left(\frac{w}{2}\right)^2 = 7^2,$$

so

$$w = \sqrt{4(7^2 - h^2)} = 2\sqrt{49 - h^2}.$$

Summing the areas of the strips gives the Riemann sum approximation

$$\text{Area of semicircle} \approx \sum w\Delta h = \sum 2\sqrt{49 - h^2}\Delta h \text{ cm}^2.$$

Taking the limit as $\Delta h \to 0$ gives the integral:

$$\text{Area of semicircle} = \lim_{\Delta h \to 0} \sum 2\sqrt{49 - h^2}\Delta h = 2\int_0^7 \sqrt{49 - h^2}\, dh \text{ cm}^2.$$

Using the table of integrals VI-28 and VI-30, or a calculator or computer, gives

$$\text{Area of semicircle} = 2 \cdot \frac{1}{2}\left(h\sqrt{49 - h^2} + 49 \arcsin\left(\frac{h}{7}\right)\right)\Bigg|_0^7 = 49 \arcsin 1 = \frac{49}{2}\pi = 76.97 \text{ cm}^2.$$

As a check, notice that the area of the whole circle of radius 7 is $\pi \cdot 7^2 = 49\pi$ cm^2.

Example 4 Set up and evaluate an integral giving the volume of the hemisphere of radius 7 cm in Figure 8.9.

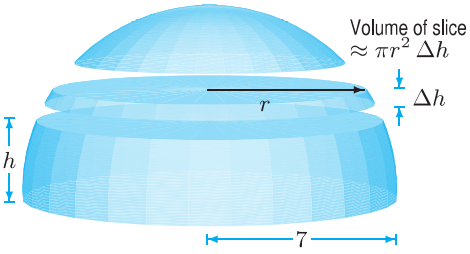

Figure 8.9: Slicing to find the volume of a hemisphere

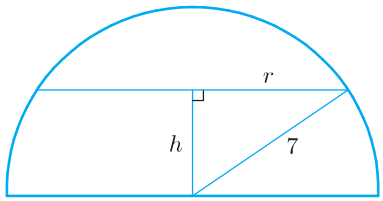

Figure 8.10: Vertical cut through center of hemisphere showing relation between r_i and h_i

Solution We will not use the formula $\frac{4}{3}\pi r^3$ for the volume of a sphere. However our approach can be used to derive that formula.

Divide the hemisphere into horizontal slices of thickness Δh cm. (See Figure 8.9.) Each slice is a circular slab. Let r be the radius of the slice at height h, so

$$\text{Volume of slice} \approx \pi r^2 \Delta h \text{ cm}^3.$$

We want to express r in terms of h. From Figure 8.10, we have $r^2 = 7^2 - h^2$, so

$$\text{Volume of slice} \approx \pi r^2\, \Delta h = \pi(7^2 - h^2)\, \Delta h \text{ cm}^3.$$

Summing the volumes of all slices gives:

$$\text{Volume} \approx \sum \pi r^2\, \Delta h = \sum \pi(7^2 - h^2)\, \Delta h \text{ cm}^3.$$

As the thickness of each slab tends to zero, the sum becomes a definite integral. Since the radius of the hemisphere is 7, we know that h varies from 0 to 7, so these are the limits of integration:

$$\text{Volume} = \int_0^7 \pi(7^2 - h^2)\, dh = \pi\left(7^2 h - \frac{1}{3}h^3\right)\Big|_0^7 = \frac{2}{3}\pi 7^3 = 718.4 \text{ cm}^3.$$

Notice that the volume of the hemisphere is half of $\frac{4}{3}\pi 7^3$ cm^3, as we expected.

We now use slicing to find the volume of a pyramid. You may already know the formula $V = \frac{1}{3}b^2 \cdot h$ for the volume V of a pyramid with height h and a square base of length b. We will not use the formula, but our approach can be used to derive it.

Example 5 Compute the volume, in cubic feet, of the Great Pyramid of Egypt, whose base is a square 755 feet by 755 feet and whose height is 410 feet.

Solution We slice the pyramid horizontally. Each slice is a square slab with thickness Δh. The bottom layer is a square slab 755 feet by 755 feet and volume about $(755)^2 \Delta h$ ft^3. As we move up the pyramid, the layers have shorter side lengths. We divide the height into n subintervals of length Δh. See Figure 8.11. Let s be the side length of the slice at height h, then

$$\text{Volume of slice} \approx s^2\, \Delta h \text{ ft}^3.$$

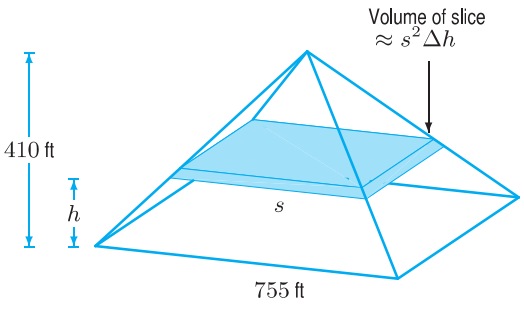

Figure 8.11: The Great Pyramid

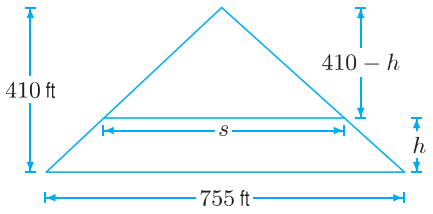

Figure 8.12: Cross-section relating s and h

We express s as a function of h using the vertical cross-section in Figure 8.12. By similar triangles, we get

$$\frac{s}{755} = \frac{(410 - h)}{410}.$$

Thus,

$$s = \left(\frac{755}{410}\right)(410 - h),$$

and the total volume, V, is approximated by adding the volumes of the n layers:

$$V \approx \sum s^2\,\Delta h = \sum \left[\left(\frac{755}{410}\right)(410-h)\right]^2 \Delta h \text{ ft}^3.$$

As the thickness of each slice tends to zero, the sum becomes a definite integral. Finally, since h varies from 0 to 410, the height of the pyramid, we have

$$V = \int_0^{410} \left[\left(\frac{755}{410}\right)(410-h)\right]^2 dh = \left(\frac{755}{410}\right)^2 \int_0^{410} (410-h)^2\,dh$$

$$= \left(\frac{755}{410}\right)^2 \left[-\frac{(410-h)^3}{3}\right]\Bigg|_0^{410} = \left(\frac{755}{410}\right)^2 \frac{(410)^3}{3} = \frac{1}{3}(755)^2(410) \approx 78 \text{ million ft}^3.$$

Note that $V = \frac{1}{3}(755)^2(410) = \frac{1}{3}b^2 \cdot h$, as expected.

Exercises and Problems for Section 8.1

Exercises

In Exercises 1–8, write a Riemann sum and then a definite integral representing the area of the region, using the strip shown. Evaluate the integral exactly.

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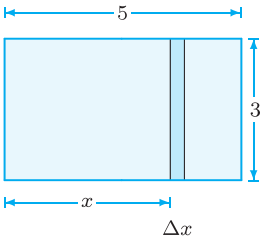

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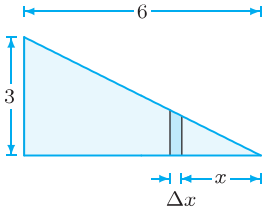

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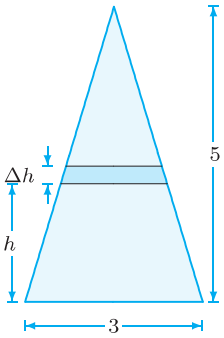

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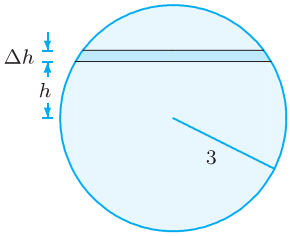

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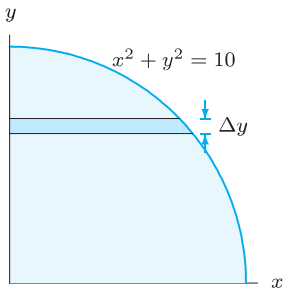

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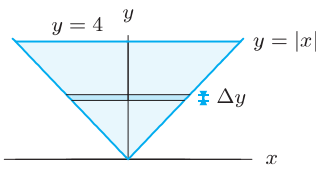

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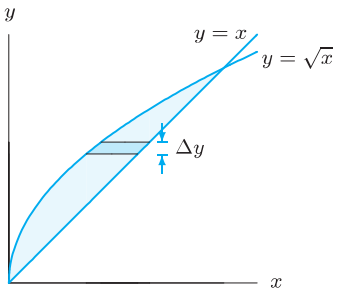

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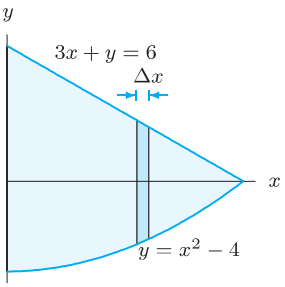

In Exercises 9–14, write a Riemann sum and then a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly. (Regions are parts of cones, cylinders, spheres, and pyramids.)

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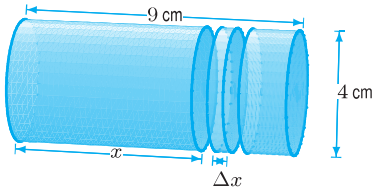

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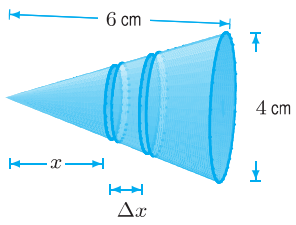

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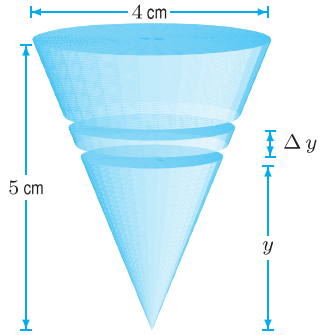

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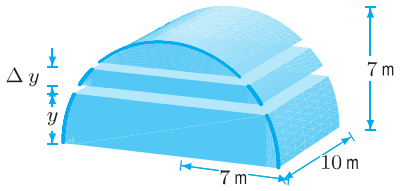

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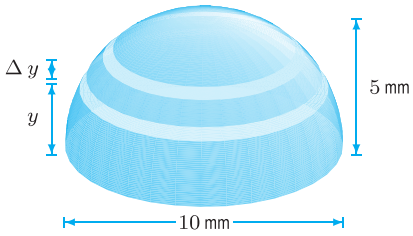

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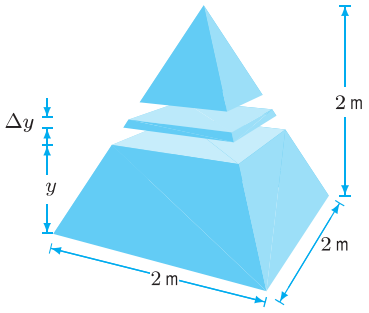


Problems

The integrals in Problems 15–18 represent the area of either a triangle or part of a circle, and the variable of integration measures a distance. In each case, say which shape is represented, and give the radius of the circle or the base and height of the triangle. Make a sketch to support your answer showing the variable and all other relevant quantities.

15. $\displaystyle\int_0^1 3x\,dx$

16. $\displaystyle\int_{-9}^{9} \sqrt{81-x^2}\,dx$

17. $\displaystyle\int_0^{\sqrt{15}} \sqrt{15-h^2}\,dh$

18. $\displaystyle\int_0^7 5\left(1-\frac{h}{7}\right)dh$

The integrals in Problems 19–22 represent the volume of either a hemisphere or a cone, and the variable of integration measures a length. In each case, say which shape is represented, and give the radius of the hemisphere or the radius and height of the cone. Make a sketch to support your answer showing the variable and all other relevant quantities.

19. $\displaystyle\int_0^{12} \pi(144-h^2)\,dh$

20. $\displaystyle\int_0^{12} \pi(x/3)^2\,dx$

21. $\displaystyle\int_0^{6} \pi(3-y/2)^2\,dy$

22. $\displaystyle\int_0^{2} \pi(2^2-(2-y)^2)\,dy$

23. Find, by slicing, the volume of a cone whose height is 3 cm and whose base radius is 1 cm. Slice the cone as shown in Figure 8.4 on page 393.

24. Find the volume of a sphere of radius r by slicing.

25. Find, by slicing, a formula for the volume of a cone of height h and base radius r.

26. Figure 8.13 shows a solid with both rectangular and triangular cross sections.

(a) Slice the solid parallel to the triangular faces. Sketch one slice and calculate its volume in terms of x, the distance of the slice from one end. Then write and evaluate an integral giving the volume of the solid.

(b) Repeat part (a) for horizontal slices. Instead of x, use h, the distance of a slice from the top.

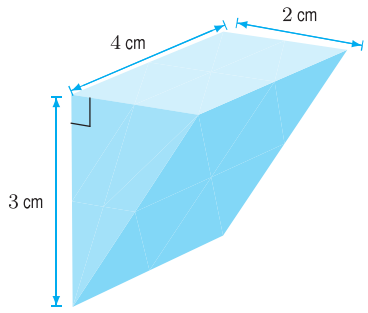


Figure 8.13

27. A rectangular lake is 150 km long and 3 km wide. The vertical cross-section through the lake in Figure 8.14 shows that the lake is 0.2 km deep at the center. (These are the approximate dimensions of Lake Mead, the largest reservoir in the US, which provides water to California, Nevada, and Arizona.) Set up and evaluate a definite integral giving the total volume of the lake.

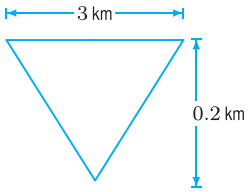


Figure 8.14: Not to scale

28. A dam has a rectangular base 1400 meters long and 160 meters wide. Its cross-section is shown in Figure 8.15. (The Grand Coulee Dam in Washington state is roughly this size.) By slicing horizontally, set up and evaluate a definite integral giving the volume of material used to build this dam.

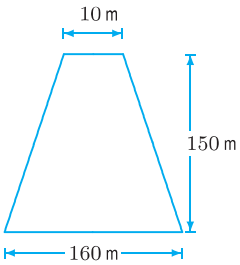


Figure 8.15: Not to scale

8.2 APPLICATIONS TO GEOMETRY

In Section 8.1, we calculated volumes using slicing and definite integrals. In this section, we use the same method to calculate the volumes of more complicated regions as well as the length of a curve. The method is summarized in the following steps:

To Compute a Volume or Length Using an Integral

- Divide the solid (or curve) into small pieces whose volume (or length) we can easily approximate;
- Add the contributions of all the pieces, obtaining a Riemann sum that approximates the total volume (or length);
- Take the limit as the number of terms in the sum tends to infinity, giving a definite integral for the total volume (or total length).

In the previous section, all the slices we created were disks or squares. We now look at different ways of generating volumes whose cross-sections include circles, rings, and squares.

Volumes of Revolution

One way to create a solid having circular cross-sections is to revolve a region in the plane around a line, giving a *solid of revolution,* as in the following examples.

Example 1 The region bounded by the curve $y = e^{-x}$ and the x-axis between $x = 0$ and $x = 1$ is revolved around the x-axis. Find the volume of this solid of revolution.

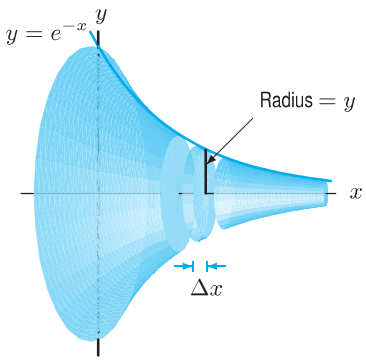


Figure 8.16: A thin strip rotated around the x-axis to form a circular slice

Solution We slice the region perpendicular to the x-axis, giving circular disks of thickness Δx. See Figure 8.16. The radius of the disk is $y = e^{-x}$, so:

$$\text{Volume of the slice} \approx \pi y^2 \, \Delta x = \pi (e^{-x})^2 \, \Delta x,$$

$$\text{Total volume} \approx \sum \pi y^2 \, \Delta x = \sum \pi \left(e^{-x}\right)^2 \, \Delta x.$$

As the thickness of each slice tends to zero, we get:

$$\text{Total volume} = \int_0^1 \pi (e^{-x})^2 \, dx = \pi \int_0^1 e^{-2x} \, dx = \pi \left(-\frac{1}{2}\right) e^{-2x} \Big|_0^1$$

$$= \pi \left(-\frac{1}{2}\right) (e^{-2} - e^0) = \frac{\pi}{2}(1 - e^{-2}) \approx 1.36.$$

Example 2 A table leg in Figure 8.17 has a circular cross section with radius r cm at a height of y cm above the ground given by $r = 3 + \cos(\pi y/25)$. Find the volume of the table leg.

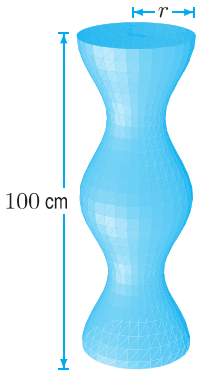


Figure 8.17

Solution The table leg is formed by rotating the curve $r = 3 + \cos(\pi y/25)$ around the y-axis. Slicing the table leg horizontally gives circular disks of thickness Δy and radius $r = 3 + \cos(\pi y/25)$.

To set up a definite integral for the volume, we find the volume of a typical slice:

$$\text{Volume of slice} \approx \pi r^2 \Delta y = \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y.$$

Summing over all slices gives the Riemann sum approximation:

$$\text{Total volume} = \sum \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y.$$

Taking the limit as $\Delta y \to 0$ gives the definite integral:

$$\text{Total volume} = \lim_{\Delta y \to 0} \sum \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 \Delta y = \int_0^{100} \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 dy.$$

Evaluating the integral numerically gives:

$$\text{Total volume} = \int_0^{100} \pi \left(3 + \cos\left(\frac{\pi}{25} y\right)\right)^2 dy = 2984.5 \text{ cm}^3.$$

Example 3 The region bounded by the curves $y = x$ and $y = x^2$ is rotated about the line $y = 3$. Compute the volume of the resulting solid.

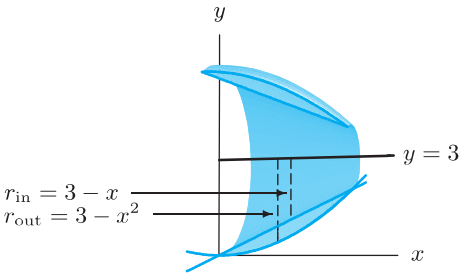


Figure 8.18: Cutaway view of volume showing inner and outer radii

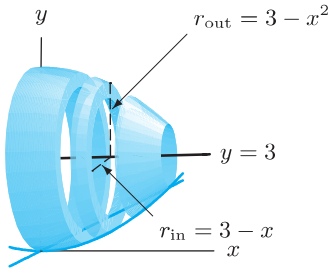


Figure 8.19: One slice (a disk-with-a-hole)

Solution The solid is shaped like a bowl with the base removed. See Figure 8.18. To compute the volume, we divide the area in the xy-plane into thin vertical strips of thickness Δx, as in Figure 8.20.

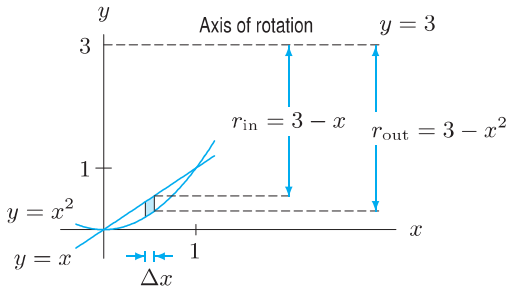


Figure 8.20: The region for Example 3

As each strip is rotated around the line $y = 3$, it sweeps out a slice shaped like a circular disk with a hole in it. See Figure 8.19. This disk-with-a-hole has an inner radius of $r_{\text{in}} = 3 - x$ and an outer radius of $r_{\text{out}} = 3 - x^2$. Think of the slice as a circular disk of radius r_{out} from which has been removed a smaller disk of radius r_{in}. Then:

$$\text{Volume of slice} \approx \pi r_{\text{out}}^2 \Delta x - \pi r_{\text{in}}^2 \Delta x = \pi (3 - x^2)^2 \Delta x - \pi (3 - x)^2 \Delta x.$$

Adding the volumes of all the slices, we have:

$$\text{Total volume} = V \approx \sum \left(\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2\right) \Delta x = \sum \left(\pi(3-x^2)^2 - \pi(3-x)^2\right) \Delta x.$$

We let Δx, the thickness of each slice, tend to zero to obtain a definite integral. Since the curves $y = x$ and $y = x^2$ intersect at $x = 0$ and $x = 1$, these are the limits of integration:

$$V = \int_0^1 \left(\pi(3-x^2)^2 - \pi(3-x)^2\right) dx = \pi \int_0^1 \left((9 - 6x^2 + x^4) - (9 - 6x + x^2)\right) dx$$
$$= \pi \int_0^1 (6x - 7x^2 + x^4)\, dx = \pi \left(3x^2 - \frac{7x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 \approx 2.72.$$

Volumes of Regions of Known Cross-Section

We now calculate the volume of a solid constructed by a different method. Starting with a region in the xy-plane as a base, the solid is built by standing squares, semicircles, or triangles vertically on edge in this region.

Example 4 Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x^2$ and $y = 8 - x^2$ and whose cross-sections perpendicular to the x-axis are squares with one side in the xy-plane. (See Figure 8.21.)

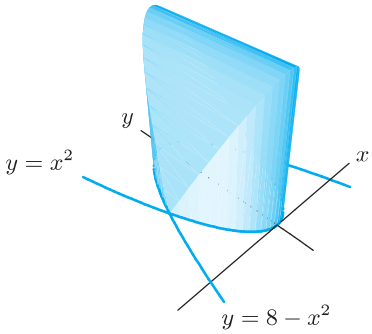


Figure 8.21: The solid for Example 4

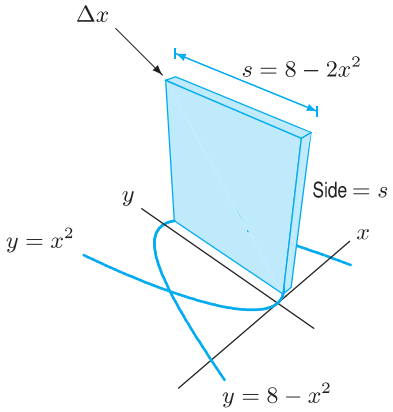


Figure 8.22: A slice of the solid for Example 4

Solution We view the solid as a loaf of bread sitting on the xy-plane and made up of square slices. A typical slice of thickness Δx is shown in Figure 8.22. The side length, s, of the square is the distance (in the y direction) between the two curves, so $s = (8 - x^2) - x^2 = 8 - 2x^2$ giving

$$\text{Volume of slice} \approx s^2\, \Delta x = (8 - 2x^2)^2\, \Delta x.$$

Thus

$$\text{Total volume} = V \approx \sum s^2\, \Delta x = \sum (8 - 2x^2)^2\, \Delta x.$$

As the thickness Δx of each slice tends to zero, the sum becomes a definite integral. Since the curves $y = x^2$ and $y = 8 - x^2$ intersect at $x = -2$ and $x = 2$, these are the limits of integration. We have

$$V = \int_{-2}^{2} (8 - 2x^2)^2\, dx = \int_{-2}^{2} (64 - 32x^2 + 4x^4)\, dx$$
$$= 64x - \frac{32}{3}x^3 + \frac{4}{5}x^5\Bigg|_{-2}^{2} = \frac{2048}{15} \approx 136.5.$$

Arc Length

A definite integral can be used to compute the *arc length*, or length, of a curve. To compute the length of the curve $y = f(x)$ from $x = a$ to $x = b$, where $a < b$, we divide the curve into small pieces, each one approximately straight.

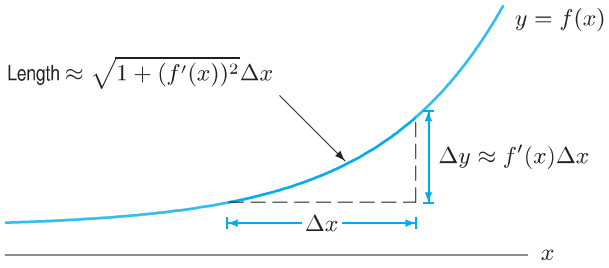


Figure 8.23: Length of a small piece of curve approximated using Pythagoras' theorem

Figure 8.23 shows that a small change Δx corresponds to a small change $\Delta y \approx f'(x)\,\Delta x$. The length of the piece of the curve is approximated by

$$\text{Length} \approx \sqrt{(\Delta x)^2 + (\Delta y)^2} \approx \sqrt{(\Delta x)^2 + (f'(x)\,\Delta x)^2} = \sqrt{1 + (f'(x))^2}\,\Delta x.$$

Thus, the arc length of the entire curve is approximated by a Riemann sum:

$$\text{Arc length} \approx \sum \sqrt{1 + (f'(x))^2}\,\Delta x.$$

Since x varies between a and b, as we let Δx tend to zero, the sum becomes the definite integral:

For $a < b$, the arc length of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$\text{Arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Example 5 Set up and evaluate an integral to compute the length of the curve $y = x^3$ from $x = 0$ to $x = 5$.

Solution If $f(x) = x^3$, then $f'(x) = 3x^2$, so

$$\text{Arc length} = \int_0^5 \sqrt{1 + (3x^2)^2}\,dx.$$

Although the formula for the arc length of a curve is easy to apply, the integrands it generates often do not have elementary antiderivatives. Evaluating the integral numerically, we find the arc length to be 125.68. To check that the answer is reasonable, notice that the curve starts at $(0, 0)$ and goes to $(5, 125)$, so its length must be at least the length of a straight line between these points, or $\sqrt{5^2 + 125^2} = 125.10$. (See Figure 8.24.)

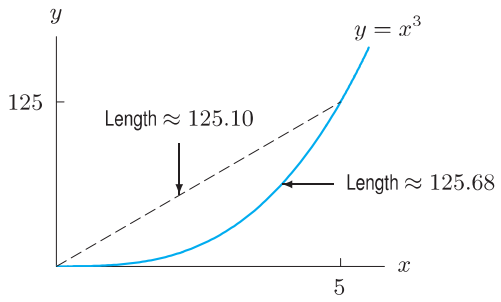


Figure 8.24: Arc length of $y = x^3$ (Note: The picture is distorted because the scales on the two axes are quite different.)

Arc Length of a Parametric Curve

A particle moving along a curve in the plane given by the parametric equations $x = f(t)$, $y = g(t)$, where t is time, has speed given by:

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

We can find the distance traveled by a particle along a curve between $t = a$ and $t = b$ by integrating its speed. Thus,

$$\text{Distance traveled} = \int_a^b v(t)\,dt.$$

If the particle never stops or reverses its direction as it moves along the curve, the distance it travels is the same as the length of the curve. This suggests the following formula:

If a curve is given parametrically for $a \le t \le b$ by differentiable functions and if the velocity $v(t)$ is not 0 for $a < t < b$, then

$$\text{Arc length of curve} = \int_a^b v(t)\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Example 6 Find the circumference of the ellipse given by the parametric equations

$$x = 2\cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$$

Solution The circumference of this curve is given by an integral which must be calculated numerically:

$$\begin{aligned}\text{Circumference} &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (\cos t)^2}\,dt \\ &= \int_0^{2\pi} \sqrt{4\sin^2 t + \cos^2 t}\,dt = 9.69.\end{aligned}$$

Since the ellipse is inscribed in a circle of radius 2 and circumscribes a circle of radius 1, we expect the length of the ellipse to be between $2\pi(2) \approx 12.57$ and $2\pi(1) \approx 6.28$, so the value of 9.69 is reasonable.

Exercises and Problems for Section 8.2

Exercises

In Exercises 1–10, the region is rotated around the x-axis. Find the volume.

1. Bounded by $y = x^2, y = 0, x = 0, x = 1$.
2. Bounded by $y = (x+1)^2, y = 0, x = 1, x = 2$.
3. Bounded by $y = 4 - x^2, y = 0, x = -2, x = 0$.
4. Bounded by $y = \sqrt{x+1}, y = 0, x = -1, x = 1$.
5. Bounded by $y = e^x, y = 0, x = -1, x = 1$.
6. Bounded by $y = \cos x, y = 0, x = 0, x = \pi/2$.
7. Bounded by $y = 1/(x+1), y = 0, x = 0, x = 1$.
8. Bounded by $y = \sqrt{\cosh 2x}, y = 0, x = 0, x = 1$.
9. Bounded by $y = x^2, y = x, x = 0, x = 1$.
10. Bounded by $y = e^{3x}, y = e^x, x = 0, x = 1$.

For Exercises 11–16, find the arc length of the graph of the function from $x = 0$ to $x = 2$.

11. $f(x) = x^2/2$
12. $f(x) = \cos x$
13. $f(x) = \ln(x+1)$
14. $f(x) = \sqrt{x^3}$
15. $f(x) = \sqrt{4 - x^2}$
16. $f(x) = \cosh x$

Find the length of the parametric curves in Exercises 17–20.

17. $x = 3 + 5t, y = 1 + 4t$ for $1 \le t \le 2$. Explain why your answer is reasonable.
18. $x = \cos(e^t), y = \sin(e^t)$ for $0 \le t \le 1$. Explain why your answer is reasonable.
19. $x = \cos(3t), y = \sin(5t)$ for $0 \le t \le 2\pi$.
20. $x = \cos^3 t, y = \sin^3 t$, for $0 \le t \le 2\pi$.

Problems

In Problems 21–24 set up, but do not evaluate, an integral that represents the volume obtained when the region is rotated about the given axis.

21. Bounded by $y = \sqrt[3]{x}, x = 4y$. Axis $x = 9$.
22. Bounded by $y = \sqrt[3]{x}, x = 4y$. Axis $y = 3$.
23. Bounded by $y = 0, x = 9, y = \frac{1}{3}x$. Axis $y = -2$.
24. Bounded by $y = 0, x = 9, y = \frac{1}{3}x$. Axis $x = -1$.
25. Find the length of one arch of $y = \cos x$.
26. Find the perimeter of the region bounded by $y = x$ and $y = x^2$.
27. Consider the hyperbola $x^2 - y^2 = 1$ in Figure 8.25.
    (a) The shaded region $2 \le x \le 3$ is rotated around the x-axis. What is the volume generated?
    (b) What is the arc length with $y \ge 0$ from $x = 2$ to $x = 3$?

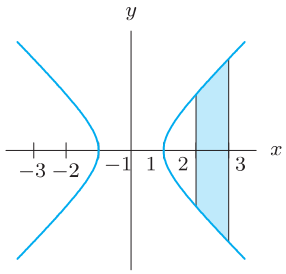


Figure 8.25

28. Rotating the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the x-axis generates an ellipsoid. Compute its volume.

For Problems 29–31, sketch the solid obtained by rotating each region around the indicated axis. Using the sketch, show how to approximate the volume of the solid by a Riemann sum, and hence find the volume.

29. Bounded by $y = x^3, x = 1, y = -1$. Axis: $y = -1$.
30. Bounded by $y = \sqrt{x}, x = 1, y = 0$. Axis: $x = 1$.
31. Bounded by the first arch of $y = \sin x, y = 0$. Axis: x axis.

Problems 32–37 concern the region bounded by $y = x^2$, $y = 1$, and the y-axis. Find the volume of the following solids.

32. The solid obtained by rotating the region around the y-axis.
33. The solid obtained by rotating the region about the x-axis.
34. The solid obtained by rotating the region about the line $y = -2$.
35. The solid whose base is the region and whose cross-sections perpendicular to the x-axis are squares.
36. The solid whose base is the region and whose cross-sections perpendicular to the x-axis are semicircles.
37. The solid whose base is the region and whose cross-sections perpendicular to the y-axis are equilateral triangles.

For Problems 38–42 consider the region bounded by $y = e^x$, the x-axis, and the lines $x = 0$ and $x = 1$. Find the volume of the following solids.

38. The solid obtained by rotating the region about the x-axis.

39. The solid obtained by rotating the region about the horizontal line $y = -3$.

40. The solid obtained by rotating the region about the horizontal line $y = 7$.

41. The solid whose base is the given region and whose cross-sections perpendicular to the x-axis are squares.

42. The solid whose base is the given region and whose cross-sections perpendicular to the x-axis are semicircles.

43. A particle starts at the origin and moves along the curve $y = 2x^{3/2}/3$ in the positive x-direction at a speed of 3 cm/sec, where x, y are in cm. Find the position of the particle at $t = 6$.

44. A tree trunk has a circular cross section at every height; its circumference is given in the following table. Estimate the volume of the tree trunk using the trapezoid rule.

Height (feet)	0	20	40	60	80	100	120
Circumference (feet)	26	22	19	14	6	3	1

45. Rotate the bell-shaped curve $y = e^{-x^2/2}$ shown in Figure 8.26 around the y-axis, forming a hill-shaped solid of revolution. By slicing horizontally, find the volume of this hill.

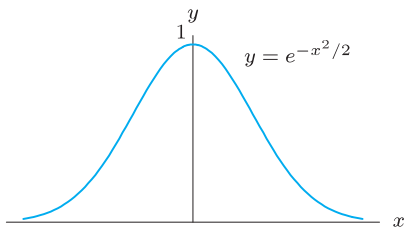


Figure 8.26

46. **(a)** A pie dish is 9 inches across the top, 7 inches across the bottom, and 3 inches deep. See Figure 8.27. Compute the volume of this dish.

(b) Make a rough estimate of the volume in cubic inches of a single cut-up apple, and estimate the number of apples that is needed to make an apple pie that fills this dish.

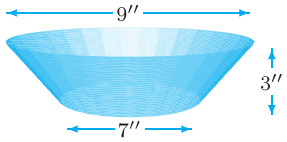


Figure 8.27

47. A 100 cm long gutter is made of three strips of metal, each 5 cm wide; Figure 8.28 shows a cross-section.

(a) Find the volume of water in the gutter when the depth is h cm.

(b) What is the maximum value of h?

(c) What is the maximum volume of water that the gutter can hold?

(d) If the gutter is filled with half the maximum volume of water, is the depth larger or smaller than half of the answer to part (b)? Explain how you can answer without any calculation.

(e) Find the depth of the water when the gutter contains half the maximum possible volume.

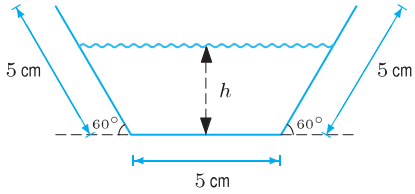


Figure 8.28

48. The design of boats is based on Archimedes' Principle, which states that the buoyant force on an object in water is equal to the weight of the water displaced. Suppose you want to build a sailboat whose hull is parabolic with cross section $y = ax^2$, where a is a constant. Your boat will have length L and its maximum draft (the maximum vertical depth of any point of the boat beneath the water line) will be H. See Figure 8.29. Every cubic meter of water weighs 10,000 newtons. What is the maximum possible weight for your boat and cargo?

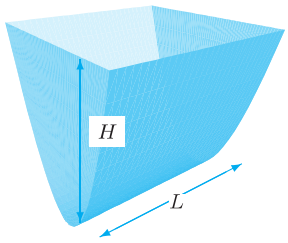


Figure 8.29

49. The circumference of a tree at different heights above the ground is given in the table below. Assume that all horizontal cross-sections of the tree are circles. Estimate the volume of the tree.

Height (inches)	0	20	40	60	80	100	120
Circumference (inches)	31	28	21	17	12	8	2

50. A bowl has the shape of the graph of $y = x^4$ between the points $(1, 1)$ and $(-1, 1)$ rotated about the y-axis. When the bowl contains water to a depth of h units, it flows out through a hole in the bottom at a rate (volume/time) proportional to $\sqrt{h}$, with constant of proportionality 6.

(a) Show that the water level falls at a constant rate.
(b) Find how long it takes to empty the bowl if it is originally full to the brim.

51. The hull of a boat has widths given by the following table. Reading across a row of the table gives widths at points 0, 10, ..., 60 feet from the front to the back at a certain level below waterline. Reading down a column of the table gives widths at levels 0, 2, 4, 6, 8 feet below waterline at a certain distance from the front. Use the trapezoidal rule to estimate the volume of the hull below waterline.

		Front of boat ⟶ Back of boat						
		0	10	20	30	40	50	60
	0	2	8	13	16	17	16	10
Depth	2	1	4	8	10	11	10	8
below	4	0	3	4	6	7	6	4
waterline	6	0	1	2	3	4	3	2
(in feet)	8	0	0	1	1	1	1	1

52. **(a)** Write an integral which represents the circumference of a circle of radius r.
(b) Evaluate the integral, and show that you get the answer you expect.

53. Compute the perimeter of the region used for the base of the solids in Problems 38–42.

54. With x and b in meters, a chain hangs in the shape of the catenary $\cosh x = \frac{1}{2}(e^x + e^{-x})$ for $-b \le x \le b$. If the chain is 10 meters long, how far apart are its ends?

55. There are very few elementary functions $y = f(x)$ for which arc length can be computed in elementary terms using the formula

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

You have seen some such functions f in Problems 14, 15, and 43, namely, $f(x) = \sqrt{4 - x^2}$, $f(x) = \sqrt{x^3}$, and $f(x) = \frac{1}{2}(e^x + e^{-x})$. Try to find some other function that "works," that is, a function whose arc length you can find using this formula and antidifferentiation.

56. After doing Problem 55, you may wonder what sort of functions can represent arc length. If $g(0) = 0$ and g is differentiable and increasing, then can $g(x)$, $x \ge 0$, represent arc length? That is, can we find a function $f(t)$ such that

$$\int_0^x \sqrt{1 + (f'(t))^2}\, dt = g(x)?$$

(a) Show that $f(x) = \int_0^x \sqrt{(g'(t))^2 - 1}\, dt$ works as long as $g'(x) \ge 1$. In other words, show that the arc length of the graph of f from 0 to x is $g(x)$.
(b) Show that if $g'(x) < 1$ for some x, then $g(x)$ cannot represent the arc length of the graph of any function.
(c) Find a function f whose arc length from 0 to x is $2x$.

57. Consider the graph of the equation

$$|x|^k + |y|^k = 1, \quad k \text{ constant.}$$

For k an even integer, the absolute values are unnecessary. For example, for $k = 2$, the equation gives the circle

$$x^2 + y^2 = 1.$$

(a) Sketch the graph of the equation for $k = 1, 2, 4$.
(b) Find the arc length of the three graphs in part (a). [Note: $k = 4$ may require a computer.]

8.3 AREA AND ARC LENGTH IN POLAR COORDINATES

Many curves and regions in the plane are easier to describe in polar coordinates than in Cartesian coordinates. Thus their areas and arc lengths are best found using integrals in polar coordinates.

A point, P, in the plane is often identified by its *Cartesian coordinates* (x, y), where x is the horizontal distance to the point from the origin and y is the vertical distance.[1] Alternatively, we can identify the point, P, by specifying its distance, r, from the origin and the angle, θ, shown in Figure 8.30. The angle θ is measured counterclockwise from the positive x-axis to the line joining P to the origin. The labels r and θ are called the *polar coordinates* of point P.

[1] Cartesian coordinates can also be called rectangular coordinates.

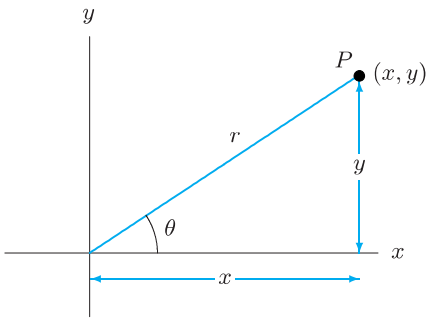


Figure 8.30: Cartesian and polar coordinates for the point P

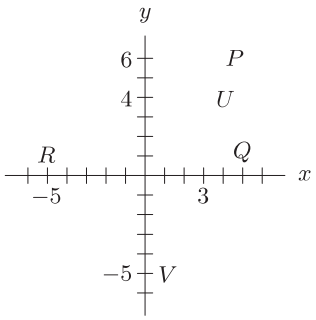


Figure 8.31

Relation Between Cartesian and Polar Coordinates

From the right triangle in Figure 8.30, we see that

- $x = r\cos\theta$ and $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$, $x \neq 0$

The angle θ is determined by the equations $\cos\theta = x/\sqrt{x^2+y^2}$ and $\sin\theta = y/\sqrt{x^2+y^2}$.

Warning: In general $\theta \neq \tan^{-1}(y/x)$. It is not possible to determine which quadrant θ is in from the value of $\tan\theta$ alone.

Example 1 (a) Give Cartesian coordinates for the points with polar coordinates (r, θ) given by $P = (7, \pi/3)$, $Q = (5, 0)$, $R = (5, \pi)$.

(b) Give polar coordinates for the points with Cartesian coordinates (x, y) given by $U = (3, 4)$ and $V = (0, -5)$.

Solution (a) See Figure 8.31. Point P is a distance of 7 from the origin. The angle θ is $\pi/3$ radians (60°). The Cartesian coordinates of P are

$$x = r\cos\theta = 7\cos\frac{\pi}{3} = \frac{7}{2} \qquad \text{and} \qquad y = r\sin\theta = 7\sin\frac{\pi}{3} = \frac{7\sqrt{3}}{2}.$$

Point Q is located a distance of 5 units along the positive x-axis with Cartesian coordinates

$$x = r\cos\theta = 5\cos 0 = 5 \qquad \text{and} \qquad y = r\sin\theta = 5\sin 0 = 0.$$

For point R, which is on the negative x-axis,

$$x = r\cos\theta = 5\cos\pi = -5 \qquad \text{and} \qquad y = r\sin\theta = 5\sin\pi = 0.$$

(b) For $U = (3, 4)$, we have $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\theta = 4/3$. A possible value for θ is $\theta = \arctan 4/3 = 0.927$ radians, or about 53°. The polar coordinates of U are $(5, 0.927)$. The point V falls on the negative y-axis, so we can choose $r = 5, \theta = 3\pi/2$ for its polar coordinates. In this case, we cannot use $\tan\theta = y/x$ to find θ, because $\tan\theta = y/x = -5/0$ is undefined.

Because the angle θ can be allowed to wrap around the origin more than once, there are many possibilities for the polar coordinates of a point. For the point V in Example 1, we can also choose $\theta = -\pi/2$ or $\theta = 7\pi/2$, so that $(5, -\pi/2)$, $(5, 7\pi/2)$, and $(5, 3\pi/2)$ are all polar coordinates for V. However, we often choose θ between 0 and 2π.

Example 2 Give three possible sets of polar coordinates for the point P in Figure 8.32.

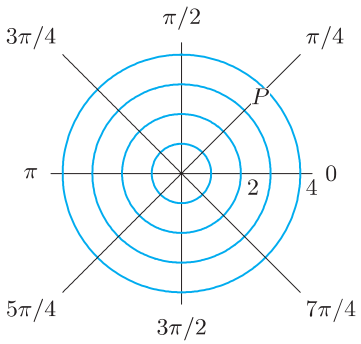


Figure 8.32

Solution Because $r = 3$ and $\theta = \pi/4$, one set of polar coordinates for P is $(3, \pi/4)$. We can also use $\theta = \pi/4 + 2\pi = 9\pi/4$ and $\theta = \pi/4 - 2\pi = -7\pi/4$, to get $(3, 9\pi/4)$ and $(3, -7\pi/4)$.

Graphing Equations in Polar Coordinates

The equations for certain graphs are much simpler when expressed in polar coordinates than in Cartesian coordinates. On the other hand, some graphs that have simple equations in Cartesian coordinates have complicated equations in polar coordinates.

Example 3 (a) Describe in words the graphs of the equation $y = 1$ (in Cartesian coordinates) and the equation $r = 1$ (in polar coordinates).

(b) Write the equation $r = 1$ using Cartesian coordinates. Write the equation $y = 1$ using polar coordinates.

Solution (a) The equation $y = 1$ describes a horizontal line. Since the equation $y = 1$ places no restrictions on the value of x, it describes every point having a y-value of 1, no matter what the value of its x-coordinate. Similarly, the equation $r = 1$ places no restrictions on the value of θ. Thus, it describes every point having an r-value of 1, that is, having a distance of 1 from the origin. This set of points is the unit circle. See Figure 8.33.

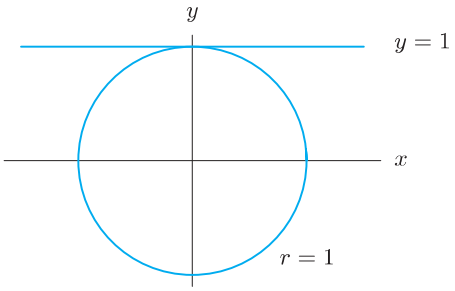


Figure 8.33: The graph of the equation $r = 1$ is the unit circle because $r = 1$ for every point regardless of the value of θ. The graph of $y = 1$ is a horizontal line since $y = 1$ for any x

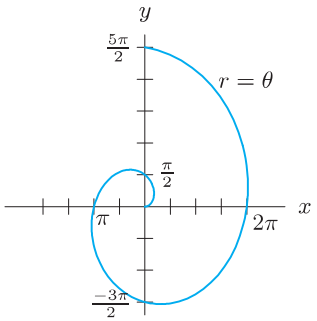


Figure 8.34: A graph of the Archimedean spiral $r = \theta$

(b) Since $r = \sqrt{x^2 + y^2}$, we rewrite the equation $r = 1$ using Cartesian coordinates as $\sqrt{x^2 + y^2} = 1$, or, squaring both sides, as $x^2 + y^2 = 1$. We see that the equation for the unit circle is simpler in polar coordinates than it is in Cartesian coordinates.

On the other hand, since $y = r \sin\theta$, we can rewrite the equation $y = 1$ in polar coordinates as $r \sin\theta = 1$, or, dividing both sides by $\sin\theta$, as $r = 1/\sin\theta$. We see that the equation for this horizontal line is simpler in Cartesian coordinates than in polar coordinates.

Example 4 Graph the equation $r = \theta$. The graph is called an *Archimedean spiral* after the Greek mathematician Archimedes who described its properties (although not by using polar coordinates).

Solution To construct this graph, use the values in Table 8.1. To help us visualize the shape of the spiral, we convert the angles in Table 8.1 to degrees and the r-values to decimals. See Table 8.2.

Table 8.1 *Points on the Archimedean spiral $r = \theta$, with θ in radians*

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$

Table 8.2 *Points on the Archimedean spiral $r = \theta$, with θ in degrees*

θ	0	30°	60°	90°	120°	150°	180°	210°	240°	270°
r	0.00	0.52	1.05	1.57	2.09	2.62	3.14	3.67	4.19	4.71

Notice that as the angle θ increases, points on the curve move farther from the origin. At 0°, the point is at the origin. At 30°, it is 0.52 units away from the origin, at 60° it is 1.05 units away, and at 90° it is 1.57 units away. As the angle winds around, the point traces out a curve that moves away from the origin, giving a spiral. (See Figure 8.34.)

In our definition, r is positive. However, graphs of curves in polar coordinates are traditionally drawn using negative values of r as well, because this makes the graphs symmetric. If an equation $r = f(\theta)$ gives a negative r-value, it is plotted in the opposite direction to θ. See Examples 5 and 6 and Figures 8.35 and 8.37.

Example 5 For $a > 0$ and n a positive integer, curves of the form $r = a \sin n\theta$ or $r = a \cos n\theta$ are called *roses*. Graph the roses

(a) $r = 3 \sin 2\theta$ (b) $r = 4 \cos 3\theta$

Solution (a) Using a calculator or making a table of values, we see that the graph is a rose with four petals, each of length 3. See Figure 8.35. Negative values of r for $\pi/2 < \theta < \pi$ and $3\pi/2 < \theta < 2\pi$ give the petals in Quadrants II and IV. For example, $\theta = 3\pi/4$ gives $r = -3$, which is plotted 3 units from the origin in the direction opposite to $\theta = 3\pi/4$, namely in Quadrant IV.

(b) The graph is a rose with three petals, each of length 4. See Figure 8.36.

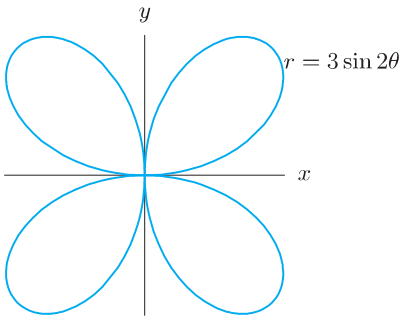


Figure 8.35: Graph of $r = 3 \sin 2\theta$
(Petals in Quadrants II and IV have $r < 0$)

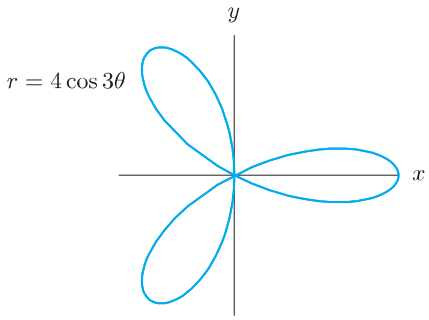


Figure 8.36: Graph of $r = 4 \cos 3\theta$

Example 6 Curves of the form $r = a + b\sin\theta$ or $r = a + b\cos\theta$ are called limaçons. Graph $r = 1 + 2\cos\theta$ and $r = 3 + 2\cos\theta$.

Solution See Figures 8.37 and 8.38. The equation $r = 1+2\cos\theta$ leads to negative r values for some θ values between $\pi/2$ and $3\pi/2$; these values give the inner loop in Figure 8.37. For example, $\theta = \pi$ gives $r = -1$, which is plotted 1 unit from the origin in the direction opposite to $\theta = \pi$, namely on the positive x-axis. The equation $r = 3 + 2\cos\theta$ does not lead to negative r-values.

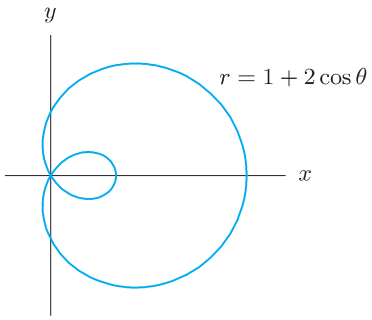


Figure 8.37: Graph of $r = 1 + 2\cos\theta$ (Inner loop has $r < 0$)

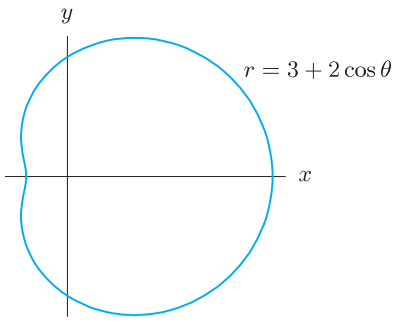


Figure 8.38: Graph of $r = 3 + 2\cos\theta$

Polar coordinates can be used with inequalities to describe regions that are obtained from circles. Such regions are often much harder to represent in Cartesian coordinates.

Example 7 Using inequalities, describe a compact disc with outer diameter 120 mm and inner diameter 15 mm.

Solution The compact disc lies between two circles of radius 7.5 mm and 60 mm. See Figure 8.39. Thus, if the origin is at the center, the disc is represented by

$$7.5 \le r \le 60 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

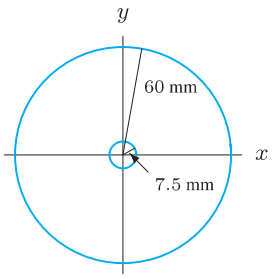


Figure 8.39: Compact disc

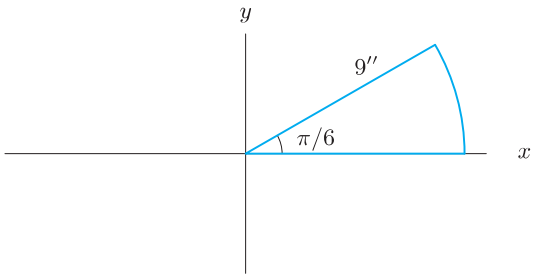


Figure 8.40: Pizza slice

Example 8 An 18 inch pizza is cut into 12 slices. Use inequalities to describe one of the slices.

Solution The pizza has radius 9 inches; the angle at the center is $2\pi/12 = \pi/6$. See Figure 8.40. Thus, if the origin is at center of the original pizza, the slice is represented by

$$0 \le r \le 9 \qquad \text{and} \qquad 0 \le \theta \le \frac{\pi}{6}.$$

Area in Polar Coordinates

We can use a definite integral to find the area of a region described in polar coordinates. As previously, we slice the region into small pieces, construct a Riemann sum, and take a limit to obtain the definite integral. In this case, the slices are approximately circular sectors.

To calculate the area of the sector in Figure 8.41, we think of the area of the sector as a fraction $\theta/2\pi$ of the area of the entire circle (for θ in radians). Then

$$\text{Area of sector } = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 \theta.$$

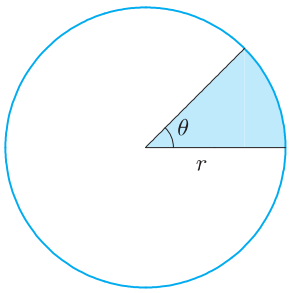


Figure 8.41: Area of shaded sector $= \frac{1}{2} r^2 \theta$ (for θ in radians)

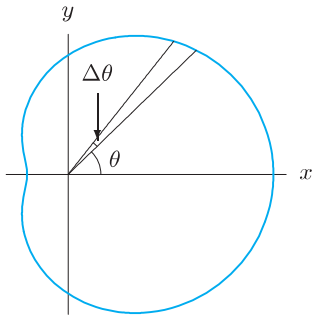


Figure 8.42: Finding the area of the limaçon $r = 3 + 2\cos\theta$

Example 9 Use circular sectors to set up a definite integral to calculate the area of the region bounded by the limaçon $r = 3 + 2\cos\theta$, for $0 \le \theta \le 2\pi$. See Figure 8.42.

Solution The slices are not exactly circular sectors because the radius r depends on θ. However,

$$\text{Area of sector } \approx \frac{1}{2} r^2 \Delta\theta = \frac{1}{2}(3 + 2\cos\theta)^2 \, \Delta\theta.$$

Thus, for the whole area,

$$\text{Area of region } \approx \sum_{i=1}^{n} \frac{1}{2}(3 + 2\cos\theta)^2 \, \Delta\theta.$$

Taking the limit as $n \to \infty$ and $\Delta\theta \to 0$ gives the integral

$$\text{Area } = \int_0^{2\pi} \frac{1}{2}(3 + 2\cos\theta)^2 \, d\theta.$$

To compute this integral, we expand the integrand and use integration by parts or formula IV-18 from the table of integrals:

$$\begin{aligned} \text{Area } &= \frac{1}{2} \int_0^{2\pi} (9 + 12\cos\theta + 4\cos^2\theta) \, d\theta \\ &= \frac{1}{2} \left(9\theta + 12\sin\theta + \frac{4}{2}(\cos\theta \sin\theta + \theta) \right) \Bigg|_0^{2\pi} \\ &= \frac{1}{2}(18\pi + 0 + 4\pi) = 11\pi. \end{aligned}$$

The reasoning in Example 9 suggests a general area formula.

For a curve $r = f(\theta)$, with $f(\theta) \geq 0$, the area in Figure 8.43 is given by

$$\text{Area of region enclosed} = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2\, d\theta.$$

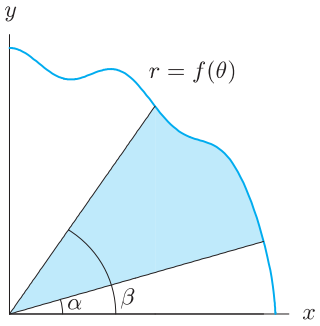


Figure 8.43: Area in polar coordinates

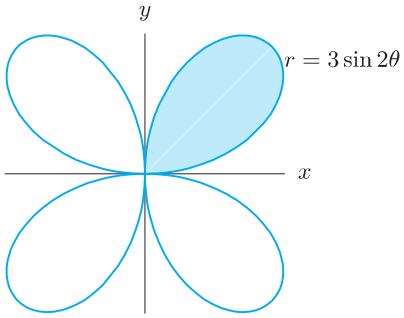


Figure 8.44: One petal of the rose $r = 3\sin 2\theta$ with $0 \leq \theta \leq \pi/2$

Example 10 Find the area of one petal of the four-petal rose $r = 3\sin 2\theta$ in Figure 8.44.

Solution The petal in the first quadrant is described by $r = 3\sin 2\theta$ for $0 \leq \theta \leq \pi/2$, so

$$\text{Area of shaded region} = \frac{1}{2}\int_0^{\pi/2} (3\sin 2\theta)^2\, d\theta = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta.$$

Using the substitution $w = 2\theta$, we rewrite the integral and use integration by parts or formula IV-17 from the table of integrals:

$$\begin{aligned}\text{Area} &= \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{9}{4}\int_0^{\pi} \sin^2 w\, dw \\ &= \frac{9}{4}\left(-\frac{1}{2}\cos w \sin w + \frac{1}{2}w\right)\Big|_0^{\pi} = \frac{9}{4}\cdot\frac{\pi}{2} = \frac{9\pi}{8}.\end{aligned}$$

Slope in Polar Coordinates

For a curve $r = f(\theta)$, we can express x and y in terms of θ as a parameter, giving

$$x = r\cos\theta = f(\theta)\cos\theta \quad \text{and} \quad y = r\sin\theta = f(\theta)\sin\theta.$$

To find the slope of the curve, we use the formula for the slope of a parametric curve

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}.$$

Example 11 Find the slope of the curve $r = 3\sin 2\theta$ at $\theta = \pi/3$.

Solution Expressing x and y in terms of θ, we have

$$x = 3\sin(2\theta)\cos\theta \quad \text{and} \quad y = 3\sin(2\theta)\sin\theta.$$

The slope is given by

$$\frac{dy}{dx} = \frac{6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta}{6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta}.$$

At $\theta = \pi/3$, we have

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{6(-1/2)(\sqrt{3}/2) + 3(\sqrt{3}/2)(1/2)}{6(-1/2)(1/2) - 3(\sqrt{3}/2)(\sqrt{3}/2)} = \frac{\sqrt{3}}{5}.$$

Arc Length in Polar Coordinates

We can calculate the arc length of the curve $r = f(\theta)$ by expressing x and y in terms of θ as a parameter

$$x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$$

and using the formula for the arc length of a parametric curve:

$$\text{Arc length} = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

The calculations may be simplified by using the alternate form of the arc length integral in Problem 41.

Example 12 Find the arc length of one petal of the rose $r = 3\sin 2\theta$ for $0 \le \theta \le \pi/2$. See Figure 8.44.

Solution The curve is given parametrically by

$$x = 3\sin(2\theta)\cos\theta \quad \text{and} \quad y = 3\sin(2\theta)\sin\theta.$$

Thus, calculating $dx/d\theta$ and $dy/d\theta$ and evaluating the integral on a calculator, we have:

$$\begin{aligned}\text{Arc length} &= \int_0^{\pi/2} \sqrt{(6\cos(2\theta)\cos\theta - 3\sin(2\theta)\sin\theta)^2 + (6\cos(2\theta)\sin\theta + 3\sin(2\theta)\cos\theta)^2}\, d\theta \\ &= 7.266.\end{aligned}$$

Exercises and Problems for Section 8.3

Exercises

Convert the polar coordinates in Exercises 1–4 to Cartesian coordinates. Give exact answers.

1. $(1, 2\pi/3)$ **2.** $(\sqrt{3}, -3\pi/4)$

3. $(2\sqrt{3}, -\pi/6)$ **4.** $(2, 5\pi/6)$

Convert the Cartesian coordinates in Exercises 5–8 to polar coordinates.

5. $(1, 1)$ **6.** $(-1, 0)$

7. $(\sqrt{6}, -\sqrt{2})$ **8.** $(-\sqrt{3}, 1)$

9. **(a)** Make a table of values for the equation $r = 1 - \sin\theta$. Include $\theta = 0, \pi/3, \pi/2, 2\pi/3, \pi, \cdots$.
(b) Use the table to graph the equation $r = 1 - \sin\theta$ in the xy-plane. This curve is called a *cardioid*.
(c) At what point(s) does the cardioid $r = 1 - \sin\theta$ intersect a circle of radius 1/2 centered at the origin?
(d) Graph the curve $r = 1 - \sin 2\theta$ in the xy-plane. Compare this graph to the cardioid $r = 1 - \sin\theta$.

10. Graph the equation $r = 1 - \sin(n\theta)$, for $n = 1, 2, 3, 4$. What is the relationship between the value of n and the shape of the graph?

11. Graph the equation $r = 1 - \sin\theta$, with $0 \le \theta \le n\pi$, for $n = 2, 3, 4$. What is the relationship between the value of n and the shape of the graph?

12. Graph the equation $r = 1 - n\sin\theta$, for $n = 2, 3, 4$. What is the relationship between the value of n and the shape of the graph?

13. Graph the equation $r = 1 - \cos\theta$. Describe its relationship to $r = 1 - \sin\theta$.

14. Give inequalities that describe the flat surface of a washer that is one inch in diameter and has an inner hole with a diameter of 3/8 inch.

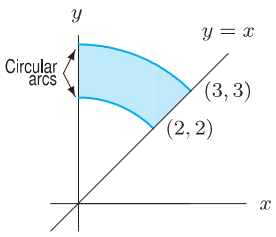
15. Graph the equation $r = 1 - \sin(2\theta)$ for $0 \le \theta \le 2\pi$. There are two loops. For each loop, give a restriction on θ that shows all of that loop and none of the other loop.

16. A slice of pizza is one eighth of a circle of radius 1 foot. The slice is in the first quadrant, with one edge along the x-axis, and the center of the pizza at the origin. Give inequalities describing this region using:

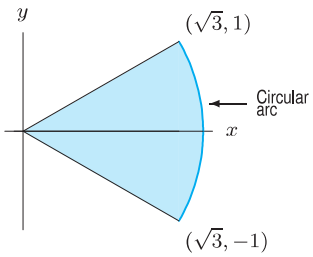
(a) Polar coordinates **(b)** Rectangular coordinates

In Exercises 17–19, give inequalities for r and θ which describe the following regions in polar coordinates.

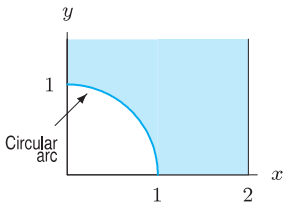
17.



18.



19.



Note: Region extends indefinitely in the y-direction.

In Exercises 20–22, say which of the points A, B, C and D in Figure 8.45 has polar coordinate θ given by the formula.

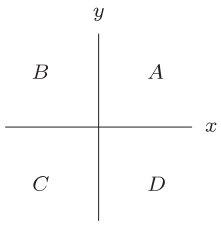


Figure 8.45

20. $\theta = \tan^{-1}(y/x)$

21. $\theta = \cos^{-1}(x/\sqrt{x^2+y^2})$

22. $\theta = \sin^{-1}(y/\sqrt{x^2+y^2})$

Problems

23. Sketch the polar region described by the following integral expression for area:

$$\frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta.$$

24. Find the area inside the spiral $r = \theta$ for $0 \le \theta \le 2\pi$.

25. Find the area between the two spirals $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le 2\pi$.

26. Find the area inside the cardioid $r = 1 + \cos\theta$ for $0 \le \theta \le 2\pi$.

27. **(a)** In polar coordinates, write equations for the line $x = 1$ and the circle of radius 2 centered at the origin.

(b) Write an integral in polar coordinates representing the area of the region to the right of $x = 1$ and inside the circle.

(c) Evaluate the integral.

28. Find the area inside the circle $r = 1$ and outside the cardioid $r = 1 + \sin\theta$.

29. Find the area inside the cardioid $r = 1 - \sin\theta$ and outside the circle $r = 1/2$.

30. Find the area lying outside $r = 2\cos\theta$ and inside $r = 1 + \cos\theta$.

31. **(a)** Graph $r = 2\cos\theta$ and $r = 2\sin\theta$ on the same axes.

(b) Using polar coordinates, find the area of the region shared by both curves.

32. For what value of a is the area enclosed by $r = \theta$, $\theta = 0$, and $\theta = a$ equal to 1?

33. **(a)** Sketch the bounded region inside the lemniscate $r^2 = 4\cos 2\theta$ and outside the circle $r = \sqrt{2}$.
(b) Compute the area of the region described in part (a).

34. Using Example 11 on page 412, find the equation of the tangent line to the curve $r = 3\sin 2\theta$ at $\theta = \pi/3$.

35. Using Example 11 on page 412 and Figure 8.35, find the points where the curve $r = 3\sin 2\theta$ has horizontal and vertical tangents.

36. For what values of θ on the polar curve $r = \theta$, with $0 \le \theta \le 2\pi$, are the tangent lines horizontal? Vertical?

37. **(a)** In Cartesian coordinates, write an equation for the tangent line to $r = 1/\theta$ at $\theta = \pi/2$.
(b) The graph of $r = 1/\theta$ has a horizontal asymptote as θ approaches 0. Find the equation of this asymptote.

38. Find the maximum value of the y-coordinate of points on the limaçon $r = 1 + 2\cos\theta$.

Find the arc length of the curves in Problems 39–40.

39. $r = \theta$, $0 \le \theta \le 2\pi$

40. $r = 1/\theta$, $\pi \le \theta \le 2\pi$

41. For the curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, show that

$$\text{Arc length} = \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta.$$

42. Find the arclength of the spiral $r = \theta$ where $0 \le \theta \le \pi$.

43. Find the arclength of part of the cardioid $r = 1 + \cos\theta$ where $0 \le \theta \le \pi/2$.

8.4 DENSITY AND CENTER OF MASS

Density and How to Slice a Region

The examples in this section involve the idea of *density*. For example,

- A population density is measured in, say, people per mile (along the edge of a road), or people per unit area (in a city), or bacteria per cubic centimeter (in a test tube).
- The density of a substance (e.g. air, wood, or metal), is the mass of a unit volume of the substance and is measured in, say, grams per cubic centimeter.

Suppose we want to calculate the total mass or total population, but the density is not constant over a region.

> **To find total quantity from density:** Divide the region into small pieces in such a way that the density is approximately constant on each piece, and add the contributions of the pieces.

Example 1 The Massachusetts Turnpike ("the Pike") starts in the middle of Boston and heads west. The number of people living next to it varies as it gets farther from the city. Suppose that, x miles out of town, the population density adjacent to the Pike is $P = f(x)$ people/mile. Express the total population living next to the Pike within 5 miles of Boston as a definite integral.

Solution Divide the Pike up into segments of length Δx. The population density at the center of Boston is $f(0)$; let's use that density for the first segment. This gives an estimate of

$$\text{People living in first segment} \approx f(0)\ \text{people/ mile} \cdot \Delta x\ \text{mile} = f(0)\Delta x\ \text{people}.$$

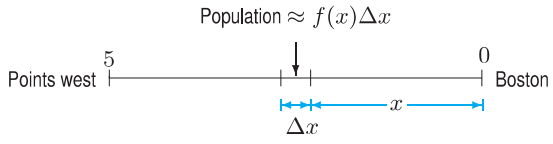


Figure 8.46: Population along the Massachusetts Turnpike

Similarly, the population in a typical segment x miles from the center of Boston is the population density times the length of the interval, or roughly $f(x)\,\Delta x$. (See Figure 8.46.) The sum of all these estimates gives the estimate

$$\text{Total population} \approx \sum f(x)\,\Delta x.$$

Letting $\Delta x \to 0$ gives

$$\text{Total population} = \lim_{\Delta x \to 0} \sum f(x)\,\Delta x = \int_0^5 f(x)\,dx.$$

The 5 and 0 in the limits of the integral are the upper and lower limits of the interval over which we are integrating.

Example 2 The air density h meters above the earth's surface is $f(h)$ kg/m^3. Find the mass of a cylindrical column of air 2 meters in diameter and 25 kilometers high, with base on the surface of the earth.

Solution The column of air is a circular cylinder 2 meters in diameter and 25 kilometers, or 25,000 meters, high. First we must decide how we are going to slice this column. Since the air density varies with altitude but remains constant horizontally, we take horizontal slices of air. That way, the density will be more or less constant over the whole slice, being close to its value at the bottom of the slice. (See Figure 8.47.)

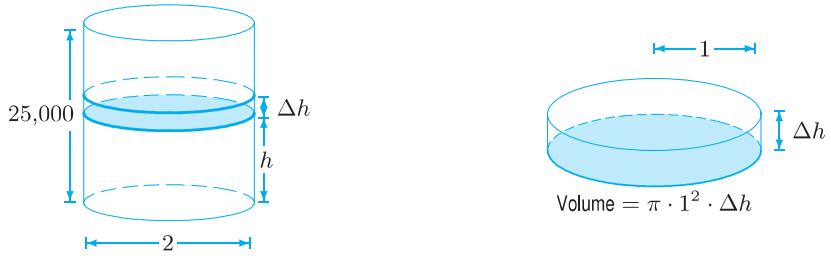


Figure 8.47: Slicing a column of air horizontally

A slice is a cylinder of height Δh and diameter 2 m, so its radius is 1 m. We find the approximate mass of the slice by multiplying its volume by its density. If the thickness of the slice is Δh, then its volume is $\pi r^2 \cdot \Delta h = \pi 1^2 \cdot \Delta h = \pi\,\Delta h$ m^3. The density of the slice is given by $f(h)$. Thus,

$$\text{Mass of slice} \approx \text{Volume} \cdot \text{ Density} = (\pi \Delta h \text{ m}^3)(f(h) \text{ kg/m}^3) = \pi\,\Delta h \cdot f(h) \text{ kg}.$$

Adding these slices up yields a Riemann sum:

$$\text{Total mass} \approx \sum \pi f(h)\,\Delta h \text{ kg}.$$

As $\Delta h \to 0$, this sum approximates the definite integral:

$$\text{Total mass} = \int_0^{25{,}000} \pi f(h)\,dh \text{ kg}.$$

In order to get a numerical value for the mass of air, we need an explicit formula for the density as a function of height, as in the next example.

Example 3 Find the mass of the column of air in Example 2 if the density of air at height h is given by

$$P = f(h) = 1.28e^{-0.000124h} \text{ kg/m}^3.$$

Solution Using the result of the previous example, we have

$$\text{Mass} = \int_0^{25{,}000} \pi 1.28 e^{-0.000124h}\,dh = \frac{-1.28\pi}{0.000124}\left(e^{-0.000124h}\Big|_0^{25{,}000}\right)$$

$$= \frac{1.28\pi}{0.000124}\left(e^0 - e^{-0.000124(25{,}000)}\right) \approx 31{,}000 \text{ kg}.$$

It requires some thought to figure out how to slice a region. The key point is that you want the density to be nearly constant within each piece.

Example 4 The population density in Ringsburg is a function of the distance from the city center. At r miles from the center, the density is $P = f(r)$ people per square mile. Ringsburg is circular with radius 5 miles. Write a definite integral that expresses the total population of Ringsburg.

Solution We want to slice Ringsburg up and estimate the population of each slice. If we were to take straight-line slices, the population density would vary on each slice, since it depends on the distance from the city center. We want the population density to be pretty close to constant on each slice. We therefore take slices that are thin rings around the center. (See Figure 8.48.) Since the ring is very thin, we can approximate its area by straightening it into a thin rectangle. (See Figure 8.49.) The width of the rectangle is Δr miles, and its length is approximately equal to the circumference of the ring, $2\pi r$ miles, so its area is about $2\pi r\,\Delta r$ mi^2. Since

$$\text{Population on ring} \approx \text{Density} \cdot \text{Area},$$

we get

$$\text{Population on ring} \approx (f(r)\ \text{people/mi}^2)(2\pi r \Delta r\ \text{mi}^2) = f(r) \cdot 2\pi r\,\Delta r\ \text{people}.$$

Adding the contributions from each ring, we get

$$\text{Total population} \approx \sum 2\pi r f(r)\,\Delta r\ \text{people}.$$

So

$$\text{Total population} = \int_0^5 2\pi r f(r)\,dr\ \text{people}.$$

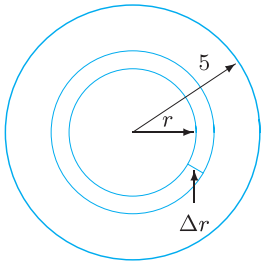


Figure 8.48: Ringsburg

Figure 8.49: Ring from Ringsburg (straightened out)

Note: You may wonder what happens if we calculate the area of the ring by subtracting the area of the inner circle (πr^2) from the area of the outer circle [$\pi(r + \Delta r)^2$], giving

$$\text{Area} = \pi(r + \Delta r)^2 - \pi r^2.$$

Multiplying out and subtracting, we get

$$\begin{aligned}\text{Area} &= \pi[r^2 + 2r\,\Delta r + (\Delta r)^2] - \pi r^2 \\ &= 2\pi r\,\Delta r + \pi(\Delta r)^2.\end{aligned}$$

This expression differs from the one we used before by the $\pi(\Delta r)^2$ term. However, as Δr becomes very small, $\pi(\Delta r)^2$ becomes much, much smaller. We say its smallness is of *second order*, since the power of the small factor, Δr, is 2. In the limit as $\Delta r \to 0$, we can ignore $\pi(\Delta r)^2$.

Center of Mass

The center of mass of a mechanical system is important for studying its behavior when in motion. For example, some sport utility vehicles and light trucks tend to tip over in accidents, because of their high centers of mass.

In this section, we first define the center of mass for a system of point masses on a line. Then we use the definite integral to extend this definition.

Point Masses

Two children on a seesaw, one twice the weight of the other, will balance if the lighter child is twice as far from the pivot as the heavier child. Thus, the balance point is 2/3 of the way from the lighter child and 1/3 of the way from the heavier child. This balance point is the *center of mass* of the mechanical system consisting of the masses of the two children (we ignore the mass of the seesaw itself). See Figure 8.50.

To find the balance point, we use the *displacement* (signed distance) of each child from the pivot to calculate the *moment*, where

$$\text{Moment of mass about pivot} = \text{Mass} \times \text{Displacement from pivot.}$$

A moment represents the tendency of a child to turn the system about the pivot point; the seesaw balances if the total moment is zero. Thus, the center of mass is the point about which the total moment is zero.

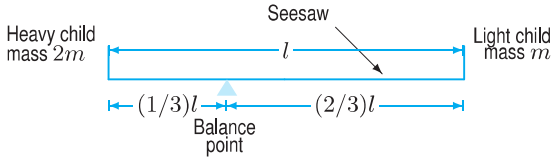


Figure 8.50: Children on seesaw

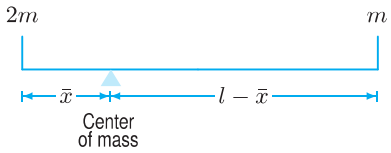


Figure 8.51: Center of mass of point masses

Example 5 Calculate the position of the center of mass of the children in Figure 8.50 using moments.

Solution Suppose the center of mass in Figure 8.51 is at a distance of $\bar{x}$ from the left end. The moment of the left mass about the center of mass is $-2m\bar{x}$ (it is negative since it is to the left of the center of mass); the moment of the right mass about the center of mass is $m(l - \bar{x})$. The system balances if

$$-2m\bar{x} + m(l - \bar{x}) = 0 \quad \text{or} \quad ml - 3m\bar{x} = 0 \quad \text{so} \quad \bar{x} = \frac{1}{3}l.$$

Thus, the center of mass is $l/3$ from the left end.

We use the same method to calculate the center of mass, $\bar{x}$, of the system in Figure 8.52. The sum of the moments of the three masses about $\bar{x}$ is 0, so

$$m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + m_3(x_3 - \bar{x}) = 0.$$

Solving for $\bar{x}$, we get

$$m_1\bar{x} + m_2\bar{x} + m_3\bar{x} = m_1x_1 + m_2x_2 + m_3x_3$$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{\sum_{i=1}^{3} m_ix_i}{\sum_{i=1}^{3} m_i}.$$

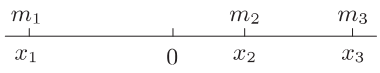


Figure 8.52: Discrete masses, m_1, m_2, m_3

Generalizing leads to the following formula:

The **center of mass** of a system of n point masses $m_1, m_2, \ldots, m_n$ located at positions $x_1, x_2, \ldots, x_n$ along the x-axis is given by

$$\overline{x} = \frac{\sum x_i m_i}{\sum m_i}.$$

The numerator is the sum of the moments of the masses about the origin; the denominator is the total mass of the system.

Example 6 Show that the definition of $\overline{x}$ gives the same answer as we found in Example 5.

Solution Suppose the origin is at the left end of the seesaw in Figure 8.50. The total mass of the system is $2m + m = 3m$. We compute

$$\overline{x} = \frac{\sum x_i m_i}{\sum m_i} = \frac{1}{3m}(2m \cdot 0 + m \cdot l) = \frac{ml}{3m} = \frac{l}{3}.$$

Continuous Mass Density

Instead of discrete masses arranged along the x-axis, suppose we have an object lying on the x-axis between $x = a$ and $x = b$. At point x, suppose the object has mass density (mass per unit length) of $\delta(x)$. To calculate the center of mass of such an object, divide it into n pieces, each of length Δx. On each piece, the density is nearly constant, so the mass of the piece is given by density times length. See Figure 8.53. Thus, if x_i is a point in the i^{th} piece,

$$\text{Mass of the } i^{\text{th}} \text{ piece, } m_i \approx \delta(x_i)\Delta x.$$

Then the formula for the center of mass, $\bar{x} = \sum x_i m_i / \sum m_i$, applied to the n pieces of the object gives

$$\bar{x} = \frac{\sum x_i \delta(x_i)\Delta x}{\sum \delta(x_i)\Delta x}.$$

In the limit as $n \to \infty$ we have the following formula:

The **center of mass** $\overline{x}$ of an object lying along the x-axis between $x = a$ and $x = b$ is

$$\overline{x} = \frac{\int_a^b x\delta(x)\,dx}{\int_a^b \delta(x)\,dx},$$

where $\delta(x)$ is the density (mass per unit length) of the object.

As in the discrete case, the denominator is the total mass of the object.

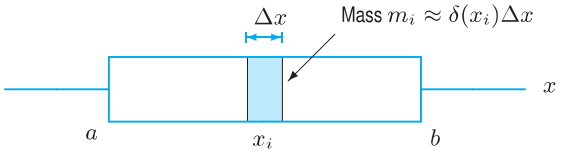


Figure 8.53: Calculating the center of mass of an object of variable density, $\delta(x)$

Example 7 Find the center of mass of a 2-meter rod lying on the x-axis with its left end at the origin if:
(a) The density is constant and the total mass is 5 kg. (b) The density is $\delta(x) = 15x^2$ kg/m.

Solution (a) Since the density is constant along the rod, we expect the balance point to be in the middle, that is, $\bar{x} = 1$. To check this, we compute $\bar{x}$. The density is the total mass divided by the length, so $\delta(x) = 5/2$ kg/m. Then

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot \frac{5}{2}\, dx}{5} = \frac{1}{5} \cdot \frac{5}{2} \cdot \frac{x^2}{2}\bigg|_0^2 = 1 \text{ meter}.$$

(b) Since more of the mass of the rod is closer to its right end (the density is greatest there), we expect the center of mass to be in the right half of the rod, that is, between $x = 1$ and $x = 2$. We have

$$\text{Total mass} = \int_0^2 15x^2\, dx = 5x^3\big|_0^2 = 40 \text{ kg}.$$

Thus,

$$\bar{x} = \frac{\text{Moment}}{\text{Mass}} = \frac{\int_0^2 x \cdot 15x^2 dx}{40} = \frac{15}{40} \cdot \frac{x^4}{4}\bigg|_0^2 = 1.5 \text{ meter}.$$

Two- and Three-Dimensional Regions

For a system of masses that lies in the plane, the center of mass is a point with coordinates $(\bar{x}, \bar{y})$. In three dimensions, the center of mass is a point with coordinates $(\bar{x}, \bar{y}, \bar{z})$. To compute the center of mass in three dimensions, we use the following formulas in which $A_x(x)$ is the area of a slice perpendicular to the x-axis at x, and $A_y(y)$ and $A_z(z)$ are defined similarly. In two dimensions, we use the same formulas for $\bar{x}$ and $\bar{y}$, but we interpret $A_x(x)$ and $A_y(y)$ as the lengths of strips perpendicular to the x- and y-axes, respectively.

For a region of constant density δ, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\, dx}{\text{Mass}} \quad \bar{y} = \frac{\int y\delta A_y(y)\, dy}{\text{Mass}} \quad \bar{z} = \frac{\int z\delta A_z(z)\, dz}{\text{Mass}}.$$

The expression $\delta A_x(x)\Delta x$ is the moment of a slice perpendicular to the x-axis. Thus, these formulas are extensions of that on page 419. In the two- and three-dimensional case, we are assuming that the density δ is constant. If the density is not constant, finding the center of mass may require a double or triple integral from multivariable calculus.

Example 8 Find the coordinates of the center of mass of the isosceles triangle in Figure 8.54. The triangle has constant density and mass m.

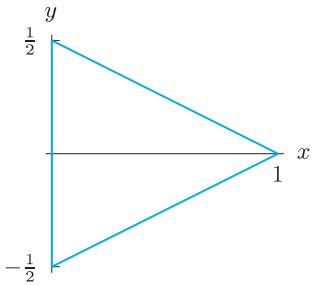


Figure 8.54: Find center of mass of this triangle

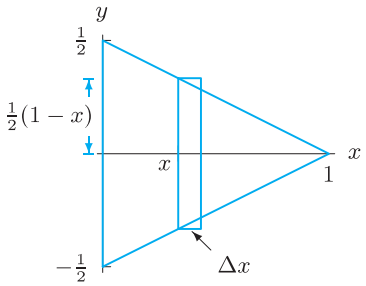


Figure 8.55: Sliced triangle

Solution Because the mass of the triangle is symmetrically distributed with respect to the x-axis, $\bar{y} = 0$. We expect $\bar{x}$ to be closer to $x = 0$ than to $x = 1$, since the triangle is wider near the origin.

The area of the triangle is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$. Thus, Density = Mass/Area $= 2m$. If we slice the triangle into strips of width Δx, then the strip at position x has length $A_x(x) = 2 \cdot \frac{1}{2}(1-x) = (1-x)$. (See Figure 8.55.) So

$$\text{Area of strip } = A_x(x)\Delta x \approx (1-x)\Delta x.$$

Since the density is $2m$, the center of mass is given by

$$\bar{x} = \frac{\int x\delta A_x(x)\,dx}{\text{Mass}} = \frac{\int_0^1 2mx(1-x)\,dx}{m} = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{1}{3}.$$

So the center of mass of this triangle is at the point $(\bar{x}, \bar{y}) = (1/3, 0)$.

Example 9 Find the center of mass of a hemisphere of radius 7 cm and constant density δ.

Solution Stand the hemisphere with its base horizontal in the xy-plane, with the center at the origin. Symmetry tells us that its center of mass lies directly above the center of the base, so $\bar{x} = \bar{y} = 0$. Since the hemisphere is wider near its base, we expect the center of mass to be nearer to the base than the top.

To calculate the center of mass, slice the hemisphere into horizontal disks, as in Figure 8.9 on page 399. A disk of thickness Δz at height z above the base has

$$\text{Volume of disk } = A_z(z)\Delta z \approx \pi(7^2 - z^2)\Delta z \text{ cm}^3.$$

So, since the density is δ,

$$\overline{z} = \frac{\int z\delta A_z(z)\,dz}{\text{Mass}} = \frac{\int_0^7 z\delta\pi(7^2 - z^2)\,dz}{\text{Mass}}.$$

Since the total mass of the hemisphere is $(\frac{2}{3}\pi 7^3)\,\delta$, we get

$$\bar{z} = \frac{\delta\pi\int_0^7 (7^2 z - z^3)\,dz}{\text{Mass}} = \frac{\delta\pi\left(7^2 z^2/2 - z^4/4\right)\Big|_0^7}{\text{Mass}} = \frac{\frac{7^4}{4}\delta\pi}{\frac{2}{3}\pi 7^3\delta} = \frac{21}{8} = 2.625 \text{ cm}.$$

The center of mass of the hemisphere is 2.625 cm above the center of its base. As expected, it is closer to the base of the hemisphere than its top.

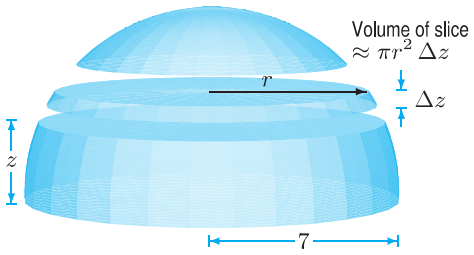


Figure 8.56: Slicing to find the center of mass of a hemisphere

Exercises and Problems for Section 8.4

Exercises

1. Find the mass of a rod of length 10 cm with density $\delta(x) = e^{-x}$ gm/cm at a distance of x cm from the left end.

2. A plate occupying the region $0 \le x \le 2, 0 \le y \le 3$ has density $\delta = 5$ gm/cm^2. Set up two integrals giving the mass of the plate, one corresponding to strips in the x-direction and one corresponding to strips in the y-direction.

3. A rod has length 2 meters. At a distance x meters from its left end, the density of the rod is given by

$$\delta(x) = 2 + 6x \text{ gm/m}.$$

(a) Write a Riemann sum approximating the total mass of the rod.
(b) Find the exact mass by converting the sum into an integral.

4. If a rod lies along the x-axis between a and b, the moment of the rod is $\int_a^b x\delta(x)\,dx$, where $\delta(x)$ is its density in grams/meter at a position x meters. Find the moment and center of mass of the rod in Problem 3.

5. The density of cars (in cars per mile) down a 20-mile stretch of the Pennsylvania Turnpike is approximated by

$$\delta(x) = 300\left(2 + \sin\left(4\sqrt{x + 0.15}\right)\right),$$

at a distance x miles from the Breezewood toll plaza.

(a) Sketch a graph of this function for $0 \le x \le 20$.
(b) Write a Riemann sum that approximates the total number of cars on this 20-mile stretch.
(c) Find the total number of cars on the 20-mile stretch.

6. **(a)** Find a Riemann sum which approximates the total mass of a 3×5 rectangular sheet, whose density per unit area at a distance x from one of the sides of length 5 is $1/(1 + x^4)$.
(b) Calculate the mass.

7. A point mass of 2 grams located 3 centimeters to the left of the origin and a point mass of 5 grams located 4 centimeters to the right of the origin are connected by a thin, light rod. Find the center of mass of the system.

8. Find the center of mass of a system containing three point masses of 5 gm, 3 gm, and 1 gm located respectively at $x = -10$, $x = 1$, and $x = 2$.

9. Find the mass of the block $0 \le x \le 10$, $0 \le y \le 3$, $0 \le z \le 1$, whose density δ, is given by

$$\delta = 2 - z \quad \text{for } 0 \le z \le 1.$$

Problems

10. Find the total mass of the triangular region in Figure 8.57, which has density $\delta(x) = 1 + x$ grams/cm^2.

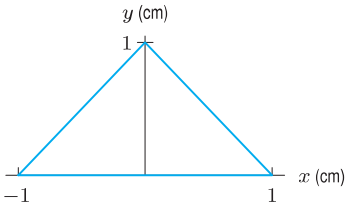


Figure 8.57

11. Circle City, a typical metropolis, is densely populated near its center, and its population gradually thins out toward the city limits. In fact, its population density is $10{,}000(3 - r)$ people/square mile at distance r miles from the center.

(a) Assuming that the population density at the city limits is zero, find the radius of the city.
(b) What is the total population of the city?

12. The density of oil in a circular oil slick on the surface of the ocean at a distance r meters from the center of the slick is given by $\delta(r) = 50/(1 + r)$ kg/m^2.

(a) If the slick extends from $r = 0$ to $r = 10{,}000$ m, find a Riemann sum approximating the total mass of oil in the slick.
(b) Find the exact value of the mass of oil in the slick by turning your sum into an integral and evaluating it.
(c) Within what distance r is half the oil of the slick contained?

13. The soot produced by a garbage incinerator spreads out in a circular pattern. The depth, $H(r)$, in millimeters, of the soot deposited each month at a distance r kilometers from the incinerator is given by $H(r) = 0.115e^{-2r}$.

(a) Write a definite integral giving the total volume of soot deposited within 5 kilometers of the incinerator each month.
(b) Evaluate the integral you found in part (a), giving your answer in cubic meters.

14. A cardboard figure has the shape shown in Figure 8.58. The region is bounded on the left by the line $x = a$, on the right by the line $x = b$, above by $f(x)$, and below by $g(x)$. If the density $\delta(x)$ gm/cm^2 varies only with x, find an expression for the total mass of the figure, in terms of $f(x)$, $g(x)$, and $\delta(x)$.

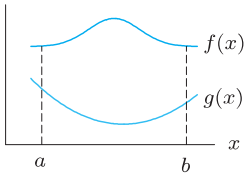


Figure 8.58

15. The storage shed in Figure 8.59 is the shape of a half-cylinder of radius r and length l.

(a) What is the volume of the shed?
(b) The shed is filled with sawdust whose density (mass/unit volume) at any point is proportional to the distance of that point from the floor. The constant of proportionality is k. Calculate the total mass of sawdust in the shed.

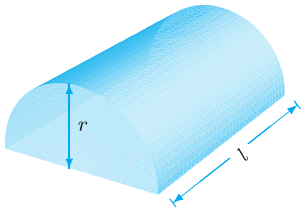


Figure 8.59

16. The following table gives the density D (in gm/cm^3) of the earth at a depth x km below the earth's surface. The radius of the earth is about 6370 km. Find an upper and a lower bound for the earth's mass such that the upper bound is less than twice the lower bound. Explain your reasoning; in particular, what assumptions have you made about the density?

x	0	1000	2000	2900	3000	4000	5000	6000	6370
D	3.3	4.5	5.1	5.6	10.1	11.4	12.6	13.0	13.0

17. Water leaks out of a tank through a square hole with 1-inch sides. At time t (in seconds) the velocity of water flowing through the hole is $v = g(t)$ ft/sec. Write a definite integral that represents the total amount of water lost in the first minute.

18. An exponential model for the density of the earth's atmosphere says that if the temperature of the atmosphere were constant, then the density of the atmosphere as a function of height, h (in meters), above the surface of the earth would be given by

$$\delta(h) = 1.28e^{-0.000124h} \text{ kg/m}^3.$$

(a) Write (but do not evaluate) a sum that approximates the mass of the portion of the atmosphere from $h = 0$ to $h = 100$ m (i.e., the first 100 meters above sea level). Assume the radius of the earth is 6400 km.
(b) Find the exact answer by turning your sum in part (a) into an integral. Evaluate the integral.

19. Three point masses of 4 gm each are placed at $x = -6, 1$ and 3. Where should a point fourth mass of 4 gm be placed to make the center of mass at the origin?

20. A rod of length 3 meters with density $\delta(x) = 1 + x^2$ grams/meter is positioned along the positive x-axis, with its left end at the origin. Find the total mass and the center of mass of the rod.

21. A rod with density $\delta(x) = 2 + \sin x$ lies on the x-axis between $x = 0$ and $x = \pi$. Find the center of mass of the rod.

22. A rod of length 1 meter has density $\delta(x) = 1 + kx^2$ grams/meter, where k is a positive constant. The rod is lying on the positive x-axis with one end at the origin.

(a) Find the center of mass as a function of k.
(b) Show that the center of mass of the rod satisfies $0.5 < \bar{x} < 0.75$.

23. A rod of length 2 meters and density $\delta(x) = 3 - e^{-x}$ kilograms per meter is placed on the x-axis with its ends at $x = \pm 1$.

(a) Will the center of mass of the rod be on the left or right of the origin? Explain.
(b) Find the coordinate of the center of mass.

24. One half of a uniform circular disk of radius 1 meter lies in the xy-plane with its diameter along the y-axis, its center at the origin, and $x > 0$. The mass of the half-disk is 3 kg. Find $(\bar{x}, \bar{y})$.

25. A metal plate, with constant density 2 gm/cm^2, has a shape bounded by the curve $y = x^2$ and the x-axis, with $0 \le x \le 1$ and x, y in cm.

(a) Find the total mass of the plate.
(b) Sketch the plate, and decide, on the basis of the shape, whether $\bar{x}$ is less than or greater than $1/2$.
(c) Find $\bar{x}$.

26. A metal plate, with constant density 5 gm/cm^2, has a shape bounded by the curve $y = \sqrt{x}$ and the x-axis, with $0 \le x \le 1$ and x, y in cm.

(a) Find the total mass of the plate.
(b) Find $\bar{x}$ and $\bar{y}$.

27. An isosceles triangle with uniform density, altitude a, and base b is placed in the xy-plane as in Figure 8.60. Show that the center of mass is at $\bar{x} = a/3$, $\bar{y} = 0$. Hence show that the center of mass is independent of the triangle's base.

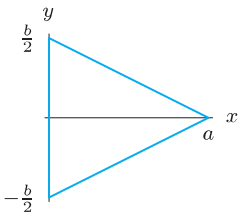


Figure 8.60

28. Find the center of mass of a cone of height 5 cm and base diameter 10 cm with constant density δ gm/cm^3.

29. A solid is formed by rotating the region bounded by the curve $y = e^{-x}$ and the x-axis between $x = 0$ and $x = 1$, around the x-axis. It was shown in Example 1 on page 399 that the volume of this solid is $\pi(1 - e^{-2})/2$. Assuming the solid has constant density δ, find $\bar{x}$ and $\bar{y}$.

30. **(a)** Find the mass of a pyramid of constant density δ gm/cm^3 with a square base of side 40 cm and height 10 cm. [That is, the vertex is 10 cm above the center of the base.]
(b) Find the center of mass of the pyramid.

8.5 APPLICATIONS TO PHYSICS

Although geometric problems were a driving force for the development of the calculus in the seventeenth century, it was Newton's spectacularly successful applications of the calculus to physics that most clearly demonstrated the power of this new mathematics.

Work

In physics the word "work" has a technical meaning which is different from its everyday meaning. Physicists say that if a constant force, F, is applied to some object to move it a distance, d, then the force has done work on the object. The force must be parallel to the motion (in the same or the opposite direction). We make the following definition:

> Work done = Force · Distance or $W = F \cdot d$.

Notice that if we walk across a room holding a book, we do no work on the book, since the force we exert on the book is vertical, but the motion of the book is horizontal. On the other hand, if we lift the book from the floor to a table, we accomplish work.

There are several sets of units in common use. To measure work, we will generally use the two sets of units, International (SI) and British, in the following table.

	Force	Distance	Work
International (SI) units	newton (nt)	meter (m)	joule (j)
British units	pound (lb)	foot (ft)	foot-pound (ft-lb)

Conversions
1 lb = 4.45 nt
1 ft = 0.305 m
1 ft-lb = 1.36 joules

One joule of work is done when a force of 1 newton moves an object through 1 meter, so 1 joule = 1 newton-meter.

Example 1 Calculate the work done on an object when

(a) A force of 2 newtons moves it 12 meters. (b) A 3-lb force moves it 4 feet.

Solution (a) Work done = 2 nt · 12 m = 24 joules. (b) Work done = 3 lb · 4 ft = 12 ft-lb.

In the previous example, the force was constant and we calculated the work by multiplication. In the next example, the force varies, so we need an integral. We divide up the distance moved and sum to get a definite integral representing the work.

Example 2 Hooke's Law says that the force, F, required to compress the spring in Figure 8.61 by a distance x, in meters, is given by $F = kx$, for some constant k. Find the work done in compressing the spring by 0.1 m if $k = 8$ nt/m.

Figure 8.61: Compression of spring: Force is kx

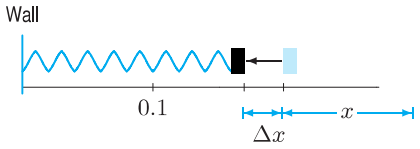


Figure 8.62: Work done in compressing spring a small distance Δx is $kx\Delta x$

Solution Since k is in newtons/meter and x is in meters, we have $F = 8x$ newtons. Since the force varies with x, we divide the distance moved into small increments, Δx, as in Figure 8.62. Then

$$\text{Work done in moving through an increment } \approx F\Delta x = 8x\Delta x \text{ joules.}$$

So, summing over all increments gives the Riemann sum approximation

$$\text{Total work done} \approx \sum 8x\Delta x.$$

Taking the limit as $\Delta x \to 0$ gives

$$\text{Total work done } = \int_0^{0.1} 8x\,dx = 4x^2\Big|_0^{0.1} = 0.04 \text{ joules.}$$

In general, if force is a function $F(x)$ of position x, then in moving from $x = a$ to $x = b$,

$$\text{Work done } = \int_a^b F(x)\,dx.$$

The Force Due to Gravity: Mass versus Weight

When an object is lifted, work is done against the force exerted by gravity on the object. By Newton's Second Law, the downward gravitational force acting on a mass m is mg, where g is the acceleration due to gravity. To lift the object, we need to exert a force equal to the gravitational force but in the opposite direction.

In International units, $g = 9.8$ m/sec^2, and we usually measure mass, m, in *kilograms*. In British units, mass is seldom used. Instead, we usually talk about the *weight* of an object, which is the force exerted by gravity on the object. Roughly speaking, the mass represents the quantity of matter in an object, whereas the weight represents the force of gravity on it. The mass of an object is the same everywhere, whereas the weight can vary if, for example, the object is moved to outer space where gravitational forces are smaller.

When we are given the weight of an object, we do not multiply by g to find the gravitational force as it has already been done. In British units, a *pound* is a unit of weight. In International units, a kilogram is a unit of mass, and the unit of weight is a newton, where 1 newton $= 1$ kg $\cdot$ m/sec^2.

Example 3 How much work is done in lifting

(a) A 5-pound book 3 feet off the floor? (b) A 1.5-kilogram book 2 meters off the floor?

Solution

(a) The force due to gravity is 5 lb, so $W = F \cdot d = (5 \text{ lb})(3 \text{ ft}) = 15$ foot-pounds.

(b) The force due to gravity is $mg = (1.5 \text{ kg})(g \text{ m/sec}^2)$, so

$$W = F \cdot d = [(1.5 \text{ kg})(9.8 \text{ m/sec}^2)] \cdot (2 \text{ m}) = 29.4 \text{ joules.}$$

In the previous example, work is found by multiplication. In the next example, different parts of the object move different distances, so an integral is needed.

Example 4 A 28-meter uniform chain with a mass 2 kilograms per meter is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building?

Solution Since 1 meter of the chain has mass density 2 kg, the gravitational force per meter of chain is $(2\,\text{kg})(9.8\,\text{m/sec}^2) = 19.6$ newtons. Let's divide the chain into small sections of length Δy, each requiring a force of $19.6\,\Delta y$ newtons to move it against gravity. See Figure 8.63. If Δy is small, all of this piece is hauled up approximately the same distance, namely y, so

$$\text{Work done on the small piece} \approx (19.6\,\Delta y \text{ newtons})(y \text{ meters}) = 19.6y\,\Delta y \text{ joules}.$$

The work done on the entire chain is given by the total of the work done on each piece:

$$\text{Work done} \approx \sum 19.6y\,\Delta y \text{ joules}.$$

As $\Delta y \to 0$, we obtain a definite integral. Since y varies from 0 to 28 meters, the total work is

$$\text{Work done} = \int_0^{28} (19.6y)\,dy = 9.8y^2\Big|_0^{28} = 7683.2 \text{ joules}.$$

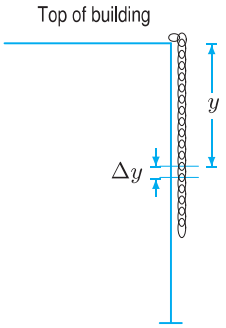


Figure 8.63: Chain for Example 4

Example 5 Calculate the work done in pumping oil from the cone-shaped tank in Figure 8.64 to the rim. The oil has density 800 kg/m^3 and its vertical depth is 10 m.

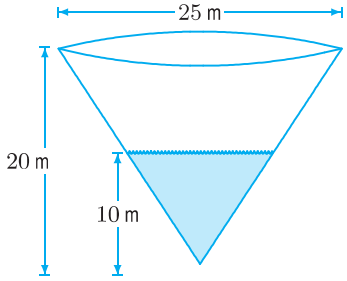


Figure 8.64: Cone-shaped tank containing oil

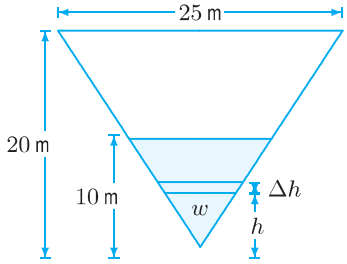


Figure 8.65: Slicing the oil horizontally to compute work

Solution We slice the oil horizontally because each part of such a slice moves the same vertical distance. Each slice is approximately a circular disk with radius $w/2$ m, so

$$\text{Volume of slice} \approx \pi\left(\frac{w}{2}\right)^2 \Delta h = \frac{\pi}{4}w^2\Delta h \text{ m}^3.$$

$$\text{Force of gravity on slice} = \text{Density} \cdot g \cdot \text{Volume} = 800g\frac{\pi}{4}w^2\Delta h = 200\pi g w^2 \Delta h \text{ nt}.$$

Since each part of the slice has to move a vertical distance of $(20 - h)$ m, we have

$$\begin{aligned}\text{Work done on slice} \approx \text{Force} \cdot \text{Distance} &= 200\pi g w^2 \Delta h \text{ nt} \cdot (20 - h) \text{ m} \\ &= 200\pi g w^2 (20 - h)\Delta h \text{ joules.}\end{aligned}$$

To find w in terms of h, we use the similar triangles in Figure 8.65:

$$\frac{w}{h} = \frac{25}{20} \quad \text{so} \quad w = \frac{5}{4}h = 1.25h.$$

Thus,

$$\text{Work done on strip} \approx 200\pi g(1.25h)^2(20 - h)\Delta h = 312.5\pi g h^2(20 - h)\Delta h \text{ joules.}$$

Summing and taking the limit as $\Delta h \to 0$ gives an integral with upper limit $h = 10$, the depth of the oil.

$$\text{Total work} = \lim_{\Delta h \to 0} \sum 312.5\pi g h^2(20 - h)\Delta h = \int_0^{10} 312.5\pi g h^2(20 - h)\, dh \text{ joules.}$$

Evaluating the integral using $g = 9.8$ m/sec^2 gives

$$\text{Total work} = 312.5\pi g \left(20\frac{h^3}{3} - \frac{h^4}{4}\right)\Big|_0^{10} = 1{,}302{,}083\pi g \approx 4.0 \cdot 10^7 \text{ joules.}$$

In the following example, information is given in British units about the weight of the pyramid, so we do not need to multiply by g to find the gravitational force.

Example 6 It is reported that the Great Pyramid of Egypt was built in 20 years. If the stone making up the pyramid has a density of 200 pounds per cubic foot, find the total amount of work done in building the pyramid. The pyramid is 410 feet high and has a square base 755 feet by 755 feet. Estimate how many workers were needed to build the pyramid.

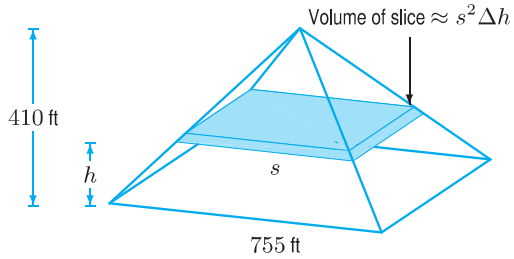


Figure 8.66: Pyramid for Example 6

Solution We assume that the stones were originally located at the approximate height of the construction site. Imagine the pyramid built in layers as we did in Example 5 on page 395.

By similar triangles, the layer at height h has a side length $s = \frac{755}{410}(410 - h)$ ft. (See Figure 8.66.) The layer at height h has a volume of $s^2\,\Delta h$ ft^3, so its weight is $200s^2\,\Delta h$ lb. This layer is lifted through a height of h, so

$$\text{Work to lift layer} = (200s^2\,\Delta h \text{ lb})(h \text{ ft}) = 200s^2 h\,\Delta h ft - lb.$$

Summing over all layers gives

$$\text{Total work} \approx \sum 200s^2\, h\Delta h = \sum 200\left(\frac{755}{410}\right)^2 (410 - h)^2 h\,\Delta h \text{ ft-lb.}$$

Since h varies from 0 to 410, as $\Delta h \to 0$, we obtain

$$\text{Total work} = \int_0^{410} 200\left(\frac{755}{410}\right)^2 (410 - h)^2 h\, dh \approx 1.6 \cdot 10^{12} \text{ foot-pounds.}$$

We have calculated the total work done in building the pyramid; now we want to estimate the total

number of workers needed. Let's assume every laborer worked 10 hours a day, 300 days a year, for 20 years. Assume that a typical worker lifted ten 50 pound blocks a distance of 4 feet every hour, thus performing 2000 foot-pounds of work per hour (this is a very rough estimate). Then each laborer performed $(10)(300)(20)(2000) = 1.2 \cdot 10^8$ foot-pounds of work over a twenty year period. Thus, the number of workers needed was about $(1.6 \cdot 10^{12})/(1.2 \cdot 10^8)$, or about 13,000.

Force and Pressure

We can use the definite integral to compute the force exerted by a liquid on a surface, for example, the force of water on a dam. The idea is to get the force from the *pressure*. The pressure in a liquid is the force per unit area exerted by the liquid. Two things you need to know about pressure are:

- At any point, pressure is exerted equally in all directions—up, down, sideways.
- Pressure increases with depth. (That is one of the reasons why deep sea divers have to take much greater precautions than scuba divers.)

At a depth of h meters, the pressure, p, exerted by the liquid, measured in newtons per square meter, is given by computing the total weight of a column of liquid h meters high with a base of 1 square meter. The volume of such a column of liquid is just h cubic meters. If the liquid has density δ (mass per unit volume), then its weight per unit volume is δg, where g is the acceleration due to gravity. The weight of the column of liquid is $\delta g h$, so

$$\text{Pressure} = \text{Mass density} \cdot g \cdot \text{Depth} \quad \text{or} \quad p = \delta g h.$$

Provided the pressure is constant over a given area, we also have the following relation:

$$\text{Force} = \text{Pressure} \cdot \text{Area}.$$

The units and data we will generally use are given in the following table:

	Density of water	Force	Area	Pressure
SI units	1000 kg/m^3 (mass)	newton (nt)	meter^2	pascal (nt/m^2)
British units	62.4 lb/ft^3 (weight)	pound (lb)	foot^2	lb/ft^2

Conversions
1 lb $= 4.45$ nt
1ft^2 $= 0.093$ m^2
1 lb/ft^2 $= 47.9$ pa

In International units, the mass density of water is 1000 kg/m^3, so the pressure at a depth of h meters is $\delta g h = 1000 \cdot 9.8h = 9800h$ nt/m^2. See Figure 8.67.

In British units, the density of the liquid is usually given as a weight per unit volume, rather than a mass per unit volume. In that case, we do not need to multiply by g because it has already been done. For example, water weighs 62.4 lb/ft^3, so the pressure at depth h feet is $62.4h$ lb/ft^2. See Figure 8.68.

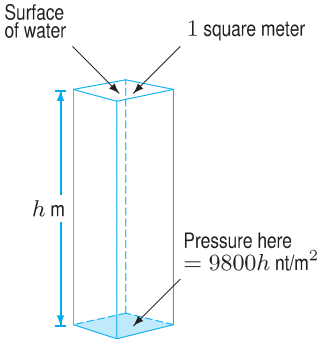


Figure 8.67: Pressure exerted by column of water (International units)

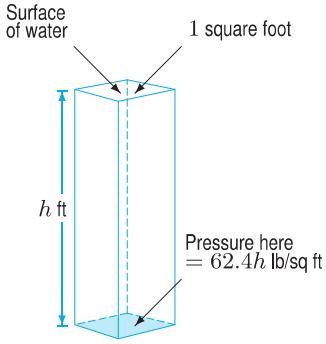


Figure 8.68: Pressure exerted by a column of water (British units)

If the pressure is constant over a surface, we calculate the force on the surface by multiplying the pressure by the area of the surface. If the pressure is not constant, we divide the surface into small pieces *in such a way that the pressure is nearly constant on each one* to obtain a definite integral for the force on the surface. Since the pressure varies with depth, we divide the surface into horizontal strips, each of which is at an approximately constant depth.

Example 7 In 1912, the ocean liner Titanic sank to the bottom of the Atlantic, 12,500 feet (nearly 2.5 miles) below the surface. Find the force on one side of a 100-foot square plate at the depth of the Titanic if the plate is: (a) Lying horizontally (b) Standing vertically.

Solution (a) When the plate is horizontal, the pressure is the same at every point on the plate, so

$$\text{Pressure } = 62.4 \text{ lb/ft}^3 \cdot 12{,}500 \text{ ft } = 780{,}000 \text{ lb/ft}^2.$$

To imagine this pressure, convert to pounds per square inch, giving $780{,}000/144 \approx 5400$ lb/in^2. For the horizontal plate

$$\text{Force } = 780{,}000 \text{ lb/ft}^2 \cdot 100^2 \text{ ft}^2 = 7.8 \cdot 10^9 \text{ pounds.}$$

(b) When the plate is vertical, only the bottom is at 12,500 feet; the top is at 12,400 feet. Dividing into horizontal strips of width Δh, as in Figure 8.69, we have

$$\text{Area of strip } = 100\Delta h \text{ ft}^2.$$

Since the pressure on a strip at a depth of h feet is $62.4h$ lb/ft^2,

$$\text{Force on strip } \approx 62.4h \cdot 100\Delta h = 6240h\Delta h \text{ pounds.}$$

Summing over all strips and taking the limit as $\Delta h \to 0$ gives a definite integral. The strips vary in depth from 12,400 to 12,500 feet, so

$$\text{Total force } = \lim_{\Delta h \to 0} \sum 6240h\Delta h = \int_{12{,}400}^{12{,}500} 6240h\, dh \text{ pounds.}$$

Evaluating the integral gives

$$\text{Total force } = 6240\frac{h^2}{2}\Big|_{12{,}400}^{12{,}500} = 3120(12{,}500^2 - 12{,}400^2) = 7.77 \cdot 10^9 \text{ pounds.}$$

Notice that the answer to part (b) is smaller than the answer to part (a). This is because part of the plate is at a smaller depth in part (b) than in part (a).

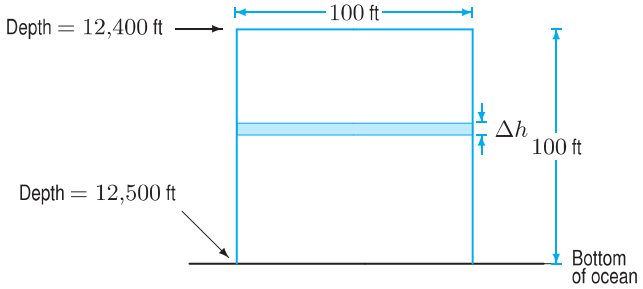


Figure 8.69: Square plate at bottom of ocean; h measured from the surface of water

Example 8 Figure 8.70 shows a dam approximately the size of Hoover Dam, which stores water for California, Nevada, and Arizona. Calculate:

(a) The water pressure at the base of the dam. (b) The total force of the water on the dam.

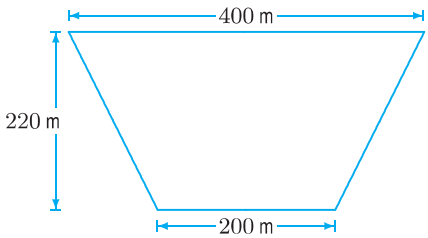


Figure 8.70: Trapezoid-shaped dam

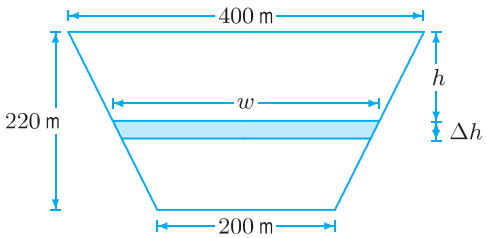


Figure 8.71: Dividing dam into horizontal strips

Solution

(a) Since the density of water is $\delta = 1000$ kg/m^3, at the base of the dam,

$$\text{Water pressure } = \delta g h = 1000 \cdot 9.8 \cdot 220 = 2.156 \cdot 10^6 \text{ nt/m}^2.$$

(b) To calculate the force on the dam, we divide the dam into horizontal strips because the pressure along each strip is approximately constant. See Figure 8.71. Since each strip is approximately rectangular,

$$\text{Area of strip } \approx w \Delta h \text{ m}^2.$$

The pressure at a depth of h meters is $\delta g h = 9800h$ nt/m^2. Thus,

$$\text{Force on strip } \approx \text{ Pressure } \cdot \text{ Area } = 9800hw\Delta h \text{ nt.}$$

To find w in terms of h, we use the fact that w decreases linearly from $w = 400$ when $h = 0$ to $w = 200$ when $h = 220$. Thus w is a linear function of h, with slope $(200 - 400)/220 = -10/11$, so

$$w = 400 - \frac{10}{11}h.$$

Thus

$$\text{Force on strip } \approx 9800h \left(400 - \frac{10}{11}h\right) \Delta h \text{ nt.}$$

Summing over all strips and taking the limit as $\Delta h \to 0$ gives

$$\begin{aligned}\text{Total force on dam } &= \lim_{\Delta h \to 0} \sum 9800h \left(400 - \frac{10}{11}h\right) \Delta h \\ &= \int_0^{220} 9800h \left(400 - \frac{10}{11}h\right) dh \text{ newtons.}\end{aligned}$$

Evaluating the integral gives

$$\text{Total force } = 9800 \left(200h^2 - \frac{10}{33}h^3\right)\Big|_0^{220} = 6.32 \cdot 10^{10} \text{ newtons.}$$

In fact, Hoover Dam is not flat, as the problem assumed, but arched, to better withstand the pressure.

Exercises and Problems for Section 8.5

Exercises

In Exercises 1–3, the force, F, required to compress a spring by a distance x meters is given by $F = 3x$ newtons.

1. Find the work done in compressing the spring from $x = 1$ to $x = 2$.
2. Find the work done to compress the spring to $x = 3$, starting at the equilibrium position, $x = 0$.
3. **(a)** Find the work done in compressing the spring from $x = 0$ to $x = 1$ and in compressing the spring from $x = 4$ to $x = 5$.
   (b) Which of the two answers is larger? Why?

4. The gravitational force on a 1 kg object at a distance r meters from the center of the earth is $F = 4 \cdot 10^{14}/r^2$ newtons. Find the work done in moving the object from the surface of the earth to a height of 10^6 meters above the surface. The radius of the earth is $6.4 \cdot 10^6$ meters.

5. How much work is required to lift a 1000-kg satellite from the surface of the earth to an altitude of $2 \cdot 10^6$ m? The gravitational force is $F = GMm/r^2$, where M is the mass of the earth, m is the mass of the satellite, and r is the distance between them. The radius of the earth is $6.4 \cdot 10^6$ m, its mass is $6 \cdot 10^{24}$ kg, and in these units the gravitational constant, G, is $6.67 \cdot 10^{-11}$.

Problems

6. A worker on a scaffolding 75 ft above the ground needs to lift a 500 lb bucket of cement from the ground to a point 30 ft above the ground by pulling on a rope weighing 0.5 lb/ft. How much work is required?

7. An anchor weighing 100 lb in water is attached to a chain weighing 3 lb/ft in water. Find the work done to haul the anchor and chain to the surface of the water from a depth of 25 ft.

8. A 1000-lb weight is being lifted to a height 10 feet off the ground. It is lifted using a rope which weighs 4 lb per foot and which is being pulled up by construction workers standing on a roof 30 feet off the ground. Find the work done to lift the weight.

9. A bucket of water of mass 20 kg is pulled at constant velocity up to a platform 40 meters above the ground. This takes 10 minutes, during which time 5 kg of water drips out at a steady rate through a hole in the bottom. Find the work needed to raise the bucket to the platform.

10. A 2000-lb cube of ice must be lifted 100 ft, and it is melting at a rate of 4 lb per minute. If it can be lifted at a rate of one foot every minute, find the work needed to get the block of ice to the desired height.

11. A rectangular water tank has length 20 ft, width 10 ft, and depth 15 ft. If the tank is full, how much work does it take to pump all the water out?

12. A rectangular swimming pool 50 ft long, 20 ft wide, and 10 ft deep is filled with water to a depth of 9 ft. Use an integral to find the work required to pump all the water out over the top.

13. A water tank is in the form of a right circular cylinder with height 20 ft and radius 6 ft. If the tank is half full of water, find the work required to pump all of it over the top rim.

14. Suppose the tank in Problem 13 is full of water. Find the work required to pump all of it to a point 10 ft above the top of the tank.

15. Water in a cylinder of height 10 ft and radius 4 ft is to be pumped out. Find the work required if

(a) The tank is full of water and the water is to pumped over the top of the tank.

(b) The tank is full of water and the water must be pumped to a height 5 ft above the top of the tank.

(c) The depth of water in the tank is 8 ft and the water must be pumped over the top of the tank.

16. A fuel oil tank is an upright cylinder, buried so that its circular top is 10 feet beneath ground level. The tank has a radius of 5 feet and is 15 feet high, although the current oil level is only 6 feet deep. Calculate the work required to pump all of the oil to the surface. Oil weighs 50 lb/ft^3.

17. A cone with height 12 ft and radius 4 ft, pointing downward, is filled with water to a depth of 9 ft. Find the work required to pump all the water out over the top.

18. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. (In other words, the tank is not standing vertically on one of its flat ends.) If the radius of the cylinder is 4 feet, its length is 12 feet, and its top is 10 feet under the ground, find the total amount of work needed to pump the gasoline out of the tank. (Gasoline weighs 42 lb/ft^3.)

19. A cylindrical barrel, standing upright on its circular end, contains muddy water. The top of the barrel, which has diameter 1 meter, is open. The height of the barrel is 1.8 meter and it is filled to a depth of 1.5 meter. The density of the water at a depth of h meters below the surface is given by $\delta(h) = 1 + kh$ kg/m^3, where k is a positive constant. Find the total work done to pump the muddy water to the top rim of the barrel. (You can leave π, k, and g in your answer.)

20. **(a)** The trough in Figure 8.72 is full of water. Find the force of the water on a triangular end.

(b) Find the work to pump all the water over the top.

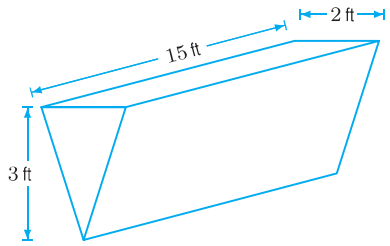


Figure 8.72

21. **(a)** A reservoir has a dam at one end. The dam is a rectangular wall, 1000 feet long and 50 feet high. Approximate the total force of the water on the dam by a Riemann sum.

(b) Write an integral which represents the force, and evaluate it.

22. Before any water was pumped out, what was the total force on the bottom and each side of the tank in Problem 11?

23. A rectangular dam is 100 ft long and 50 ft high. If the water is 40 ft deep, find the force of the water on the dam.

24. A lobster tank in a restaurant is 4 ft long by 3 ft wide by 2 ft deep. Find the water force on the bottom and on each of the four sides.

25. The Three Gorges Dam is currently being built in China. When it is finished in 2009, it will be the largest dam in the world: about 2000 m long and 180 m high, creating a lake the length of Lake Superior. Assume the dam is rectangular in shape.

(a) Estimate the water pressure at the base of the dam.
(b) Set up and evaluate a definite integral giving the total force of the water on the dam.

26. On August 12, 2000, the Russian submarine Kursk sank to the bottom of the sea, 350 feet below the surface. Find the following at the depth of the Kursk.

(a) The water pressure in pounds per square foot and pounds per square inch.
(b) The force on a 5-foot square metal sheet held
(i) Horizontally.
(ii) Vertically.

27. The ocean liner Titanic lies under 12,500 feet of water at the bottom of the Atlantic Ocean.

(a) What is the water pressure at the Titanic? Give your answer in pounds per square foot and pounds per square inch.
(b) Set up and calculate an integral giving the total force on a circular porthole (window) of diameter 6 feet standing vertically with its center at the depth of the Titanic.

28. Set up and calculate a definite integral giving the total force on the dam shown in Figure 8.73, which is about the size of the Aswan Dam in Egypt.

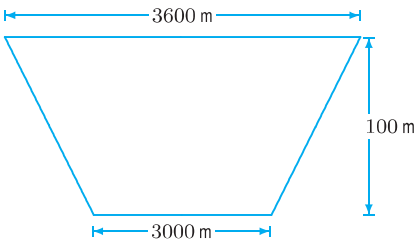


Figure 8.73

29. We define the electric potential at a distance r from an electric charge q by q/r. The electric potential of a charge distribution is obtained by adding up the potential from each point. Electric charge is sprayed (with constant density σ in units of charge/unit area) on to a circular disk of radius a. Consider the axis perpendicular to the disk and through its center. Find the electric potential at the point P on this axis at a distance R from the center. (See Figure 8.74.)

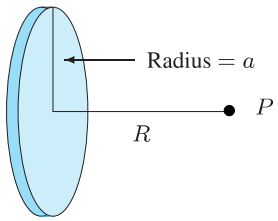


Figure 8.74

For Problems 30–31, find the kinetic energy of the rotating body. Use the fact that the kinetic energy of a particle of mass m moving at a speed v is $\frac{1}{2}mv^2$. Slice the object into pieces in such a way that the velocity is approximately constant on each piece.

30. Find the kinetic energy of a rod of mass 10 kg and length 6 m rotating about an axis perpendicular to the rod at its midpoint, with an angular velocity of 2 radians per second. (Imagine a helicopter blade of uniform thickness.)

31. Find the kinetic energy of a phonograph record of uniform density, mass 50 gm and radius 10 cm rotating at $33\frac{1}{3}$ revolutions per minute.

For Problems 32–34, find the gravitational force between two objects. Use the fact that the gravitational attraction between particles of mass m_1 and m_2 at a distance r apart is Gm_1m_2/r^2. Slice the objects into pieces, use this formula for the pieces, and sum using a definite integral.

32. What is the force of gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end?

33. Two long, thin, uniform rods of lengths l_1 and l_2 lie on a straight line with a gap between them of length a. Suppose their masses are M_1 and M_2, respectively, and the constant of the gravitation is G. What is the force of attraction between the rods? (Use the result of Problem 32.)

34. Find the gravitational force exerted by a thin uniform ring of mass M and radius a on a particle of mass m lying on a line perpendicular to the ring through its center. Assume m is at a distance y from the center of the ring.

35. A uniform, thin, circular disk of radius a and mass M lies on a horizontal plane. The point P lies a distance y directly above O, the center of the disk. Calculate the gravitational force on a mass m at the point P. (See Figure 8.75.) Use the fact that the gravitational force exerted on the mass m by a thin horizontal ring of radius r, mass μ, and center O is toward O and given by

$$F = \frac{G\mu m y}{(r^2 + y^2)^{3/2}}, \quad \text{where } G \text{ is constant.}$$

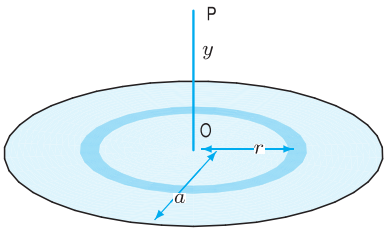


Figure 8.75

8.6 APPLICATIONS TO ECONOMICS

Present and Future Value

Many business deals involve payments in the future. If you buy a car or furniture, for example, you may buy it on credit and pay over a period of time. If you are going to accept payment in the future under such a deal, you obviously need to know how much you should be paid. Being paid \$100 in the future is clearly worse than being paid \$100 today for many reasons. If you are given the money today, you can do something else with it—for example, put it in the bank, invest it somewhere, or spend it. Thus, even without considering inflation, if you are to accept payment in the future, you would expect to be paid more to compensate for this loss of potential earnings. The question we will consider now is, how much more?

To simplify matters, we consider only what we would lose by not earning interest; we will not consider the effect of inflation. Let's look at some specific numbers. Suppose you deposit \$100 in an account which earns 7% interest compounded annually, so that in a year's time you will have \$107. Thus, \$100 today will be worth \$107 a year from now. We say that the \$107 is the *future value* of the \$100, and that the \$100 is the *present value* of the \$107. Observe that the present value is smaller than the future value. In general, we say the following:

- The **future value**, $\$B$, of a payment, $\$P$, is the amount to which the $\$P$ would have grown if deposited in an interest bearing bank account.
- The **present value**, $\$P$, of a future payment, $\$B$, is the amount which would have to be deposited in a bank account today to produce exactly $\$B$ in the account at the relevant time in the future.

With an interest rate of r, compounded annually, and a time period of t years, a deposit of $\$P$ grows to a future balance of $\$B$, where

$$B = P(1+r)^t, \quad \text{or equivalently,} \quad P = \frac{B}{(1+r)^t}.$$

Note that for a 7% interest rate, $r = 0.07$. If instead of annual compounding, we have continuous compounding, we get the following result:

$$B = Pe^{rt}, \quad \text{or equivalently,} \quad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

Example 1 You win the lottery and are offered the choice between \$1 million in four yearly installments of \$250,000 each, starting now, and a lump-sum payment of \$920,000 now. Assuming a 6% interest rate, compounded continuously, and ignoring taxes, which should you choose?

Solution We will do the problem in two ways. First, we assume that you pick the option with the largest present value. The first of the four $250,000 payments is made now, so

$$\text{Present value of first payment} = \$250{,}000.$$

The second payment is made one year from now, so

$$\text{Present value of second payment} = \$250{,}000e^{-0.06(1)}.$$

Calculating the present value of the third and fourth payments similarly, we find:

$$\begin{aligned}\text{Total present value} &= \$250{,}000 + \$250{,}000e^{-0.06(1)} + \$250{,}000e^{-0.06(2)} + \$250{,}000e^{-0.06(3)} \\ &\approx \$250{,}000 + \$235{,}441 + \$221{,}730 + \$208{,}818 \\ &= \$915{,}989.\end{aligned}$$

Since the present value of the four payments is less than $920,000, you are better off taking the $920,000 right now.

Alternatively, we can compare the future values of the two pay schemes. The scheme with the highest future value is the best from a purely financial point of view. We calculate the future value of both schemes three years from now, on the date of the last $250,000 payment. At that time,

$$\text{Future value of the lump sum payment} = \$920{,}000e^{0.06(3)} \approx \$1{,}101{,}440.$$

Now we calculate the future value of the first $250,000 payment:

$$\text{Future value of the first payment} = \$250{,}000e^{0.06(3)}.$$

Calculating the future value of the other payments similarly, we find:

$$\begin{aligned}\text{Total future value} &= \$250{,}000e^{0.06(3)} + \$250{,}000e^{0.06(2)} + \$250{,}000e^{0.06(1)} + \$250{,}000 \\ &\approx \$299{,}304 + \$281{,}874 + \$265{,}459 + \$250{,}000 \\ &= \$1{,}096{,}637.\end{aligned}$$

The future value of the $920,000 payment is greater, so you are better off taking the $920,000 right now. Of course, since the present value of the $920,000 payment is greater than the present value of the four separate payments, you would expect the future value of the $920,000 payment to be greater than the future value of the four separate payments.

(Note: If you read the fine print, you will find that many lotteries do not make their payments right away, but often spread them out, sometimes far into the future. This is to reduce the present value of the payments made, so that the value of the prizes is much less than it might first appear!)

Income Stream

When we consider payments made to or by an individual, we usually think of *discrete* payments, that is, payments made at specific moments in time. However, we may think of payments made by a company as being *continuous*. The revenues earned by a huge corporation, for example, come in essentially all the time and can be represented by a continuous *income stream*, written

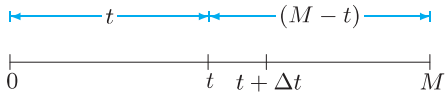
$$P(t) \text{ dollars/year.}$$

Notice that $P(t)$ is the *rate* at which deposits are made (its units are dollars per year, for example) and that this rate may vary with time, t.

Present and Future Values of an Income Stream

Just as we can find the present and future values of a single payment, so we can find the present and future values of a stream of payments. We will assume that interest is compounded continuously.

Suppose that we want to calculate the present value of the income stream described by a rate of $P(t)$ dollars per year, and that we are interested in the period from now until M years in the future. We divide the stream into many small deposits, each of which is made at approximately one instant. We divide the interval $0 \le t \le M$ into subintervals, each of length Δt:



Assuming Δt is small, the rate, $P(t)$, at which deposits are being made will not vary much within one subinterval. Thus, between t and $t + \Delta t$:

$$\begin{aligned}\text{Amount deposited} &\approx \text{Rate of deposits} \times \text{Time} \\ &\approx (P(t) \text{ dollars/year})(\Delta t \text{ years}) \\ &= P(t)\Delta t \text{ dollars.}\end{aligned}$$

Measured from the present, the deposit of $P(t)\Delta t$ is made t years in the future. Thus,

$$\begin{pmatrix}\text{Present value of money deposited} \\ \text{in interval } t \text{ to } t + \Delta t\end{pmatrix} \approx P(t)\Delta t e^{-rt}.$$

Summing over all subintervals gives

$$\text{Total present value} \approx \sum P(t)e^{-rt}\Delta t \text{ dollars.}$$

In the limit as $\Delta t \to 0$, we get the following integral:

$$\boxed{\text{Present value} = \int_0^M P(t)e^{-rt}dt \text{ dollars.}}$$

In computing future value, the deposit of $P(t)\Delta t$ has a period of $(M - t)$ years to earn interest, and therefore

$$\begin{pmatrix}\text{Future value of money deposited} \\ \text{in interval } t \text{ to } t + \Delta t\end{pmatrix} \approx [P(t)\Delta t]\, e^{r(M-t)}.$$

Summing over all subintervals, we get:

$$\text{Total future value} \approx \sum P(t)\Delta t e^{r(M-t)} \text{ dollars.}$$

As the length of the subdivisions tends toward zero, the sum becomes an integral:

$$\boxed{\text{Future value} = \int_0^M P(t)e^{r(M-t)}dt \text{ dollars.}}$$

In addition, by writing $e^{r(M-t)} = e^{rM} \cdot e^{-rt}$ and factoring out e^{rM}, we see that

$$\text{Future value} = e^{rM} \cdot \text{ Present value.}$$

Example 2 Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 10% compounded continuously.

Solution Using $P(t) = 1000$ and $r = 0.1$, we have

$$\text{Present value} = \int_0^{20} 1000e^{-0.1t}dt = 1000\left(-\frac{e^{-0.1t}}{0.1}\right)\Bigg|_0^{20} = 10{,}000(1-e^{-2}) \approx 8646.65 \text{ dollars.}$$

There are two ways to compute the future value. Using the present value of \$8646.65, we have

$$\text{Future value} = 8646.65e^{0.1(20)} = 63{,}890.58 \text{ dollars.}$$

Alternatively, we can use the integral formula:

$$\text{Future value} = \int_0^{20} 1000e^{0.1(20-t)}dt = \int_0^{20} 1000e^2e^{-0.1t}dt$$
$$= 1000e^2\left(-\frac{e^{-0.1t}}{0.1}\right)\Bigg|_0^{20} = 10{,}000e^2(1-e^{-2}) \approx 63890.58 \text{ dollars.}$$

Notice that the total amount deposited is \$1000 per year for 20 years, or \$20,000. The additional \$43,895.58 of the future value comes from interest earned.

Supply and Demand Curves

In a free market, the quantity of a certain item produced and sold can be described by the supply and demand curves of the item. The *supply curve* shows the quantity of the item the producers will supply at different price levels. It is usually assumed that as the price increases, the quantity supplied will increase. The consumers' behavior is reflected in the *demand curve*, which shows what quantity of goods are bought at various prices. An increase in price is usually assumed to cause a decrease in the quantity purchased. See Figure 8.76.

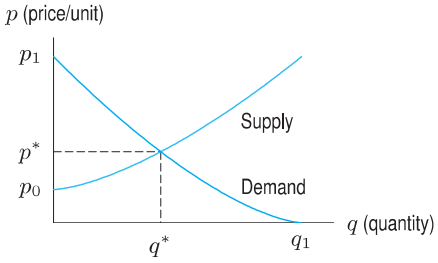


Figure 8.76: Supply and demand curves

It is assumed that the market settles to the *equilibrium price* and *quantity*, p^* and q^*, where the graphs cross. At equilibrium, a quantity q^* of an item is produced and sold for a price of p^* each.

Consumer and Producer Surplus

Notice that at equilibrium, a number of consumers have bought the item at a lower price than they would have been willing to pay. (For example, there are some consumers who would have been willing to pay prices up to p_1.) Similarly, there are some suppliers who would have been willing to produce the item at a lower price (down to p_0, in fact). We define the following terms:

- The **consumer surplus** measures the consumers' gain from trade. It is the total amount gained by consumers by buying the item at the current price rather than at the price they would have been willing to pay.
- The **producer surplus** measures the suppliers' gain from trade. It is the total amount gained by producers by selling at the current price, rather than at the price they would have been willing to accept.

In the absence of price controls, the current price is assumed to be the equilibrium price.

Both consumers and producers are richer for having traded. The consumer and producer surplus measure how much richer they are.

Suppose that all consumers buy the good at the maximum price they are willing to pay. Divide the interval from 0 to q^* into subintervals of length Δq. Figure 8.77 shows that a quantity Δq of items are sold at a price of about p_1, another Δq are sold for a slightly lower price of about p_2, the next Δq for a price of about p_3, and so on. Thus,

$$\text{Consumers' total expenditure} \approx p_1\Delta q + p_2\Delta q + p_3\Delta q + \cdots = \sum p_i \Delta q.$$

If D is the demand function given by $p = D(q)$, and if all consumers who were willing to pay more than p^* paid as much as they were willing, then as $\Delta q \to 0$, we would have

$$\text{Consumer expenditure} = \int_0^{q^*} D(q)dq = \begin{array}{l}\text{Area under demand}\\ \text{curve from 0 to } q^*.\end{array}$$

Now if all goods are sold at the equilibrium price, the consumers' actual expenditure is p^*q^*, the area of the rectangle between the axes and the lines $q = q^*$ and $p = p^*$. Thus, if p^* and q^* are equilibrium price and quantity, the consumer surplus is calculated as follows:

$$\text{Consumer surplus} = \left(\int_0^{q^*} D(q)dq\right) - p^*q^* = \begin{array}{l}\text{Area under demand}\\ \text{curve above } p = p^*.\end{array}$$

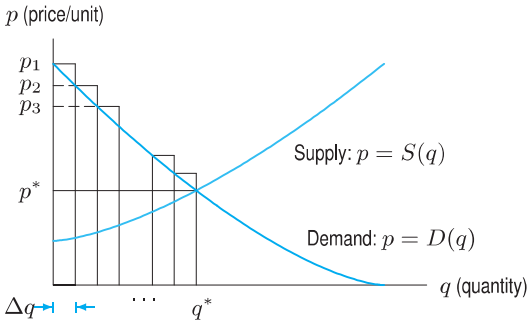


Figure 8.77: Calculation of consumer surplus

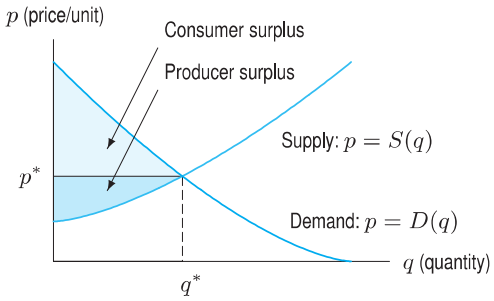


Figure 8.78: Consumer and producer surplus

See Figure 8.78. Similarly, if the supply curve is given by the function $p = S(q)$ and p^* and q^* are equilibrium price and quantity, the producer surplus is calculated as follows:

$$\text{Producer surplus} = p^*q^* - \left(\int_0^{q^*} S(q)dq\right) = \begin{array}{l}\text{Area between supply}\\ \text{curve and line } p = p^*.\end{array}$$

Exercises and Problems for Section 8.6

Exercises

1. Find the future value of an income stream of \$1000 per year, deposited into an account paying 8% interest, compounded continuously, over a 10-year period.
2. Find the present and future values of an income stream of \$3000 per year over a 15-year period, assuming a 6% annual interest rate compounded continuously.
3. Find the present and future values of an income stream of \$2000 a year, for a period of 5 years, if the continuous interest rate is 8%.
4. A person deposits money into a retirement account, which pays 7% interest compounded continuously, at a rate of \$1000 per year for 20 years. Calculate:
   (a) The balance in the account at the end of the 20 years.
   (b) The amount of money actually deposited into the account.
   (c) The interest earned during the 20 years.

Problems

5. Draw a graph, with time in years on the horizontal axis, of what an income stream might look like for a company that sells sunscreen in the northeast United States.
6. On March 6, 2007, the Associated Press reported that Ed Nabors had won half of a \$390 million jackpot, the largest lottery prize in US History. Suppose he was given the choice of receiving his \$195 million share paid out continuously over 20 years or one lump sum of \$120 million paid immediately.
   (a) Which option is better if the interest rate is 6%, compounded continuously? An interest rate of 3%?
   (b) If Mr. Nabors chose the lump sum option, what assumption was he making about interest rates?
7. **(a)** A bank account earns 10% interest compounded continuously. At what (constant, continuous) rate must a parent deposit money into such an account in order to save \$100,000 in 10 years for a child's college expenses?
   (b) If the parent decides instead to deposit a lump sum now in order to attain the goal of \$100,000 in 10 years, how much must be deposited now?
8. **(a)** If you deposit money continuously at a constant rate of \$1000 per year into a bank account that earns 5% interest, how many years will it take for the balance to reach \$10,000?
   (b) How many years would it take if the account had \$2000 in it initially?
9. A business associate who owes you \$3000 offers to pay you \$2800 now, or else pay you three yearly installments of \$1000 each, with the first installment paid now. If you use only financial reasons to make your decision, which option should you choose? Justify your answer, assuming a 6% interest rate per year, compounded continuously.
10. Big Tree McGee is negotiating his rookie contract with a professional basketball team. They have agreed to a three-year deal which will pay Big Tree a fixed amount at the end of each of the three years, plus a signing bonus at the beginning of his first year. They are still haggling about the amounts and Big Tree must decide between a big signing bonus and fixed payments per year, or a smaller bonus with payments increasing each year. The two options are summarized in the table. All values are payments in millions of dollars.

	Signing bonus	Year 1	Year 2	Year 3
Option #1	6.0	2.0	2.0	2.0
Option #2	1.0	2.0	4.0	6.0

   (a) Big Tree decides to invest all income in stock funds which he expects to grow at a rate of 10% per year, compounded continuously. He would like to choose the contract option which gives him the greater future value at the end of the three years when the last payment is made. Which option should he choose?
   (b) Calculate the present value of each contract offer.
11. Sales of Version 6.0 of a computer software package start out high and decrease exponentially. At time t, in years, the sales are $s(t) = 50e^{-t}$ thousands of dollars per year. After two years, Version 7.0 of the software is released and replaces Version 6.0. Assume that all income from software sales is immediately invested in government bonds which pay interest at a 6% rate compounded continuously, calculate the total value of sales of Version 6.0 over the two year period.
12. The value of good wine increases with age. Thus, if you are a wine dealer, you have the problem of deciding whether to sell your wine now, at a price of \$$P$ a bottle, or to sell it later at a higher price. Suppose you know that the amount a wine-drinker is willing to pay for a bottle of this wine t years from now is $\$P(1 + 20\sqrt{t})$. Assuming continuous compounding and a prevailing interest rate of 5% per year, when is the best time to sell your wine?
13. An oil company discovered an oil reserve of 100 million barrels. For time $t > 0$, in years, the company's extraction plan is a linear declining function of time as follows:

$$q(t) = a - bt,$$

where $q(t)$ is the rate of extraction of oil in millions of barrels per year at time t and $b = 0.1$ and $a = 10$.

   (a) How long does it take to exhaust the entire reserve?

(b) The oil price is a constant \$20 per barrel, the extraction cost per barrel is a constant \$10, and the market interest rate is 10% per year, compounded continuously. What is the present value of the company's profit?

14. In 1980 West Germany made a loan of 20 billion Deutsche Marks to the Soviet Union, to be used for the construction of a natural gas pipeline connecting Siberia to Western Russia, and continuing to West Germany (Urengoi–Uschgorod–Berlin). Assume that the deal was as follows: In 1985, upon completion of the pipeline, the Soviet Union would deliver natural gas to West Germany, at a constant rate, for all future times. Assuming a constant price of natural gas of 0.10 Deutsche Mark per cubic meter, and assuming West Germany expects 10% annual interest on its investment (compounded continuously), at what rate does the Soviet Union have to deliver the gas, in billions of cubic meters per year? Keep in mind that delivery of gas could not begin until the pipeline was completed. Thus, West Germany received no return on its investment until after five years had passed. (Note: A more complex deal of this type was actually made between the two countries.)

15. In May 1991, *Car and Driver* described a Jaguar that sold for \$980,000. At that price only 50 have been sold. It is estimated that 350 could have been sold if the price had been \$560,000. Assuming that the demand curve is a straight line, and that \$560,000 and 350 are the equilibrium price and quantity, find the consumer surplus at the equilibrium price.

16. Using Riemann sums, explain the economic significance of $\int_0^{q^*} S(q)\,dq$ to the producers.

17. Using Riemann sums, give an interpretation of producer surplus, $\int_0^{q^*} (p^* - S(q))\,dq$ analogous to the interpretation of consumer surplus.

18. In Figure 8.78, page 437, mark the regions representing the following quantities and explain their economic meaning:

(a) p^*q^*

(b) $\displaystyle\int_0^{q^*} D(q)\,dq$

(c) $\displaystyle\int_0^{q^*} S(q)\,dq$

(d) $\displaystyle\int_0^{q^*} D(q)\,dq - p^*q^*$

(e) $\displaystyle p^*q^* - \int_0^{q^*} S(q)\,dq$

(f) $\displaystyle\int_0^{q^*} (D(q) - S(q))\,dq$

19. The dairy industry is an example of cartel pricing: the government has set milk prices artificially high. On a supply and demand graph, label p^+, a price above the equilibrium price. Using the graph, describe the effect of forcing the price up to p^+ on:

(a) The consumer surplus.
(b) The producer surplus.
(c) The total gains from trade (Consumer surplus + Producer surplus).

20. Rent controls on apartments are an example of price controls on a commodity. They keep the price artificially low (below the equilibrium price). Sketch a graph of supply and demand curves, and label on it a price p^- below the equilibrium price. What effect does forcing the price down to p^- have on:

(a) The producer surplus?
(b) The consumer surplus?
(c) The total gains from trade (Consumer surplus + Producer surplus)?

8.7 DISTRIBUTION FUNCTIONS

Understanding the distribution of various quantities through the population is important to decision makers. For example, the income distribution gives useful information about the economic structure of a society. In this section we will look at the distribution of ages in the US. To allocate funding for education, health care, and social security, the government needs to know how many people are in each age group. We will see how to represent such information by a density function.

US Age Distribution

Table 8.3 *Distribution of ages in the US in 1995*

Age group	Fraction of total population
0–20	$29\% = 0.29$
20–40	$31\% = 0.31$
40–60	$24\% = 0.24$
60–80	$13\% = 0.13$
80–100	$3\% = 0.03$

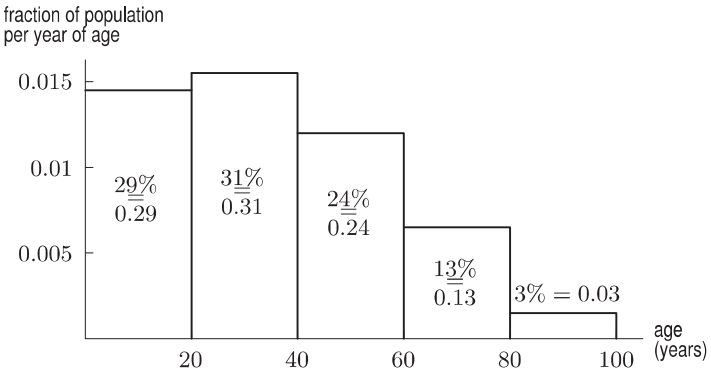


Figure 8.79: How ages were distributed in the US in 1995

The data in Table 8.3 shows how the ages of the US population were distributed in 1995. To represent this information graphically we use a type[2] of *histogram*, putting a vertical bar above each age group in such a way that the *area* of each bar represents the fraction of the population in that age group. The total area of all the rectangles is $100\% = 1$. We only consider people who are less than 100 years old.[3] For the 0–20 age group, the base of the rectangle is 20, and we want the area to be 0.29, so the height must be $0.29/20 = 0.0145$. We treat ages as though they were continuously distributed. The category 0–20, for example, contains people who are just one day short of their twentieth birthday. (See Figure 8.79.)

Example 1 In 1995, estimate what fraction of the US population was:

(a) Between 20 and 60 years old.
(b) Less than 10 years old.
(c) Between 75 and 80 years old.
(d) Between 80 and 85 years old.

Solution

(a) We add the fractions, so $0.31 + 0.24 = 0.55$; that is, 55% of the US population was in this age group.
(b) To find the fraction less than 10 years old, we could assume, for example, that the population was distributed evenly over the 0–20 group. (This means we are assuming that babies were born at a fairly constant rate over the last 20 years, which is probably reasonable.) If we make this assumption, then we can say that the population less than 10 years old was about half that in the 0–20 group, that is, 0.145 of the total population. Notice that we get the same result by computing the area of the rectangle from 0 to 10. (See Figure 8.80.)
(c) To find the population between 75 and 80 years old, since 0.13 of Americans in 1990 were in the 60-80 group, we might apply the same reasoning and say that $\frac{1}{4}(0.13) = 0.0325$ of the population was in this age group. This result is represented as an area in Figure 8.80. The assumption that the population was evenly distributed is not a good one here; certainly there were more people between the ages of 60 and 65 than between 75 and 80. Thus, the estimate of 0.0325 is certainly too high.
(d) Again using the (faulty) assumption that ages in each group were distributed uniformly, we would find that the fraction between 80 and 85 was $\frac{1}{4}(0.03) = 0.0075$. (See Figure 8.80.) This estimate is also poor—there were certainly more people in the 80–85 group than, say, the 95–100 group, and so the 0.0075 estimate is too low.

[2] There are other types of histogram which have frequency on the vertical axis.
[3] In fact, 0.02% of the population is over 100, but this is too small to be visible on the histogram.

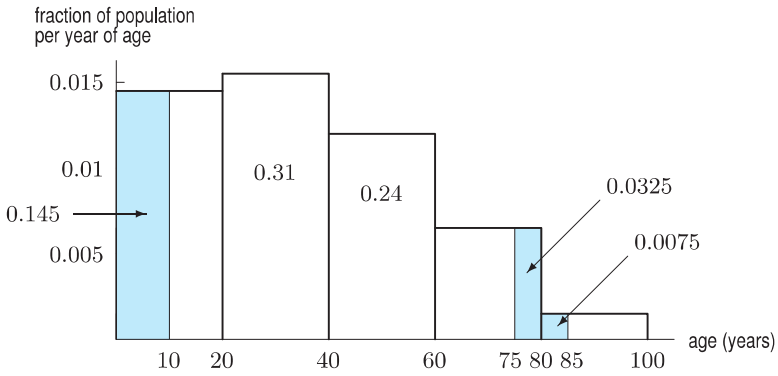


Figure 8.80: Ages in the US in 1995 — various subgroups (for Example 1)

Smoothing Out the Histogram

We could get better estimates if we had smaller age groups (each age group in Figure 8.79 is 20 years, which is quite large). The more detailed data in Table 8.4 leads to the new histogram in Figure 8.81. As we get more detailed information, the upper silhouette of the histogram becomes smoother, but the area of any of the bars still represents the percentage of the population in that age group. Imagine, in the limit, replacing the upper silhouette of the histogram by a smooth curve in such a way that area under the curve above one age group is the same as the area in the corresponding rectangle. The total area under the whole curve is again 100% = 1. (See Figure 8.81.)

The Age Density Function

If t is age in years, we define $p(t)$, the age *density function*, to be a function which "smooths out" the age histogram. This function has the property that

$$\begin{array}{c}\text{Fraction of population}\\ \text{between ages } a \text{ and } b\end{array} = \begin{array}{c}\text{Area under graph of } p\\ \text{between } a \text{ and } b\end{array} = \int_a^b p(t)dt.$$

If a and b are the smallest and largest possible ages (say, $a = 0$ and $b = 100$), so that the ages

Table 8.4 *Ages in the US in 1995 (more detailed)*

Age group	Fraction of total population
0–10	15% = 0.15
10–20	14% = 0.14
20–30	14% = 0.14
30–40	17% = 0.17
40–50	14% = 0.14
50–60	10% = 0.10
60–70	8% = 0.08
70–80	5% = 0.05
80–90	2% = 0.02
90–100	1% = 0.01

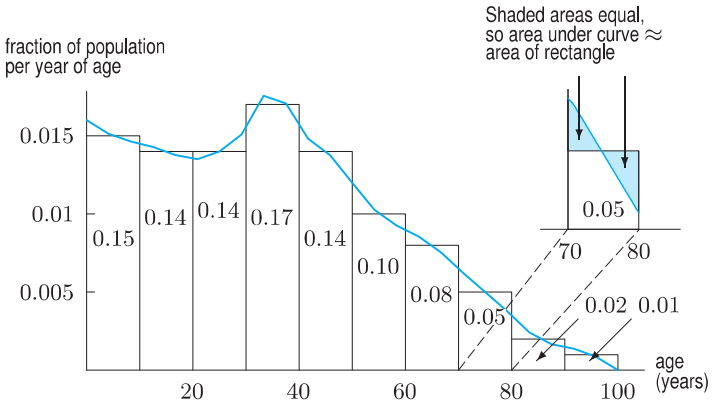


Figure 8.81: Smoothing out the age histogram

of all of the population are between a and b, then

$$\int_a^b p(t)dt = \int_0^{100} p(t)dt = 1.$$

What does the age density function p tell us? Notice that we have not talked about the meaning of $p(t)$ itself, but *only* of the integral $\int_a^b p(t)\,dt$. Let's look at this in a bit more detail. Suppose, for example, that $p(10) = 0.015$ per year. This is *not* telling us that 0.015 of the population is precisely 10 years old (where 10 years old means exactly 10, not $10\frac{1}{2}$, not $10\frac{1}{4}$, not 10.1). However, $p(10) = 0.015$ does tell us that for some small interval Δt around 10, the fraction of the population with ages in this interval is approximately $p(10)\,\Delta t = 0.015\,\Delta t$.

The Density Function

Suppose we are interested in how a certain characteristic, x, is distributed through a population. For example, x might be height or age if the population is people, or might be wattage for a population of light bulbs. Then we define a general density function with the following properties:

The function, $p(x)$, is a **density function** if

$$\begin{array}{c}\text{Fraction of population for which} \\ x \text{ is between } a \text{ and } b\end{array} = \begin{array}{c}\text{Area under graph of } p \\ \text{between } a \text{ and } b\end{array} = \int_a^b p(x)dx.$$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1 \qquad \text{and} \qquad p(x) \geq 0 \quad \text{for all } x.$$

The density function must be nonnegative because its integral always gives a fraction of the population. Also, the fraction of the population with x between $-\infty$ and ∞ is 1 because the entire population has the characteristic x between $-\infty$ and ∞. The function p that was used to smooth out the age histogram satisfies this definition of a density function. We do not assign a meaning to the value $p(x)$ directly, but rather interpret $p(x)\,\Delta x$ as the fraction of the population with the characteristic in a short interval of length Δx around x.

The density function is often approximated by formulas, as in the next example.

Example 2 Find formulas to approximate the density function, p, for the US age distribution. To reflect Figure 8.81, use a continuous function, constant at 0.015 up to age 40 and then dropping linearly.

Solution We have $p(t) = 0.015$ for $0 \leq t < 40$. For $t \geq 40$, we need a linear function sloping downward. Because p is continuous, we have $p(40) = 0.015$. Because p is a density function we have $\int_0^{100} p(t)dt = 1$. Suppose b is as in Figure 8.82 then

$$\int_0^{100} p(t)dt = \int_0^{40} p(t)dt + \int_{40}^{100} p(t)dt = 40(0.015) + \frac{1}{2}(0.015)b = 1,$$

where $\int_{40}^{100} p(t)dt$ is given by the area of the triangle. This gives

$$\frac{0.015}{2}b = 0.4, \quad \text{and so} \quad b \approx 53.3.$$

Thus the slope of the line is $-0.015/53.3 \approx -0.00028$, so for $40 \le t \le 40+53.3 = 93.3$, we have

$$p(t) - 0.015 = -0.00028(t - 40),$$
$$p(t) = 0.0262 - 0.00028t.$$

According to this way of smoothing the data, there is no one over 93.3 years old, so $p(t) = 0$ for $t > 93.3$.

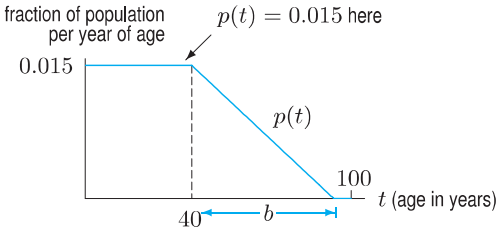


Figure 8.82: Age density function

Cumulative Distribution Function for Ages

Another way of showing how ages are distributed in the US is by using the *cumulative distribution function* $P(t)$, defined by

$$P(t) = \begin{matrix}\text{Fraction of population} \\ \text{of age less than } t\end{matrix} = \int_0^t p(x)dx.$$

Thus, P is the antiderivative of p with $P(0) = 0$, and $P(t)$ gives the area under the density curve between 0 and t.

Notice that the cumulative distribution function is nonnegative and increasing (or at least nondecreasing), since the number of people younger than age t increases as t increases. Another way of seeing this is to notice that $P' = p$, and p is positive (or nonnegative). Thus the cumulative age distribution is a function which starts with $P(0) = 0$ and increases as t increases. $P(t) = 0$ for $t < 0$ because, when $t < 0$, there is no one whose age is less than t. The limiting value of P, as $t \to \infty$, is 1 since as t becomes very large (100 say), everyone is younger than age t, so the fraction of people with age less than t tends toward 1. (See Figure 8.83.) For t less than 40, the graph of P is a straight line, because p is constant there. For $t > 40$, the graph of P levels off as p tends to 0.

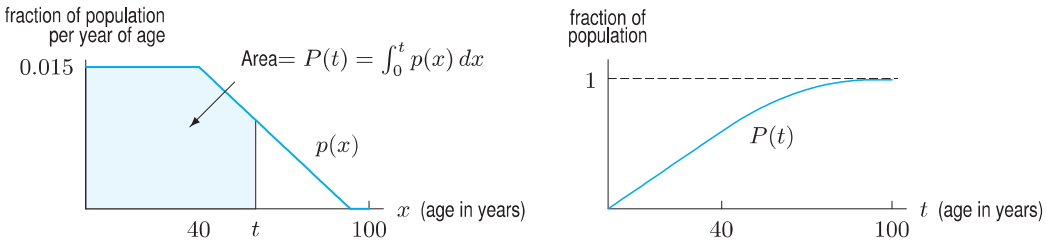


Figure 8.83: $P(t)$, the cumulative age distribution function, and its relation to $p(x)$, the age density function

Cumulative Distribution Function

A **cumulative distribution function**, $P(t)$, of a density function p, is defined by

$$P(t) = \int_{-\infty}^{t} p(x)\,dx = \begin{array}{c}\text{Fraction of population having}\\ \text{values of } x \text{ below } t.\end{array}$$

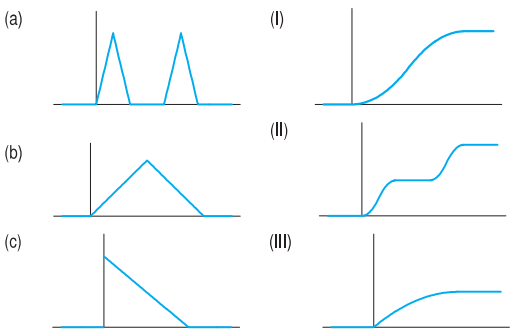
Thus, P is an antiderivative of p, that is, $P' = p$.
Any cumulative distribution has the following properties:

- P is increasing (or nondecreasing).
- $\lim_{t\to\infty} P(t) = 1$ and $\lim_{t\to-\infty} P(t) = 0$.
- $\begin{array}{c}\text{Fraction of population having}\\ \text{values of } x \text{ between } a \text{ and } b\end{array} = \int_a^b p(x)\,dx = P(b) - P(a)$.

Exercises and Problems for Section 8.7

Exercises

1. Match the graphs of the density functions (a), (b), and (c) with the graphs of the cumulative distribution functions I, II, and III.



In Exercises 2–4, graph a density function and a cumulative distribution function which could represent the distribution of income through a population with the given characteristics.

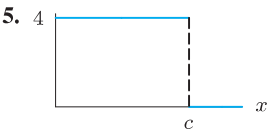
2. A large middle class.

3. Small middle and upper classes and many poor people.

4. Small middle class, many poor and many rich people.

Decide if the function graphed in Exercises 5–10 is a probability density function (pdf) or a cumulative distribution function (cdf). Give reasons. Find the value of c. Sketch and label the other function. (That is, sketch and label the cdf if the problem shows a pdf, and the pdf if the problem shows a cdf.)

5.



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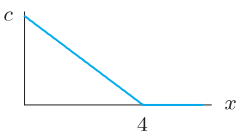

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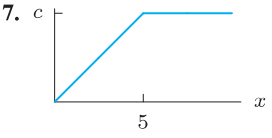

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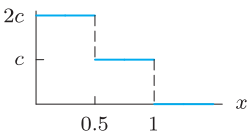

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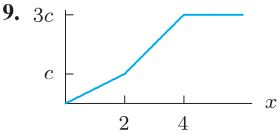

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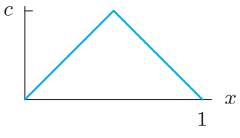


Problems

11. A large number of people take a standardized test, receiving scores described by the density function p graphed in Figure 8.84. Does the density function imply that most people receive a score near 50? Explain why or why not.

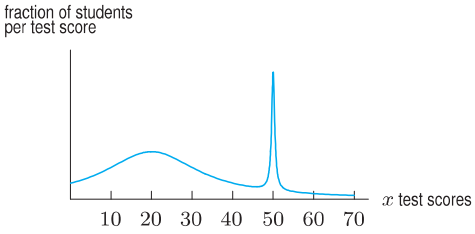


Figure 8.84

12. An experiment is done to determine the effect of two new fertilizers A and B on the growth of a species of peas. The cumulative distribution functions of the heights of the mature peas without treatment and treated with each of A and B are graphed in Figure 8.85.

(a) About what height are most of the unfertilized plants?
(b) Explain in words the effect of the fertilizers A and B on the mature height of the plants.

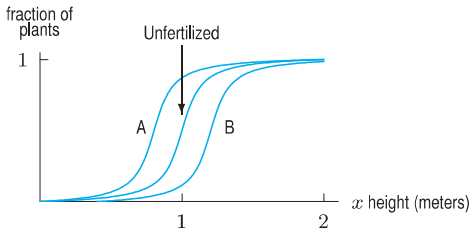


Figure 8.85

13. Suppose $F(x)$ is the cumulative distribution function for heights (in meters) of trees in a forest.

(a) Explain in terms of trees the meaning of the statement $F(7) = 0.6$.
(b) Which is greater, $F(6)$ or $F(7)$? Justify your answer in terms of trees.

14. Suppose that $p(x)$ is the density function for heights of American men, in inches. What is the meaning of the statement $p(68) = 0.2$?

15. Suppose $P(t)$ is the fraction of the US population of age less than t. Using Table 8.4 on page 441, make a table of values for $P(t)$.

16. Figure 8.86 shows a density function and the corresponding cumulative distribution function.[4]

(a) Which curve represents the density function and which represents the cumulative distribution function? Give a reason for your choice.
(b) Put reasonable values on the tick marks on each of the axes.

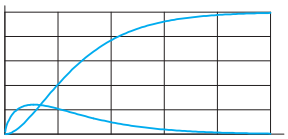


Figure 8.86

17. The density function and cumulative distribution function of heights of grass plants in a meadow are in Figures 8.87 and 8.88, respectively.

(a) There are two species of grass in the meadow, a short grass and a tall grass. Explain how the graph of the density function reflects this fact.
(b) Explain how the graph of the cumulative distribution functions reflects the fact that there are two species of grass in the meadow.
(c) About what percentage of the grasses in the meadow belong to the short grass species?

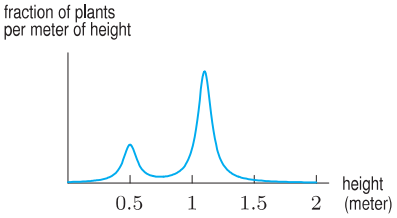


Figure 8.87

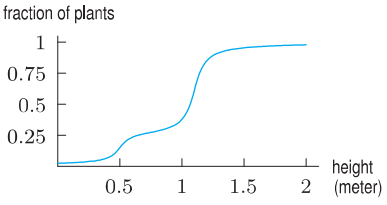


Figure 8.88

[4] Adapted from *Calculus*, by David A. Smith and Lawerence C. Moore (Lexington: D.C. Heath, 1994).

18. After measuring the duration of many telephone calls, the telephone company found their data was well-approximated by the density function $p(x) = 0.4e^{-0.4x}$, where x is the duration of a call, in minutes.

(a) What percentage of calls last between 1 and 2 minutes?
(b) What percentage of calls last 1 minute or less?
(c) What percentage of calls last 3 minutes or more?
(d) Find the cumulative distribution function.

19. Students at the University of California were surveyed and asked their grade point average . (The GPA ranges from 0 to 4, where 2 is just passing.) The distribution of GPAs is shown in Figure 8.89.[5]

(a) Roughly what fraction of students are passing?
(b) Roughly what fraction of the students have honor grades (GPAs above 3)?
(c) Why do you think there is a peak around 2?
(d) Sketch the cumulative distribution function.

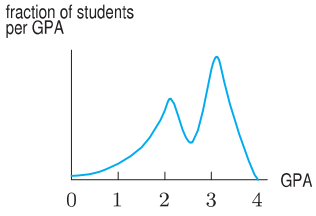


Figure 8.89

20. Figure 8.90[5] shows the distribution of elevation, in miles, across the earth's surface. Positive elevation denotes land above sea level; negative elevation shows land below sea level (i.e., the ocean floor).

(a) Describe in words the elevation of most of the earth's surface.
(b) Approximately what fraction of the earth's surface is below sea level?

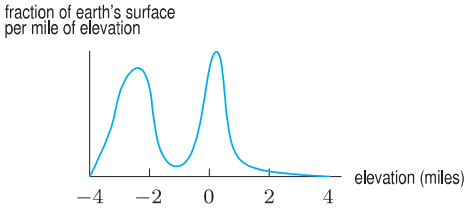


Figure 8.90

21. Consider a population of individuals with a disease. Suppose that t is the number of years since the onset of the disease. The death density function, $f(t) = cte^{-kt}$, approximates the fraction of the sick individuals who die in the time interval $[t, t + \Delta t]$ as follows:

$$\text{Fraction who die } \approx f(t)\Delta t = cte^{-kt}\Delta t$$

where c and k are positive constants whose values depend on the particular disease.

(a) Find the value of c in terms of k.
(b) If 40% of the population dies within 5 years, find c and k.
(c) Find the cumulative death distribution function, $C(t)$. Give your answer in terms of k.

8.8 PROBABILITY, MEAN, AND MEDIAN

Probability

Suppose we pick a member of the US population at random and ask what is the probability that the person is between, say, the ages of 70 and 80. We saw in Table 8.4 on page 441 that $5\% = 0.05$ of the population is in this age group. We say that the probability, or chance, that the person is between 70 and 80 is 0.05. Using any age density function $p(t)$, we can define probabilities as follows:

$$\text{Probability that a person is between ages } a \text{ and } b = \text{Fraction of population between ages } a \text{ and } b = \int_a^b p(t)\,dt.$$

Since the cumulative distribution function gives the fraction of the population younger than age t, the cumulative distribution can also be used to calculate the probability that a randomly selected person is in a given age group.

[5] Adapted from *Statistics*, by Freedman, Pisani, Purves, and Adikhari (New York: Norton, 1991).

$$\text{Probability that a person is younger than age } t = \text{Fraction of population younger than age } t = P(t) = \int_0^t p(x)\,dx.$$

In the next example, both a density function and a cumulative distribution function are used to describe the same situation.

Example 1 Suppose you want to analyze the fishing industry in a small town. Each day, the boats bring back at least 2 tons of fish, and never more than 8 tons.

(a) Using the density function describing the daily catch in Figure 8.91, find and graph the corresponding cumulative distribution function and explain its meaning.

(b) What is the probability that the catch will be between 5 and 7 tons?

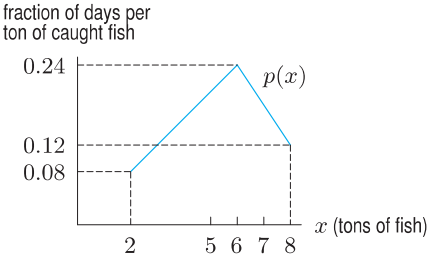


Figure 8.91: Density function of daily catch

Solution

(a) The cumulative distribution function $P(t)$ is equal to the fraction of days on which the catch is less than t tons of fish. Since the catch is never less than 2 tons, we have $P(t) = 0$ for $t \le 2$. Since the catch is always less than 8 tons, we have $P(t) = 1$ for $t \ge 8$. For t in the range $2 < t < 8$, we must evaluate the integral

$$P(t) = \int_{-\infty}^{t} p(x)dx = \int_2^t p(x)dx.$$

This integral equals the area under the graph of $p(x)$ between $x = 2$ and $x = t$. It can be calculated by noting that $p(x)$ is given by the formula

$$p(x) = \begin{cases} 0.04x & \text{for } 2 \le x \le 6 \\ -0.06x + 0.6 & \text{for } 6 < x \le 8 \end{cases}$$

and $p(x) = 0$ for $x < 2$ or $x > 8$. Thus, for $2 \le t \le 6$,

$$P(t) = \int_2^t 0.04x\,dx = 0.04\frac{x^2}{2}\Big|_2^t = 0.02t^2 - 0.08.$$

And for $6 \le t \le 8$,

$$\begin{aligned} P(t) &= \int_2^t p(x)\,dx = \int_2^6 p(x)\,dx + \int_6^t p(x)\,dx \\ &= 0.64 + \int_6^t (-0.06x + 0.6)\,dx = 0.64 + \left(-0.06\frac{x^2}{2} + 0.6x\right)\Big|_6^t \\ &= -0.03t^2 + 0.6t - 1.88. \end{aligned}$$

Thus

$$P(t) = \begin{cases} 0.02t^2 - 0.08 & \text{for } 2 \le t \le 6 \\ -0.03t^2 + 0.6t - 1.88 & \text{for } 6 < t \le 8. \end{cases}$$

In addition $P(t) = 0$ for $t < 2$ and $P(t) = 1$ for $8 < t$. (See Figure 8.92.)

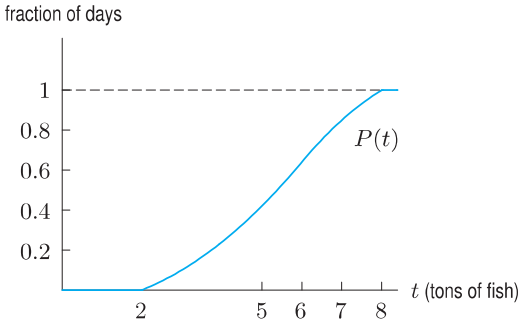


Figure 8.92: Cumulative distribution of daily catch

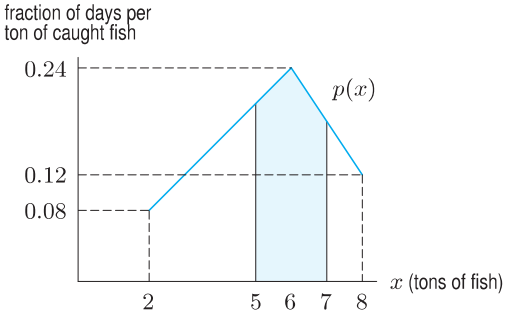


Figure 8.93: Shaded area represents the probability the catch is between 5 and 7 tons

(b) The probability that the catch is between 5 and 7 tons can be found using either the density function, p, or the cumulative distribution function, P. If we use the density function, this probability can be represented by the shaded area in Figure 8.93, which is about 0.43.

$$\text{Probability catch is between 5 and 7 tons} = \int_5^7 p(x)\,dx = 0.43.$$

The probability can be found from the cumulative distribution as follows:

$$\text{Probability catch is between 5 and 7 tons} = P(7) - P(5) = 0.85 - 0.42 = 0.43.$$

The Median and Mean

It is often useful to be able to give an "average" value for a distribution. Two measures that are in common use are the *median* and the *mean*.

The Median

A **median** of a quantity x distributed through a population is a value T such that half the population has values of x less than (or equal to) T, and half the population has values of x greater than (or equal to) T. Thus, a median T satisfies

$$\int_{-\infty}^{T} p(x)\,dx = 0.5,$$

where p is the density function. In other words, half the area under the graph of p lies to the left of T.

Example 2 Find the median age in the US in 1995, using the age density function given by

$$p(t) = \begin{cases} 0.015 & \text{for } 0 \le t \le 40 \\ 0.0262 - 0.00028t & \text{for } 40 < t \le 93.3. \end{cases}$$

Solution We want to find the value of T such that

$$\int_{-\infty}^{T} p(t)\,dt = \int_{0}^{T} p(t)\,dt = 0.5.$$

Since $p(t) = 0.015$ up to age 40, we have

$$\text{Median} = T = \frac{0.5}{0.015} \approx 33 \text{ years}.$$

(See Figure 8.94.)

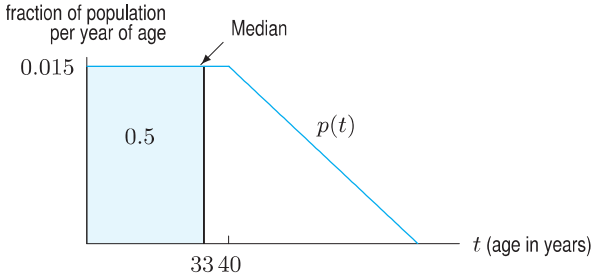


Figure 8.94: Median of age distribution

The Mean

Another commonly used average value is the *mean*. To find the mean of N numbers, you add the numbers and divide the sum by N. For example, the mean of the numbers 1, 2, 7, and 10 is $(1 + 2 + 7 + 10)/4 = 5$. The mean age of the entire US population is therefore defined as

$$\text{Mean age} = \frac{\sum \text{Ages of all people in the US}}{\text{Total number of people in the US}}.$$

Calculating the sum of all the ages directly would be an enormous task; we will approximate the sum by an integral. The idea is to "slice up" the age axis and consider the people whose age is between t and $t + \Delta t$. How many are there?

The fraction of the population between t and $t + \Delta t$ is the area under the graph of p between these points, which is well approximated by the area of the rectangle, $p(t)\Delta t$. (See Figure 8.95.) If the total number of people in the population is N, then

$$\text{Number of people with age between } t \text{ and } t + \Delta t \approx p(t)\Delta t N.$$

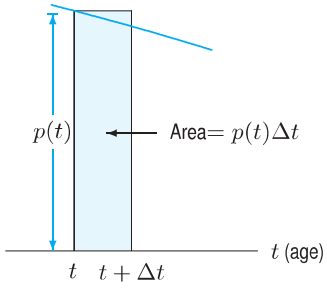


Figure 8.95: Shaded area is percentage of population with age between t and $t + \Delta t$

The age of all of these people is approximately t:

$$\begin{matrix}\text{Sum of ages of people} \\ \text{between age } t \text{ and } t+\Delta t\end{matrix} \approx tp(t)\Delta t N.$$

Therefore, adding and factoring out an N gives us

$$\text{Sum of ages of all people} \approx \left(\sum tp(t)\Delta t\right) N.$$

In the limit, as we allow Δt to shrink to 0, the sum becomes an integral, so

$$\text{Sum of ages of all people} = \left(\int_0^{100} tp(t)dt\right) N.$$

Therefore, with N equal to the total number of people in the US, and assuming no person is over 100 years old,

$$\text{Mean age} = \frac{\text{Sum of ages of all people in US}}{N} = \int_0^{100} tp(t)dt.$$

We can give the same argument for any[6] density function $p(x)$.

If a quantity has density function $p(x)$,

$$\textbf{Mean value} \text{ of the quantity} = \int_{-\infty}^{\infty} xp(x)\,dx.$$

It can be shown that the mean is the point on the horizontal axis where the region under the graph of the density function, if it were made out of cardboard, would balance.

Example 3 Find the mean age of the US population, using the density function of Example 2.

Solution The formula for p is

$$p(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.015 & \text{for } 0 \le t \le 40 \\ 0.0262 - 0.00028t & \text{for } 40 < t \le 93.3 \\ 0 & \text{for } t > 93.3. \end{cases}$$

Using these formulas, we compute

$$\begin{aligned}\text{Mean age} &= \int_0^{100} tp(t)dt = \int_0^{40} t(0.015)dt + \int_{40}^{93.3} t(0.0262 - 0.00028t)dt \\ &= 0.015\frac{t^2}{2}\Big|_0^{40} + 0.0262\frac{t^2}{2}\Big|_{40}^{93.3} - 0.00028\frac{t^3}{3}\Big|_{40}^{93.3} \approx 35 \text{ years.}\end{aligned}$$

The mean is shown is Figure 8.96.

[6] Provided all the relevant improper integrals converge.

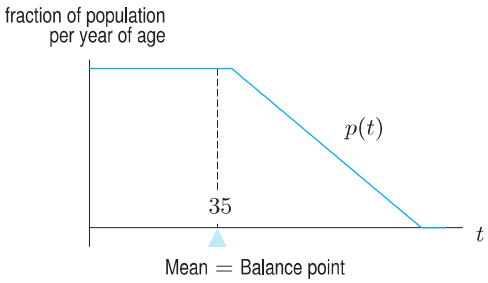


Figure 8.96: Mean of age distribution

Normal Distributions

How much rain do you expect to fall in your home town this year? If you live in Anchorage, Alaska, the answer is something close to 15 inches (including the snow). Of course, you don't expect exactly 15 inches. Some years there are more than 15 inches, and some years there is less. Most years, however, the amount of rainfall is close to 15 inches; only rarely is it well above or well below 15 inches. What does the density function for the rainfall look like? To answer this question, we look at rainfall data over many years. Records show that the distribution of rainfall is well-approximated by a *normal distribution*. The graph of its density function is a bell-shaped curve which peaks at 15 inches and slopes downward approximately symmetrically on either side.

Normal distributions are frequently used to model real phenomena, from grades on an exam to the number of airline passengers on a particular flight. A normal distribution is characterized by its *mean*, μ, and its *standard deviation*, σ. The mean tells us the location of the central peak. The standard deviation tells us how closely the data is clustered around the mean. A small value of σ tells us that the data is close to the mean; a large σ tells us the data is spread out. In the following formula for a normal distribution, the factor of $1/(\sigma\sqrt{2\pi})$ makes the area under the graph equal to 1.

A **normal distribution** has a density function of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)},$$

where μ is the mean of the distribution and σ is the standard deviation, with $\sigma > 0$.

To model the rainfall in Anchorage, we use a normal distribution with $\mu = 15$ and $\sigma = 1$. (See Figure 8.97.)

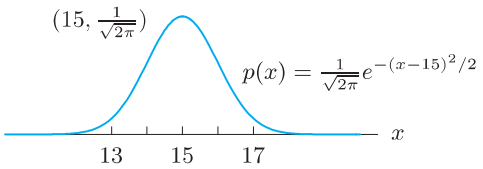


Figure 8.97: Normal distribution with $\mu = 15$ and $\sigma = 1$

Example 4 For Anchorage's rainfall, use the normal distribution with the density function with $\mu = 15$ and $\sigma = 1$ to compute the fraction of the years with rainfall between
(a) 14 and 16 inches, (b) 13 and 17 inches, (c) 12 and 18 inches.

Solution (a) The fraction of the years with annual rainfall between 14 and 16 inches is $\int_{14}^{16} \frac{1}{\sqrt{2\pi}} e^{-(x-15)^2/2}\,dx$. Since there is no elementary antiderivative for $e^{-(x-15)^2/2}$, we find the integral numerically. Its value is about 0.68.

$$\text{Fraction of years with rainfall between 14 and 16 inches} = \int_{14}^{16} \frac{1}{\sqrt{2\pi}} e^{-(x-15)^2/2}\,dx \approx 0.68.$$

(b) Finding the integral numerically again:

$$\text{Fraction of years with rainfall between 13 and 17 inches} = \int_{13}^{17} \frac{1}{\sqrt{2\pi}} e^{-(x-15)^2/2}\,dx \approx 0.95.$$

(c)

$$\text{Fraction of years with rainfall between 12 and 18 inches} = \int_{12}^{18} \frac{1}{\sqrt{2\pi}} e^{-(x-15)^2/2}\,dx \approx 0.997.$$

Since 0.95 is so close to 1, we expect that most of the time the rainfall will be between 13 and 17 inches a year.

Among the normal distributions, the one having $\mu = 0$, $\sigma = 1$ is called the *standard normal distribution*. Values of the corresponding cumulative distribution function are published in tables.

Exercises and Problems for Section 8.8

Exercises

1. Show that the area under the fishing density function in Figure 8.91 on page 447 is 1. Why is this to be expected?

2. Find the mean daily catch for the fishing data in Figure 8.91, page 447.

3. **(a)** Using a calculator or computer, sketch graphs of the density function of the normal distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

(i) For fixed μ (say, $\mu = 5$) and varying σ (say, $\sigma = 1, 2, 3$).

(ii) For varying μ (say, $\mu = 4, 5, 6$) and fixed σ (say, $\sigma = 1$).

(b) Explain how the graphs confirm that μ is the mean of the distribution and that σ is a measure of how closely the data is clustered around the mean.

Problems

4. The probability of a transistor failing between $t = a$ months and $t = b$ months is given by $c\int_a^b e^{-ct}dt$, for some constant c.

(a) If the probability of failure within the first six months is 10%, what is c?

(b) Given the value of c in part (a), what is the probability the transistor fails within the second six months?

5. Suppose that x measures the time (in hours) it takes for a student to complete an exam. All students are done within two hours and the density function for x is

$$p(x) = \begin{cases} x^3/4 & \text{if } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What proportion of students take between 1.5 and 2.0 hours to finish the exam?

(b) What is the mean time for students to complete the exam?

(c) Compute the median of this distribution.

6. In 1950 an experiment was done observing the time gaps between successive cars on the Arroyo Seco Freeway.[7] The data show that the density function of these time gaps was given approximately by

$$p(x) = ae^{-0.122x}$$

where x is the time in seconds and a is a constant.

(a) Find a.
(b) Find P, the cumulative distribution function.
(c) Find the median and mean time gap.
(d) Sketch rough graphs of p and P.

7. Consider a group of people who have received treatment for a disease such as cancer. Let t be the *survival time*, the number of years a person lives after receiving treatment. The density function giving the distribution of t is $p(t) = Ce^{-Ct}$ for some positive constant C.

(a) What is the practical meaning for the cumulative distribution function $P(t) = \int_0^t p(x)\,dx$?
(b) The survival function, $S(t)$, is the probability that a randomly selected person survives for at least t years. Find $S(t)$.
(c) Suppose a patient has a 70% probability of surviving at least two years. Find C.

8. While taking a walk along the road where you live, you accidentally drop your glove, but you don't know where. The probability density $p(x)$ for having dropped the glove x kilometers from home (along the road) is

$$p(x) = 2e^{-2x} \quad \text{for } x \geq 0.$$

(a) What is the probability that you dropped it within 1 kilometer of home?
(b) At what distance y from home is the probability that you dropped it within y km of home equal to 0.95?

9. The distribution of IQ scores can be modeled by a normal distribution with mean 100 and standard deviation 15.

(a) Write the formula for the density function of IQ scores.
(b) Estimate the fraction of the population with IQ between 115 and 120.

10. The speeds of cars on a road are approximately normally distributed with a mean $\mu = 58$ km/hr and standard deviation $\sigma = 4$ km/hr.

(a) What is the probability that a randomly selected car is going between 60 and 65 km/hr?
(b) What fraction of all cars are going slower than 52 km/hr?

11. Consider the normal distribution, $p(x)$.

(a) Show that $p(x)$ is a maximum when $x = \mu$. What is that maximum value?
(b) Show that $p(x)$ has points of inflection where $x = \mu + \sigma$ and $x = \mu - \sigma$.
(c) Describe in your own words what μ and σ tell you about the distribution.

12. For a normal population of mean 0, show that the fraction of the population within one standard deviation of the mean does not depend on the standard deviation.

[Hint: Use the substitution $w = x/\sigma$.]

13. Which of the following functions makes the most sense as a model for the probability density representing the time (in minutes, starting from $t = 0$) that the next customer walks into a store?

(a) $p(t) = \begin{cases} \cos t & 0 \leq t \leq 2\pi \\ e^{t-2\pi} & t \geq 2\pi \end{cases}$
(b) $p(t) = 3e^{-3t}$ for $t \geq 0$
(c) $p(t) = e^{-3t}$ for $t \geq 0$
(d) $p(t) = 1/4$ for $0 \leq t \leq 4$

14. Let $P(x)$ be the cumulative distribution function for the household income distribution in the US in 2006. Values of $P(x)$ are in the following table:

Income x (thousand \$)	20	40	60	80	100	150
$P(x)$ (%)	21.7	45.4	63.0	75.8	84.0	94.0

(a) What percent of the households made between \$40,000 and \$60,000? More than \$150,000?
(b) Approximately what was the median income?
(c) Is the statement "More than one-third of households made between \$40,000 and \$80,000" true or false?

15. If we think of an electron as a particle, the function

$$P(r) = 1 - (2r^2 + 2r + 1)e^{-2r}$$

is the cumulative distribution function of the distance, r, of the electron in a hydrogen atom from the center of the atom. The distance is measured in Bohr radii. (1 Bohr radius = 5.29×10^{-11} m. Niels Bohr (1885–1962) was a Danish physicist.)

For example, $P(1) = 1 - 5e^{-2} \approx 0.32$ means that the electron is within 1 Bohr radius from the center of the atom 32% of the time.

(a) Find a formula for the density function of this distribution. Sketch the density function and the cumulative distribution function.
(b) Find the median distance and the mean distance. Near what value of r is an electron most likely to be found?
(c) The Bohr radius is sometimes called the "radius of the hydrogen atom." Why?

[7] Reported by Daniel Furlough and Frank Barnes.

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Geometry**
  Area, volume, arc length.
- **Density**
  Finding total quantity from density, center of mass.
- **Physics**
  Work, force and pressure.
- **Economics**
  Present and future value of income stream, consumer and producer surplus.
- **Probability**
  Density function, cumulative distribution function, mean, median, normal distribution.
- **Polar coordinates**
  Area, slope, arc length.

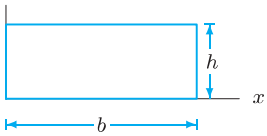
REVIEW EXERCISES AND PROBLEMS FOR CHAPTER EIGHT

Exercises

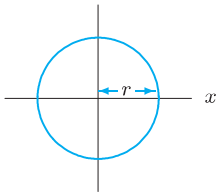
1. Imagine a hard-boiled egg lying on its side cut into thin slices. First think about vertical slices and then horizontal ones. What would these slices look like? Sketch them.

For each region in Exercises 2–4, write a definite integral which represents its area. Evaluate the integral to derive a formula for the area.

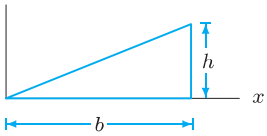
2. A rectangle with base b and height h:



3. A circle of radius r:



4. A right triangle of base b and height h:



In Exercises 5–9, the region is rotated about the x-axis. Find the volume.

5. Bounded by $y = x^2 + 1$, the x-axis, $x = 0$, $x = 4$.

6. Bounded by $y = \sqrt{x}$, x-axis, $x = 1$, $x = 2$.

7. Bounded by $y = e^{-2x}$, the x-axis, $x = 0$, $x = 1$.

8. Bounded by $y = 4 - x^2$ and the x-axis.

9. Bounded by $y = 2x$, $y = x$, $x = 0$, $x = 3$.

Exercises 10–15 refer to the regions marked in Figure 8.98. Set up, but do not evaluate, an integral that represents the volume obtained when the region is rotated about the given axis.

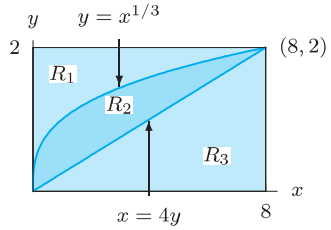


Figure 8.98

10. R_2 about the x-axis

11. R_1 about the y-axis

12. R_1 about the line $y = -2$

13. R_3 about the line $x = 10$

14. R_3 about the line $y = 3$

15. R_2 about the line $x = -3$

16. Find the volume of the region in Figure 8.99, given that the radius, r of the circular slice at h is $r = \sqrt{h}$.

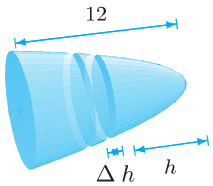


Figure 8.99

17. Set up and evaluate an integral to find the volume of a cone of height 12 m and radius 3 m at the open end.

18. **(a)** Set up and evaluate an integral giving the volume of a pyramid of height 10 m and square base 8 m by 8 m.
(b) The pyramid in part (a) is cut off at a height of 6 m. See Figure 8.100. Find the volume.

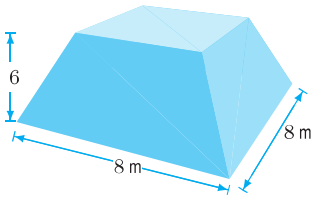


Figure 8.100

19. The exterior of a holding tank is a cylinder with radius 3 m and height 6 m; the interior is cone-shaped; Figure 8.101 shows its cross-section. Using an integral, find the volume of material needed to make the tank.

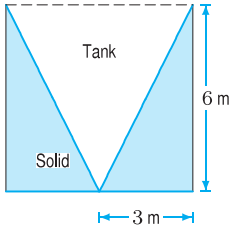


Figure 8.101

For the curves described in Exercises 20–21, write the integral that gives the exact length of the curve; do not evaluate it.

20. One arch of the sine curve, from $x = 0$ to $x = \pi$.

21. The ellipse with equation $(x^2/a^2) + (y^2/b^2) = 1$.

In Exercises 22–23, find the arclength of the function from $x = 0$ to $x = 3$. Use a graph to explain why your answer is reasonable.

22. $f(x) = \sin x$ **23.** $f(x) = 5x^2$

For Problems 24–26, find the arc lengths.

24. $f(x) = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$

25. $f(x) = e^x$ from $x = 1$ to $x = 2$

26. $f(x) = \frac{1}{3}x^3 + \frac{1}{4x}$ from $x = 1$ to $x = 2$.

In Exercises 27–28, find the length of the parametric curves. Give exact answers if possible.

27. $x = 3\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$.

28. $x = 1 + \cos(2t)$, $y = 3 + \sin(2t)$, for $0 \le t \le \pi$.

Problems

29. **(a)** Find the area of the region between $y = x^2$ and $y = 2x$.
(b) Find the volume of the solid of revolution if this region is rotated about the x-axis.
(c) Find the length of the perimeter of this region.

30. **(a)** Sketch the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 1$, and $y = 0$ around the line $y = 0$.
(b) Approximate its volume by Riemann sums, showing the volume represented by each term in your sum on the sketch.
(c) Now find the volume of this solid using an integral.

31. Using the region of Problem 30, find the volume when it is rotated around
(a) The line $y = 1$. **(b)** The y-axis.

32. **(a)** Find (in terms of a) the area of the region bounded by $y = ax^2$, the x-axis, and $x = 2$. Assume $a > 0$.
(b) If this region is rotated about the x-axis, find the volume of the solid of revolution in terms of a.

33. **(a)** Find (in terms of b) the area of the region between $y = e^{-bx}$ and the x-axis, between $x = 0$ and $x = 1$. Assume $b > 0$.
(b) If this region is rotated about the x-axis, find the volume of the solid of revolution in terms of b.

For Problems 34–36, set up and compute an integral giving the volume of the solid of revolution.

34. Bounded by $y = \sin x$, $y = 0.5x$, $x = 0$, $x = 1.9$;
(a) Rotated about the x-axis.
(b) Rotated about $y = 5$.

35. Bounded by $y = 2x$, the x-axis, $x = 0$, $x = 4$. Axis: $y = -5$.

36. Bounded by $y = x^2$, the x-axis, $x = 0$, $x = 3$;
(a) Rotated about $y = -2$.
(b) Rotated about $y = 10$.

Problems 37–42 concern the region bounded by the quarter circle $x^2 + y^2 = 1$, with $x \ge 0$, $y \ge 0$. Find the volume of the following solids.

37. The solid obtained by rotating the region about the x-axis.

38. The solid obtained by rotating the region about the line $x = -2$.

39. The solid obtained by rotating the region about the line $x = 1$.

40. The solid whose base is the region and whose cross-sections perpendicular to the x-axis are squares.

41. The solid whose base is the region and whose cross-sections perpendicular to the y-axis are semicircles.

42. The solid whose base is the region and whose cross-section perpendicular to the y-axis is an isosceles right triangle with one leg in the region.

43. The catenary $\cosh x = \frac{1}{2}(e^x + e^{-x})$ represents the shape of a hanging cable. Find the exact length of this catenary between $x = -1$ and $x = 1$.

44. The reflector behind a car headlight is made in the shape of the parabola, $x = \frac{4}{9}y^2$, with a circular cross-section, as shown in Figure 8.102.

(a) Find a Riemann sum approximating the volume contained by this headlight.
(b) Find the volume exactly.

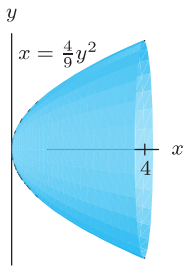


Figure 8.102

45. In this problem, you will derive the formula for the volume of a right circular cone with height l and base radius b by rotating the line $y = ax$ from $x = 0$ to $x = l$ around the x-axis. See Figure 8.103.

(a) What value should you choose for a such that the cone will have height l and base radius b?
(b) Given this value of a, find the volume of the cone.

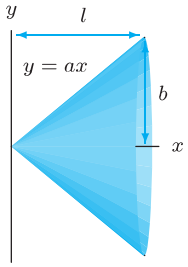


Figure 8.103

46. Figure 8.104 shows a cross section through an apple. (Scale: One division = 1/2 inch.)

(a) Give a rough estimate for the volume of this apple (in cubic inches).
(b) The density of these apples is about 0.03 lb/in^3 (a little less than the density of water—as you might expect, since apples float). Estimate how much this apple would cost. (They go for 80 cents a pound.)

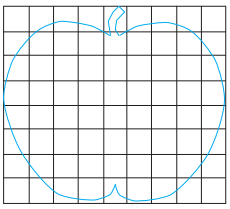


Figure 8.104

47. The circle $x^2 + y^2 = 1$ is rotated about the line $y = 3$ forming a torus (a doughnut-shaped figure). Find the volume of this torus.

48. Water is flowing in a cylindrical pipe of radius 1 inch. Because water is viscous and sticks to the pipe, the rate of flow varies with the distance from the center. The speed of the water at a distance r inches from the center is $10(1 - r^2)$ inches per second. What is the rate (in cubic inches per second) at which water is flowing through the pipe?

Problems 49–53 concern, C, the circle $r = 2a\cos\theta$, for $-\pi/2 \le \theta \le \pi/2$, of radius $a > 0$ centered at the point $(x, y) = (a, 0)$ on the x-axis.

49. By converting to Cartesian coordinates, show that $r = 2a\cos\theta$ gives the circle described.

50. Find the area of the circle C by integrating in polar coordinates.

51. Find the area of the region enclosed by C and outside the circle of radius a centered at the origin. What percent is this of the area of C?

52. **(a)** Find the slope of C at the angle θ.
(b) At what value of θ does the maximum y-value occur?

53. Calculate the arc length of C using polar coordinates.

54. Find the center of mass of a system containing four identical point masses of 3 gm, located at $x = -5, -3, 2, 7$.

55. A metal plate, with constant density 2 gm/cm^2, has a shape bounded by the two curves $y = x^2$ and $y = \sqrt{x}$, with $0 \le x \le 1$, and x, y in cm.

(a) Find the total mass of the plate.
(b) Because of the symmetry of the plate about the line $y = x$, we have $\bar{x} = \bar{y}$. Sketch the plate, and decide, on the basis of the shape, whether $\bar{x}$ is less than or greater than $1/2$.
(c) Find $\bar{x}$ and $\bar{y}$.

56. A 200-lb weight is attached to a 20-foot rope and dangling from the roof of a building. The rope weighs 2 lb/ft. Find the work done in lifting the weight to the roof.

57. Water is raised from a well 40 ft deep by a bucket attached to a rope. When the bucket is full, it weighs 30 lb. However, a leak in the bucket causes it to lose water at a rate of 1/4 lb for each foot that the bucket is raised. Neglecting the weight of the rope, find the work done in raising the bucket to the top.

58. A cylindrical garbage can of depth 3 ft and radius 1 ft fills with rainwater up to a depth of 2 ft. How much work would be done in pumping the water up to the top edge of the can? (Water weighs 62.4 lb/ft^3.)

59. A water tank is in the shape of a right circular cone with height 18 ft and radius 12 ft at the top. If it is filled with water to a depth of 15 ft, find the work done in pumping all of the water over the top of the tank. (The density of water is $\delta = 62.4$ lb/ft^3.)

60. An underground tank filled with gasoline of density 42 lb/ft^3 is a hemisphere of radius 5 ft, as in Figure 8.105. Use an integral to find the work to pump the gasoline over the top of the tank.

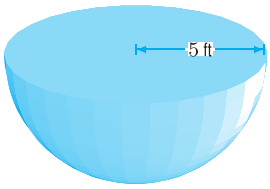


Figure 8.105

61. The dam in Hannawa Falls, NY, on the Raquette River is approximately 60 feet across and 25 feet high. Find the water force on the dam.

62. A crane lifts a 1000 lb object to a height of 20 ft using chain that weighs 2 lb/ft. If the crane arm is at a height of 50 ft, find the work required.

63. **(a)** Find the present and future values of a constant income stream of \$100 per year over a period of 20 years, assuming a 10% annual interest rate compounded continuously.
(b) How many years will it take for the balance to reach \$5000?

64. You are manufacturing a particular item. After t years, the rate at which you earn a profit on the item is $(2-0.1t)$ thousand dollars per year. (A negative profit represents a loss.) Interest is 10%, compounded continuously,

(a) Write a Riemann sum approximating the present value of the total profit earned up to a time M years in the future.
(b) Write an integral representing the present value in part (a). (You need not evaluate this integral.)
(c) For what M is the present value of the stream of profits on this item maximized? What is the present value of the total profit earned up to that time?

65. A nuclear power plant produces strontium-90 at a rate of 3 kg/yr. How much of the strontium produced since 1971 (when the plant opened) was still around in 1992? (The half-life of strontium-90 is 28 years.)

66. Mt. Shasta is a cone-like volcano whose radius at an elevation of h feet above sea level is approximately $(3.5 \cdot 10^5)/\sqrt{h+600}$ feet. Its bottom is 400 feet above sea level, and its top is 14,400 feet above sea level. See Figure 8.106. (Note: Mt. Shasta is in northern California, and for some time was thought to be the highest point in the US outside Alaska.)

(a) Give a Riemann sum approximating the volume of Mt. Shasta.
(b) Find the volume in cubic feet.

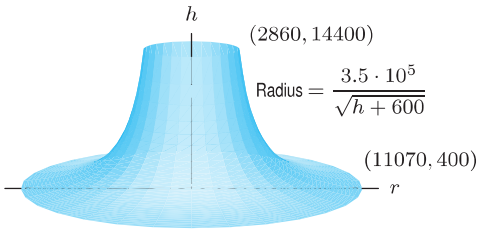


Figure 8.106: Mt. Shasta

67. Figure 8.107 shows an ancient Greek water clock called a clepsydra, which is designed so that the depth of the water decreases at a constant rate as the water runs out a hole in the bottom. This design allows the hours to be marked by a uniform scale. The tank of the clepsydra is a volume of revolution about a vertical axis. According to Torricelli's law, the exit speed of the water flowing through the hole is proportional to the square root of the depth of the water. Use this to find the formula $y = f(x)$ for this profile, assuming that $f(1) = 1$.

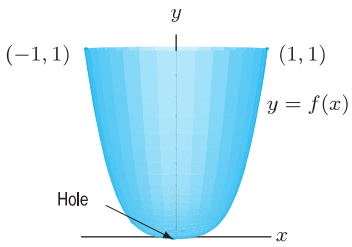
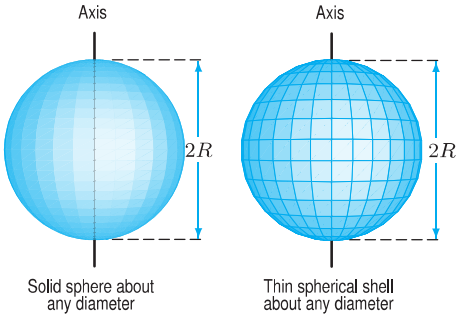


Figure 8.107

68. Figure 8.108 shows the distribution of kinetic energy of molecules in a gas at temperatures 300 kelvins and 500 kelvins. At higher temperatures, more of the molecules in a gas have higher kinetic energies. Which graph corresponds to which temperature?

In Problems 75–76, you are given two objects which have the same mass M, the same radius R, and the same angular velocity about the indicated axes (say, one revolution per minute). For each problem, determine which of the two objects has the greater kinetic energy. (The kinetic energy of a particle of mass m with speed v is $\frac{1}{2}mv^2$.) Don't compute the kinetic energy of the objects to do this; just use reasoning.

75.



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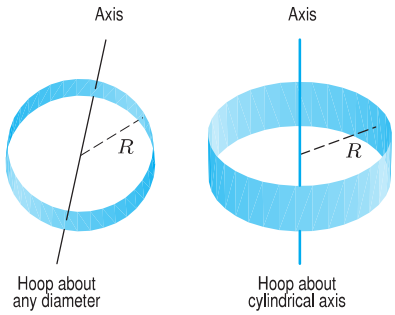

CAS Challenge Problems

77. For a positive constant a, consider the curve

$$y = \sqrt{\frac{x^3}{a-x}}, \qquad 0 \le x < a.$$

(a) Using a computer algebra system, show that for $0 \le t < \pi/2$, the point with coordinates (x, y) lies on the curve if:

$$x = a\sin^2 t, \quad y = \frac{a\sin^3 t}{\cos t}.$$

(b) A solid is obtained by rotating the curve about its asymptote at $x = a$. Use horizontal slicing to write an integral in terms of x and y that represents the volume of this solid.

(c) Use part (a) to substitute in the integral for both x and y in terms of t. Use a computer algebra system or trigonometric identities to calculate the volume of the solid.

For Problems 78–79, define $A(t)$ to be the arc length of the graph of $y = f(x)$ from $x = 0$ to $x = t$, for $t \ge 0$.

(a) Use the integral expression for arc length and a computer algebra system to obtain a formula for $A(t)$.

(b) Graph $A(t)$ for $0 \le t \le 10$. What simple function does $A(t)$ look like? What does this tell you about the approximate value of $A(t)$ for large t?

(c) In order to estimate arc length visually, you need the same scales on both axes, so that the lengths are not distorted in one direction. Draw a graph of $f(x)$ with viewing window $0 \le x \le 100$, $0 \le y \le 100$. Explain what you noticed in part (b) in terms of this graph.

78. $f(x) = x^2$

79. $f(x) = \sqrt{x}$

80. A bead is formed by drilling a cylindrical hole of circular cross section and radius a through a sphere of radius $r > a$, the axis of the hole passing through the center of the sphere.

(a) Write a definite integral expressing the volume of the bead.

(b) Find a formula for the bead by evaluating the definite integral in part (a).

CHECK YOUR UNDERSTANDING

In Problems 1–21, are the statements true or false? Give an explanation for your answer.

1. The integral $\int_{-3}^{3} \pi(9 - x^2)\,dx$ represents the volume of a sphere of radius 3.

2. The integral $\int_0^h \pi(r - y)\,dy$ gives the volume of a cone of radius r and height h.

3. The integral $\int_0^r \pi\sqrt{r^2 - y^2}\,dy$ gives the volume of a hemisphere of radius r.

4. A cylinder of radius r and length l is lying on its side. Horizontal slicing tells us that the volume is given by $\int_{-r}^{r} 2l\sqrt{r^2 - y^2}\,dy$.

5. If a region in the xy-plane lies below the x-axis, then revolving that region around the x-axis gives a solid of negative volume.

6. To find the total population in a circular city, we always slice it into concentric rings, no matter what the population density function.

7. A city occupies a region in the xy-plane, with population density $\delta(y) = 1 + y$. To set up an integral representing the total population in the city, we should slice the region parallel to the y-axis.

8. The population density in a circular city of radius 2 depends on the distance r from the center by $f(r) = 10 - 3r$, so that the density is greatest at the center. Then the population of the inner city, $0 \le r \le 1$, is greater than the population of the suburbs, $1 \le r \le 2$.

9. The location of the center of mass of a system of three masses on the x-axis does not change if all the three masses are doubled.

10. The center of mass of a region in the plane cannot be outside the region.

11. Particles are shot at a circular target. The density of particles hitting the target decreases with the distance from the center. To set up a definite integral to calculate the total number of particles hitting the target, we should slice the region into concentric rings.

12. A metal rod of density $f(x)$ lying along the x-axis from $x = 0$ to $x = 4$ has its center of mass at $x = 2$. Then the two halves of the rod on either side of $x = 2$ have equal mass.

13. It takes more work to lift a 20 lb weight 10 ft slowly than to lift it the same distance quickly.

14. Work can be negative or positive.

15. The force on a rectangular dam is doubled if its length stays the same and its depth is doubled.

16. To find the force of water on a vertical wall, we always slice the wall horizontally, no matter what the shape of the wall.

17. The force of a liquid on a wall can be negative or positive.

18. If the average value of the force $F(x)$ is 7 on the interval $1 \le x \le 4$, then the work done by the force in moving from $x = 1$ to $x = 4$ is 21.

19. The present value of an income stream is always less than its future value.

20. If $p(x) = xe^{-x^2}$ for all x, then $p(x)$ is a probability density function.

21. If $p(x) = xe^{-x^2}$ for all $x > 0$ and $p(x) = 0$ for $x \le 0$, then $p(x)$ is a probability density function.

Are the statements in Problems 22–25 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

22. Of two solids of revolution, the one with the greater volume is obtained by revolving the region in the plane with the greater area.

23. If f is differentiable on the interval $[0, 10]$, then the arc length of the graph of f on the interval $[0, 1]$ is less than the arc length of the graph of f on the interval $[1, 10]$.

24. If f is concave up for all x and $f'(0) = 3/4$, then the arc length of the graph of f on the interval $[0, 4]$ is at least 5.

25. If f is concave down for all x and $f'(0) = 3/4$, then the arc length of the graph of f on the interval $[0, 4]$ is at most 5.

In Problems 26–30, a quantity x is distributed through a population with probability density function $p(x)$ and cumulative distribution function $P(x)$. Decide if the statements in Problems 26–30 are true or false. Give an explanation for your answer.

26. If $p(10) = 1/2$, then half the population has $x < 10$.

27. If $P(10) = 1/2$, then half the population has $x < 10$.

28. If $p(10) = 1/2$, then the fraction of the population lying between $x = 9.98$ and $x = 10.04$ is about 0.03.

29. If $p(10) = p(20)$, then none of the population has x values lying between 10 and 20.

30. If $P(10) = P(20)$, then none of the population has x values lying between 10 and 20.

PROJECTS FOR CHAPTER EIGHT

1. Volume Enclosed by Two Cylinders

Two cylinders are inscribed in a cube of side length 2, as shown in Figure 8.111. What is the volume of the solid that the two cylinders enclose? [Hint: Use horizontal slices.] Note: The solution was known to Archimedes. The Chinese mathematician Liu Hui (third century A.D.) tried to find this volume, but he failed; he wrote a poem about his efforts calling the enclosed volume a "box-lid:"

Look inside the cube
And outside the box-lid;
Though the dimension increases,
It doesn't quite fit.
The marriage preparations are complete;
But square and circle wrangle,
Thick and thin are treacherous plots,
They are incompatible.
I wish to give my humble reflections,
But fear that I will miss the correct principle;
I dare to let the doubtful points stand,
Waiting
For one who can expound them.

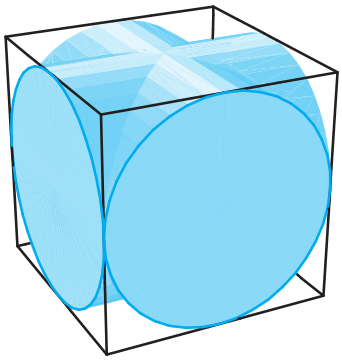

Figure 8.111

2. Length of a Hanging Cable

The distance between the towers of the main span of the Golden Gate Bridge is about 1280 m; the sag of the cable halfway between the towers on a cold winter day is about 143 m. See Figure 8.112.

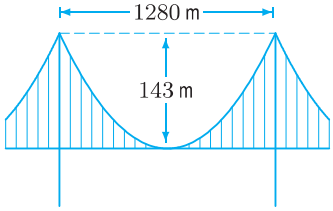

Figure 8.112

(a) How long is the cable, assuming it has an approximately parabolic shape? (Represent the cable as a parabola of the form $y = kx^2$ and determine k to at least 1 decimal place.)

(b) On a hot summer day the cable is about 0.05% longer, due to thermal expansion. By how much does the sag increase? Assume no movement of the towers.

3. Surface Area of an Unpaintable Can of Paint

This project introduces the formula for the surface area of a volume of revolution and shows that it is possible to have a solid with finite volume but infinite surface area.

We know that the arclength of the curve $y = f(x)$ from a to b can be found using the integral

$$\text{Arc length } = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Figure 8.113 shows the corresponding arc length of a small piece of the curve. Similarly, it can be shown that the surface area from a to b of the solid obtained by revolving $y = f(x)$ around the x-axis is given by

$$\text{Surface area } = 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

We can see this why this might be true by looking at Figure 8.114. We approximate the small piece of the surface by a slanted cylinder of radius y and "height" equal to the arclength of the curve, so that

$$\text{Surface area of edge of slice } \approx 2\pi y\sqrt{1 + (f'(x))^2}\, \Delta x.$$

Integrating this expression from $x = a$ to $x = b$ gives the surface area of the solid.

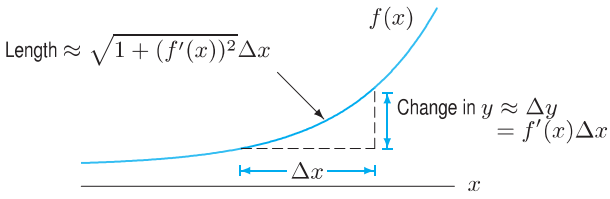

Figure 8.113

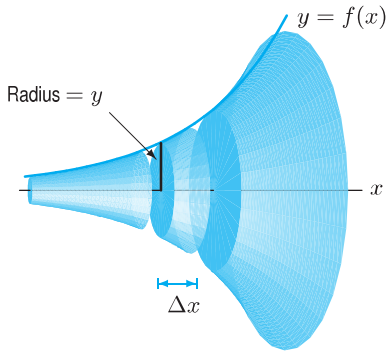

Figure 8.114

(a) Calculate the surface area of a sphere of radius r.
(b) Calculate the surface area of a cone of radius r and height h.
(c) Rotate the curve $y = 1/x$ for $x \geq 1$ around the x-axis. Find the volume of this solid.
(d) Show that the surface area of the solid in part (c) is infinite. [Hint: You might not be able to find an antiderivative of the integrand in the surface area formula; can you get a lower bound on the integral?]
(e) (Optional. Requires Chapter 11.) Find a curve such that when the portion of the curve from $x = a$ to $x = b$ is rotated around the x-axis (for any a and b), the volume of the solid of revolution is equal to its surface area. You may assume $dy/dx \geq 0$.

4. Maxwell's Distribution of Molecular Velocities

Let v be the speed, in meters/second, of an oxygen molecule, and let $p(v)$ be the density function of the speed distribution of oxygen molecules at room temperature. Maxwell showed that

$$p(v) = av^2 e^{-mv^2/(2kT)},$$

where $k = 1.4 \times 10^{-23}$ is the Boltzmann constant, T is the temperature in Kelvin (at room temperature, $T = 293$), and $m = 5 \times 10^{-26}$ is the mass of the oxygen molecule in kilograms.

(a) Find the value of a.
(b) Estimate the median and the mean speed. Find the maximum of $p(v)$.
(c) How do your answers in part (b) for the mean and the maximum of $p(v)$ change as T changes?