Chapter Ten

APPROXIMATING FUNCTIONS USING SERIES

Contents

10.1 TAYLOR POLYNOMIALS

In this section, we see how to approximate a function by polynomials.

Linear Approximations

We already know how to approximate a function using a degree 1 polynomial, namely the tangent line approximation given in Section 3.9 :

$$f(x) \approx f(a) + f'(a)(x-a).$$

The tangent line and the curve have the same slope at $x = a$. As Figure 10.1 suggests, the tangent line approximation to the function is generally more accurate for values of x close to a.

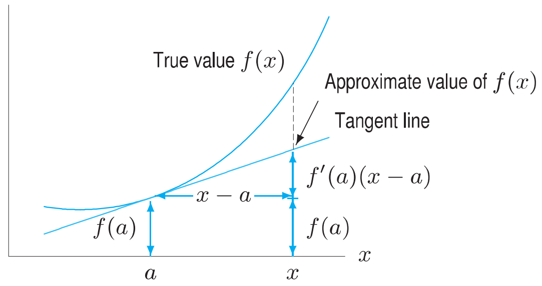


Figure 10.1: Tangent line approximation of $f(x)$ for x near a

We first focus on $a = 0$. The tangent line approximation at $x = 0$ is referred to as the *first Taylor approximation* at $x = 0$, or as follows:

Taylor Polynomial of Degree 1 Approximating $f(x)$ for x near 0

$$f(x) \approx P_1(x) = f(0) + f'(0)x$$

Example 1 Find the Taylor Polynomial of degree 1 for $g(x) = \cos x$, with x in radians, for x near 0.

Solution The tangent line at $x = 0$ is just the horizontal line $y = 1$, as shown in Figure 10.2, so

$$g(x) = \cos x \approx 1, \quad \text{for } x \text{ near } 0.$$

If we take $x = 0.05$, then

$$g(0.05) = \cos(0.05) = 0.998\ldots,$$

which is quite close to the approximation $\cos x \approx 1$. Similarly, if $x = -0.1$, then

$$g(-0.1) = \cos(-0.1) = 0.995\ldots$$

is close to the approximation $\cos x \approx 1$. However, if $x = 0.4$, then

$$g(0.4) = \cos(0.4) = 0.921\ldots,$$

so the approximation $\cos x \approx 1$ is less accurate. The graph suggests that the farther a point x is away from 0, the worse the approximation is likely to be.

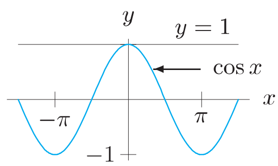


Figure 10.2: Graph of $\cos x$ and its tangent line at $x = 0$

The previous example shows that the Taylor polynomial of degree 1 might actually have degree less than 1.

Quadratic Approximations

To get a more accurate approximation, we use a quadratic function instead of a linear function.

Example 2 Find the quadratic approximation to $g(x) = \cos x$ for x near 0.

Solution To ensure that the quadratic, $P_2(x)$, is a good approximation to $g(x) = \cos x$ at $x = 0$, we require that $\cos x$ and the quadratic have the same value, the same slope, and the same second derivative at $x = 0$. That is, we require $P_2(0) = g(0)$, $P_2'(0) = g'(0)$, and $P_2''(0) = g''(0)$. We take the quadratic polynomial

$$P_2(x) = C_0 + C_1 x + C_2 x^2,$$

and determine C_0, C_1, and C_2. Since

$$\begin{aligned} P_2(x) &= C_0 + C_1 x + C_2 x^2 & \text{and} \qquad g(x) &= \cos x \\ P_2'(x) &= C_1 + 2C_2 x & g'(x) &= -\sin x \\ P_2''(x) &= 2C_2 & g''(x) &= -\cos x, \end{aligned}$$

we have

$$\begin{aligned} C_0 &= P_2(0) = g(0) = \cos 0 = 1 & \text{so} \qquad C_0 &= 1 \\ C_1 &= P_2'(0) = g'(0) = -\sin 0 = 0 & C_1 &= 0 \\ 2C_2 &= P_2''(0) = g''(0) = -\cos 0 = -1, & C_2 &= -\tfrac{1}{2}. \end{aligned}$$

Consequently, the quadratic approximation is

$$\cos x \approx P_2(x) = 1 + 0 \cdot x - \frac{1}{2}x^2 = 1 - \frac{x^2}{2}, \quad \text{for } x \text{ near } 0.$$

Figure 10.3 suggests that the quadratic approximation $\cos x \approx P_2(x)$ is better than the linear approximation $\cos x \approx P_1(x)$ for x near 0. Let's compare the accuracy of the two approximations. Recall that $P_1(x) = 1$ for all x. At $x = 0.4$, we have $\cos(0.4) = 0.921\ldots$ and $P_2(0.4) = 0.920$, so the quadratic approximation is a significant improvement over the linear approximation. The magnitude of the error is about 0.001 instead of 0.08.

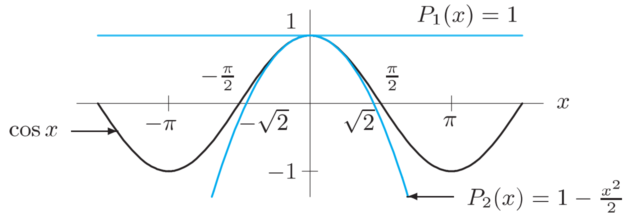


Figure 10.3: Graph of $\cos x$ and its linear, $P_1(x)$, and quadratic, $P_2(x)$, approximations for x near 0

Generalizing the computations in Example 2, we define the *second Taylor approximation* at $x = 0$.

Taylor Polynomial of Degree 2 Approximating $f(x)$ for x near 0

$$f(x) \approx P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

Higher-Degree Polynomials

In a small interval around $x = 0$, the quadratic approximation to a function is usually a better approximation than the linear (tangent line) approximation. However, Figure 10.3 shows that the quadratic can still bend away from the original function for large x. We can attempt to fix this by using an approximating polynomial of higher degree. Suppose that we approximate a function $f(x)$ for x near 0 by a polynomial of degree n:

$$f(x) \approx P_n(x) = C_0 + C_1x + C_2x^2 + \cdots + C_{n-1}x^{n-1} + C_nx^n.$$

We need to find the values of the constants: $C_0,\ C_1,\ C_2, \dots,\ C_n$. To do this, we require that the function $f(x)$ and each of its first n derivatives agree with those of the polynomial $P_n(x)$ at the point $x = 0$. In general, the more derivatives there are that agree at $x = 0$, the larger the interval on which the function and the polynomial remain close to each other.

To see how to find the constants, let's take $n = 3$ as an example:

$$f(x) \approx P_3(x) = C_0 + C_1x + C_2x^2 + C_3x^3.$$

Substituting $x = 0$ gives

$$f(0) = P_3(0) = C_0.$$

Differentiating $P_3(x)$ yields

$$P_3'(x) = C_1 + 2C_2x + 3C_3x^2,$$

so substituting $x = 0$ shows that

$$f'(0) = P_3'(0) = C_1.$$

Differentiating and substituting again, we get

$$P_3''(x) = 2 \cdot 1C_2 + 3 \cdot 2 \cdot 1C_3x,$$

which gives

$$f''(0) = P_3''(0) = 2 \cdot 1C_2,$$

so that

$$C_2 = \frac{f''(0)}{2 \cdot 1}.$$

The third derivative, denoted by P_3''', is

$$P_3'''(x) = 3 \cdot 2 \cdot 1C_3,$$

so

$$f'''(0) = P_3'''(0) = 3 \cdot 2 \cdot 1C_3,$$

and then

$$C_3 = \frac{f'''(0)}{3 \cdot 2 \cdot 1}.$$

You can imagine a similar calculation starting with $P_4(x)$, using the fourth derivative $f^{(4)}$, which would give

$$C_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2 \cdot 1},$$

and so on. Using factorial notation,[1] we write these expressions as

$$C_3 = \frac{f'''(0)}{3!}, \quad C_4 = \frac{f^{(4)}(0)}{4!}.$$

Writing $f^{(n)}$ for the n^{th} derivative of f, we have, for any positive integer n

$$C_n = \frac{f^{(n)}(0)}{n!}.$$

So we define the n^{th} *Taylor approximation* at $x = 0$:

[1]Recall that $k! = k(k-1)\cdots 2 \cdot 1$. In addition, $1! = 1$, and $0! = 1$.

Taylor Polynomial of Degree n Approximating $f(x)$ for x near 0

$$\begin{aligned}f(x) &\approx P_n(x)\\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots + \frac{f^{(n)}(0)}{n!}x^n\end{aligned}$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = 0$, or the Taylor polynomial about (or around) $x = 0$.

Example 3 Construct the Taylor polynomial of degree 7 approximating the function $f(x) = \sin x$ for x near 0. Compare the value of the Taylor approximation with the true value of f at $x = \pi/3$.

Solution We have

$$\begin{array}{rlcrl}
f(x) = & \sin x & \text{giving} & f(0) = & 0\\
f'(x) = & \cos x & & f'(0) = & 1\\
f''(x) = & -\sin x & & f''(0) = & 0\\
f'''(x) = & -\cos x & & f'''(0) = & -1\\
f^{(4)}(x) = & \sin x & & f^{(4)}(0) = & 0\\
f^{(5)}(x) = & \cos x & & f^{(5)}(0) = & 1\\
f^{(6)}(x) = & -\sin x & & f^{(6)}(0) = & 0\\
f^{(7)}(x) = & -\cos x, & & f^{(7)}(0) = & -1.
\end{array}$$

Using these values, we see that the Taylor polynomial approximation of degree 7 is

$$\begin{aligned}\sin x \approx P_7(x) &= 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} - 1\cdot\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + 1\cdot\frac{x^5}{5!} + 0\cdot\frac{x^6}{6!} - 1\cdot\frac{x^7}{7!}\\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad \text{for } x \text{ near } 0.\end{aligned}$$

Notice that since $f^{(8)}(0) = 0$, the seventh and eighth Taylor approximations to $\sin x$ are the same.

In Figure 10.4 we show the graphs of the sine function and the approximating polynomial of degree 7 for x near 0. They are indistinguishable where x is close to 0. However, as we look at values of x farther away from 0 in either direction, the two graphs move apart. To check the accuracy of this approximation numerically, we see how well it approximates $\sin(\pi/3) = \sqrt{3}/2 = 0.8660254\ldots$.

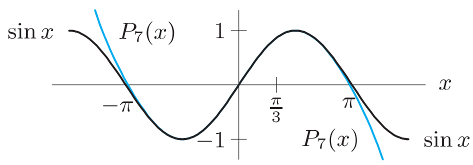


Figure 10.4: Graph of $\sin x$ and its seventh degree Taylor polynomial, $P_7(x)$, for x near 0

When we substitute $\pi/3 = 1.0471976\ldots$ into the polynomial approximation, we obtain $P_7(\pi/3) = 0.8660213\ldots$, which is extremely accurate—to about four parts in a million.

Example 4 Graph the Taylor polynomial of degree 8 approximating $g(x) = \cos x$ for x near 0.

Solution We find the coefficients of the Taylor polynomial by the method of the preceding example, giving

$$\cos x \approx P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.$$

Figure 10.5 shows that $P_8(x)$ is close to the cosine function for a larger interval of x-values than the quadratic approximation $P_2(x) = 1 - x^2/2$ in Example 2 on page 507.

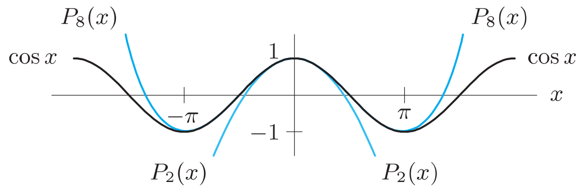


Figure 10.5: $P_8(x)$ approximates $\cos x$ better than $P_2(x)$ for x near 0

Example 5 Construct the Taylor polynomial of degree 10 about $x = 0$ for the function $f(x) = e^x$.

Solution We have $f(0) = 1$. Since the derivative of e^x is equal to e^x, all the higher-order derivatives are equal to e^x. Consequently, for any $k = 1, 2, \ldots, 10$, $f^{(k)}(x) = e^x$ and $f^{(k)}(0) = e^0 = 1$. Therefore, the Taylor polynomial approximation of degree 10 is given by

$$e^x \approx P_{10}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{10}}{10!}, \quad \text{for } x \text{ near } 0.$$

To check the accuracy of this approximation, we use it to approximate $e = e^1 = 2.718281828\ldots$. Substituting $x = 1$ gives $P_{10}(1) = 2.718281801$. Thus, P_{10} yields the first seven decimal places for e. For large values of x, however, the accuracy diminishes because e^x grows faster than any polynomial as $x \to \infty$. Figure 10.6 shows graphs of $f(x) = e^x$ and the Taylor polynomials of degree $n = 0, 1, 2, 3, 4$. Notice that each successive approximation remains close to the exponential curve for a larger interval of x-values.

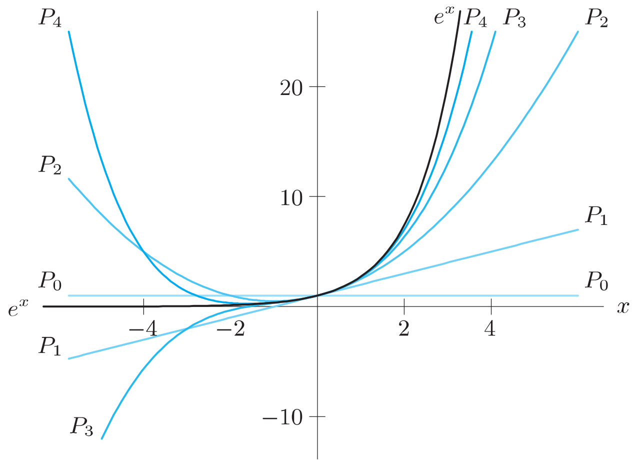


Figure 10.6: For x near 0, the value of e^x is more closely approximated by higher-degree Taylor polynomials

Example 6 Construct the Taylor polynomial of degree n approximating $f(x) = \dfrac{1}{1-x}$ for x near 0.

Solution Differentiating gives $f(0) = 1$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 3!$, $f^{(4)}(0) = 4!$, and so on. This means

$$\frac{1}{1-x} \approx P_n(x) = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n, \quad \text{for } x \text{ near } 0,$$

Let us compare the Taylor polynomial with the formula obtained from the sum of a finite geometric series on page 472:

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

If x is close to 0 and x^{n+1} is small enough to neglect, the formula for the sum of a finite geometric series gives us the Taylor approximation of degree n:

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 + \cdots + x^n.$$

Taylor Polynomials Around $x = a$

Suppose we want to approximate $f(x) = \ln x$ by a Taylor polynomial. This function has no Taylor polynomial about $x = 0$ because the function is not defined for $x \leq 0$. However, it turns out that we can construct a polynomial centered about some other point, $x = a$.

First, let's look at the equation of the tangent line at $x = a$:

$$y = f(a) + f'(a)(x-a).$$

This gives the first Taylor approximation

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ near } a.$$

The $f'(a)(x-a)$ term is a correction term which approximates the change in f as x moves away from a.

Similarly, the Taylor polynomial $P_n(x)$ centered at $x = a$ is set up as $f(a)$ plus correction terms which are zero for $x = a$. This is achieved by writing the polynomial in powers of $(x-a)$ instead of powers of x:

$$f(x) \approx P_n(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + C_n(x-a)^n.$$

If we require n derivatives of the approximating polynomial $P_n(x)$ and the original function $f(x)$ to agree at $x = a$, we get the following result for the n^{th} Taylor approximations at $x = a$:

Taylor Polynomial of Degree n Approximating $f(x)$ for x near a

$$\begin{aligned} f(x) &\approx P_n(x) \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

We call $P_n(x)$ the Taylor polynomial of degree n centered at $x = a$, or the Taylor polynomial about $x = a$.

You can derive the formula for these coefficients in the same way that we did for $a = 0$. (See Problem 30, page 513.)

Example 7 Construct the Taylor polynomial of degree 4 approximating the function $f(x) = \ln x$ for x near 1.

Solution We have

$$\begin{aligned} f(x) &= \ln x & \text{so} \quad f(1) &= \ln(1) = 0 \\ f'(x) &= 1/x & f'(1) &= 1 \\ f''(x) &= -1/x^2 & f''(1) &= -1 \\ f'''(x) &= 2/x^3 & f'''(1) &= 2 \\ f^{(4)}(x) &= -6/x^4, & f^{(4)}(1) &= -6. \end{aligned}$$

The Taylor polynomial is therefore

$$\begin{aligned} \ln x \approx P_4(x) &= 0 + (x-1) - \frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!} \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}, \qquad \text{for } x \text{ near } 1. \end{aligned}$$

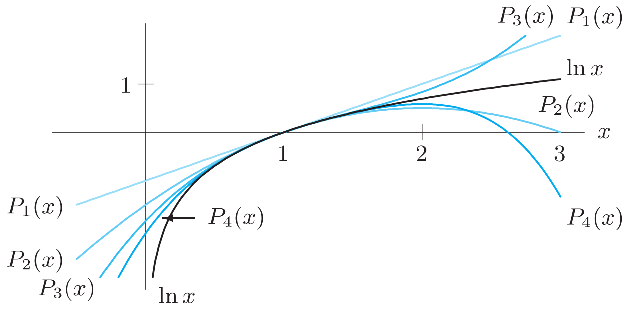


Figure 10.7: Taylor polynomials approximate $\ln x$ closely for x near 1, but not necessarily farther away

Graphs of $\ln x$ and several of its Taylor polynomials are shown in Figure 10.7. Notice that $P_4(x)$ stays reasonably close to $\ln x$ for x near 1, but bends away as x gets farther from 1. Also, note that the Taylor polynomials are defined for $x \le 0$, but $\ln x$ is not.

The examples in this section suggest that the following results are true for common functions:

- Taylor polynomials centered at $x = a$ give good approximations to $f(x)$ for x near a. Farther away, they may or may not be good.
- The higher the degree of the Taylor polynomial, the larger the interval over which it fits the function closely.

Exercises and Problems for Section 10.1

Exercises

For Exercises 1–10, find the Taylor polynomials of degree n approximating the functions for x near 0. (Assume p is a constant.)

1. $\dfrac{1}{1-x}$, $\quad n = 3, 5, 7$

2. $\dfrac{1}{1+x}$, $\quad n = 4, 6, 8$

3. $\sqrt{1+x}$, $\quad n = 2, 3, 4$

4. $\sqrt[3]{1-x}$, $\quad n = 2, 3, 4$

5. $\cos x$, $\quad n = 2, 4, 6$

6. $\ln(1+x)$, $\quad n = 5, 7, 9$

7. $\arctan x$, $\quad n = 3, 4$

8. $\tan x$, $\quad n = 3, 4$

9. $\dfrac{1}{\sqrt{1+x}}$, $\quad n = 2, 3, 4$

10. $(1+x)^p$, $\quad n = 2, 3, 4$

For Exercises 11–16, find the Taylor polynomial of degree n for x near the given point a.

11. $\sqrt{1-x}$, $\quad a = 0$, $\quad n = 3$

12. e^x, $\quad a = 1$, $\quad n = 4$

13. $\sqrt{1+x}$, $\quad a = 1$, $\quad n = 3$

14. $\cos x$, $\quad a = \pi/2$, $\quad n = 4$

15. $\sin x$, $\quad a = -\pi/4$, $\quad n = 3$

16. $\ln(x^2)$, $\quad a = 1$, $\quad n = 4$

Problems

17. The function $f(x)$ is approximated near $x = 0$ by the third degree Taylor polynomial

$$P_3(x) = 2 - x - x^2/3 + 2x^3.$$

Give the value of

(a) $f(0)$ **(b)** $f'(0)$
(c) $f''(0)$ **(d)** $f'''(0)$

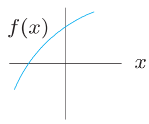
18. Find the second-degree Taylor polynomial for $f(x) = 4x^2 - 7x + 2$ about $x = 0$. What do you notice?

19. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ about $x = 0$. What do you notice?

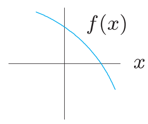
20. **(a)** Based on your observations in Problems 18–19, make a conjecture about Taylor approximations in the case when f is itself a polynomial.
(b) Show that your conjecture is true.

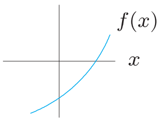
For Problems 21–24, suppose $P_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function f about $x = 0$. What can you say about the signs of a, b, c if f has the graph given below?

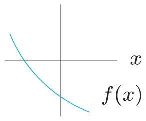
21.



22.



23. 

24. 

25. Show how you can use the Taylor approximation $\sin x \approx x - \frac{x^3}{3!}$, for x near 0, to explain why $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

26. Use the fourth-degree Taylor approximation $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for x near 0 to explain why $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

27. Use a fourth degree Taylor approximation for e^h, for h near 0, to evaluate the following limits. Would your answer be different if you used a Taylor polynomial of higher degree?

(a) $\lim_{h\to 0} \frac{e^h - 1 - h}{h^2}$

(b) $\lim_{h\to 0} \frac{e^h - 1 - h - \frac{h^2}{2}}{h^3}$

28. If $f(2) = g(2) = h(2) = 0$, and $f'(2) = h'(2) = 0$, $g'(2) = 22$, and $f''(2) = 3$, $g''(2) = 5$, $h''(2) = 7$, calculate the following limits. Explain your reasoning.

(a) $\lim_{x\to 2} \frac{f(x)}{h(x)}$ **(b)** $\lim_{x\to 2} \frac{f(x)}{g(x)}$

29. One of the two sets of functions, f_1, f_2, f_3, or g_1, g_2, g_3, is graphed in Figure 10.8; the other set is graphed in Figure 10.9. Points A and B each have $x = 0$. Taylor polynomials of degree 2 approximating these functions near $x = 0$ are as follows:

$$f_1(x) \approx 2 + x + 2x^2 \qquad g_1(x) \approx 1 + x + 2x^2$$
$$f_2(x) \approx 2 + x - x^2 \qquad g_2(x) \approx 1 + x + x^2$$
$$f_3(x) \approx 2 + x + x^2 \qquad g_3(x) \approx 1 - x + x^2.$$

(a) Which group of functions, the fs or the gs, is represented by each figure?
(b) What are the coordinates of the points A and B?
(c) Match each function with the graphs (I)–(III) in the appropriate figure.

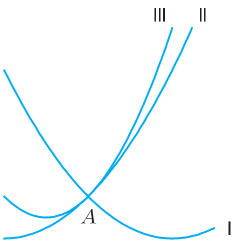


Figure 10.8

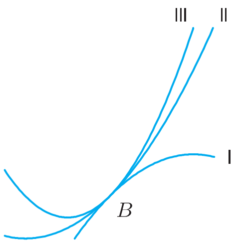


Figure 10.9

30. Derive the formulas given in the box on page 511 for the coefficients of the Taylor polynomial approximating a function f for x near a.

31. (a) Find the Taylor polynomial approximation of degree 4 about $x = 0$ for the function $f(x) = e^{x^2}$.
(b) Compare this result to the Taylor polynomial approximation of degree 2 for the function $f(x) = e^x$ about $x = 0$. What do you notice?
(c) Use your observation in part (b) to write out the Taylor polynomial approximation of degree 20 for the function in part (a).
(d) What is the Taylor polynomial approximation of degree 5 for the function $f(x) = e^{-2x}$?

32. The integral $\int_0^1 (\sin t/t)\, dt$ is difficult to approximate using, for example, left Riemann sums or the trapezoid rule because the integrand $(\sin t)/t$ is not defined at $t = 0$. However, this integral converges; its value is $0.94608\ldots$. Estimate the integral using Taylor polynomials for $\sin t$ about $t = 0$ of

(a) Degree 3 **(b)** Degree 5

33. Consider the equations $\sin x = 0.2$ and $x - \frac{x^3}{3!} = 0.2$

(a) How many solutions does each equation have?
(b) Which of the solutions of the two equations are approximately equal? Explain.

34. When we model the motion of a pendulum, we replace the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \quad \text{by} \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta,$$

where θ is the angle between the pendulum and the vertical. Explain why, and under what circumstances, it is reasonable to make this replacement.

35. (a) Using a graph, explain why the following equation has a solution at $x = 0$ and another just to the right of $x = 0$:

$$\cos x = 1 - 0.1x$$

(b) Replace $\cos x$ by its second-degree Taylor polynomial near 0 and solve the equation. Your answers are approximations to the solutions to the original equation at or near 0.

10.2 TAYLOR SERIES

In the previous section we saw how to approximate a function near a point by Taylor polynomials. Now we define a Taylor series, which is a power series that can be thought of as a Taylor polynomial that goes on forever.

Taylor Series for cos x, sin x, e^x

We have the following Taylor polynomials centered at $x = 0$ for $\cos x$:

$$\begin{aligned}
\cos x &\approx P_0(x) = 1 \\
\cos x &\approx P_2(x) = 1 - \frac{x^2}{2!} \\
\cos x &\approx P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \\
\cos x &\approx P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \\
\cos x &\approx P_8(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}.
\end{aligned}$$

Here we have a sequence of polynomials, $P_0(x)$, $P_2(x)$, $P_4(x)$, $P_6(x)$, $P_8(x)$, ..., each of which is a better approximation to $\cos x$ than the last, for x near 0. When we go to a higher-degree polynomial (say from P_6 to P_8), we add more terms ($x^8/8!$, for example), but the terms of lower degree don't change. Thus, each polynomial includes the information from all the previous ones. We represent the whole sequence of Taylor polynomials by writing the *Taylor series* for $\cos x$:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots.$$

Notice that the partial sums of this series are the Taylor polynomials, $P_n(x)$.

We define the Taylor series for $\sin x$ and e^x similarly. It turns out that, for these functions, the Taylor series converges to the function for all x, so we can write the following:

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\end{aligned}$$

These series are also called *Taylor expansions* of the functions $\sin x$, $\cos x$, and e^x about $x = 0$. The *general term* of a Taylor series is a formula which gives any term in the series. For example, $x^n/n!$ is the general term in the Taylor expansion for e^x, and $(-1)^k x^{2k}/(2k)!$ is the general term in the expansion for $\cos x$. We call n or k the *index*.

Taylor Series in General

Any function f, all of whose derivatives exist at 0, has a Taylor series. However, the Taylor series for f does not necessarily converge to $f(x)$ for all values of x. For the values of x for which the series does converge to $f(x)$, we have the following formula:

Taylor Series for $f(x)$ about $x = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

In addition, just as we have Taylor polynomials centered at points other than 0, we can also have a Taylor series centered at $x = a$ (provided all the derivatives of f exist at $x = a$). For the values of x for which the series converges to $f(x)$, we have the following formula:

Taylor Series for $f(x)$ about $x = a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series is a power series whose partial sums are the Taylor polynomials. As we saw in Section 9.5, power series generally converge on an interval centered at $x = a$. The Taylor series for such a function can be interpreted when x is replaced by a complex number. This extends the domain of the function. See Problem 33.

For a given function f and a given x, even if the Taylor series converges, it might not converge to $f(x)$. However,the Taylor series for most commonly encountered functions, including e^x, $\cos x$, and $\sin x$, do converge to the original function for all x. See Section 10.4.

Intervals of Convergence of Taylor Series

Let us look again at the Taylor polynomial for $\ln x$ about $x = 1$ that we derived in Example 7 on page 511. A similar calculation gives the Taylor Series

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots.$$

Example 4 on page 493 and Example 6 on page 495 show that this power series has interval of convergence $0 < x \leq 2$. However, although we know that the series converges in this interval, we do not yet know that its sum is $\ln x$. The fact that in Figure 10.10 the polynomials fit the curve well for $0 < x < 2$ suggests that the Taylor series does converge to $\ln x$ for $0 < x \leq 2$. For such x-values, a higher-degree polynomial gives, in general, a better approximation.

However, when $x > 2$, the polynomials move away from the curve and the approximations get worse as the degree of the polynomial increases. Thus, the Taylor polynomials are effective only as approximations to $\ln x$ for values of x between 0 and 2; outside that interval, they should not be used. Inside the interval, but near the ends, 0 or 2, the polynomials converge very slowly. This means we might have to take a polynomial of very high degree to get an accurate value for $\ln x$.

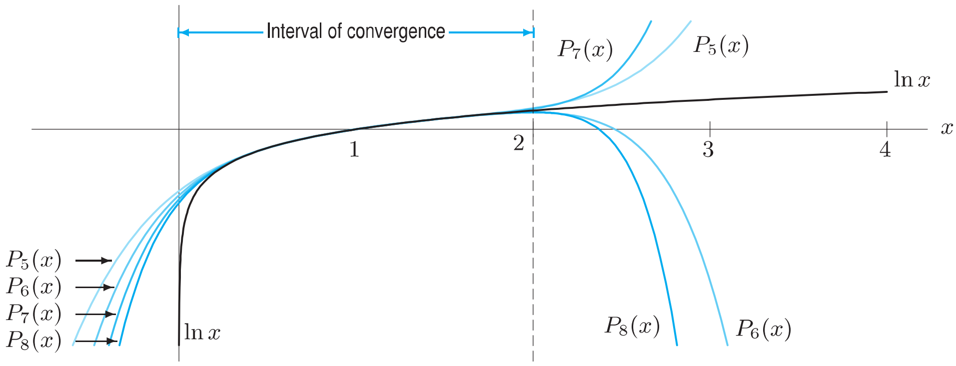


Figure 10.10: Taylor polynomials $P_5(x), P_6(x), P_7(x), P_8(x), \ldots$ converge to $\ln x$ for $0 < x \leq 2$ and diverge outside that interval

To compute the interval of convergence exactly, we first compute the radius of convergence using the method on page 493. Convergence at the endpoints, $x = 0$ and $x = 2$, has to be determined separately. However, proving that the series converges to $\ln x$ on its interval of convergence, as Figure 10.10 suggests, requires the error term introduced in Section 10.4.

Example 1 Find the Taylor series for $\ln(1+x)$ about $x=0$, and calculate its interval of convergence.

Solution Taking derivatives of $\ln(1+x)$ and substituting $x=0$ leads to the Taylor series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

Notice that this is the same series that we get by substituting $(1+x)$ for x in the series for $\ln x$:

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad \text{for } 0 < x \le 2.$$

Since the series for $\ln x$ about $x=1$ converges for $0 < x \le 2$, the interval of convergence for the Taylor series for $\ln(1+x)$ about $x=0$ is $-1 < x \le 1$. Thus we write

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \text{for } -1 < x \le 1.$$

Notice that the series could not possibly converge to $\ln(1+x)$ for $x \le -1$ since $\ln(1+x)$ is not defined there.

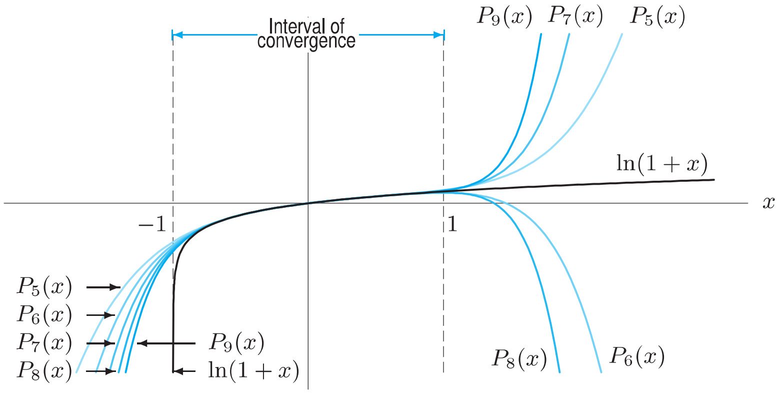


Figure 10.11: Interval of convergence for the Taylor series for $\ln(1+x)$ is $-1 < x \le 1$

The Binomial Series Expansion

We now find the Taylor series about $x=0$ for the function $f(x) = (1+x)^p$, with p a constant, but not necessarily a positive integer. Taking derivatives:

$$\begin{aligned}
f(x) &= (1+x)^p & \text{so} \quad f(0) &= 1 \\
f'(x) &= p(1+x)^{p-1} & f'(0) &= p \\
f''(x) &= p(p-1)(1+x)^{p-2} & f''(0) &= p(p-1) \\
f'''(x) &= p(p-1)(p-2)(1+x)^{p-3}, & f'''(0) &= p(p-1)(p-2).
\end{aligned}$$

Thus, the third-degree Taylor polynomial for x near 0 is

$$(1+x)^p \approx P_3(x) = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3.$$

Graphing $P_3(x), P_4(x), \ldots$ for various specific values of p suggests that the Taylor polynomials converge to $f(x)$ for $-1 < x < 1$. (See Problems 26–27, page 518.) This can be confirmed using the radius of convergence test. The Taylor series for $f(x) = (1+x)^p$ about $x=0$ is as follows:

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad \text{for } -1 < x < 1.$$

In fact the binomial series gives the same result as multiplying $(1+x)^p$ out when p is a positive integer. (Newton discovered that the binomial series can be used for noninteger exponents.)

Example 2 Use the binomial series with $p = 3$ to expand $(1+x)^3$.

Solution The series is

$$(1+x)^3 = 1 + 3x + \frac{3\cdot 2}{2!}x^2 + \frac{3\cdot 2\cdot 1}{3!}x^3 + \frac{3\cdot 2\cdot 1\cdot 0}{4!}x^4 + \cdots.$$

The x^4 term and all terms beyond it turn out to be zero, because each coefficient contains a factor of 0. Simplifying gives

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3,$$

which is the usual expansion obtained by multiplying out $(1+x)^3$.

Example 3 Find the Taylor series about $x = 0$ for $\dfrac{1}{1+x}$.

Solution Since $\dfrac{1}{1+x} = (1+x)^{-1}$, use the binomial series with $p = -1$. Then

$$\begin{aligned}\frac{1}{1+x} = (1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \quad \text{for } -1 < x < 1.\end{aligned}$$

This series is both a special case of the binomial series and an example of a geometric series. It converges for $-1 < x < 1$.

Exercises and Problems for Section 10.2

Exercises

For Exercises 1–8, find the first four nonzero terms of the Taylor series for the function about 0.

1. $(1+x)^{3/2}$

2. $\sqrt[4]{x+1}$

3. $\sin(-x)$

4. $\ln(1-x)$

5. $\dfrac{1}{1-x}$

6. $\dfrac{1}{\sqrt{1+x}}$

7. $\sqrt{1+x}$

8. $\sqrt[3]{1-y}$

For Exercises 9–16, find the first four terms of the Taylor series for the function about the point a.

9. $\sin x,\quad a = \pi/4$

10. $\cos\theta,\quad a = \pi/4$

11. $\cos t,\quad a = \pi/6$

12. $\sin\theta,\quad a = -\pi/4$

13. $\tan x,\quad a = \pi/4$

14. $1/x,\quad a = 1$

15. $1/x,\quad a = 2$

16. $1/x,\quad a = -1$

Find an expression for the general term of the series in Exercises 17–24 and give the starting value of the index (n or k, for example).

17. $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

18. $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

19. $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$

20. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \cdots$

21. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

22. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$

23. $e^{x^2} = 1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

24. $x^2\cos x^2 = x^2 - \dfrac{x^6}{2!} + \dfrac{x^{10}}{4!} - \dfrac{x^{14}}{6!} + \cdots$

Problems

25. By graphing the function $f(x) = \dfrac{1}{\sqrt{1+x}}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 6.

26. By graphing the function $f(x) = \sqrt{1+x}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 7.

27. By graphing the function $f(x) = \dfrac{1}{1-x}$ and several of its Taylor polynomials, estimate the interval of convergence of the series you found in Problem 5. Compute the radius of convergence analytically.

28. Find the radius of convergence of the Taylor series around $x = 0$ for $\ln(1-x)$.

29. **(a)** Write the general term of the binomial series for $(1+x)^p$ about $x = 0$.

(b) Find the radius of convergence of this series.

30. Use the fact that the Taylor series of $g(x) = \sin(x^2)$ is

$$x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

to find $g''(0)$, $g'''(0)$, and $g^{(10)}(0)$. (There is an easy way and a hard way to do this!)

31. The Taylor series of $f(x) = x^2 e^{x^2}$ about $x = 0$ is

$$x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots.$$

Find $\left.\dfrac{d}{dx}\left(x^2 e^{x^2}\right)\right|_{x=0}$ and $\left.\dfrac{d^6}{dx^6}\left(x^2 e^{x^2}\right)\right|_{x=0}$.

32. One of the two sets of functions, f_1, f_2, f_3, or g_1, g_2, g_3 is graphed in Figure 10.12; the other set is graphed in Figure 10.13. Taylor series for the functions about a point corresponding to either A or B are as follows:

$$\begin{aligned}
f_1(x) &= 3 + (x-1) - (x-1)^2 + \cdots \\
f_2(x) &= 3 - (x-1) + (x-1)^2 + \cdots \\
f_3(x) &= 3 - 2(x-1) + (x-1)^2 + \cdots \\
g_1(x) &= 5 - (x-4) - (x-4)^2 + \cdots \\
g_2(x) &= 5 - (x-4) + (x-4)^2 + \cdots \\
g_3(x) &= 5 + (x-4) + (x-4)^2 + \cdots.
\end{aligned}$$

(a) Which group of functions is represented in each figure?

(b) What are the coordinates of the points A and B?

(c) Match each function with the graphs (I)–(III) in the appropriate figure.

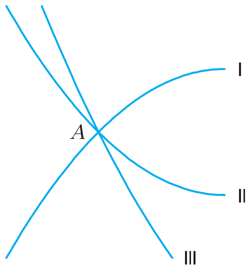


Figure 10.12

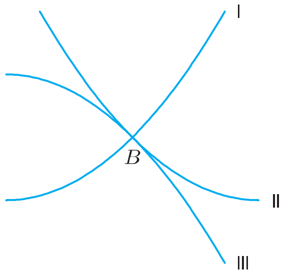


Figure 10.13

33. Let $i = \sqrt{-1}$. We define $e^{i\theta}$ by substituting $i\theta$ in the Taylor series for e^x. Use this definition[2] to explain Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

By recognizing each series in Problems 34–42 as a Taylor series evaluated at a particular value of x, find the sum of each of the following convergent series.

34. $1 + \dfrac{2}{1!} + \dfrac{4}{2!} + \dfrac{8}{3!} + \cdots + \dfrac{2^n}{n!} + \cdots$

35. $1 - \dfrac{1}{3!} + \dfrac{1}{5!} - \dfrac{1}{7!} + \cdots + \dfrac{(-1)^n}{(2n+1)!} + \cdots$

36. $1 + \dfrac{1}{4} + \left(\dfrac{1}{4}\right)^2 + \left(\dfrac{1}{4}\right)^3 + \cdots + \left(\dfrac{1}{4}\right)^n + \cdots$

37. $1 - \dfrac{100}{2!} + \dfrac{10000}{4!} + \cdots + \dfrac{(-1)^n \cdot 10^{2n}}{(2n)!} + \cdots$

38. $\dfrac{1}{2} - \dfrac{(\frac{1}{2})^2}{2} + \dfrac{(\frac{1}{2})^3}{3} - \dfrac{(\frac{1}{2})^4}{4} + \cdots + \dfrac{(-1)^n \cdot (\frac{1}{2})^{n+1}}{(n+1)} + \cdots$

39. $1 - 0.1 + 0.1^2 - 0.1^3 + \cdots$

40. $1 + 3 + \dfrac{9}{2!} + \dfrac{27}{3!} + \dfrac{81}{4!} + \cdots$

41. $1 - \dfrac{1}{2!} + \dfrac{1}{4!} - \dfrac{1}{6!} + \cdots$

42. $1 - 0.1 + \dfrac{0.01}{2!} - \dfrac{0.001}{3!} + \cdots$

In Problems 43–44 solve exactly for the variable.

43. $1 + x + x^2 + x^3 + \cdots = 5$

44. $x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \cdots = 0.2$

[2]Complex numbers are discussed in Appendix B.

10.3 FINDING AND USING TAYLOR SERIES

Finding a Taylor series for a function means finding the coefficients. Assuming the function has all its derivatives defined, finding the coefficients can always be done, in theory at least, by differentiation. That is how we derived the four most important Taylor series, those for the functions e^x, $\sin x$, $\cos x$, and $(1+x)^p$.

For many functions, however, computing Taylor series coefficients by differentiation can be a very laborious business. We now introduce easier ways of finding Taylor series, if the series we want is closely related to a series that we already know.

New Series by Substitution

Suppose we want to find the Taylor series for e^{-x^2} about $x = 0$. We could find the coefficients by differentiation. Differentiating e^{-x^2} by the chain rule gives $-2xe^{-x^2}$, and differentiating again gives $-2e^{-x^2} + 4x^2e^{-x^2}$. Each time we differentiate we use the product rule, and the number of terms grows. Finding the tenth or twentieth derivative of e^{-x^2}, and thus the series for e^{-x^2} up to the x^{10} or x^{20} terms, by this method is tiresome (at least without a computer or calculator that can differentiate).

Fortunately, there's a quicker way. Recall that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \quad \text{for all } y.$$

Substituting $y = -x^2$ tells us that

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \quad \text{for all } x. \end{aligned}$$

Using this method, it is easy to find the series up to the x^{10} or x^{20} terms. It can be shown that this is the Taylor series for e^{-x^2}.

Example 1 Find the Taylor series about $x = 0$ for $f(x) = \dfrac{1}{1+x^2}$.

Solution The binomial series tells us that

$$\frac{1}{1+y} = (1+y)^{-1} = 1 - y + y^2 - y^3 + y^4 + \cdots \quad \text{for } -1 < y < 1.$$

Substituting $y = x^2$ gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots \quad \text{for } -1 < x < 1,$$

which is the Taylor series for $\dfrac{1}{1+x^2}$.

These examples demonstrate that we can get new series from old ones by substitution. More advanced texts show that series obtained by substituting $g(x)$ into a Taylor series for $f(x)$ gives a Taylor series for $f(g(x))$.

In Example 1, we made the substitution $y = x^2$. We can also substitute an entire series into another one, as in the next example.

Example 2 Find the Taylor series about $\theta = 0$ for $g(\theta) = e^{\sin\theta}$.

Solution For all y and θ, we know that

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots.$$

Let's substitute the series for $\sin\theta$ for y:

$$e^{\sin\theta} = 1 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) + \frac{1}{2!}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^2 + \frac{1}{3!}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)^3 + \cdots.$$

To simplify, we multiply out and collect terms. The only constant term is the 1, and there's only one θ term. The only θ^2 term is the first term we get by multiplying out the square, namely $\theta^2/2!$. There are two contributors to the θ^3 term: the $-\theta^3/3!$ from within the first parentheses, and the first term we get from multiplying out the cube, which is $\theta^3/3!$. Thus the series starts

$$\begin{aligned} e^{\sin\theta} &= 1 + \theta + \frac{\theta^2}{2!} + \left(-\frac{\theta^3}{3!} + \frac{\theta^3}{3!}\right) + \cdots \\ &= 1 + \theta + \frac{\theta^2}{2!} + 0\cdot\theta^3 + \cdots \quad \text{for all } \theta. \end{aligned}$$

New Series by Differentiation and Integration

Just as we can get new series by substitution, we can also get new series by differentiation and integration. It can be shown that term-by-term differentiation of a Taylor series for $f(x)$ gives a Taylor series for $f'(x)$ and that the two series have the same radius of convergence. Integration works similarly.

Example 3 Find the Taylor Series about $x = 0$ for $\dfrac{1}{(1-x)^2}$ from the series for $\dfrac{1}{1-x}$.

Solution We know that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$, so we start with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{for } -1 < x < 1.$$

Differentiation term by term gives the binomial series

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad \text{for } -1 < x < 1.$$

Example 4 Find the Taylor series about $x = 0$ for $\arctan x$ from the series for $\dfrac{1}{1+x^2}$.

Solution We know that $\dfrac{d(\arctan x)}{dx} = \dfrac{1}{1+x^2}$, so we use the series from Example 1 on page 519:

$$\frac{d(\arctan x)}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \quad \text{for } -1 < x < 1.$$

Antidifferentiating term by term gives

$$\arctan x = \int \frac{1}{1+x^2}\,dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{for } -1 < x < 1,$$

where C is the constant of integration. Since $\arctan 0 = 0$, we have $C = 0$, so

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \quad \text{for} \quad -1 < x < 1.$$

The series for $\arctan x$ was discovered by James Gregory (1638–1675).

Applications of Taylor Series

Example 5 Use the series for $\arctan x$ to estimate the numerical value of π.

Solution Since $\arctan 1 = \pi/4$, we use the series for $\arctan x$ from Example 4. We assume—as is the case—that the series does converge to $\pi/4$ at $x = 1$, the endpoint of its interval of convergence. Substituting $x = 1$ into the series for $\arctan x$ gives

$$\pi = 4 \arctan 1 = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right).$$

Table 10.1 *Approximating π using the series for $\arctan x$*

n	4	5	25	100	500	1000	10,000
S_n	2.895	3.340	3.182	3.132	3.140	3.141	3.141

Table 10.1 shows the value of the n^{th} partial sum, S_n, obtained by summing the nonzero terms from 1 through n. The values of S_n do seem to converge to $\pi = 3.141\ldots$. However, this series converges very slowly, meaning that we have to take a large number of terms to get an accurate estimate for π. So this way of calculating π is not particularly practical. (A better one is given in Problem 2, page 547.) However, the expression for π given by this series is surprising and elegant.

A basic question we can ask about two functions is which one gives larger values. Taylor series can often be used to answer this question over a small interval. If the constant terms of the series for two functions are the same, compare the linear terms; if the linear terms are the same, compare the quadratic terms, and so on.

Example 6 By looking at their Taylor series, decide which of the following functions is largest, and which is smallest, for θ near 0. (a) $1 + \sin\theta$ (b) e^θ (c) $\dfrac{1}{\sqrt{1-2\theta}}$

Solution The Taylor expansion about $\theta = 0$ for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

So

$$1 + \sin\theta = 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

The Taylor expansion about $\theta = 0$ for e^θ is

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots.$$

The Taylor expansion about $\theta = 0$ for $1/\sqrt{1+\theta}$ is

$$\begin{aligned}\frac{1}{\sqrt{1+\theta}} = (1+\theta)^{-1/2} &= 1 - \frac{1}{2}\theta + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\theta^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\theta^3 + \cdots \\ &= 1 - \frac{1}{2}\theta + \frac{3}{8}\theta^2 - \frac{5}{16}\theta^3 + \cdots.\end{aligned}$$

So, substituting -2θ for θ:

$$\begin{aligned}\frac{1}{\sqrt{1-2\theta}} &= 1 - \frac{1}{2}(-2\theta) + \frac{3}{8}(-2\theta)^2 - \frac{5}{16}(-2\theta)^3 + \cdots \\ &= 1 + \theta + \frac{3}{2}\theta^2 + \frac{5}{2}\theta^3 + \cdots.\end{aligned}$$

For θ near 0, we can neglect terms beyond the second degree. We are left with the approximations:

$$\begin{aligned}1 + \sin\theta &\approx 1 + \theta \\ e^\theta &\approx 1 + \theta + \frac{\theta^2}{2} \\ \frac{1}{\sqrt{1-2\theta}} &\approx 1 + \theta + \frac{3}{2}\theta^2.\end{aligned}$$

Since

$$1 + \theta < 1 + \theta + \frac{1}{2}\theta^2 < 1 + \theta + \frac{3}{2}\theta^2,$$

and since the approximations are valid for θ near 0, we conclude that, for θ near 0,

$$1 + \sin\theta < e^\theta < \frac{1}{\sqrt{1-2\theta}}.$$

Example 7 Two electrical charges of equal magnitude and opposite signs located near one another are called an electrical dipole. The charges Q and $-Q$ are a distance r apart. (See Figure 10.14.) The electric field, E, at the point P is given by

$$E = \frac{Q}{R^2} - \frac{Q}{(R+r)^2}.$$

Use series to investigate the behavior of the electric field far away from the dipole. Show that when R is large in comparison to r, the electric field is approximately proportional to $1/R^3$.

Figure 10.14: Approximating the electric field at P due to a dipole consisting of charges Q and $-Q$ a distance r apart

Solution In order to use a series approximation, we need a variable whose value is small. Although we know that r is much smaller than R, we do not know that r itself is small. The quantity r/R is, however, very small. Hence we expand $1/(R+r)^2$ in powers of r/R so that we can safely use only the first few terms of the Taylor series. First we rewrite using algebra:

$$\frac{1}{(R+r)^2} = \frac{1}{R^2(1+r/R)^2} = \frac{1}{R^2}\left(1+\frac{r}{R}\right)^{-2}.$$

Now we use the binomial expansion for $(1+x)^p$ with $x = r/R$ and $p = -2$:

$$\begin{aligned}\frac{1}{R^2}\left(1+\frac{r}{R}\right)^{-2} &= \frac{1}{R^2}\left(1 + (-2)\left(\frac{r}{R}\right) + \frac{(-2)(-3)}{2!}\left(\frac{r}{R}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{r}{R}\right)^3 + \cdots\right) \\ &= \frac{1}{R^2}\left(1 - 2\frac{r}{R} + 3\frac{r^2}{R^2} - 4\frac{r^3}{R^3} + \cdots\right).\end{aligned}$$

So, substituting the series into the expression for E, we have

$$\begin{aligned}E = \frac{Q}{R^2} - \frac{Q}{(R+r)^2} &= Q\left[\frac{1}{R^2} - \frac{1}{R^2}\left(1 - 2\frac{r}{R} + 3\frac{r^2}{R^2} - 4\frac{r^3}{R^3} + \cdots\right)\right] \\ &= \frac{Q}{R^2}\left(2\frac{r}{R} - 3\frac{r^2}{R^2} + 4\frac{r^3}{R^3} - \cdots\right).\end{aligned}$$

Since r/R is smaller than 1, the binomial expansion for $(1+r/R)^{-2}$ converges. We are interested in the electric field far away from the dipole. The quantity r/R is small there, and $(r/R)^2$ and higher powers are smaller still. Thus, we approximate by disregarding all terms except the first, giving

$$E \approx \frac{Q}{R^2}\left(\frac{2r}{R}\right), \quad \text{so} \quad E \approx \frac{2Qr}{R^3}.$$

Since Q and r are constants, this means that E is approximately proportional to $1/R^3$.

In the previous example, we say that E is *expanded in terms of* r/R, meaning that the variable in the expansion is r/R.

Exercises and Problems for Section 10.3

Exercises

Using known Taylor series, find the first four nonzero terms of the Taylor series about 0 for the functions in Exercises 1–12.

1. e^{-x}

2. $\sqrt{1-2x}$

3. $\cos(\theta^2)$

4. $\ln(1-2y)$

5. $\arcsin x$

6. $t\sin(3t)$

7. $\dfrac{1}{\sqrt{1-z^2}}$

8. $\dfrac{z}{e^{z^2}}$

9. $\phi^3\cos(\phi^2)$

10. $\sqrt{1+\sin\theta}$

11. $\sqrt{(1+t)}\sin t$

12. $e^t\cos t$

Find the Taylor series about 0 for the functions in Exercises 13–15, including the general term.

13. $(1+x)^3$

14. $t\sin(t^2)-t^3$

15. $\dfrac{1}{\sqrt{1-y^2}}$

For Exercises 16–21, expand the quantity about 0 in terms of the variable given. Give four nonzero terms.

16. $\dfrac{1}{2+x}$ in terms of $\dfrac{x}{2}$

17. $\sqrt{T+h}$ in terms of $\dfrac{h}{T}$

18. $\dfrac{1}{a-r}$ in terms of $\dfrac{r}{a}$

19. $\dfrac{1}{(a+r)^2}$ in terms of $\dfrac{r}{a}$

20. $\sqrt[3]{P+t}$ in terms of $\dfrac{t}{P}$

21. $\dfrac{a}{\sqrt{a^2+x^2}}$ in terms of $\dfrac{x}{a}$, where $a>0$

Problems

22. **(a)** Find the first three nonzero terms of the Taylor series for e^x+e^{-x}.
(b) Explain why the graph of e^x+e^{-x} looks like a parabola near $x=0$. What is the equation of this parabola?

23. **(a)** Find the first three nonzero terms of the Taylor series for e^x-e^{-x}.
(b) Explain why the graph of e^x-e^{-x} near $x=0$ looks like the graph of a cubic polynomial symmetric about the origin. What is the equation for this cubic?

24. Find the sum of $\displaystyle\sum_{p=1}^{\infty} px^{p-1}$ for $|x|<1$.

25. For values of y near 0, put the following functions in increasing order, using their Taylor expansions.

(a) $\ln(1+y^2)$ **(b)** $\sin(y^2)$ **(c)** $1-\cos y$

26. Figure 10.15 shows the graphs of the four functions below for values of x near 0. Use Taylor series to match graphs and formulas.

(a) $\dfrac{1}{1-x^2}$

(b) $(1+x)^{1/4}$

(c) $\sqrt{1+\dfrac{x}{2}}$

(d) $\dfrac{1}{\sqrt{1-x}}$

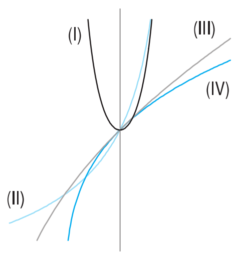


Figure 10.15

27. Find the sum of $\sum_{n=1}^{\infty} \frac{k^{n-1}}{(n-1)!} e^{-k}$.

28. The hyperbolic sine and cosine are differentiable and satisfy the conditions $\cosh 0 = 1$ and $\sinh 0 = 0$, and

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sinh x) = \cosh x.$$

(a) Using only this information, find the Taylor approximation of degree $n = 8$ about $x = 0$ for $f(x) = \cosh x$.

(b) Estimate the value of $\cosh 1$.

(c) Use the result from part (a) to find a Taylor polynomial approximation of degree $n = 7$ about $x = 0$ for $g(x) = \sinh x$.

29. Use the series for e^x to find the Taylor series for $\sinh 2x$ and $\cosh 2x$.

30. Use Taylor series to explain the patterns in the digits in the following expansions:

(a) $\frac{1}{0.98} = 1.02040816\ldots$

(b) $\left(\frac{1}{0.99}\right)^2 = 1.020304050607\ldots$

31. Padé approximants are rational functions used to approximate more complicated functions. In this problem, you will derive the Padé approximant to the exponential function.

(a) Let $f(x) = (1 + ax)/(1 + bx)$, where a and b are constants. Write down the first three terms of the Taylor series for $f(x)$ about $x = 0$.

(b) By equating the first three terms of the Taylor series about $x = 0$ for $f(x)$ and for e^x, find a and b so that $f(x)$ approximates e^x as closely as possible near $x = 0$.

32. One of Einstein's most amazing predictions was that light traveling from distant stars would bend around the sun on the way to earth. His calculations involved solving for ϕ in the equation

$$\sin\phi + b(1 + \cos^2\phi + \cos\phi) = 0$$

where b is a very small positive constant.

(a) Explain why the equation could have a solution for ϕ which is near 0.

(b) Expand the left-hand side of the equation in Taylor series about $\phi = 0$, disregarding terms of order ϕ^2 and higher. Solve for ϕ. (Your answer will involve b.)

33. A hydrogen atom consists of an electron, of mass m, orbiting a proton, of mass M, where m is much smaller than M. The *reduced mass*, μ, of the hydrogen atom is defined by

$$\mu = \frac{mM}{m + M}.$$

(a) Show that $\mu \approx m$.

(b) To get a more accurate approximation for μ, express μ as m times a series in m/M.

(c) The approximation $\mu \approx m$ is obtained by disregarding all but the constant term in the series. The first-order correction is obtained by including the linear term but no higher terms. If $m \approx M/1836$, by what percentage does including the first-order correction change the estimate $\mu \approx m$?

34. Resonance in electric circuits leads to the expression

$$\left(\omega L - \frac{1}{\omega C}\right)^2,$$

where ω is the variable and L and C are constants.

(a) Find ω_0, the value of ω making the expression zero.

(b) In practice, ω fluctuates about ω_0, so we are interested in the behavior of this expression for values of ω near ω_0. Let $\omega = \omega_0 + \Delta\omega$ and expand the expression in terms of $\Delta\omega$ up to the first nonzero term. Give your answer in terms of $\Delta\omega$ and L but not C.

35. The Michelson-Morley experiment, which contributed to the formulation of the Theory of Relativity, involved the difference between the two times t_1 and t_2 that light took to travel between two points. If v is the velocity of light; l_1, l_2, and c are constants; and $v < c$, then the times t_1 and t_2 are given by

$$t_1 = \frac{2l_2}{c(1 - v^2/c^2)} - \frac{2l_1}{c\sqrt{1 - v^2/c^2}}$$
$$t_2 = \frac{2l_2}{c\sqrt{1 - v^2/c^2}} - \frac{2l_1}{c(1 - v^2/c^2)}.$$

(a) Find an expression for $\Delta t = t_1 - t_2$, and give its Taylor expansion in terms of v^2/c^2 up to the second nonzero term.

(b) For small v, to what power of v is Δt proportional? What is the constant of proportionality?

36. The theory of relativity predicts that when an object moves at speeds close to the speed of light, the object appears heavier. The apparent, or relativistic, mass, m, of the object when it is moving at speed v is given by the formula

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where c is the speed of light and m_0 is the mass of the object when it is at rest.

(a) Use the formula for m to decide what values of v are possible.

(b) Sketch a rough graph of m against v, labeling intercepts and asymptotes.

(c) Write the first three nonzero terms of the Taylor series for m in terms of v.

(d) For what values of v do you expect the series to converge?

37. The potential energy, V, of two gas molecules separated by a distance r is given by

$$V = -V_0\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right),$$

where V_0 and r_0 are positive constants.

(a) Show that if $r = r_0$, then V takes on its minimum value, $-V_0$.
(b) Write V as a series in $(r - r_0)$ up through the quadratic term.
(c) For r near r_0, show that the difference between V and its minimum value is approximately proportional to $(r - r_0)^2$. In other words, show that $V - (-V_0) = V + V_0$ is approximately proportional to $(r - r_0)^2$.
(d) The force, F, between the molecules is given by $F = -dV/dr$. What is F when $r = r_0$? For r near r_0, show that F is approximately proportional to $(r - r_0)$.

38. Van der Waal's equation relates the pressure, P, and the volume, V, of a fixed quantity of a gas at constant temperature T:

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \qquad \text{where } a, b, n, R \text{ are constants.}$$

Find the first two nonzero terms for the Taylor series of P in terms of $1/V$.

10.4 THE ERROR IN TAYLOR POLYNOMIAL APPROXIMATIONS

In order to use an approximation with confidence, we need to be able to estimate the size of the error, which is the difference between the exact answer (which we do not know) and the approximate value.

When we use $P_n(x)$, the n^{th} degree Taylor polynomial, to approximate $f(x)$, the error is the difference

$$E_n(x) = f(x) - P_n(x).$$

We are interested in finding a *bound* on the magnitude of the error, $|E_n|$; that is, we want a number which we are sure is bigger than $|E_n|$. In practice, we want a bound which is reasonably close to the maximum value of $|E_n|$.

Finding an Error Bound

Recall that we constructed $P_n(x)$, the Taylor polynomial of f about 0, so that its first n derivatives equal the corresponding derivatives of $f(x)$. Therefore, $E_n(0) = 0$, $E_n'(0) = 0$, $E_n''(0) = 0$, $\cdots$, $E_n^{(n)}(0) = 0$. Since $P_n(x)$ is an n^{th} degree polynomial, its $(n+1)^{\text{st}}$ derivative is 0, so $E_n^{(n+1)}(x) = f^{(n+1)}(x)$. In addition, suppose that $\left|f^{(n+1)}(x)\right|$ is bounded by a positive constant M, for all positive values of x near 0, say for $0 \le x \le d$, so that

$$-M \le f^{(n+1)}(x) \le M \quad \text{for } 0 \le x \le d.$$

This means that

$$-M \le E_n^{(n+1)}(x) \le M \quad \text{for } 0 \le x \le d.$$

Writing t for the variable, we integrate this inequality from 0 to x, giving

$$-\int_0^x M\,dt \le \int_0^x E_n^{(n+1)}(t)\,dt \le \int_0^x M\,dt \quad \text{for } 0 \le x \le d,$$

so

$$-Mx \le E_n^{(n)}(x) \le Mx \quad \text{for } 0 \le x \le d.$$

We integrate this inequality again from 0 to x, giving

$$-\int_0^x Mt\,dt \le \int_0^x E_n^{(n)}(t)\,dt \le \int_0^x Mt\,dt \quad \text{for } 0 \le x \le d,$$

so

$$-\frac{1}{2}Mx^2 \le E_n^{(n-1)}(x) \le \frac{1}{2}Mx^2 \quad \text{for } 0 \le x \le d.$$

By repeated integration, we obtain the following bound:

$$-\frac{1}{(n+1)!}Mx^{n+1} \leq E_n(x) \leq \frac{1}{(n+1)!}Mx^{n+1} \quad \text{for } 0 \leq x \leq d,$$

which means that

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{1}{(n+1)!}Mx^{n+1} \quad \text{for } 0 \leq x \leq d.$$

When x is to the left of 0, so $-d \leq x \leq 0$, and when the Taylor series is centered at $a \neq 0$, similar calculations lead to the following result:

Theorem 10.1: The Lagrange Error Bound for $P_n(x)$

Suppose f and all its derivatives are continuous. If $P_n(x)$ is the n^{th} Taylor polynomial for to $f(x)$ about a, then

$$|E_n(x)| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1},$$

where $\max \left|f^{(n+1)}\right| \leq M$ on the interval between a and x.

Using the Lagrange Error Bound for Taylor Polynomials

Example 1 Give a bound on the error, E_4, when e^x is approximated by its fourth-degree Taylor polynomial about 0 for $-0.5 \leq x \leq 0.5$.

Solution Let $f(x) = e^x$. We need to find a bound for the fifth derivative, $f^{(5)}(x) = e^x$. Since e^x is increasing,

$$|f^{(5)}(x)| \leq e^{0.5} = \sqrt{e} < 2 \quad \text{for} \quad -0.5 \leq x \leq 0.5.$$

So we can take $M = 2$. Then

$$|E_4| = |f(x) - P_4(x)| \leq \frac{2}{5!}|x|^5.$$

This means, for example, that on $-0.5 \leq x \leq 0.5$, the approximation

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

has an error of at most $\frac{2}{120}(0.5)^5 < 0.0006$.

The Lagrange error bound for Taylor polynomials can be used to see how the accuracy of the approximation depends on the value of x or the value of n. Observe that the error for a Taylor polynomial of degree n depends on the $(n+1)^{\text{st}}$ power of $(x-a)$. That means, for example, with a Taylor polynomial of degree n centered at 0, if we decrease x by a factor of 2, the error bound decreases by a factor of 2^{n+1}.

Example 2 Compare the errors in the approximations

$$e^{0.1} \approx 1 + 0.1 + \frac{1}{2!}(0.1)^2 \qquad \text{and} \qquad e^{0.05} \approx 1 + (0.05) + \frac{1}{2!}(0.05)^2.$$

Solution We are approximating e^x by its second-degree Taylor polynomial about 0. We evaluate the polynomial first at $x = 0.1$, and then at $x = 0.05$. Since we have decreased x by a factor of 2, the error bound decreases by a factor of about $2^3 = 8$. To see what actually happens to the errors, we compute them:

$$e^{0.1} - \left(1 + 0.1 + \frac{1}{2!}(0.1)^2\right) = 1.105171 - 1.105000 = 0.000171$$
$$e^{0.05} - \left(1 + 0.05 + \frac{1}{2!}(0.05)^2\right) = 1.051271 - 1.051250 = 0.000021$$

Since $(0.000171)/(0.000021) = 8.1$, the error has also decreased by a factor of about 8.

Convergence of Taylor Series for $\cos x$

We have already seen that the Taylor polynomials centered at $x = 0$ for $\cos x$ are good approximations for x near 0. (See Figure 10.16.) In fact, for any value of x, if we take a Taylor polynomial centered at $x = 0$ of high enough degree, its graph is nearly indistinguishable from the graph of the cosine near that point.

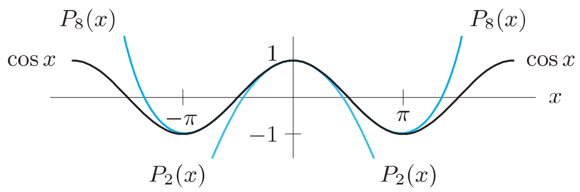


Figure 10.16: Graph of $\cos x$ and two Taylor polynomials for x near 0

Let's see what happens numerically. Let $x = \pi/2$. The successive Taylor polynomial approximations to $\cos(\pi/2) = 0$ about $x = 0$ are

$$\begin{aligned}
P_2(\pi/2) &= \quad 1 - (\pi/2)^2/2! &&= -0.23370\ldots\\
P_4(\pi/2) &= 1 - (\pi/2)^2/2! + (\pi/2)^4/4! &&= 0.01997\ldots\\
P_6(\pi/2) &= \qquad \cdots &&= -0.00089\ldots\\
P_8(\pi/2) &= \qquad \cdots &&= 0.00002\ldots.
\end{aligned}$$

It appears that the approximations converge to the true value, $\cos(\pi/2) = 0$, very rapidly. Now take a value of x somewhat farther away from 0, say $x = \pi$, then $\cos\pi = -1$ and

$$\begin{aligned}
P_2(\pi) &= 1 - (\pi)^2/2! &&= -3.93480\ldots\\
P_4(\pi) &= \cdots &&= 0.12391\ldots\\
P_6(\pi) &= \cdots &&= -1.21135\ldots\\
P_8(\pi) &= \cdots &&= -0.97602\ldots\\
P_{10}(\pi) &= \cdots &&= -1.00183\ldots\\
P_{12}(\pi) &= \cdots &&= -0.99990\ldots\\
P_{14}(\pi) &= \cdots &&= -1.000004\ldots
\end{aligned}$$

We see that the rate of convergence is somewhat slower; it takes a 14^{th} degree polynomial to approximate $\cos\pi$ as accurately as an 8^{th} degree polynomial approximates $\cos(\pi/2)$. If x were taken still farther away from 0, then we would need still more terms to obtain as accurate an approximation of $\cos x$.

Using the ratio test, we can show the Taylor series for $\cos x$ converges for all values of x. In addition, we will prove that it converges to $\cos x$ using Theorem 10.1. Thus, we are justified in writing the equality:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad \text{for all } x.$$

Showing the Taylor Series for cos x Converges to cos x

The Lagrange error bound in Theorem 10.1 allows us to see if the Taylor series for a function converges to that function. In the series for $\cos x$, the odd powers are missing, so we assume n is even and write

$$E_n(x) = \cos x - P_n(x) = \cos x - \left(1 - \frac{x^2}{2!} + \cdots + (-1)^{n/2}\frac{x^n}{n!}\right),$$

giving

$$\cos x = 1 - \frac{x^2}{2!} + \cdots + (-1)^{n/2}\frac{x^n}{n!} + E_n(x).$$

Thus, for the Taylor series to converge to $\cos x$, we must have $E_n(x) \to 0$ as $n \to \infty$.

Showing $E_n(x) \to 0$ as $n \to \infty$

Proof Since $f(x) = \cos x$, the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ is $\pm\cos x$ or $\pm\sin x$, no matter what n is. So for all n, we have $|f^{(n+1)}(x)| \le 1$ on the interval between 0 and x.

By the Lagrange error bound with $M = 1$, we have

$$|E_n(x)| = |\cos x - P_n(x)| \le \frac{|x|^{n+1}}{(n+1)!} \quad \text{for every } n.$$

To show that the errors go to zero, we must show that for a fixed x,

$$\frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n \to \infty.$$

To see why this is true, think about what happens when n is much larger than x. Suppose, for example, that $x = 17.3$. Let's look at the value of the sequence for n more than twice as big as 17.3, say $n = 36$, or $n = 37$, or $n = 38$:

$$\begin{aligned}
\text{For } n = 36:&\quad \frac{1}{37!}(17.3)^{37}\\
\text{For } n = 37:&\quad \frac{1}{38!}(17.3)^{38} = \frac{17.3}{38}\cdot\frac{1}{37!}(17.3)^{37},\\
\text{For } n = 38:&\quad \frac{1}{39!}(17.3)^{39} = \frac{17.3}{39}\cdot\frac{17.3}{38}\cdot\frac{1}{37!}(17.3)^{37}, \quad \ldots
\end{aligned}$$

Since $17.3/36$ is less than $\frac{1}{2}$, each time we increase n by 1, the term is multiplied by a number less than $\frac{1}{2}$. No matter what the value of $\frac{1}{37!}(17.3)^{37}$ is, if we keep on dividing it by two, the result gets closer to zero. Thus $\frac{1}{(n+1)!}(17.3)^{n+1}$ goes to 0 as n goes to infinity.

We can generalize this by replacing 17.3 by an arbitrary $|x|$. For $n > 2|x|$, the following sequence converges to 0 because each term is obtained from its predecessor by multiplying by a number less than $\frac{1}{2}$:

$$\frac{x^{n+1}}{(n+1)!}, \quad \frac{x^{n+2}}{(n+2)!}, \quad \frac{x^{n+3}}{(n+3)!}, \ldots.$$

Therefore, the Taylor series $1 - x^2/2! + x^4/4! - \cdots$ does converge to $\cos x$.

Problems 21 and 20 ask you to show that the Taylor series for $\sin x$ and e^x converge to the original function for all x. In each case, you again need the following limit:

$$\boxed{\lim_{n\to\infty}\frac{x^n}{n!} = 0.}$$

Exercises and Problems for Section 10.4

Exercises

In Exercises 1–8, use Theorem 10.1 to find a bound for the error in approximating the quantity with a third-degree Taylor polynomial for the given function $f(x)$ about $x = 0$. Compare the bound with the actual error.

1. $e^{0.1}$, $f(x) = e^x$

2. $\sin(0.2)$, $f(x) = \sin x$

3. $\cos(-0.3)$, $f(x) = \cos x$

4. $\sqrt{0.9}$, $f(x) = \sqrt{1+x}$

5. $\ln(1.5)$, $f(x) = \ln(1+x)$

6. $1/\sqrt{3}$, $f(x) = (1+x)^{-1/2}$

7. $\tan 1$, $f(x) = \tan x$

8. $0.5^{1/3}$, $f(x) = (1-x)^{1/3}$

Problems

9. Find a bound on the magnitude of the error if we approximate $\sqrt{2}$ using the Taylor approximation of degree three for $\sqrt{1+x}$ about $x = 0$.

10. Consider the error in using the approximation $\sin\theta \approx \theta$ on the interval $[-1, 1]$.

(a) Reasoning informally, say where the approximation is an overestimate, and where is it an underestimate.

(b) Use Theorem 10.1 to bound the error. Check your answer graphically on a computer or calculator.

11. Repeat Problem 10 for the approximation $\sin\theta \approx \theta - \theta^3/3!$.

12. You approximate $f(t) = e^t$ by a Taylor polynomial of degree 0 about $t = 0$ on the interval $[0, 0.5]$.

(a) Reasoning informally, say whether the approximation is an overestimate or an underestimate.

(b) Use Theorem 10.1 to bound the error. Check your answer graphically on a computer or calculator.

13. Repeat Problem 12 using the second-degree Taylor approximation, $P_2(t)$, to e^t.

14. **(a)** Use the graphs of $y = \cos x$ and its Taylor polynomials, $P_{10}(x)$ and $P_{20}(x)$, in Figure 10.17 to bound:

(i) The error in approximating $\cos 6$ by $P_{10}(6)$ and by $P_{20}(6)$.

(ii) The error in approximating $\cos x$ by $P_{20}(x)$ for $|x| \leq 9$.

(b) If we want to approximate $\cos x$ by $P_{10}(x)$ to an accuracy of within 0.1, what is the largest interval of x-values on which we can work? Give your answer to the nearest integer.

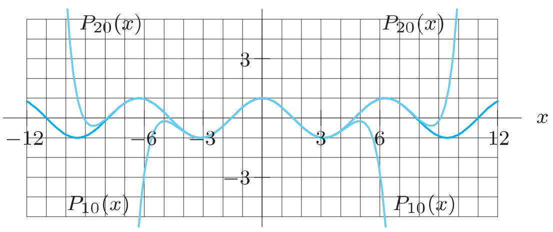


Figure 10.17

15. Give a bound for the error for the n^{th} degree Taylor polynomial about $x = 0$ approximating $\cos x$ on the interval $[0, 1]$. What is the bound for $\sin x$?

16. What degree Taylor polynomial about $x = 0$ do you need to calculate $\cos 1$ to four decimal places? To six decimal places? Justify your answer using the results of Problem 15.

17. **(a)** Using a calculator, make a table of the values to four decimal places of $\sin x$ for

$$x = -0.5, -0.4, \ldots, -0.1, 0, 0.1, \ldots, 0.4, 0.5.$$

(b) Add to your table the values of the error $E_1 = \sin x - x$ for these x-values.

(c) Using a calculator or computer, draw a graph of the quantity $E_1 = \sin x - x$ showing that

$$|E_1| < 0.03 \quad \text{for} \quad -0.5 \leq x \leq 0.5.$$

18. In this problem, you will investigate the error in the n^{th} degree Taylor approximation to e^x about 0 for various values of n.

(a) Let $E_1 = e^x - P_1(x) = e^x - (1+x)$. Using a calculator or computer, graph E_1 for $-0.1 \leq x \leq 0.1$. What shape is the graph of E_1? Use the graph to confirm that

$$|E_1| \leq x^2 \quad \text{for} \quad -0.1 \leq x \leq 0.1.$$

(b) Let $E_2 = e^x - P_2(x) = e^x - (1+x+x^2/2)$. Choose a suitable range and graph E_2 for $-0.1 \leq x \leq 0.1$. What shape is the graph of E_2? Use the graph to confirm that

$$|E_2| \leq x^3 \quad \text{for} \quad -0.1 \leq x \leq 0.1.$$

(c) Explain why the graphs of E_1 and E_2 have the shapes they do.

19. For $|x| \leq 0.1$, graph the error
$$E_0 = \cos x - P_0(x) = \cos x - 1.$$
Explain the shape of the graph, using the Taylor expansion of $\cos x$. Find a bound for $|E_0|$ for $|x| \leq 0.1$.

20. Show that the Taylor series about 0 for e^x converges to e^x for every x. Do this by showing that the error $E_n(x) \to 0$ as $n \to \infty$.

21. Show that the Taylor series about 0 for $\sin x$ converges to $\sin x$ for every x.

22. To approximate π using a Taylor polynomial, we could use the series for the arctangent or the series for the arcsine. In this problem, we compare the two methods.

(a) Using the fact that $d(\arctan x)/dx = 1/(1+x^2)$ and $\arctan 1 = \pi/4$, approximate the value of π using the third-degree Taylor polynomial of i $4\arctan x$ about $x = 0$.

(b) Using the fact that $d(\arcsin x)/dx = 1/\sqrt{1-x^2}$ and $\arcsin 1 = \pi/2$, approximate the value of π using the third-degree Taylor polynomial of $2\arcsin x$ about $x = 0$.

(c) Estimate the maximum error of the approximation you found in part (a).

(d) Explain the problem in estimating the error in the arcsine approximation.

10.5 FOURIER SERIES

We have seen how to approximate a function by a Taylor polynomial of fixed degree. Such a polynomial is usually very close to the true value of the function near one point (the point at which the Taylor polynomial is centered), but not necessarily at all close anywhere else. In other words, Taylor polynomials are good approximations of a function *locally*, but not necessarily *globally*. In this section, we take another approach: we approximate the function by trigonometric functions, called *Fourier approximations*. The resulting approximation may not be as close to the original function at some points as the Taylor polynomial. However, the Fourier approximation is, in general, close over a larger interval. In other words, a Fourier approximation can be a better approximation globally. In addition, Fourier approximations are useful even for functions that are not continuous. Unlike Taylor approximations, Fourier approximations are periodic, so they are particularly useful for approximating periodic functions.

Many processes in nature are periodic or repeating, so it makes sense to approximate them by periodic functions. For example, sound waves are made up of periodic oscillations of air molecules. Heartbeats, the movement of the lungs, and the electrical current that powers our homes are all periodic phenomena. Two of the simplest periodic functions are the square wave in Figure 10.18 and the triangular wave in Figure 10.19. Electrical engineers use the square wave as the model for the flow of electricity when a switch is repeatedly flicked on and off.

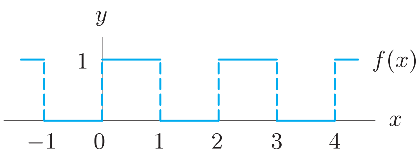


Figure 10.18: Square wave

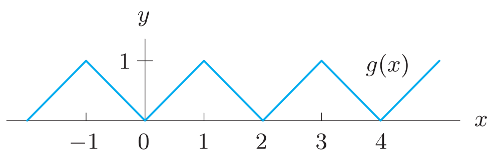


Figure 10.19: Triangular wave

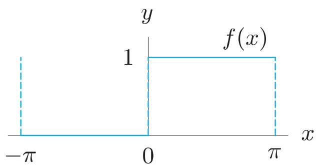
Fourier Polynomials

We can express the square wave and the triangular wave by the formulas

$$f(x) = \begin{cases} \vdots & \vdots \\ 0 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1 \\ 0 & 1 \leq x < 2 \\ 1 & 2 \leq x < 3 \\ 0 & 3 \leq x < 4 \\ \vdots & \vdots \end{cases} \qquad g(x) = \begin{cases} \vdots & \vdots \\ -x & -1 \leq x < 0 \\ x & 0 \leq x < 1 \\ 2-x & 1 \leq x < 2 \\ x-2 & 2 \leq x < 3 \\ 4-x & 3 \leq x < 4 \\ \vdots & \vdots \end{cases}$$

However, these formulas are not particularly easy to work with. Worse, the functions are not differentiable at various points. Here we show how to approximate such functions by differentiable, periodic functions.

Since the sine and cosine are the simplest periodic functions, they are the building blocks we use. Because they repeat every 2π, we assume that the function f we want to approximate repeats every 2π. (Later, we deal with the case where f has some other period.) We start by considering the square wave in Figure 10.20. Because of the periodicity of all the functions concerned, we only have to consider what happens in the course of a single period; the same behavior repeats in any other period.



$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

Figure 10.20: Square wave on $[-\pi, \pi]$

We will attempt to approximate f with a sum of trigonometric functions of the form

$$\begin{aligned} f(x) &\approx F_n(x) \\ &= a_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \cdots + a_n \cos(nx) \\ &\qquad + b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \cdots + b_n \sin(nx) \\ &= a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx). \end{aligned}$$

$F_n(x)$ is known as a *Fourier polynomial of degree* n, named after the French mathematician Joseph Fourier (1768–1830), who was one of the first to investigate it.[3] The coefficients a_k and b_k are called *Fourier coefficients*. Since each of the component functions $\cos(kx)$ and $\sin(kx)$, $k = 1, 2, \ldots, n$, repeats every 2π, $F_n(x)$ must repeat every 2π and so is a potentially good match for $f(x)$, which also repeats every 2π. The problem is to determine values for the Fourier coefficients that achieve a close match between $f(x)$ and $F_n(x)$. We choose the following values:

The Fourier Coefficients for a Periodic Function f of Period 2π

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad \text{for } k > 0, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \quad \text{for } k > 0. \end{aligned}$$

Notice that a_0 is just the average value of f over the interval $[-\pi, \pi]$.

For an informal justification for the use of these values, see page 538. In addition, the integrals over $[-\pi, \pi]$ for a_k and b_k can be replaced by integrals over any interval of length 2π.

Example 1 Construct successive Fourier polynomials for the square wave function f, with period 2π, given by

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi. \end{cases}$$

[3] The Fourier polynomials are not polynomials in the usual sense of the word.

Solution

Since a_0 is the average value of f on $[-\pi, \pi]$, we suspect from the graph of f that $a_0 = \frac{1}{2}$. We can verify this analytically:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}.$$

Furthermore,

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos x\,dx = 0$$

and

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin x\,dx = \frac{2}{\pi}.$$

Therefore, the Fourier polynomial of degree 1 is given by

$$f(x) \approx F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x$$

and the graphs of the function and the first Fourier approximation are shown in Figure 10.21.

We next construct the Fourier polynomial of degree 2. The coefficients a_0, a_1, b_1 are the same as before. In addition,

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos(2x)\,dx = 0$$

and

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin(2x)\,dx = 0.$$

Since $a_2 = b_2 = 0$, the Fourier polynomial of degree 2 is identical to the Fourier polynomial of degree 1. Let's look at the Fourier polynomial of degree 3:

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(3x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos(3x)\,dx = 0$$

and

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(3x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin(3x)\,dx = \frac{2}{3\pi}.$$

So the approximation is given by

$$f(x) \approx F_3(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x).$$

The graph of F_3 is shown in Figure 10.22. This approximation is better than $F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x$, as comparing Figure 10.22 to Figure 10.21 shows.

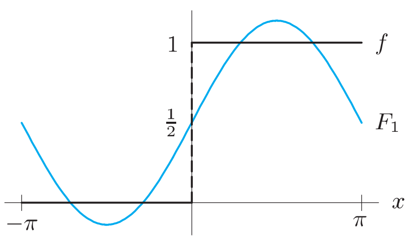


Figure 10.21: First Fourier approximation to the square wave

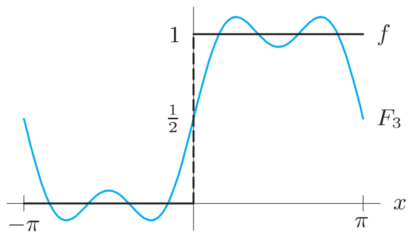
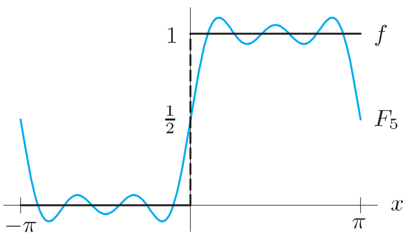


Figure 10.22: Third Fourier approximation to the square wave



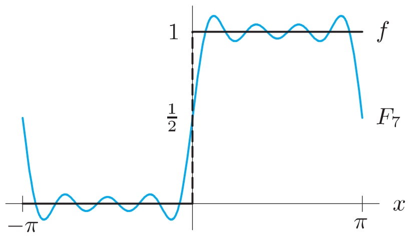


Figure 10.23: Fifth and seventh Fourier approximations to the square wave

Without going through the details, we calculate the coefficients for higher-degree Fourier approximations:

$$F_5(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x)$$
$$F_7(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \frac{2}{7\pi}\sin(7x).$$

Figure 10.23 shows that higher-degree approximations match the step-like nature of the square wave function more and more closely.

We could have used a Taylor series to approximate the square wave, provided we did not center the series at a point of discontinuity. Since the square wave is a constant function on each interval, all its derivatives are zero, and so its Taylor series approximations are the constant functions: 0 or 1, depending on where the Taylor series is centered. They approximate the square wave perfectly on each piece, but they do not do a good job over the whole interval of length 2π. That is what Fourier polynomials succeed in doing: they approximate a curve fairly well everywhere, rather than just near a particular point. The Fourier approximations above look a lot like square waves, so they approximate well *globally*. However, they may not give good values near points of discontinuity. (For example, near $x = 0$, they all give values near $1/2$, which are incorrect.) Thus Fourier polynomials may not be good *local* approximations.

Taylor polynomials give good *local* approximations to a function;
Fourier polynomials give good *global* approximations to a function.

Fourier Series

As with Taylor polynomials, the higher the degree of the Fourier approximation, the more accurate it is. Therefore, we carry this procedure on indefinitely by letting $n \to \infty$, and we call the resulting infinite series a *Fourier series*.

The Fourier Series for f on $[-\pi, \pi]$

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$$
$$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$

where a_k and b_k are the Fourier coefficients.

Thus, the Fourier series for the square wave is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \frac{2}{7\pi}\sin 7x + \cdots.$$

Harmonics

Let us start with a function $f(x)$ that is periodic with period 2π, expanded in a Fourier series:

$$\begin{aligned} f(x) = a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \end{aligned}$$

The function

$$a_k \cos kx + b_k \sin kx$$

is referred to as the k^{th} *harmonic* of f, and it is customary to say that the Fourier series expresses f in terms of its harmonics. The first harmonic, $a_1 \cos x + b_1 \sin x$, is sometimes called the *fundamental harmonic* of f.

Example 2 Find a_0 and the first four harmonics of a *pulse train* function f of period 2π shown in Figure 10.24:

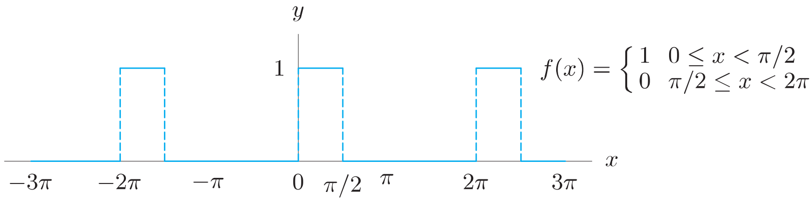


Figure 10.24: A train of pulses with period 2π

Solution First, a_0 is the average value of the function, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} 1\,dx = \frac{1}{4}.$$

Next, we compute a_k and b_k, $k = 1, 2, 3$, and 4. The formulas

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_0^{\pi/2} \cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi/2} \sin(kx)\,dx$$

lead to the harmonics

$$\begin{aligned} a_1 \cos x + b_1 \sin x &= \frac{1}{\pi}\cos x + \frac{1}{\pi}\sin x \\ a_2 \cos(2x) + b_2 \sin(2x) &= \frac{1}{\pi}\sin(2x) \\ a_3 \cos(3x) + b_3 \sin(3x) &= -\frac{1}{3\pi}\cos(3x) + \frac{1}{3\pi}\sin(3x) \\ a_4 \cos(4x) + b_4 \sin(4x) &= 0. \end{aligned}$$

Figure 10.25 shows the graph of the sum of a_0 and these harmonics, which is the fourth Fourier approximation of f.

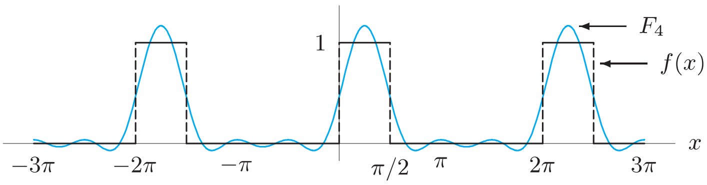


Figure 10.25: Fourth Fourier approximation to pulse train f equals the sum of a_0 and the first four harmonics

Energy and the Energy Theorem

The quantity $A_k = \sqrt{a_k^2 + b_k^2}$ is called the amplitude of the k^{th} harmonic. The square of the amplitude has a useful interpretation. Adopting terminology from the study of periodic waves, we define the *energy* E of a periodic function f of period 2π to be the number

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx.$$

Problem 17 on page 541 asks you to check that for all positive integers k,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (a_k \cos(kx) + b_k \sin(kx))^2 \, dx = a_k^2 + b_k^2 = A_k^2.$$

This shows that the k^{th} harmonic of f has energy A_k^2. The energy of the constant term a_0 of the Fourier series is $\frac{1}{\pi} \int_{-\pi}^{\pi} a_0^2 \, dx = 2a_0^2$, so we make the definition

$$A_0 = \sqrt{2} a_0.$$

It turns out that for all reasonable periodic functions f, the energy of f equals the sum of the energy of its harmonics:

The Energy Theorem for a Periodic Function f of Period 2π

$$E = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = A_0^2 + A_1^2 + A_2^2 + \cdots$$

where $A_0 = \sqrt{2} a_0$ and $A_k = \sqrt{a_k^2 + b_k^2}$ (for all integers $k \geq 1$).

The graph of A_k^2 against k is called the *energy spectrum* of f. It shows how the energy of f is distributed among its harmonics.

Example 3 (a) Graph the energy spectrum of the square wave of Example 1.
(b) What fraction of the energy of the square wave is contained in the constant term and first three harmonics of its Fourier series?

Solution (a) We know from Example 1 that $a_0 = 1/2$, $a_k = 0$ for $k \geq 1$, $b_k = 0$ for k even, and $b_k = 2/(k\pi)$ for k odd. Thus

$$A_0^2 = 2a_0^2 = \frac{1}{2}$$

$$A_k^2 = 0 \quad \text{if } k \text{ is even}, \quad k \geq 1,$$

$$A_k^2 = \left(\frac{2}{k\pi}\right)^2 = \frac{4}{k^2\pi^2} \quad \text{if } k \text{ is odd}, \quad k \geq 1.$$

The energy spectrum is graphed in Figure 10.26. Notice that it is customary to represent the energy A_k^2 of the k^{th} harmonic by a vertical line of length A_k^2. The graph shows that the constant term and first harmonic carry most of the energy of f.

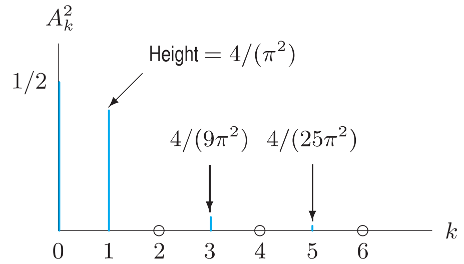


Figure 10.26: The energy spectrum of a square wave

(b) The energy of the square wave $f(x)$ is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1.$$

The energy in the constant term and the first three harmonics of the Fourier series is

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 = \frac{1}{2} + \frac{4}{\pi^2} + 0 + \frac{4}{9\pi^2} = 0.950.$$

The fraction of energy carried by the constant term and the first three harmonics is

$$0.95/1 = 0.95, \text{ or } 95\%.$$

Musical Instruments

You may have wondered why different musical instruments sound different, even when playing the same note. A first step might be to graph the periodic deviations from the average air pressure that form the sound waves they produce. This has been done for clarinet and trumpet in Figure 10.27.[4] However, it is more revealing to graph the energy spectra of these functions, as in Figure 10.28. The most striking difference is the relative weakness of the second, fourth, and sixth harmonics for the clarinet, with the second harmonic completely absent. The trumpet sounds the second harmonic with as much energy as it does the fundamental.

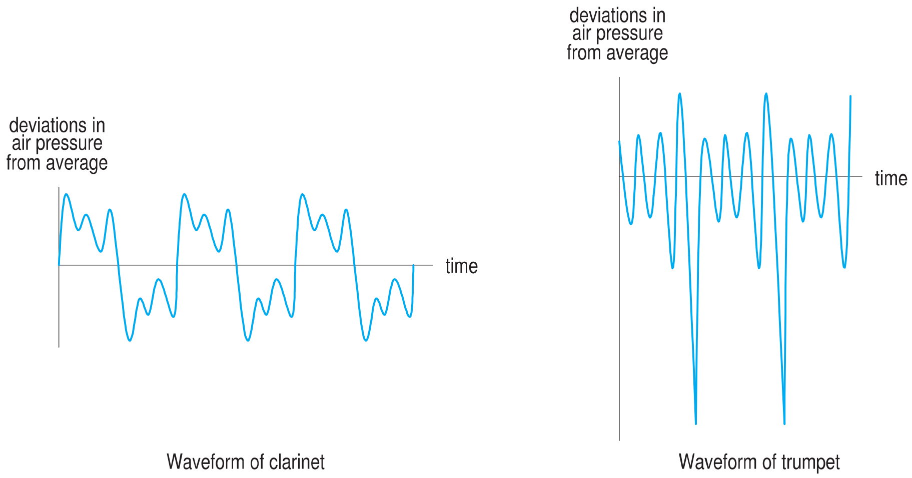


Figure 10.27: Sound waves of a clarinet and trumpet

[4]Adapted from C.A. Culver, *Musical Acoustics* (New York: McGraw-Hill, 1956) pp. 204, 220.

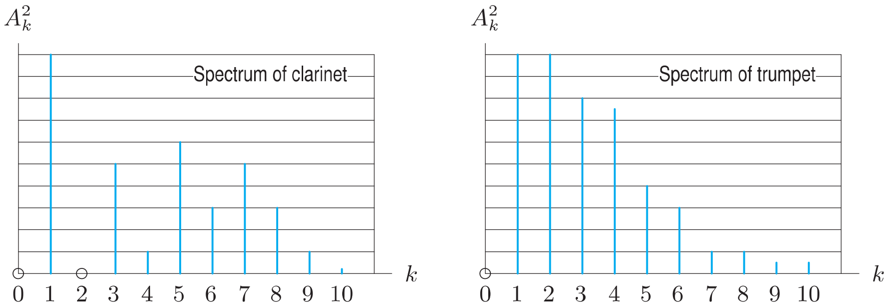


Figure 10.28: Energy spectra of a clarinet and trumpet

What Do We Do if Our Function Does Not Have Period 2π?

We can adapt our previous work to find the Fourier series for a function of period b. Suppose $f(x)$ is given on the interval $[-b/2, b/2]$. In Problem 29, we see how to use a change of variable to get the following result:

> **The Fourier Series for f on $[-b/2, b/2]$**
>
> $$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi kx}{b}\right) + b_k \sin\left(\frac{2\pi kx}{b}\right)\right)$$
>
> where $a_0 = \frac{1}{b}\int_{-b/2}^{b/2} f(x)\,dx$ and, for $k \geq 1$,
>
> $$a_k = \frac{2}{b}\int_{-b/2}^{b/2} f(x)\cos\left(\frac{2\pi kx}{b}\right)dx \qquad b_k = \frac{2}{b}\int_{-b/2}^{b/2} f(x)\sin\left(\frac{2\pi kx}{b}\right)dx$$
>
> The constant $2\pi k/b$ is called the angular frequency of the k^{th}-harmonic; b is the period of f.

Note that the integrals over $[-b/2, b/2]$ can be replaced by integrals over any interval of length b.

Example 4 Find the fifth-degree Fourier polynomial of the square wave $f(x)$ graphed in Figure 10.29.

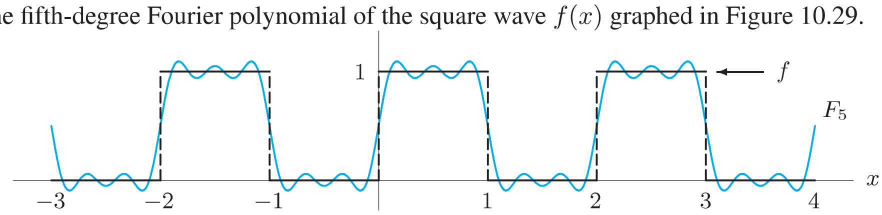


Figure 10.29: Square wave f and its fifth Fourier approximation F_5

Solution Since $f(x)$ repeats outside the interval $[-1, 1]$, we have $b = 2$. As an example of how the coefficients are computed, we find b_1. Since $f(x) = 0$ for $-1 < x < 0$,

$$b_1 = \frac{2}{2}\int_{-1}^{1} f(x)\sin\left(\frac{2\pi x}{2}\right)dx = \int_0^1 \sin(\pi x)dx = -\frac{1}{\pi}\cos(\pi x)\bigg|_0^1 = \frac{2}{\pi}.$$

Finding the other coefficients by a similar method, we have

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x).$$

Notice that the coefficients in this series are the same as those in Example 1. This is because the graphs in Figures 10.23 and 10.29 are the same except with different scales on the x-axes.

Seasonal Variation in the Incidence of Measles

Example 5 Fourier approximations have been used to analyze the seasonal variation in the incidence of diseases. One study[5] done in Baltimore, Maryland, for the years 1901–1931, studied $I(t)$, the average number of cases of measles per 10,000 susceptible children in the t^{th} month of the year. The data points in Figure 10.30 show $f(t) = \log I(t)$. The curve in Figure 10.30 shows the second Fourier approximation of $f(t)$. Figure 10.31 contains the graphs of the first and second harmonics of $f(t)$, plotted separately as deviations about a_0, the average logarithmic incidence rate. Describe what these two harmonics tell you about incidence of measles.

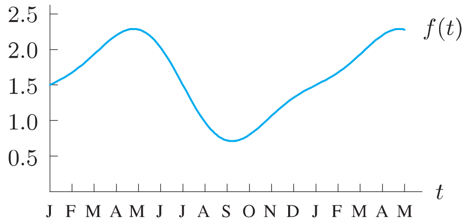


Figure 10.30: Logarithm of incidence of measles per month (dots) and second Fourier approximation (curve)

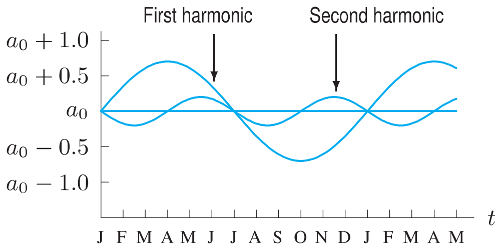


Figure 10.31: First and second harmonics of $f(t)$ plotted as deviations from average log incidence rate, a_0

Solution Taking the log of $I(t)$ has the effect of reducing the amplitude of the oscillations. However, since the log of a function increases when the function increases, and decreases when it decreases, oscillations in $f(t)$ correspond to oscillations in $I(t)$.

Figure 10.31 shows that the first harmonic in the Fourier series has a period of one year (the same period as the original function); the second harmonic has a period of six months. The graph in Figure 10.31 shows that the first harmonic is approximately a sine with amplitude about 0.7; the second harmonic is approximately the negative of a sine with amplitude about 0.2. Thus, for t in months ($t = 0$ in January),

$$\log I(t) = f(t) \approx a_0 + 0.7\sin\left(\frac{\pi}{6}t\right) - 0.2\sin\left(\frac{\pi}{3}t\right),$$

where $\pi/6$ and $\pi/3$ are introduced to make the periods 12 and 6 months, respectively. We can estimate a_0 from the original graph of f: it is the average value of f, approximately 1.5. Thus

$$f(t) \approx 1.5 + 0.7\sin\left(\frac{\pi}{6}t\right) - 0.2\sin\left(\frac{\pi}{3}t\right).$$

Figure 10.30 shows that the second Fourier approximation of $f(t)$ is quite good. The harmonics of $f(t)$ beyond the second must be rather insignificant. This suggests that the variation in incidence in measles comes from two sources, one with a yearly cycle that is reflected in the first harmonic and one with a half-yearly cycle reflected in the second harmonic. At this point the mathematics can tell us no more; we must turn to the epidemiologists for further explanation.

Informal Justification of the Formulas for the Fourier Coefficients

Recall that the coefficients in a Taylor series (which is a good approximation locally) are found by differentiation. In contrast, the coefficients in a Fourier series (which is a good approximation globally) are found by integration.

We want to find the constants $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in the expression

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx).$$

[5]From C. I. Bliss and D. L. Blevins, *The Analysis of Seasonal Variation in Measles* (Am. J. Hyg. 70, 1959), reported by Edward Batschelet, *Introduction to Mathematics for the Life Sciences* (Springer-Verlag, Berlin, 1979).

Consider the integral

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx) \right] dx.$$

Splitting the integral into separate terms, and assuming we can interchange integration and summation, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} a_k \cos(kx)\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} b_k \sin(kx)\,dx \\ &= \int_{-\pi}^{\pi} a_0\,dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} a_k \cos(kx)\,dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} b_k \sin(kx)\,dx.\end{aligned}$$

But for $k \geq 1$, thinking of the integral as an area shows that

$$\int_{-\pi}^{\pi} \sin(kx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos(kx)\,dx = 0,$$

so all terms drop out except the first, giving

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx = a_0 x \Big|_{-\pi}^{\pi} = 2\pi a_0$$

and so we get the following result:

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx.}$$

Thus a_0 is the average value of f on the interval $[-\pi, \pi]$.

To determine the values of any of the other a_k or b_k (for positive k), we use a rather clever method which depends on the following facts. For all integers k and m,

$$\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = 0,$$

and, provided $k \neq m$,

$$\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = 0.$$

(See Problems 24–28 on page 542.) In addition, provided $m \neq 0$, we have

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(mx)\,dx = \pi.$$

To determine a_k, we multiply the Fourier series by $\cos(mx)$, where m is any positive integer:

$$f(x)\cos(mx) = a_0 \cos(mx) + \sum_{k=1}^{\infty} a_k \cos(kx)\cos(mx) + \sum_{k=1}^{\infty} b_k \sin(kx)\cos(mx).$$

We integrate this between $-\pi$ and π, term by term:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-\pi}^{\pi}\left(a_0\cos(mx) + \sum_{k=1}^{\infty} a_k\cos(kx)\cos(mx) + \sum_{k=1}^{\infty} b_k\sin(kx)\cos(mx)\right)dx$$

$$= a_0\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos(kx)\cos(mx)\,dx\right)$$

$$+ \sum_{k=1}^{\infty}\left(b_k\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\,dx\right).$$

Provided $m \neq 0$, we have $\int_{-\pi}^{\pi}\cos(mx)\,dx = 0$. Since the integral $\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\,dx = 0$, all the terms in the second sum are zero. Since $\int_{-\pi}^{\pi}\cos kx\cos mx\,dx = 0$ provided $k \neq m$, all the terms in the first sum are zero except where $k = m$. Thus the right-hand side reduces to one term:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\int_{-\pi}^{\pi}\cos(mx)\cos(mx)\,dx = \pi a_m.$$

This leads, for each value of $m = 1, 2, 3\ldots,$ to the following formula:

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx.}$$

To determine b_k, we multiply through by $\sin(mx)$ instead of $\cos(mx)$ and eventually obtain, for each value of $m = 1, 2, 3\ldots,$ the following result:

$$\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx.}$$

Exercises and Problems for Section 10.5

Exercises

Which of the series in Exercises 1–4 are Fourier series?

1. $1 + \cos x + \cos^2 x + \cos^3 x + \cos^4 x + \cdots$

2. $\sin x + \sin(x+1) + \sin(x+2) + \cdots$

3. $\dfrac{\cos x}{2} + \sin x - \dfrac{\cos(2x)}{4} - \dfrac{\sin(2x)}{2} + \dfrac{\cos(3x)}{8} + \dfrac{\sin(3x)}{3} - \cdots$

4. $\dfrac{1}{2} - \dfrac{1}{3}\sin x + \dfrac{1}{4}\sin(2x) - \dfrac{1}{5}\sin(3x) + \cdots$

5. Construct the first three Fourier approximations to the square wave function

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi. \end{cases}$$

Use a calculator or computer to draw the graph of each approximation.

6. Repeat Problem 5 with the function

$$f(x) = \begin{cases} -x & -\pi \leq x < 0 \\ x & 0 \leq x < \pi. \end{cases}$$

7. What fraction of the energy of the function in Problem 6 is contained in the constant term and first three harmonics of its Fourier series?

For Exercises 8–10, find the n^{th} Fourier polynomial for the given functions, assuming them to be periodic with period 2π. Graph the first three approximations with the original function.

8. $f(x) = x^2, \quad -\pi < x \leq \pi.$

9. $h(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x \leq \pi. \end{cases}$

10. $g(x) = x, \quad -\pi < x \leq \pi.$

Problems

11. **(a)** For $-2\pi \le x \le 2\pi$, use a calculator to sketch:
i) $y = \sin x + \frac{1}{3}\sin 3x$
ii) $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$

(b) Each of the functions in part (a) is a Fourier approximation to a function whose graph is a square wave. What term would you add to the right-hand side of the second function in part (a) to get a better approximation to the square wave?

(c) What is the equation of the square wave function? Is this function continuous?

12. **(a)** Find and graph the third Fourier approximation of the square wave $g(x)$ of period 2π:

$$g(x) = \begin{cases} 0 & -\pi \le x < -\pi/2 \\ 1 & -\pi/2 \le x < \pi/2 \\ 0 & \pi/2 \le x < \pi. \end{cases}$$

(b) How does the result of part (a) differ from that of the square wave in Example 1?

13. Suppose we have a periodic function f with period 1 defined by $f(x) = x$ for $0 \le x < 1$. Find the fourth degree Fourier polynomial for f and graph it on the interval $0 \le x < 1$. [Hint: Remember that since the period is not 2π, you will have to start by doing a substitution. Notice that the terms in the sum are not $\sin(nx)$ and $\cos(nx)$, but instead turn out to be $\sin(2\pi nx)$ and $\cos(2\pi nx)$.]

14. Suppose f has period 2 and $f(x) = x$ for $0 \le x < 2$. Find the fourth-degree Fourier polynomial and graph it on $0 \le x < 2$. [Hint: See Problem 13.]

15. Suppose that a spacecraft near Neptune has measured a quantity A and sent it to earth in the form of a periodic signal $A\cos t$ of amplitude A. On its way to earth, the signal picks up periodic noise, containing only second and higher harmonics. Suppose that the signal $h(t)$ actually received on earth is graphed below in Figure 10.32. Determine the signal that the spacecraft originally sent and hence the value A of the measurement.

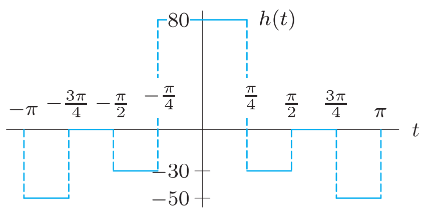


Figure 10.32

16. Figures 10.33 and 10.34 show the waveforms and energy spectra for notes produced by flute and bassoon.[6] Describe the principal differences between the two spectra.

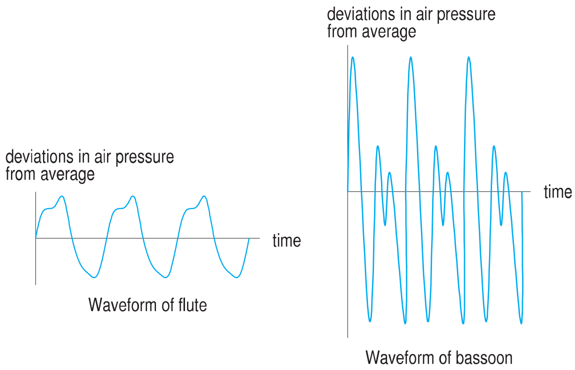


Figure 10.33

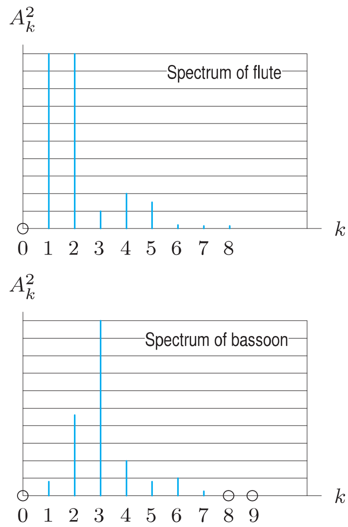


Figure 10.34

17. Show that for positive integers k, the periodic function $f(x) = a_k \cos kx + b_k \sin kx$ of period 2π has energy $a_k^2 + b_k^2$.

18. Given the graph of f in Figure 10.35, find the first two Fourier approximations numerically.

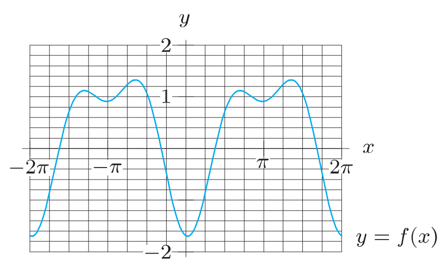


Figure 10.35

[6]Adapted from C.A. Culver, *Musical Acoustics* (New York: McGraw-Hill, 1956), pp. 200, 213.

19. Justify the formula $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$ for the Fourier coefficients, b_k, of a periodic function of period 2π. The argument is similar to that in the text for a_k. In addition to some of the formulas used there, you will need the formulas: $\int_{-\pi}^{\pi} \sin^2(kx)\,dx = \pi$ and $\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = 0$ for $k \neq m$.

In Problems 20–23, the pulse train of width c is the periodic function f of period 2π given by

$$f(x) = \begin{cases} 0 & -\pi \le x < -c/2 \\ 1 & -c/2 \le x < c/2 \\ 0 & c/2 \le x < \pi. \end{cases}$$

20. Suppose that f is the pulse train of width 1.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?

(b) Find a formula for the energy of the k^{th} harmonic of f. Use it to sketch the energy spectrum of f.

(c) How many terms of the Fourier series of f are needed to capture 90% of the energy of f?

(d) Graph f and its fifth Fourier approximation on the interval $[-3\pi, 3\pi]$.

21. Suppose that f is the pulse train of width 0.4.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?

(b) Find a formula for the energy of the k^{th} harmonic of f. Use it to sketch the energy spectrum of f.

(c) What fraction of the energy of f is contained in the constant term and the first five harmonics of f? (The constant term and the first thirteen harmonics are needed to capture 90% of the energy of f.)

(d) Graph f and its fifth Fourier approximation on the interval $[-3\pi, 3\pi]$.

22. Suppose that f is the pulse train of width 2.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?

(b) How many terms of the Fourier series of f are needed to capture 90% of the energy of f?

(c) Graph f and its third Fourier approximation on the interval $[-3\pi, 3\pi]$.

23. After working Problems 20– 22, write a paragraph about the approximation of pulse trains by Fourier polynomials. Explain how the energy spectrum of a pulse train of width c changes as c gets closer and closer to 0 and how this affects the number of terms required for an accurate approximation.

For Problems 24–28, use the table of integrals inside the back cover to show that the following statements are true for positive integers k and m.

24. $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = 0, \quad \text{if } k \neq m.$

25. $\displaystyle\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi.$

26. $\displaystyle\int_{-\pi}^{\pi} \sin^2(mx)\,dx = \pi.$

27. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = 0.$

28. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = 0, \quad \text{if } k \neq m.$

29. Suppose that $f(x)$ is a periodic function with period b. Show that

(a) $g(t) = f(bt/2\pi)$ is periodic with period 2π and $f(x) = g(2\pi x/b)$.

(b) The Fourier series for g is given by

$$g(t) = a_0 + \sum_{k=1}^{\infty} \big(a_k \cos(kt) + b_k \sin(kt)\big)$$

where the coefficients a_0, a_k, b_k are given in the box on page 537.

(c) The Fourier series for f is given by

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k x}{b}\right) + b_k \sin\left(\frac{2\pi k x}{b}\right)\right)$$

where the coefficients are the same as in part (b).

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Taylor series and polynomials**
  General expansion about $x = 0$ or $x = a$; specific series for e^x, $\sin x$, $\cos x$, $(1+x)^p$; using known Taylor series to find others by substitution, integration, and differentiation; interval of convergence; error in Taylor polynomial expansion
- **Fourier series**
  Formula for coefficients on $[-\pi, \pi]$, $[-b, b]$; Energy theorem

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TEN

Exercises

For Exercises 1–4, find the second-degree Taylor polynomial about the given point.

1. e^x, $x = 1$

2. $\ln x$, $x = 2$

3. $\sin x$, $x = -\pi/4$

4. $\tan\theta$, $\theta = \pi/4$

5. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ at $x = 1$.

In Exercises 6–13, find the first four nonzero terms of the Taylor series about the origin of the given functions.

6. $t^2 e^t$

7. $\cos(3y)$

8. $\theta^2 \cos\theta^2$

9. $\sin t^2$

10. $\dfrac{1}{\sqrt{4-x}}$

11. $\dfrac{1}{1-4z^2}$

12. $\dfrac{t}{1+t}$

13. $\dfrac{z^2}{\sqrt{1-z^2}}$

For Problems 14–16, find the Taylor polynomial of degree n for x near the given point a.

14. $\dfrac{1}{1-x}$, $a = 2$, $n = 4$

15. $\dfrac{1}{1+x}$, $a = 2$, $n = 4$

16. $\ln x$, $a = 2$, $n = 4$

For Exercises 17–20, expand the quantity in a Taylor series around 0 in terms of the variable given. Give four nonzero terms.

17. $\dfrac{a}{a+b}$ in terms of $\dfrac{b}{a}$

18. $\dfrac{1}{(a+r)^{3/2}}$ in terms of $\dfrac{r}{a}$

19. $(B^2+y^2)^{3/2}$ in terms of $\dfrac{y}{B}$, where $B > 0$

20. $\sqrt{R-r}$ in terms of $\dfrac{r}{R}$

Problems

Find the exact value of the sums in Problems 21–26.

21. $3 + 3 + \dfrac{3}{2!} + \dfrac{3}{3!} + \dfrac{3}{4!} + \dfrac{3}{5!} + \cdots$

22. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \dfrac{1}{81} - \cdots$

23. $8 + 4 + 2 + 1 + \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^{10}}$

24. $1 - 2 + \dfrac{4}{2!} - \dfrac{8}{3!} + \dfrac{16}{4!} + \cdots$

25. $2 - \dfrac{8}{3!} + \dfrac{32}{5!} - \dfrac{128}{7!} + \cdots$

26. $(0.1)^2 - \dfrac{(0.1)^4}{3!} + \dfrac{(0.1)^6}{5!} - \dfrac{(0.1)^8}{7!} + \cdots$

27. Find an exact value for each of the following sums.

(a) $7(1.02)^3 + 7(1.02)^2 + 7(1.02) + 7 + \dfrac{7}{(1.02)} + \dfrac{7}{(1.02)^2} + \cdots + \dfrac{7}{(1.02)^{100}}$.

(b) $7 + 7(0.1)^2 + \dfrac{7(0.1)^4}{2!} + \dfrac{7(0.1)^6}{3!} + \cdots$.

28. All the derivatives of some function f exist at 0, and that Taylor series for f about $x = 0$ is

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \cdots.$$

Find $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(10)}(0)$.

29. A function f has $f(3) = 1$, $f'(3) = 5$ and $f''(3) = -10$. Find the best estimate you can for $f(3.1)$.

30. Suppose x is positive but very small. Arrange the following expressions in increasing order:

$$x\,,\quad \sin x,\quad \ln(1+x),\quad 1-\cos x,$$
$$e^x - 1,\quad \arctan x,\quad x\sqrt{1-x}.$$

31. By plotting several of its Taylor polynomials and the function $f(x) = 1/(1+x)$, estimate graphically the interval of convergence of the series expansions for this function about $x = 0$. Compute the radius of convergence analytically.

32. Use Taylor series to evaluate $\displaystyle\lim_{x\to 0} \frac{\ln(1+x+x^2) - x}{\sin^2 x}$.

33. **(a)** Find $\displaystyle\lim_{\theta\to 0} \frac{\sin(2\theta)}{\theta}$. Explain your reasoning.

(b) Use series to explain why $f(\theta) = \dfrac{\sin(2\theta)}{\theta}$ looks like a parabola near $\theta = 0$. What is the equation of the parabola?

34. **(a)** Find the Taylor series for $f(t) = te^t$ about $t = 0$.

(b) Using your answer to part (a), find a Taylor series expansion about $x = 0$ for

$$\int_0^x te^t\,dt.$$

(c) Using your answer to part (b), show that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4(2!)} + \frac{1}{5(3!)} + \frac{1}{6(4!)} + \cdots = 1.$$

35. **(a)** Find the Taylor series expansion of $\arcsin x$.
(b) Use Taylor series to find the limit of $\dfrac{\arctan x}{\arcsin x}$ as $x \to 0$.

36. A particle moving along the x-axis has potential energy at the point x given by $V(x)$. The potential energy has a minimum at $x = 0$.

(a) Write the Taylor polynomial of degree 2 for V about $x = 0$. What can you say about the signs of the coefficients of each of the terms of the Taylor polynomial?
(b) The force on the particle at the point x is given by $-V'(x)$. For small x, show that the force on the particle is approximately proportional to its distance from the origin. What is the sign of the proportionality constant? Describe the direction in which the force points.

37. Consider the functions $y = e^{-x^2}$ and $y = 1/(1 + x^2)$.

(a) Write the Taylor expansions for the two functions about $x = 0$. What is similar about the two series? What is different?
(b) Looking at the series, which function do you predict will be greater over the interval $(-1, 1)$? Graph both and see.
(c) Are these functions even or odd? How might you see this by looking at the series expansions?
(d) By looking at the coefficients, explain why it is reasonable that the series for $y = e^{-x^2}$ converges for all values of x, but the series for $y = 1/(1 + x^2)$ converges only on $(-1, 1)$.

38. The electric potential, V, at a distance R along the axis perpendicular to the center of a charged disc with radius a and constant charge density σ, is given by

$$V = 2\pi\sigma(\sqrt{R^2 + a^2} - R).$$

Show that, for large R,

$$V \approx \frac{\pi a^2 \sigma}{R}.$$

39. The *gravitational field* at a point in space is the gravitational force that would be exerted on a unit mass placed there. We will assume that the gravitational field strength at a distance d away from a mass M is

$$\frac{GM}{d^2}$$

where G is constant. In this problem you will investigate the gravitational field strength, F, exerted by a system consisting of a large mass M and a small mass m, with a distance r between them. (See Figure 10.36.)

Figure 10.36

(a) Write an expression for the gravitational field strength, F, at the point P.
(b) Assuming r is small in comparison to R, expand F in a series in r/R.
(c) By discarding terms in $(r/R)^2$ and higher powers, explain why you can view the field as resulting from a single particle of mass $M + m$, plus a correction term. What is the position of the particle of mass $M + m$? Explain the sign of the correction term.

40. A thin disk of radius a and mass M lies horizontally; a particle of mass m is at a height h directly above the center of the disk. The gravitational force, F, exerted by the disk on the mass m is given by

$$F = \frac{2GMmh}{a^2}\left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}}\right).$$

Assume $a < h$ and think of F as a function of a, with the other quantities constant.

(a) Expand F as a series in a/h. Give the first two nonzero terms.
(b) Show that the approximation for F obtained by using only the first nonzero term in the series is independent of the radius, a.
(c) If $a = 0.02h$, by what percentage does the approximation in part (a) differ from the approximation in part (b)?

41. When a body is near the surface of the earth, we usually assume that the force due to gravity on it is a constant mg, where m is the mass of the body and g is the acceleration due to gravity at sea level. For a body at a distance h above the surface of the earth, a more accurate expression for the force F is

$$F = \frac{mgR^2}{(R + h)^2}$$

where R is the radius of the earth. We will consider the situation in which the body is close to the surface of the earth so that h is much smaller than R.

(a) Show that $F \approx mg$.
(b) Express F as mg multiplied by a series in h/R.
(c) The first-order correction to the approximation $F \approx mg$ is obtained by taking the linear term in the series but no higher terms. How far above the surface of the earth can you go before the first-order correction changes the estimate $F \approx mg$ by more than 10%? (Assume $R = 6400$ km.)

42. Expand $f(x + h)$ and $g(x + h)$ in Taylor series and take a limit to confirm the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

43. Use Taylor expansions for $f(y+k)$ and $g(x+h)$ to confirm the chain rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x).$$

44. Suppose all the derivatives of g exist at $x = 0$ and that g has a critical point at $x = 0$.

(a) Write the n^{th} Taylor polynomial for g at $x = 0$.
(b) What does the Second Derivative test for local maxima and minima say?
(c) Use the Taylor polynomial to explain why the Second Derivative test works.

45. (Continuation of Problem 44) You may remember that the Second Derivative test tells us nothing when the second derivative is zero at the critical point. In this problem you will investigate that special case.

Assume g has the same properties as in Problem 44, and that, in addition, $g''(0) = 0$. What does the Taylor polynomial tell you about whether g has a local maximum or minimum at $x = 0$?

46. Use the Fourier polynomials for the square wave

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi \end{cases}$$

to explain why the following sum must approach $\pi/4$ as $n \to \infty$:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1}.$$

47. Suppose that $f(x)$ is a differentiable periodic function of period 2π. Assume the Fourier series of f is differentiable term by term.

(a) If the Fourier coefficients of f are a_k and b_k, show that the Fourier coefficients of its derivative f' are kb_k and $-ka_k$.
(b) How are the amplitudes of the harmonics of f and f' related?
(c) How are the energy spectra of f and f' related?

48. If the Fourier coefficients of f are a_k and b_k, and the Fourier coefficients of g are c_k and d_k, and if A and B are real, show that the Fourier coefficients of $Af + Bg$ are $Aa_k + Bc_k$ and $Ab_k + Bd_k$.

49. Suppose that f is a periodic function of period 2π and that g is a horizontal shift of f, say $g(x) = f(x + c)$. Show that f and g have the same energy.

CAS Challenge Problems

50. (a) Use a computer algebra system to find $P_{10}(x)$ and $Q_{10}(x)$, the Taylor polynomials of degree 10 about $x = 0$ for $\sin^2 x$ and $\cos^2 x$.
(b) What similarities do you observe between the two polynomials? Explain your observation in terms of properties of sine and cosine.

51. (a) Use your computer algebra system to find $P_7(x)$ and $Q_7(x)$, the Taylor polynomials of degree 7 about $x = 0$ for $f(x) = \sin x$ and $g(x) = \sin x \cos x$.
(b) Find the ratio between the coefficient of x^3 in the two polynomials. Do the same for the coefficients of x^5 and x^7.
(c) Describe the pattern in the ratios that you computed in part (b). Explain it using the identity $\sin(2x) = 2\sin x \cos x$.

52. (a) Calculate the equation of the tangent line to the function $f(x) = x^2$ at $x = 2$. Do the same calculation for $g(x) = x^3 - 4x^2 + 8x - 7$ at $x = 1$ and for $h(x) = 2x^3 + 4x^2 - 3x + 7$ at $x = -1$.
(b) Use a computer algebra system to divide $f(x)$ by $(x-2)^2$, giving your result in the form

$$\frac{f(x)}{(x-2)^2} = q(x) + \frac{r(x)}{(x-2)^2},$$

where $q(x)$ is the quotient and $r(x)$ is the remainder. In addition, divide $g(x)$ by $(x-1)^2$ and $h(x)$ by $(x+1)^2$.
(c) For each of the functions, f, g, h, compare your answers to part (a) with the remainder, $r(x)$. What do you notice? Make a conjecture about the tangent line to a polynomial $p(x)$ at the point $x = a$ and the remainder, $r(x)$, obtained from dividing $p(x)$ by $(x-a)^2$.
(d) Use the Taylor expansion of $p(x)$ about $x = a$ to prove your conjecture.[7]

53. Let $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2}$. Although the formula for f is not defined at $x = 0$, we can make f continuous by setting $f(0) = 1$. If we do this, f has a Taylor series about $x = 0$.

(a) Use a computer algebra system to find $P_{10}(x)$, the Taylor polynomial of degree 10 about $x = 0$ for f.
(b) What do you notice about the degrees of the terms in the polynomial? What property of f does this suggest?
(c) Prove that f has the property suggested by part (b).

54. Let $S(x) = \int_0^x \sin(t^2)\,dt$.

(a) Use a computer algebra system to find $P_{11}(x)$, the Taylor polynomial of degree 11 about $x = 0$, for $S(x)$.
(b) What is the percentage error in the approximation of $S(1)$ by $P_{11}(1)$? What about the approximation of $S(2)$ by $P_{11}(2)$?

[7]See "Tangents without Calculus" by Jorge Aarao, in *The College Mathematics Journal* Vol 31, No 5, Nov 2000 (Mathematical Association of America).

CHECK YOUR UNDERSTANDING

Decide if the statements in Problems 1–24 are true or false. Assume that the Taylor series for a function converges to that function. Give an explanation for your answer.

1. If $f(x)$ and $g(x)$ have the same Taylor polynomial of degree 2 near $x = 0$, then $f(x) = g(x)$.

2. Using $\sin\theta \approx \theta - \theta^3/3!$ with $\theta = 1°$, we have $\sin(1°) \approx 1 - 1^3/6 = 5/6$.

3. The Taylor polynomial of degree 2 for e^x near $x = 5$ is $1 + (x-5) + (x-5)^2/2$.

4. If the Taylor polynomial of degree 2 for $f(x)$ near $x = 0$ is $P_2(x) = 1 + x - x^2$, then $f(x)$ is concave up near $x = 0$.

5. The Taylor series for $\sin x$ about $x = \pi$ is

$$(x-\pi) - \frac{(x-\pi)^3}{3!} + \frac{(x-\pi)^5}{5!} - \cdots.$$

6. If f is an even function, then the Taylor series for f near $x = 0$ has only terms with even exponents.

7. The Taylor series for $x^3 \cos x$ about $x = 0$ has only odd powers.

8. If f has the following Taylor series about $x = 0$, then $f^{(7)}(0) = -8$:

$$f(x) = 1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \cdots.$$

(Assume the pattern of the coefficients continues.)

9. The Taylor series for $f(x)g(x)$ about $x = 0$ is

$$f(0)g(0) + f'(0)g'(0)x + \frac{f''(0)g''(0)}{2!}x^2 + \cdots.$$

10. To find the Taylor series for $\sin x + \cos x$ about any point, add the Taylor series for $\sin x$ and $\cos x$ about that point.

11. The quadratic approximation to $f(x)$ for x near 0 is better than the linear approximation for all values of x.

12. The Taylor series for f converges everywhere f is defined.

13. The graphs of e^x and its Taylor polynomial $P_{10}(x)$ get further and further apart as $x \to \infty$.

14. A Taylor polynomial for f near $x = a$ touches the graph of f only at $x = a$.

15. Let $P_n(x)$ be the n^{th} Taylor polynomial for a function f near $x = a$. Although $P_n(x)$ is a good approximation to f near $x = a$, it is not possible to have $P_n(x) = f(x)$ for all x.

16. If $|f^{(n)}(x)| < 10$ for all $n > 0$ and all x, then the Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

17. If f is an even function, then the Fourier series for f on $[-\pi, \pi]$ has only cosines.

18. The linear approximation to $f(x)$ near $x = -1$ shows that if $f(-1) = g(-1)$ and $f'(-1) < g'(-1)$, then $f(x) < g(x)$ for all x sufficiently close to -1 (but not equal to -1).

19. The quadratic approximation to $f(x)$ near $x = -1$ shows that if $f(-1) = g(-1)$, $f'(-1) = g'(-1)$, and $f''(-1) < g''(-1)$, then $f(x) < g(x)$ for all x sufficiently close to -1 (but not equal to -1).

20. If $L_1(x)$ is the linear approximation to $f_1(x)$ near $x = 0$ and $L_2(x)$ is the linear approximation to $f_2(x)$ near $x = 0$, then $L_1(x) + L_2(x)$ is the linear approximation to $f_1(x) + f_2(x)$ near $x = 0$

21. If $L_1(x)$ is the linear approximation to $f_1(x)$ near $x = 0$ and $L_2(x)$ is the linear approximation to $f_2(x)$ near $x = 0$, then $L_1(x)L_2(x)$ is the quadratic approximation to $f_1(x)f_2(x)$ near $x = 0$.

22. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 0$.

23. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 1$.

24. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 1/2$.

PROJECTS FOR CHAPTER TEN

1. Shape of Planets
Rotation causes planets to bulge at the equator. Let α be the angle between the direction downward perpendicular to the surface and the direction toward the center of the planet. At a point on the surface with latitude θ we have

$$\cos\alpha = \frac{1 - A\cos^2\theta}{(1 - 2A\cos^2\theta + A^2\cos^2\theta)^{1/2}},$$

where A is a small positive constant that depends on the particular planet. (For earth, $A = 0.0034$.)

(a) Expand $\cos\alpha$ in powers of A to show that $\cos\alpha \approx 1 - \frac{1}{2}A^2\cos^2\theta\sin^2\theta$.

(b) Show that $\alpha \approx \frac{1}{2} A \sin(2\theta)$.

(c) By what percentage is the approximation in part (b) in error for the earth at latitudes $\theta = 0°$, $20°$, $40°$, $60°$, $80°$?

2. Machin's Formula and the Value of π

(a) In the 17^{th} century, Machin obtained the formula: $\pi/4 = 4\arctan(1/5) - \arctan(1/239)$. Use a calculator to check this formula.

(b) Use the Taylor polynomial approximation of degree 5 to the arctangent function to approximate the value of π. (Note: In 1873 William Shanks used this approach to calculate π to 707 decimal places. Unfortunately, in 1946 it was found that he made an error in the 528^{th} place. Currently, several billion decimal places are known.)

(c) Why do the two series for arctangent converge so rapidly here while the series used in Example 5 on page 521 converges so slowly?

(d) Now we prove Machin's formula using the tangent addition formula

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

(i) Let $A = \arctan(120/119)$ and $B = -\arctan(1/239)$ and show that

$$\arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1.$$

(ii) Let $A = B = \arctan(1/5)$ and show that

$$2\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{5}{12}\right).$$

Use a similar method to show that

$$4\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{120}{119}\right).$$

(iii) Derive Machin's formula.

3. Approximating the Derivative

In applications, the values of a function $f(x)$ are frequently known only at discrete values x_0, $x_0 \pm h$, $x_0 \pm 2h, \ldots$. Suppose we are interested in approximating the derivative $f'(x_0)$. The definition

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

suggests that for small h we can approximate $f'(x)$ as follows:

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}.$$

Such *finite-difference approximations* are used frequently in programming a computer to solve differential equations.[8]

Taylor series can be used to analyze the error in this approximation. Substituting

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \cdots$$

into the approximation for $f'(x_0)$, we find

$$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{f''(x_0)}{2}h + \cdots$$

[8]From Mark Kunka

This suggests (and it can be proved) that the error in the approximation is bounded as follows:

$$\left|\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)\right| \le \frac{Mh}{2},$$

where

$$|f''(x)| \le M \qquad \text{for} \qquad |x-x_0| \le |h|.$$

Notice that as $h \to 0$, the error also goes to zero, provided M is bounded.

As an example, we take $f(x)=e^x$ and $x_0=0$, so $f'(x_0)=1$. The error for various values of h are given in Table 10.2. We see that decreasing h by a factor of 10 decreases the error by a factor of about 10, as predicted by the error bound $Mh/2$.

Table 10.2

h	$(f(x_0+h)-f(x_0))/h$	Error
10^{-1}	1.05171	5.171×10^{-2}
10^{-2}	1.00502	5.02×10^{-3}
10^{-3}	1.00050	5.0×10^{-4}
10^{-4}	1.00005	5.0×10^{-5}

(a) Using Taylor series, suggest an error bound for each of the following finite-difference approximations.

(i) $f'(x_0) \approx \dfrac{f(x_0)-f(x_0-h)}{h}$

(ii) $f'(x_0) \approx \dfrac{f(x_0+h)-f(x_0-h)}{2h}$

(iii) $f'(x_0) \approx \dfrac{-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h)}{12h}$

(b) Use each of the formulas in part (a) to approximate the first derivative of e^x at $x=0$ for $h=10^{-1},10^{-2},10^{-3},10^{-4}$. As h is decreased by a factor of 10, how does the error decrease? Does this agree with the error bounds found in part (a)? Which is the most accurate formula?

(c) Repeat part (b) using $f(x)=1/x$ and $x_0=10^{-5}$. Why are these formulas not good approximations anymore? Continue to decrease h by factors of 10. How small does h have to be before formula (iii) is the best approximation? At these smaller values of h, what changed to make the formulas accurate again?