Chapter I

The Complex Number System

§1. The real numbers

We denote the set of all real numbers by $\mathbb{R}$. It is assumed that each reader is acquainted with the real number system and all its properties. In particular we assume a knowledge of the ordering of $\mathbb{R}$, the definitions and properties of the supremum and infimum (sup and inf), and the completeness of $\mathbb{R}$ (every set in $\mathbb{R}$ which is bounded above has a supremum). It is also assumed that every reader is familiar with sequential convergence in $\mathbb{R}$ and with infinite series. Finally, no one should undertake a study of Complex Variables unless he has a thorough grounding in functions of one real variable. Although it has been traditional to study functions of several real variables before studying analytic function theory, this is not an essential prerequisite for this book. There will not be any occasion when the deep results of this area are needed.

§2. The field of complex numbers

We define $\mathbb{C}$, the complex numbers, to be the set of all ordered pairs (a, b) where a and b are real numbers and where addition and multiplication are defined by:

$$(a, b) + (c, d) = (a+c, b+d)$$

$$(a, b)(c, d) = (ac-bd, bc+ad)$$

It is easily checked that with these definitions $\mathbb{C}$ satisfies all the axioms for a field. That is, $\mathbb{C}$ satisfies the associative, commutative and distributive laws for addition and multiplication; $(0,0)$ and $(1,0)$ are identities for addition and multiplication respectively, and there are additive and multiplicative inverses for each nonzero element in $\mathbb{C}$.

We will write a for the complex number $(a, 0)$. This mapping $a \to (a, 0)$ defines a field isomorphism of $\mathbb{R}$ into $\mathbb{C}$ so we may consider $\mathbb{R}$ as a subset of $\mathbb{C}$. If we put $i = (0, 1)$ then $(a, b) = a + bi$. From this point on we abandon the ordered pair notation for complex numbers.

Note that $i^2 = -1$, so that the equation $z^2 + 1 = 0$ has a root in $\mathbb{C}$. In fact, for each z in $\mathbb{C}$, $z^2 + 1 = (z+i)(z-i)$. More generally, if z and w are complex numbers we obtain

$$z^2 + w^2 = (z + iw)(z - iw)$$

By letting z and w be real numbers a and b we can obtain (with both a and $b \neq 0$)

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\left(\frac{b}{a^2+b^2}\right)$$

so that we have a formula for the reciprocal of a complex number.

When we write $z = a+ib$ ($a, b \in \mathbb{R}$) we call a and b the *real* and *imaginary parts* of z and denote this by $a = \operatorname{Re} z$, $b = \operatorname{Im} z$.

We conclude this section by introducing two operations on $\mathbb{C}$ which are not field operations. If $z = x+iy$ ($x, y \in \mathbb{R}$) then we define $|z| = (x^2+y^2)^{\frac{1}{2}}$ to be the *absolute value* of z and $\bar{z} = x-iy$ is the *conjugate* of z. Note that

2.1 $$|z|^2 = z\bar{z}$$

In particular, if $z \neq 0$ then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

The following are basic properties of absolute values and conjugates whose verifications are left to the reader.

2.2 $$\operatorname{Re} z = \tfrac{1}{2}(z+\bar{z}) \quad \text{and} \quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z}).$$

2.3 $$(\overline{z+w}) = \bar{z}+\bar{w} \quad \text{and} \quad \overline{zw} = \bar{z}\bar{w}.$$

2.4 $$|zw| = |z|\,|w|.$$

2.5 $$|z/w| = |z|/|w|.$$

2.6 $$|\bar{z}| = |z|.$$

The reader should try to avoid expanding z and w into their real and imaginary parts when he tries to prove these last three. Rather, use (2.1), (2.2), and (2.3).

Exercises

1. Find the real and imaginary parts of each of the following:

$$\frac{1}{z}\,;\; \frac{z-a}{z+a}\,(a \in \mathbb{R});\; z^3;\; \frac{3+5i}{7i+1}\,;\; \left(\frac{-1+i\sqrt{3}}{2}\right)^3;$$

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^6;\; i^n;\; \left(\frac{1+i}{\sqrt{2}}\right)^n \quad \text{for} \quad 2 \le n \le 8.$$

2. Find the absolute value and conjugate of each of the following:

$$-2+i;\; -3;\; (2+i)(4+3i);\; \frac{3-i}{\sqrt{2}+3i}\,;\; \frac{i}{i+3}\,;$$

$$(1+i)^6;\; i^{17}.$$

3. Show that z is a real number if and only if $z = \bar{z}$.

4. If z and w are complex numbers, prove the following equations:

$$|z+w|^2 = |z|^2 + 2\operatorname{Re} z\bar{w} + |w|^2.$$

$$|z-w|^2 = |z|^2 - 2\operatorname{Re} z\bar{w} + |w|^2.$$

$$|z+w|^2 + |z-w|^2 = 2(|z|^2+|w|^2).$$

5. Use induction to prove that for $z = z_1 + \ldots + z_n$; $w = w_1 w_2 \ldots w_n$: $|w| = |w_1| \ldots |w_n|$; $\bar{z} = \bar{z}_1 + \ldots + \bar{z}_n$; $\bar{w} = \bar{w}_1 \ldots \bar{w}_n$.

6. Let $R(z)$ be a rational function of z. Show that $\overline{R(z)} = R(\bar{z})$ if all the coefficients in $R(z)$ are real.

§3. The complex plane

From the definition of complex numbers it is clear that each z in $\mathbb{C}$ can be identified with the unique point $(\operatorname{Re} z, \operatorname{Im} z)$ in the plane $\mathbb{R}^2$. The addition of complex numbers is exactly the addition law of the vector space $\mathbb{R}^2$. If z and w are in $\mathbb{C}$ then draw the straight lines from z and w to 0 ($=(0,0)$). These form two sides of a parallelogram with 0, z and w as three vertices. The fourth vertex turns out to be $z+w$.

Note also that $|z-w|$ is exactly the distance between z and w. With this in mind the last equation of Exercise 4 in the preceding section states the *parallelogram law*: The sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of its diagonals.

A fundamental property of a distance function is that it satisfies the triangle inequality (see the next chapter). In this case this inequality becomes

$$|z_1 - z_2| \le |z_1 - z_3| + |z_3 - z_2|$$

for complex numbers z_1, z_2, z_3. By using $z_1 - z_2 = (z_1 - z_3) + (z_3 - z_2)$, it is easy to see that we need only show

3.1 $$|z+w| \le |z| + |w| \quad (z, w \in \mathbb{C}).$$

To show this first observe that for any z in $\mathbb{C}$,

3.2 $$-|z| \le \operatorname{Re} z \le |z|$$

$$-|z| \le \operatorname{Im} z \le |z|$$

Hence, $\operatorname{Re}(z\bar{w}) \le |z\bar{w}| = |z|\,|w|$. Thus,

$$|z+w|^2 = |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2$$

$$\le |z|^2 + 2|z|\,|w| + |w|^2$$

$$= (|z|+|w|)^2,$$

from which (3.1) follows. (This is called the *triangle inequality* because, if we represent z and w in the plane, (3.1) says that the length of one side of the triangle $[0, z, z+w]$ is less than the sum of the lengths of the other two sides. Or, the shortest distance between two points is a straight line.) On encounter-

ing an inequality one should always ask for necessary and sufficient conditions that equality obtains. From looking at a triangle and considering the geometrical significance of (3.1) we are led to consider the condition $z = tw$ for some $t \in \mathbb{R}$, $t \geq 0$. (or $w = tz$ if $w = 0$). It is clear that equality will occur when the two points are colinear with the origin. In fact, if we look at the proof of (3.1) we see that a necessary and sufficient condition for $|z+w| = |z|+|w|$ is that $|z\bar{w}| = \text{Re}\,(z\bar{w})$. Equivalently, this is $z\bar{w} \geq 0$ (i.e., $z\bar{w}$ is a real number and is non negative). Multiplying this by w/w we get $|w|^2(z/w) \geq 0$ if $w \neq 0$. If

$$t = z/w = \left(\frac{1}{|w|^2}\right)|w|^2(z/w)$$

then $t \geq 0$ and $z = tw$.

By induction we also get

3.3 $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|+\ldots+|z_n|$

Also useful is the inequality

3.4 $\left| |z|-|w| \right| \leq |z-w|$

Now that we have given a geometric interpretation of the absolute value let us see what taking a complex conjugate does to a point in the plane. This is also easy; in fact, $\bar{z}$ is the point obtained by reflecting z across the x-axis (i.e., the real axis).

Exercises

1. Prove (3.4) and give necessary and sufficient conditions for equality.
2. Show that equality occurs in (3.3) if and only if $z_k/z_l \geq 0$ for any integers k and l, $1 \leq k, l \leq n$, for which $z_l \neq 0$.
3. Let $a \in \mathbb{R}$ and $c > 0$ be fixed. Describe the set of points z satisfying

$$|z-a|-|z+a| = 2c$$

for every possible choice of a and c. Now let a be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.

§4. Polar representation and roots of complex numbers

Consider the point $z = x+iy$ in the complex plane $\mathbb{C}$. This point has polar coordinates (r, θ): $x = r \cos \theta$, $y = r \sin \theta$. Clearly $r = |z|$ and θ is the angle between the positive real axis and the line segment from 0 to z. Notice that θ plus any multiple of 2π can be substituted for θ in the above equations. The angle θ is called the *argument of z* and is denoted by $\theta = \arg z$. Because of the ambiguity of θ, "arg" is not a function. We introduce the notation

4.1 $\text{cis}\,\theta = \cos \theta + i \sin \theta.$

Let $z_1 = r_1 \text{ cis } \theta_1$, $z_2 = r_2 \text{ cis } \theta_2$. Then $z_1 z_2 = r_1 r_2 \text{ cis } \theta_1 \text{ cis } \theta_2 = r_1 r_2$ $[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 + \sin \theta_2 \cos \theta_1)]$. By the formulas for the sine and cosine of the sum of two angles we get

4.2 $z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

Alternately, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. (What function of a real variable takes products into sums?) By induction we get for $z_k = r_k \text{ cis } \theta_k$, $1 \leq k \leq n$.

4.3 $z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n \text{ cis } (\theta_1 + \ldots + \theta_n)$

In particular,

4.4 $z^n = r^n \text{ cis } (n\theta),$

for every integer $n \geq 0$. Moreover if $z \neq 0$, $z \cdot [r^{-1} \text{ cis } (-\theta)] = 1$; so that (4.4) also holds for all integers n, positive, negative, and zero, if $z \neq 0$. As a special case of (4.4) we get *de Moivre's formula*:

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

We are now in a position to consider the following problem: For a given complex number $a \neq 0$ and an integer $n \geq 2$, can you find a number z satisfying $z^n = a$? How many such z can you find? In light of (4.4) the solution is easy. Let $a = |a| \text{ cis } \alpha$; by (4.4), $z = |a|^{1/n} \text{ cis } (\alpha/n)$ fills the bill. However this is not the only solution because $z' = |a|^{1/n} \text{ cis } \frac{1}{n}(\alpha + 2\pi)$ also satisfies $(z')^n = a$. In fact each of the numbers

4.5 $|a|^{1/n} \text{ cis } \frac{1}{n}(\alpha + 2\pi k), 0 \leq k \leq n-1,$

in an nth root of a. By means of (4.4) we arrive at the following: for each non zero number a in $\mathbb{C}$ there are n *distinct* nth roots of a; they are given by formula (4.5).

Example

Calculate the nth roots of unity. Since $1 = \text{cis } 0$, (4.5) gives these roots as

$$1, \text{ cis } \frac{2\pi}{n}, \text{ cis } \frac{4\pi}{n}, \ldots, \text{ cis } \frac{2\pi}{n}(n-1).$$

In particular, the cube roots of unity are

$$1, \frac{1}{\sqrt{2}}(1+i\sqrt{3}), \frac{1}{\sqrt{2}}(1-i\sqrt{3}).$$

Exercises

1. Find the sixth roots of unity.

2. Calculate the following:
   (a) the square roots of i
   (b) the cube roots of i
   (c) the square roots of $\sqrt{3} + 3i$

3. A *primitive* nth *root of unity* is a complex number a such that $1, a, a^2, \ldots, a^{n-1}$ are distinct nth roots of unity. Show that if a and b are primitive nth and mth roots of unity, respectively, then ab is a kth root of unity for some integer k. What is the smallest value of k? What can be said if a and b are nonprimitive roots of unity?

4. Use the binomial equation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and compare the real and imaginary parts of each side of de Moivre's formula to obtain the formulas:

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \ldots$$

$$\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \ldots$$

5. Let $z = \operatorname{cis} \dfrac{2\pi}{n}$ for an integer $n \geq 2$. Show that $1 + z + \ldots + z^{n-1} = 0$.

6. Show that $\varphi(t) = \operatorname{cis} t$ is a group homomorphism of the additive group $\mathbb{R}$ onto the multiplicative group $T = \{z : |z| = 1\}$.

7. If $z \in \mathbb{C}$ and $\operatorname{Re}(z^n) \geq 0$ for every positive integer n, show that z is a positive real number.

§5. *Lines and half planes in the complex plane*

Let L denote a straight line in $\mathbb{C}$. From elementary analytic geometry, L is determined by a point in L and a direction vector. Thus if a is any point in L and b is its direction vector then

$$L = \{z = a + tb : -\infty < t < \infty\}.$$

Since $b \neq 0$ this gives, for z in L,

$$\operatorname{Im}\left(\frac{z-a}{b}\right) = 0.$$

In fact if z is such that

$$0 = \operatorname{Im}\left(\frac{z-a}{b}\right)$$

then

$$t \quad \left(\frac{z-a}{b}\right)$$

implies that $z = a + tb$, $-\infty < t < \infty$. That is

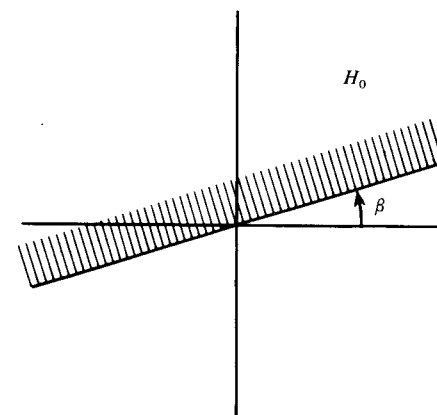$$5.1 \qquad\qquad L = \left\{z : \operatorname{Im}\left(\frac{z-a}{b}\right) = 0\right\}.$$

What is the locus of each of the sets

$$\left\{z : \operatorname{Im}\left(\frac{z-a}{b}\right) > 0\right\},$$

$$\left\{z : \operatorname{Im}\left(\frac{z-a}{b}\right) < 0\right\}?$$

As a first step in answering this question, observe that since b is a direction we may assume $|b| = 1$. For the moment, let us consider the case where $a = 0$, and put $H_0 = \{z : \operatorname{Im}(z/b) > 0\}$, $b = \operatorname{cis} \beta$. If $z = r \operatorname{cis} \theta$ then $z/b = r \operatorname{cis}(\theta - \beta)$. Thus, z is in H_0 if and only if $\sin(\theta - \beta) > 0$; that is, when $\beta < \theta < \pi + \beta$. Hence H_0 is the half plane lying to the left of the line L if

we are "walking along L in the direction of b." If we put

$$H_a = \left\{z : \operatorname{Im}\left(\frac{z-a}{b}\right) > 0\right\}$$

then it is easy to see that $H_a = a + H_0 \equiv \{a + w : w \in H_0\}$; that is, H_a is the translation of H_0 by a. Hence, H_a is the half plane lying to the left of L. Similarly,

$$K_a = \left\{z : \operatorname{Im}\left(\frac{z-a}{b}\right) < 0\right\}$$

is the half plane on the right of L.

Exercise

1. Let C be the circle $\{z : |z - c| = r\}$, $r > 0$; let $a = c + r \operatorname{cis} \alpha$ and put

$$L_\beta = \left\{ z : \text{Im}\left(\frac{z-a}{b}\right) = 0 \right\}$$

where $b = \text{cis } \beta$. Find necessary and sufficient conditions in terms of β that L_β be tangent to C at a.

§6. The extended plane and its spherical representation

Often in complex analysis we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the *extended plane* which is $\mathbb{C} \cup \{\infty\} \equiv \mathbb{C}_\infty$. We also wish to introduce a distance function on $\mathbb{C}_\infty$ in order to discuss continuity properties of functions assuming the value infinity. To accomplish this and to give a concrete picture of $\mathbb{C}_\infty$ we represent $\mathbb{C}_\infty$ as the unit sphere in $\mathbb{R}^3$,

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Let $N = (0, 0, 1)$; that is, N is the north pole on S. Also, identify $\mathbb{C}$ with $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ so that $\mathbb{C}$ cuts S along the equator. Now for each point z in $\mathbb{C}$ consider the straght line in $\mathbb{R}^3$ through z and N. This intersects
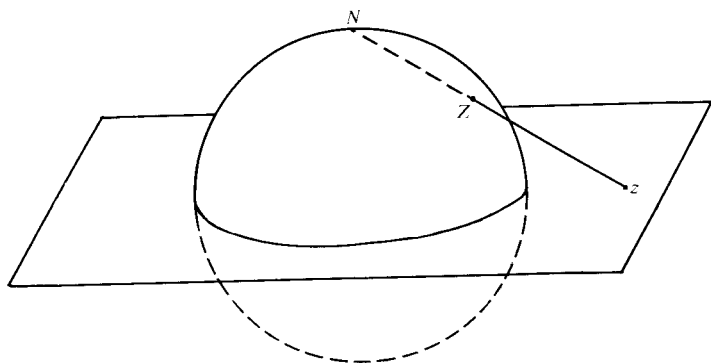
the sphere in exactly one point $Z \neq N$. If $|z| > 1$ then Z is in the northern hemisphere and if $|z| < 1$ then Z is in the southern hemisphere; also, for $|z| = 1$, $Z = z$. What happens to Z as $|z| \to \infty$? Clearly Z approaches N; hence, we identify N and the point ∞ in $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ is represented as the sphere S.

Let us explore this representation. Put $z = x + iy$ and let $Z = (x_1, x_2, x_3)$ be the corresponding point on S. We will find equations expressing x_1, x_2, and x_3 in terms of x and y. The line in $\mathbb{R}^3$ through z and N is given by $\{tN + (1-t)z : -\infty < t < \infty\}$, or by

6.1 $$\{((1-t)x, (1-t)y, t) : -\infty < t < \infty\}.$$

Hence, we can find the coordinates of Z if we can find the value of t at

which this line intersects S. If t is this value then

$$1 = (1-t)^2 x^2 + (1-t)^2 y^2 + t^2$$
$$= (1-t)^2 |z|^2 + t^2$$

From which we get

$$1 - t^2 = (1-t)^2 |z|^2.$$

Since $t \neq 1$ ($z \neq \infty$) we arrive at

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Thus

6.2 $$x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

But this gives

6.3 $$x_1 = \frac{z + \bar{z}}{|z|^2 + 1} \quad x_2 = \frac{-i(z - \bar{z})}{|z|^2 + 1} \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

If the point Z is given ($Z \neq N$) and we wish to find z then by setting $t = x_3$ and using (6.1), we arrive at

6.4 $$z = \frac{x_1 + ix_2}{1 - x_3}$$

Now let us define a distance function between points in the extended plane in the following manner: for z, z' in $\mathbb{C}_\infty$ define the distance from z to z', $d(z, z')$, to be the distance between the corresponding points Z and Z' in $\mathbb{R}^3$. If $Z = (x_1, x_2, x_3)$ and $Z' = (x_1', x_2', x_3')$ then

6.5 $$d(z, z') = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2]^{\frac{1}{2}}$$

Using the fact that Z and Z' are on S, (6.5) gives

6.6 $$[d(z, z')]^2 = 2 - 2(x_1 x_1' + x_2 x_2' + x_3 x_3')$$

By using equation (6.3) we get

6.7 $$d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{\frac{1}{2}}}, \quad (z, z' \in \mathbb{C})$$

In a similar manner we get for z in $\mathbb{C}$

6.8 $$d(z, \infty) = \frac{2}{(1 + |z|^2)^{\frac{1}{2}}}$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the *stereographic projection*.

Exercises

1. Give the details in the derivation of (6.7) and (6.8).
2. For each of the following points in $\mathbb{C}$, give the corresponding point of S: $0, 1+i, 3+2i$.
3. Which subsets of S correspond to the real and imaginary axes in $\mathbb{C}$.
4. Let Λ be a circle lying in S. Then there is a unique plane P in $\mathbb{R}^3$ such that $P \cap S = \Lambda$. Recall from analytic geometry that

$$P = \{(x_1, x_2, x_3): x_1\beta_1 + x_2\beta_2 + x_3\beta_3 = l\}$$

where $(\beta_1, \beta_2, \beta_3)$ is a vector orthogonal to P and l is some real number. It can be assumed that $\beta_1^2 + \beta_2^2 + \beta_3^2 = 1$. Use this information to show that if Λ contains the point N then its projection on $\mathbb{C}$ is a straight line. Otherwise, Λ projects onto a circle in C.
5. Let Z and Z' be points on S corresponding to z and z' respectively. Let W be the point on S corresponding to $z+z'$. Find the coordinates of W in terms of the coordinates of Z and Z'.

Chapter II

Metric Spaces and the Topology of $\mathbb{C}$

§1. Definition and examples of metric spaces

A *metric space* is a pair (X, d) where X is a set and d is a function from $X \times X$ into $\mathbb{R}$, called a *distance function* or *metric*, which satisfies the following conditions for x, y, and z in X:

$$d(x, y) \geq 0$$

$$d(x, y) = 0 \text{ if and only if } x = y$$

$$d(x, y) = d(y, x) \text{ (symmetry)}$$

$$d(x, z) \leq d(x, y) + d(y, z) \text{ (triangle inequality)}$$

If x and $r > 0$ are fixed then define

$$B(x; r) = \{y \in X: d(x, y) < r\}$$
$$\bar{B}(x; r) = \{y \in X: d(x, y) \leq r\}.$$

$B(x; r)$ and $\bar{B}(x; r)$ are called the *open* and *closed balls*, respectively, with center x and radius r.

Examples

1.1 Let $X = \mathbb{R}$ or $\mathbb{C}$ and define $d(z, w) = |z - w|$. This makes both $(\mathbb{R}, d)$ and $(\mathbb{C}, d)$ metric spaces. In fact, $(\mathbb{C}, d)$ will be the example of principal interest to us. If the reader has never encountered the concept of a metric space before this, he should continually keep $(\mathbb{C}, d)$ in mind during the study of this chapter.
1.2 Let (X, d) be a metric space and let $Y \subset X$; then (Y, d) is also a metric space.
1.3 Let $X = \mathbb{C}$ and define $d(x+iy, a+ib) = |x-a| + |y-b|$. Then $(\mathbb{C}, d)$ is a metric space.
1.4 Let $X = \mathbb{C}$ and define $d(x+iy, a+ib) = \max\{|x-a|, |y-b|\}$.
1.5 Let X be any set and define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. To show that the function d satisfies the triangle inequality one merely considers all possibilities of equality among x, y, and z. Notice here that $B(x; \epsilon)$ consists only of the point x if $\epsilon \leq 1$ and $B(x; \epsilon) = X$ if $\epsilon > 1$. This metric space does not appear in the study of analytic function theory.
1.6 Let $X = \mathbb{R}^n$ and for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ define

$$d(x, y) = \left[\sum_{j=1}^{n} (x_j - y_j)^2 \right]^{\frac{1}{2}}$$