Chapter III

Elementary Properties and Examples of Analytic Functions

§1. Power series

In this section the definition and basic properties of a power series will be given. The power series will then be used to give examples of analytic functions. Before doing this it is necessary to give some elementary facts on infinite series in $\mathbb{C}$ whose statements for infinite series in $\mathbb{R}$ should be well known to the reader. If a_n is in $\mathbb{C}$ for every $n \geq 0$ then the series $\sum_{n=0}^{\infty} a_n$ *converges* to z iff for every $\epsilon > 0$ there is an integer N such that $|\sum_{n=0}^{m} a_n - z| < \epsilon$ whenever $m \geq N$. The series $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.

1.1 Proposition. *If $\sum a_n$ converges absolutely then $\sum a_n$ converges.*

Proof. Let $\epsilon > 0$ and put $z_n = a_0 + a_1 + \ldots + a_n$. Since $\sum |a_n|$ converges there is an integer N such that $\sum_{n=N}^{\infty} |a_n| < \epsilon$. Thus, if $m > k \geq N$,

$$|z_m - z_k| = \left|\sum_{n=k+1}^{m} a_n\right| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{\infty} |a_n| < \epsilon.$$

That is, $\{z_n\}$ is a Cauchy sequence and so there is a z in $\mathbb{C}$ with $z = \lim z_n$. Hence $\sum a_n = z$. ■

Also recall the definitions of limit inferior and superior of a sequence in $\mathbb{R}$. If $\{a_n\}$ is a sequence in $\mathbb{R}$ then define

$$\liminf a_n = \lim_{n \to \infty} [\inf \{a_n, a_{n+1}, \ldots\}]$$

$$\limsup a_n = \lim_{n \to \infty} [\sup \{a_n, a_{n+1}, \ldots\}]$$

An alternate notation for $\liminf a_n$ and $\limsup a_n$ is $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$. If $b_n = \inf \{a_n, a_{n+1}, \ldots\}$ then $\{b_n\}$ is an increasing sequence of real numbers or $\{-\infty\}$. Hence, $\liminf a_n$ always exists although it may be $\pm\infty$. Similarly $\limsup a_n$ always exists although it may be $\pm\infty$.

A number of properties of lim inf and lim sup are included in the exercises of this section.

A *power series* about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$. One of the easiest examples of a power series (and one of the most useful) is the *geometric series* $\sum_{n=0}^{\infty} z^n$. For which values of z does this series converge and

when does it diverge? It is easy to see that $1 - z^{n+1} = (1-z)(1+z+\ldots+z^n)$, so that

1.2
$$1+z+\ldots+z^n = \frac{1-z^{n+1}}{1-z}.$$

If $|z| < 1$ then $0 = \lim z^n$ and so the geometric series is convergent with

$$\sum_{0}^{\infty} z^n = \frac{1}{1-z}.$$

If $|z| > 1$ then $\lim |z|^n = \infty$ and the series diverges. Not only is this result an archetype for what happens to a general power series, but it can be used to explore the convergence properties of power series.

1.3 Theorem. *For a given power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ define the number R, $0 \leq R \leq \infty$, by*

$$\frac{1}{R} = \limsup |a_n|^{1/n},$$

then:

(a) *if $|z-a| < R$, the series converges absolutely*:
(b) *if $|z-a| > R$, the terms of the series become unbounded and so the series diverges*;
(c) *if $0 < r < R$ then the series converges uniformly on $\{z\colon |z| \leq r\}$.*

Moreover, the number R is the only number having properties (a) *and* (b).

Proof. We may suppose that $a = 0$. If $|z| < R$ there is an r with $|z| < r < R$. Thus, there is an integer N such that $|a_n|^{1/n} < \frac{1}{r}$ for all $n \geq N$ $\left(\text{because } \frac{1}{r} > \frac{1}{R}\right)$. But then $|a_n| < \frac{1}{r^n}$ and so $|a_n z^n| < \left(\frac{|z|}{r}\right)^n$ for all $n \geq N$. This says that the tail end $\sum_{n=N}^{\infty} a_n z^n$ is dominated by the series $\sum \left(\frac{|z|}{r}\right)^n$, and since $\frac{|z|}{r} < 1$ the power series converges absolutely for each $|z| < R$.

Now suppose $r < R$ and choose ρ such that $r < \rho < R$. As above, let N be such that $|a_n| < \frac{1}{\rho^n}$ for all $n \geq N$. Then if $|z| \leq r$, $|a_n z^n| \leq \left(\frac{r}{\rho}\right)^n$ and $\left(\frac{r}{\rho}\right) < 1$. Hence the Weierstrass M-test gives that the power series converges uniformly on $\{z\colon |z| \leq r\}$. This proves parts (a) and (c).

To prove (b), let $|z| > R$ and choose r with $|z| > r > R$. Hence $\frac{1}{r} < \frac{1}{R}$; from the definition of lim sup, this gives infinitely many integers n with $\frac{1}{r} < |a_n|^{1/n}$. It follows that $|a_n z^n| > \left(\frac{|z|}{r}\right)^n$ and, since $\left(\frac{|z|}{r}\right) > 1$, these terms become unbounded. ■

The number R is called the *radius of convergence* of the power series.

1.4 Proposition. *If $\sum a_n(z-a)^n$ is a given power series with radius of convergence R, then*

$$R = \lim |a_n/a_{n+1}|$$

if this limit exists.

Proof. Again assume that $a = 0$ and let $\alpha = \lim |a_n/a_{n+1}|$, which we suppose to exist. Suppose that $|z| < r < \alpha$ and find an integer N such that $r < |a_n/a_{n+1}|$ for all $n \geq N$. Let $B = |a_N|r^N$; then $|a_{N+1}|r^{N+1} = |a_{N+1}|rr^N < |a_N|r^N = B$; $|a_{N+2}|r^{N+2} = |a_{N+2}|rr^{N+1} < |a_{N+1}|r^{N+1} < B$; continuing we get $|a_n r^n| \leq B$ for all $n \geq N$. But then $|a_n z^n| = |a_n r^n| \dfrac{|z|^n}{r^n} \leq B\dfrac{|z|^n}{r^n}$ for all $n \geq N$. Since $|z| < r$ we get that $\sum_{n=1}^{\infty} |a_n z^n|$ is dominated by a convergent series and hence converges. Since $r < \alpha$ was arbitrary this gives that $\alpha \leq R$.

On the other hand if $|z| > r > \alpha$, then $|a_n| < r|a_{n+1}|$ for all n larger than some integer N. As before, we get $|a_n r^n| \geq B = |a_N r^N|$ for $n \geq N$. This gives $|a_n z^n| \geq B\dfrac{|z|^n}{|r|^n}$ which approaches ∞ as n does. Hence, $\sum a_n z^n$ diverges and so $R \leq \alpha$. Thus $R = \alpha$. ■

Consider the series $\sum_{n=0}^{\infty} \dfrac{z^n}{n!}$; by Proposition 1.4 we have that this series has radius of convergence ∞. Hence it converges at every complex number and the convergence is uniform on each compact subset of $\mathbb{C}$. Maintaining a parallel with calculus, we designate this series by

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

the *exponential series* or *function.*

Recall the following proposition from the theory of infinite series (the proof will not be given).

1.5 Proposition. *Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series and put*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Then $\sum c_n$ is absolutely convergent with sum

$$(\sum a_n)(\sum b_n).$$

1.6 Proposition. *Let $\sum a_n(z-a)^n$ and $\sum b_n(z-a)^n$ be power series with radius of convergence $\geq r > 0$. Put*

$$c_n = \sum_{k=0}^{n} a_k b_{n-k};$$

then both power series $\sum (a_n+b_n)(z-a)^n$ and $\sum c_n(z-a)^n$ have radius of convergence $\geq r$, and

$$\sum (a_n+b_n)(z-a)^n = [\sum a_n(z-a)^n + \sum b_n(z-a)^n]$$
$$\sum c_n(z-a)^n = [\sum a_n(z-a)^n][\sum b_n(z-a)^n]$$

for $|z-a| < r$.

Proof. We only give an outline of the proof. If $0 < s < r$ then for $|z| \leq s$, we get $\sum |a_n+b_n|\ |z|^n \leq \sum |a_n|s^n + \sum |b_n|s^n < \infty$; $\sum |c_n|\ |z|^n \leq (\sum |a_n|s^n)(\sum |b_n|s^n) < \infty$. From here the proof can easily be completed. ■

Exercises

1. Prove Proposition 1.5.
2. Give the details of the proof of Proposition 1.6.
3. Prove that $\limsup (a_n+b_n) \leq \limsup a_n + \limsup b_n$ and $\liminf (a_n+b_n) \geq \liminf a_n + \liminf b_n$ for $\{a_n\}$ and $\{b_n\}$ sequences of real numbers.
4. Show that $\liminf a_n \leq \limsup a_n$ for any sequence in $\mathbb{R}$.
5. If $\{a_n\}$ is a convergent sequence in $\mathbb{R}$ and $a = \lim a_n$, show that $a = \liminf a_n = \limsup a_n$.
6. Find the radius of convergence for each of the following power series: (a) $\sum_{n=0}^{\infty} a^n z^n$, $a \in \mathbb{C}$; (b) $\sum_{n=0}^{\infty} a^{n^2} z^n$, $a \in \mathbb{C}$; (c) $\sum_{n=0}^{\infty} k^n z^n$, k an integer $\neq 0$; (d) $\sum_{n=0}^{\infty} z^{n!}$.
7. Show that the radius of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$
is 1, and discuss convergence for $z = 1$, -1, and i. (Hint: The nth coefficient of this series is not $(-1)^n/n$.)

§2. *Analytic functions*

In this section analytic functions are defined and some examples are given. It is also shown that the Cauchy-Riemann equations hold for the real and imaginary parts of an analytic function.

2.1 Definition. If G is an open set in $\mathbb{C}$ and $f\colon G \to \mathbb{C}$ then f is *differentiable at a point a* in G if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists; the value of this limit is denoted by $f'(a)$ and is called the *derivative* of f at a. If f is differentiable at each point of G we say that f is *differentiable on G*. Notice that if f is differentiable on G then $f'(a)$ defines a function $f'\colon G \to \mathbb{C}$. If f' is continuous then we say that f is *continuously differentiable.* If f' is differentiable then f is *twice differentiable*; continuing, a differentiable function such that each successive derivative is again differentiable is called *infinitely differentiable*.

(Henceforward, all functions will be assumed to take their values in $\mathbb{C}$ unless it is stated to the contrary.)

The following was surely predicted by the reader.

2.2 Proposition. *If $f\colon G \to \mathbb{C}$ is differentiable at a point a in G then f is continuous at a.*

Proof. In fact,

$$\lim_{z\to a} |f(z)-f(a)| = \left[\lim_{z\to a} \frac{|f(z)-f(a)|}{|z-a|}\right] \cdot \left[\lim_{z\to a} |z-a|\right] = f'(a)\cdot 0 = 0. \blacksquare$$

2.3 Definition. A function $f\colon G\to\mathbb{C}$ is *analytic* if f is continuously differentiable on G.

It follows readily, as in calculus, that sums and products of functions analytic on G are analytic. Also, if f and g are analytic on G and G_1 is the set of points in G where g doesn't vanish, then f/g is analytic on G_1.

Since constant functions and the function z are clearly analytic it follows that all rational functions are analytic on the complement of the set of zeros of the denominator.

Moreover, the usual laws for differentiating sums, products, and quotients remain valid.

2.4 Chain Rule. *Let f and g be analytic on G and Ω respectively and suppose $f(G)\subset\Omega$. Then $g\circ f$ is analytic on G and*

$$(g\circ f)'(z) = g'(f(z))f'(z)$$

for all z in G.

Proof. Fix z_0 in G and choose a positive number r such that $B(z_0;r)\subset G$. We must show that if $0<|h_n|<r$ and $\lim h_n=0$ then $\lim\{h_n^{-1}[g(f(z_0+h_n)) - g(f(z_0))]\}$ exists and equals $g'(f(z_0))f'(z_0)$. (Why will this suffice for a proof?)

Case 1 Suppose $f(z_0)\neq f(z_0+h_n)$ for all n.

In this case

$$\frac{g\circ f(z_0+h_n)-g\circ f(z_0)}{h_n} = \frac{g(f(z_0+h_n))-g(f(z_0))}{f(z_0+h_n)-f(z_0)}\cdot\frac{f(z_0+h_n)-f(z_0)}{h_n}.$$

Since $\lim[f(z_0+h_n)-f(z_0)]=0$ by (2.2) we have that

$$\lim h_n^{-1}[g\circ f(z_0+h_n)-g\circ f(z_0)] = g'(f(z_0))f'(z_0)$$

Case 2 $f(z_0)=f(z_0+h_n)$ for infinitely many values of n.

Write $\{h_n\}$ as the union of two sequences $\{k_n\}$ and $\{l_n\}$ where $f(z_0)\neq f(z_0+k_n)$ and $f(z_0)=f(z_0+l_n)$ for all n. Since f is differentiable, $f'(z_0)=\lim l_n^{-1}[f(z_0+l_n)-f(z_0)]=0$. Also $\lim l_n^{-1}[g\circ f(z_0+l_n)-g\circ f(z_0)]=0$. By Case 1, $\lim k_n^{-1}[g\circ f(z_0+k_n)-g\circ f(z_0)]=g'(f(z_0))f'(z_0)=0$. Therefore $\lim h_n^{-1}[g\circ f(z_0+h_n)-g\circ f(z_0)]=0=g'(f(z_0))f'(z_0)$.

The general case easily follows from the preceding two. $\blacksquare$

In order to define the derivative, the function was assumed to be defined on an open set. If we say f is analytic on a set A and A is not open, we mean that f is analytic on an open set containing A.

Perhaps the definition of analytic function has been anticlimatic to many readers. After seeing books written on analytic functions and year-long courses and seminars on the theory of analytic functions, one can excuse a certain degree of disappointment in discovering that the definition has already been encountered in calculus. Is this theory to be a simple generalization of calculus? The answer is a resounding no. To show how vastly different the two subjects are let us mention that we will show that *a differentiable function is analytic.* This is truly a remarkable result and one for which there is no analogue in the theory of functions of a real variable (e.g., consider $x^2 \sin \frac{1}{x}$). Another equally remarkable result is that *every analytic function is infinitely differentiable and, furthermore, has a power series expansion about each point of its domain.* How can such a humble hypothesis give such far-reaching results? One can get come indication of what produces this phenomenon if one considers the definition of derivative.

In the complex variable case there are an infinity of directions in which a variable can approach a point a. In the real case, however, there are only two avenues of approach. Continuity, for example, of a function defined on $\mathbb{R}$ can be discussed in terms of right and left continuity; this is far from the case for functions of a complex variable. So the statement that a function of a complex variable has a derivative is stronger than the same statement about a function of a real variable. Even more, if we consider a function f defined on $G \subset \mathbb{C}$ as a function of two real variables by putting $g(x, y) = f(x+iy)$ for $(x, y) \in G$, then requiring that f be Frechet differentiable will not ensure that f has a derivative in our sense.

In an exercise we ask the reader to show that $f(z) = |z|^2$ has a derivative only at $z = 0$; but, $g(x, y) = f(x+iy) = x^2+y^2$ is Frechet differentiable.

That differentiability implies analyticity is proved in Chapter IV; but right now we prove that power series are analytic functions.

2.5 Proposition. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ have radius of convergence $R > 0$. Then:*

(a) *For each $k \geq 1$ the series*

2.6 $$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-a)^{n-k}$$

has radius of convergence R;

(b) *The function f is infinitely differentiable on $B(a;R)$ and, furthermore, $f^{(k)}(z)$ is given by the series (2.6) for all $k \geq 1$ and $|z-a| < R$;*

(c) *For $n \geq 0$,*

2.7 $$a_n = \frac{1}{n!}f^{(n)}(a).$$

Proof. Again assume that $a = 0$.

(a) We first remark that if (a) is proved for $k = 1$ then the cases $k = 2, \dots$ will follow. In fact, the case $k = 2$ can be obtained by applying part (a) for $k = 1$ to the series $\sum na_n(z-a)^{n-1}$. We have that $R^{-1} = \limsup |a_n|^{1/n}$; we wish to show that $R^{-1} = \limsup |na_n|^{1/(n-1)}$. Now it follows from l'Hôpital's rule that $\lim\limits_{n\to\infty} \frac{\log n}{n-1} = 0$, so that $\lim\limits_{n\to\infty} n^{1/(n-1)} = 1$. The result will follow from Exercise 2 if it can be shown that $\limsup |a_n|^{1/(n-1)} = R^{-1}$.

Let $(R')^{-1} = \limsup |a_n|^{1/(n-1)}$; then R' is the radius of convergence of $\sum\limits_1^\infty a_n z^{n-1} = \sum\limits_0^\infty a_{n+1} z^n$. Notice that $z \sum a_{n+1} z^n + a_0 = \sum a_n z^n$; hence if $|z| < R'$ then $\sum |a_n z^n| \leq |a_0| + |z| \sum |a_{n+1} z^n| < \infty$. This gives $R' \leq R$. If $|z| < R$ and $z \neq 0$ then $\sum |a_n z^n| < \infty$ and $\sum |a_{n+1} z^n| \leq \frac{1}{|z|} \cdot \sum |a_n z^n| + \frac{1}{|z|}|a_0| < \infty$, so that $R \leq R'$. This gives that $R = R'$ and completes the proof of part (a).

(b) For $|z| < R$ put $g(z) = \sum\limits_{n=1}^\infty n a_n z^{n-1}$, $s_n(z) = \sum\limits_{k=0}^n a_k z^k$, and $R_n(z) = \sum\limits_{k=n+1}^\infty a_k z^k$. Fix a point w in $B(0; R)$ and fix r with $|w| < r < R$; we wish to show that $f'(w)$ exists and is equal to $g(w)$. To do this let $\delta > 0$ be arbitrary except for the restriction that $\bar{B}(w; \delta) \subset B(0; r)$. (We will further restrict δ later in the proof.) Let $z \in B(w; \delta)$; then

2.8
$$\frac{f(z)-f(w)}{z-w} - g(w) = \left[\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right] + [s_n'(w) - g(w)] + \left[\frac{R_n(z)-R_n(w)}{z-w}\right]$$

Now
$$\frac{R_n(z)-R_n(w)}{z-w} = \frac{1}{z-w} \sum_{k=n+1}^\infty a_k(z^k - w^k) = \sum_{k=n+1}^\infty a_k\left(\frac{z^k-w^k}{z-w}\right)$$

But
$$\frac{|z^k - w^k|}{|z-w|} = |z^{k-1} + z^{k-2}w + \ldots + zw^{k-2} + w^{k-1}| \leq kr^{k-1}.$$

Hence,
$$\left|\frac{R_n(z)-R_n(w)}{z-w}\right| \leq \sum_{k=n+1}^\infty |a_k| k r^{k-1}$$

Since $r < R$, $\sum\limits_{k=1}^\infty |a_k| k r^{k-1}$ converges and so for any $\epsilon > 0$ there is an integer N_1 such that for $n \geq N_1$
$$\left|\frac{R_n(z)-R_n(w)}{z-w}\right| < \frac{\epsilon}{3} \qquad (z \in B(w; \delta)).$$

Also, $\lim s_n'(w) = g(w)$ so there is an integer N_2 such that $|s_n'(w) - g(w)| < \frac{\epsilon}{3}$ whenever $n \geq N_2$. Let $n \geq$ the maximum of the two integers N_1 and N_2.

Then we can choose $\delta > 0$ such that
$$\left|\frac{s_n(z)-s_n(w)}{z-w} - s_n'(w)\right| < \frac{\epsilon}{3}$$
whenever $0 < |z-w| < \delta$. Putting these inequalities together with equation (2.8) we have that
$$\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \epsilon$$
for $0 < |z-w| < \delta$. That is, $f'(w) = g(w)$.

(c) By a straightforward evaluation we get $f(0) = f^{(0)}(0) = a_0$. Using (2.6) (for $a = 0$), we get $f^{(k)}(0) = k! a_k$ and this gives formula (2.7). ■

2.9 Corollary. *If the series $\sum\limits_{n=0}^\infty a_n(z-a)^n$ has radius of convergence $R > 0$ then $f(z) = \sum\limits_{n=0}^\infty a_n(z-a)^n$ is analytic in $B(a; R)$.*

Hence, $\exp z = \sum\limits_{n=0}^\infty z^n/n!$ is analytic in $\mathbb{C}$. Before further examining the exponential function and defining $\cos z$ and $\sin z$, the following result must be proved.

2.10 Proposition. *If G is open and connected and $f: G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G, then f is constant.*

Proof. Fix z_0 in G and let $\omega_0 = f(z_0)$. Put $A = \{z \in G: f(z) = \omega_0\}$; we will show that $A = G$ by showing that A is both open and closed in G. Let $z \in G$ and let $\{z_n\} \subset A$ be such that $z = \lim z_n$. Since $f(z_n) = \omega_0$ for each $n \geq 1$ and f is continuous we get $f(z) = \omega_0$, or $z \in A$. Thus, A is closed in G. Now fix a in A, and let $\epsilon > 0$ be such that $B(a; \epsilon) \subset G$. If $z \in B(a; \epsilon)$, set $g(t) = f(tz + (1-t)a)$, $0 \leq t \leq 1$. Then

2.11
$$\frac{g(t)-g(s)}{t-s} = \frac{g(t)-g(s)}{(t-s)z+(s-t)a} \cdot \frac{(t-s)z+(s-t)a}{t-s}.$$

Thus, if we let $t \to s$ we get (A.4(b), Appendix A)
$$\lim_{t\to s} \frac{g(t)-g(s)}{t-s} = f'(sz+(1-s)a)\cdot(z-a) = 0.$$

That is, $g'(s) = 0$ for $0 \leq s \leq 1$, implying that g is a constant. Hence, $f(z) = g(1) = g(0) = f(a) = \omega_0$. That is, $B(a; \epsilon) \subset A$ and A is also open. ■

Now differentiate $f(z) = e^z$; we do this by Proposition 2.5. This gives that
$$f'(z) = \sum_{n=1}^\infty \frac{n}{n!} z^{n-1} = \sum_{n=1}^\infty \frac{1}{(n-1)!} z^{n-1} = \sum_{n=0}^\infty \frac{1}{n!} z^n = f(z).$$

Thus the complex exponential function has the same property as its real counterpart. That is

2.12
$$\frac{d}{dz} e^z = e^z$$

Put $g(z) = e^z e^{a-z}$ for some fixed a in $\mathbb{C}$; then $g'(z) = e^z e^{a-z} + e^z(-e^{a-z}) = 0$. Hence $g(z) = \omega$ for all z in $\mathbb{C}$ and some constant ω. In particular, using $e^0 = 1$ we get $\omega = g(0) = e^a$. Then $e^z e^{a-z} = e^a$ for all z. Thus $e^{a+b} = e^a e^b$ for all a and b in $\mathbb{C}$. This also gives $1 = e^z e^{-z}$ which implies that $e^z \neq 0$ for any z and $e^{-z} = 1/e^z$. Returning to the power series expansion of e^z, since all the coefficients of this series are real we have $\exp \bar{z} = \overline{\exp z}$. In particular, for θ a real number we get $|e^{i\theta}|^2 = e^{i\theta}e^{-i\theta} = e^0 = 1$. More generally, $|e^z|^2 = e^z e^{\bar{z}} = e^{z+\bar{z}} = \exp(2 \operatorname{Re} z)$. Thus,

2.13 $$|\exp z| = \exp(\operatorname{Re} z).$$

We see, therefore, that e^z has the same properties that the real function e^x has. Again by analogy with the real power series we define the functions $\cos z$ and $\sin z$ by the power series

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + (-1)^n \frac{z^{2n}}{(2n)!} + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \ldots$$

Each of the series has infinite radius of convergence and so $\cos z$ and $\sin z$ are analytic in $\mathbb{C}$. By using Proposition 2.5 we find that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$. By manipulating power series (which is justified since these series converge absolutely)

2.14 $$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

This gives for z in $\mathbb{C}$, $\cos^2 z + \sin^2 z = 1$ and

2.15 $$e^{iz} = \cos z + i \sin z.$$

In particular if we let $z =$ a real number θ in (2.15) we get $e^{i\theta} = \operatorname{cis} \theta$. Hence, for z in $\mathbb{C}$

2.16 $$z = |z|e^{i\theta}$$

where $\theta = \arg z$. Since $e^{x+iy} = e^x e^{iy}$ we have $|e^z| = \exp(\operatorname{Re} z)$ and $\arg e^z = \operatorname{Im} z$.

A function f is *periodic* with *period* c if $f(z+c) = f(z)$ for all z in $\mathbb{C}$. If c is a period of e^z then $e^z = e^{z+c} = e^z e^c$ implies that $e^c = 1$. Since $1 = |e^c| = \exp \operatorname{Re}(c)$, $\operatorname{Re}(c) = 0$. Thus $c = i\theta$ for some θ in $\mathbb{R}$. But $1 = e^c = e^{i\theta} = \cos\theta + i \sin\theta$ gives that the periods of e^z are the multiples of $2\pi i$. Thus, if we divide the plane into infinitely many horizontal strips by the lines $\operatorname{Im} z = \pi(2k-1)$, k any integer, the exponential function behaves the same in each of these strips. This property of periodicity is one which is not present in the real exponential function. Notice that by examining complex functions we have demonstrated a relationship (2.15) between the exponential function and the trigonometric functions which was not expected from our knowledge of the real case.

Now let us define $\log z$. We could adopt the same procedure as before and let $\log z$ be the power series expansion of the real logarithm about some point. But this only gives $\log z$ in some disk. The method of defining the logarithm as the integral of t^{-1} from 1 to x, $x > 0$, is a possibility, but proves to be risky and unsatisfying in the complex case. Also, since e^z is not a one-one map as in the real case, $\log z$ cannot be defined as the inverse of e^z. We can, however, do something similar.

We want to define $\log w$ so that it satisfies $w = e^z$ when $z = \log w$. Now since $e^z \neq 0$ for any z we cannot define $\log 0$. Therefore, suppose $e^z = w$ and $w \neq 0$; if $z = x + iy$ then $|w| = e^x$ and $y = \arg w + 2\pi k$, for some k. Hence

2.17 $$\{\log |w| + i(\arg w + 2\pi k) : k \text{ is any integer}\}$$

is the solution set for $e^z = w$. (Note that $\log |w|$ is the usual real logarithm.)

2.18 Definition. If G is an open connected set in $\mathbb{C}$ and $f: G \to \mathbb{C}$ is a continuous function such that $z = \exp f(z)$ for all z in G then f is a *branch of the logarithm.*

Notice that $0 \notin G$.

Suppose f is a given branch of the logarithm on the connected set G and suppose k is an integer. Let $g(z) = f(z) + 2\pi k i$. Then $\exp g(z) = \exp f(z) = z$, so g is also a branch of the logarithm. Conversely, if f and g are both branches of $\log z$ then for each z in G, $f(z) = g(z) + 2\pi k i$ for some integer k, where k depends on z. Does the same k work for each z in G? The answer is yes. In fact, if $h(z) = \dfrac{1}{2\pi i}[f(z) - g(z)]$ then h is continuous on G and $h(G) \subset \mathbb{Z}$, the integers. Since G is connected, $h(G)$ must also be connected (Theorem II. 5.8). Hence there is a k in $\mathbb{Z}$ with $f(z) + 2\pi k i = g(z)$ for all z in G. This gives

2.19 Proposition. *If $G \subset \mathbb{C}$ is open and connected and f is a branch of* $\log z$ *on G then the totality of branches of* $\log z$ *are the functions $f(z) + 2\pi k i$, $k \in \mathbb{Z}$.*

Now let us manufacture at least one branch of $\log z$ on some open connected set. Let

$$G = \mathbb{C} - \{z : z \leq 0\};$$

that is, "slit" the plane along the negative real axis. Clearly G is connected and each z in G can be uniquely represented by $z = |z|e^{i\theta}$ where $-\pi < \theta < \pi$. For θ in this range, define $f(re^{i\theta}) = \log r + i\theta$. We leave the proof of continuity to the reader (Exercise 9). It follows that f is a branch of the logarithm on G.

Is f analytic? To answer this we first prove a general fact.

2.20 Proposition. *Let G and Ω be open subsets of $\mathbb{C}$. Suppose that $f: G \to \mathbb{C}$ and $g: \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ and $g(f(z)) = z$ for all z in G. If g is differentiable and $g'(z) \neq 0$, f is differentiable and*

$$f'(z) = \frac{1}{g'(f(z))}$$

If g is analytic, f is analytic

Proof. Fix a in G and let $h \in \mathbb{C}$ such that $h \neq 0$ and $a+h \in G$. Hence $a = g(f(a))$ and $a+h = g(f(a+h))$ implies $f(a) \neq f(a+h)$. Also

$$1 = \frac{g(f(a+h))-g(f(a))}{h}$$

$$= \frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)} \cdot \frac{f(a+h)-f(a)}{h}.$$

Now the limit of the left hand side as $h \to 0$ is, of course, 1; so the limit of the right hand side exists. Since $\lim_{h\to 0} [f(a+h)-f(a)] = 0$,

$$\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)} = g'(f(a)).$$

Hence we get that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists since $g'(f(a)) \neq 0$, and $1 = g'(f(a))f'(a)$.

Thus, $f'(z) = [g'(f(z))]^{-1}$. If g is analytic then g' is continuous and this gives that f is analytic. ■

2.21 Corollary. *A branch of the logarithm function is analytic and its derivative is z^{-1}.*

We designate the particular branch of the logarithm defined above on $\mathbb{C}-\{z: z \leq 0\}$ to be the *principal branch of the logarithm*. If we write $\log z$ as a function we will always take it to be the principal branch of the logarithm unless otherwise stated.

If f is a branch of the logarithm on an open connected set G and if b in $\mathbb{C}$ is fixed then define $g: G \to \mathbb{C}$ by $g(z) = \exp(bf(z))$. If b is an integer, then $g(z) = z^b$. In this manner we define a branch of z^b, b in $\mathbb{C}$, for an open connected set on which there is a branch of $\log z$. If we write $g(z) = z^b$ as a function we will always understand that $z^b = \exp(b \log z)$ where $\log z$ is the principal branch of the logarithm; z^b is analytic since $\log z$ is.

As is evident from the considerations just concluded, connectedness plays an important role in analytic function theory. For example, Proposition 2.10 is false unless G is connected. This is analogous to the role played by intervals in calculus. Because of this it is convenient to introduce the term "region." A *region* is an open connected subset of the plane.

This section concludes with a discussion of the Cauchy-Riemann equations. Let $f: G \to \mathbb{C}$ be analytic and let $u(x, y) = \text{Re}\, f(x+iy)$, $v(x, y) = \text{Im}\, f(x+iy)$ for $x+iy$ in G. Let us evaluate the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

in two different ways. First let $h \to 0$ through real values of h. For $h \neq 0$

$$\frac{f(z+h)-f(z)}{h} = \frac{f(x+h+iy)-f(x+iy)}{h}$$

$$= \frac{u(x+h, y)-u(x, y)}{h} + i\,\frac{v(x+h, y)-v(x, y)}{h}$$

Letting $h \to 0$ gives

2.22 $$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\,\frac{\partial v}{\partial x}(x, y)$$

Now let $h \to 0$ through purely imaginary values; that is, for $h \neq 0$ and h real,

$$\frac{f(z+ih)-f(z)}{ih} = -i\,\frac{u(x, y+h)-u(x, y)}{h} + \frac{v(x, y+h)-v(x, y)}{h}$$

Thus,

2.23 $$f'(z) = -i\,\frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y)$$

Equating the real and imaginary parts of (2.22) and (2.23) we get the *Cauchy-Riemann equations*

2.24 $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Suppose that u and v have continuous second partial derivatives (we will eventually show that they are infinitely differentiable). Differentiating the Cauchy-Riemann equations again we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Hence,

2.25 $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Any function with continuous second derivatives satisfying (2.25) is said to be *harmonic*. In a similar fashion, v is also harmonic. We will study harmonic functions in Chapter X.

Let G be a region in the plane and let u and v be functions defined on G with continuous partial derivatives. Furthermore, suppose that u and v satisfy the Cauchy-Riemann equations. If $f(z) = u(z)+iv(z)$ then f can be shown to be analytic in G. To see this, let $z = x+iy \in G$ and let $B(z; r) \subset G$. If $h = s+it \in B(0; r)$ then

$$u(x+s, y+t)-u(x, y) = [u(x+s, y+t)-u(x, y+t)]+[u(x, y+t)-u(x, y)]$$

Applying the mean value theorem for the derivative of a function of one variable to each of these bracketed expressions, yields for each $s+it$ in $B(0; r)$ numbers s_1 and t_1 such that $|s_1| < |s|$ and $|t_1| < |t|$ and

2.26 $$\begin{cases} u(x+s, y+t)-u(x, y+t) = u_x(x+s_1, y+t)s \\ u(x, y+t)-u(x, y) = u_y(x, y+t_1)t \end{cases}$$

Letting

$$\varphi(s, t) = [u(x+s, y+t)-u(x, y)]-[u_x(x, y)s+u_y(x, y)t]$$

(2.26) gives that

$$\frac{\varphi(s,t)}{s+it} = \frac{s}{s+it}\left[u_x(x+s_1,y+t)-u_x(x,y)\right]+\frac{t}{s+it}\left[u_y(x,y+t_1)-u_y(x,y)\right]$$

But $|s| \leq |s+it|$, $|t| \leq |s+it|$, $|s_1| < |s|$, $|t_1| < |t|$, and the fact that u_x and u_y are continuous gives that

2.27
$$\lim_{s+it\to 0} \frac{\varphi(s, t)}{s+it} = 0$$

Hence

$$u(x+s, y+t)-u(x, y) = u_x(x, y)s+u_y(x, y)t+\varphi(s, t)$$

where φ satisfies (2.27). Similarly

$$v(x+s, y+t)-v(x, y) = v_x(x, y)s+v_y(x, y)t+\psi(s, t)$$

where ψ satisfies

2.28
$$\lim_{s+it\to 0} \frac{\psi(s, t)}{s+it} = 0$$

Using the fact that u and v satisfy the Cauchy-Riemann equations it is easy to see that

$$\frac{f(z+s+it)-f(z)}{s+it} = u_x(z)+iv_x(z) + \frac{\varphi(s, t)+\psi(x, t)}{s+it}$$

In light of (2.27) and (2.28), f is differentiable and $f'(z) = u_x(z)+iv_x(z)$. Since u_x and v_x are continuous, f' is continuous and f is analytic. These results are summarized as follows.

2.29. Theorem. *Let u and v be real-valued functions defined on a region G and suppose that u and v have continuous partial derivatives. Then $f\colon G \to \mathbb{C}$ defined by $f(z) = u(z)+iv(z)$ is analytic iff u and v satisfy the Cauchy-Riemann equations.*

Example. Is $u(x, y) = \log (x^2+y^2)^{\frac{1}{2}}$ harmonic on $G = \mathbb{C}-\{0\}$? The answer is yes! This could be shown by differentiating u to see that it satisfies (2.25). However, it can also be shown by observing that in a neighborhood of each point of G, u is the real part of an analytic function defined in that neighborhood. (Which function?)

Another problem concerning harmonic functions which will be taken up in more detail in Section VIII. 3, is the following. Suppose G is a region in the plane and $u\colon G \to \mathbb{R}$ is harmonic. Does there exist a harmonic function $v\colon G \to \mathbb{R}$ such that $f = u+iv$ is analytic in G? If such a function v exists it is called a *harmonic conjugate* of u. If v_1 and v_2 are two harmonic conjugates of u then $i(v_1-v_2) = (u+iv_1)-(u+iv_2)$ is analytic on G and only takes on

purely imaginary values. It follows that two harmonic conjugates of a harmonic function differ by a constant (see Exercise 14).

Returning to the question of the existence of a harmonic conjugate, the above example $u(z) = \log |z|$ of a harmonic function on the region $G = \mathbb{C}-\{0\}$ has no harmonic conjugate. Indeed, if it did then it would be possible to define an analytic branch of the logarithm on G and this cannot be done. (Exercise 21.) However, there are some regions for which every harmonic function has a conjugate. In particular, it will now be shown that this is the case when G is any disk or the whole plane.

2.30 Theorem. *Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u\colon G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.*

Proof. To carry out the proof of this theorem, Leibniz's rule for differentiating under the integral sign is needed (this is stated and proved in Proposition IV. 2.1). Let $G = B(0; R)$, $0 < R \leq \infty$, and let $u\colon G \to \mathbb{R}$ be a harmonic function. The proof will be accomplished by finding a harmonic function v such that u and v satisfy the Cauchy-Riemann equations. So define

$$v(x, y) = \int_0^y u_x(x, t)dt+\varphi(x)$$

and determine φ so that $v_x = -u_y$. Differentiating both sides of this equation with respect to x gives

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\, dt+\varphi'(x) \\ &= -\int_0^y u_{yy}(x, t)\, dt+\varphi'(x) \\ &= -u_y(x, y)+u_y(x, 0)+\varphi'(x) \end{aligned}$$

So it must be that $\varphi'(x) = -u_y(x, 0)$. It is easily checked that u and

$$v(x, y) = \int_0^y u_x(x, t)dt-\int_0^x u_y(s, 0)ds$$

do satisfy the Cauchy-Riemann equations. ■

Where was the fact that G is a disk or $\mathbb{C}$ used? Why can't this method of proof be doctored sufficiently that it holds for general regions G? Where does the proof break down when $G = \mathbb{C}-\{0\}$ and $u(z) = \log |z|$?

Exercises

1. Show that $f(z) = |z|^2 = x^2+y^2$ has a derivative only at the origin.
2. Prove that if b_n, a_n are real and positive and $0 < b = \lim b_n$, $a = \lim \sup a_n$ then $ab = \lim \sup (a_n b_n)$. Does this remain true if the requirement of positivity is dropped?
3. Show that $\lim n^{1/n} = 1$.

4. Show that $(\cos z)' = -\sin z$ and $(\sin z)' = \cos z$.
5. Derive formulas (2.14).
6. Describe the following sets: $\{z: e^z = i\}$, $\{z: e^z = -1\}$, $\{z: e^z = -i\}$, $\{z: \cos z = 0\}$, $\{z: \sin z = 0\}$.
7. Prove formulas for $\cos(z+w)$ and $\sin(z+w)$.
8. Define $\tan z = \dfrac{\sin z}{\cos z}$; where is this function defined and analytic?
9. Suppose that z_n, $z \in G = \mathbb{C} - \{z: z \le 0\}$ and $z_n = r_n e^{i\theta_n}$, $z = re^{i\theta}$ where $-\pi < \theta, \theta_n < \pi$. Prove that if $z_n \to z$ then $\theta_n \to \theta$ and $r_n \to r$.
10. Prove the following generalization of Proposition 2.20. Let G and Ω be open in $\mathbb{C}$ and suppose f and h are functions defined on G, $g: \Omega \to \mathbb{C}$ and suppose that $f(G) \subset \Omega$. Suppose that g and h are analytic, $g'(\omega) \ne 0$ for any ω, that f is continuous, h is one-one, and that they satisfy $h(z) = g(f(z))$ for z in G. Show that f is analytic. Give a formula for $f'(z)$.
11. Suppose that $f: G \to \mathbb{C}$ is a branch of the logarithm and that n is an integer. Prove that $z^n = \exp(nf(z))$ for all z in G.
12. Show that the real part of the function $z^{\frac{1}{2}}$ is always positive.
13. Let $G = \mathbb{C} - \{z: z \le 0\}$ and let n be a positive integer. Find all analytic functions $f: G \to \mathbb{C}$ such that $z = (f(z))^n$ for all $z \in G$.
14. Suppose $f: G \to \mathbb{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.
15. For $r > 0$ let $A = \left\{\omega: \omega = \exp\left(\dfrac{1}{z}\right) \text{ where } 0 < |z| < r\right\}$; determine the set A.
16. Find an open connected set $G \subset \mathbb{C}$ and two continuous functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all z in G. Can you make G maximal? Are f and g analytic?
17. Give the principal branch of $\sqrt{1-z}$.
18. Let $f: G \to \mathbb{C}$ and $g: G \to \mathbb{C}$ be branches of z^a and z^b respectively. Show that fg is a branch of z^{a+b} and f/g is a branch of z^{a-b}. Suppose that $f(G) \subset G$ and $g(G) \subset G$ and prove that both $f \circ g$ and $g \circ f$ are branches of z^{ab}.
19. Let G be a region and define $G^* = \{z: \bar{z} \in G\}$. If $f: G \to \mathbb{C}$ is analytic prove that $f^*: G^* \to C$, defined by $f^*(z) = \overline{f(\bar{z})}$, is also analytic.
20. Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $\operatorname{Re} z_k > 0$ and $\operatorname{Re}(z_1 \ldots z_k) > 0$ for $1 \le k \le n$. Show that $\log(z_1 \ldots z_n) = \log z_1 + \ldots + \log z_n$, where $\log z$ is the principal branch of the logarithm. If the restrictions on the z_k are removed, does the formula remain valid?
21. Prove that there is no branch of the logarithm defined on $G = \mathbb{C} - \{0\}$. (Hint: Suppose such a branch exists and compare this with the principal branch.)

§3. *Analytic functions as mappings. Möbius transformations*

Consider the function defined by $f(z) = z^2$. If $z = x+iy$ and $\mu + i\nu = f(z)$ then $\mu = x^2 - y^2$, $\nu = 2xy$. Hence, the hyperbolas $x^2 - y^2 = c$ and $2xy = d$ are mapped by f into the straight lines $\mu = c$, $\nu = d$. One interesting fact is

that for c and d not zero, these hyperbolas intersect at right angles, just as their images do. This is not an isolated phenomenon and this property will be explored in general later in this section.

Now examine what happens to the lines $x = c$ and $y = d$. First consider $x = c$ (y arbitrary); f maps this line into $\mu = c^2 - y^2$ and $\nu = 2cy$. Eliminating y we get that $x = c$ is mapped onto the parabola $\nu^2 = -4c^2(\mu - c^2)$. Similarly, f takes the line $y = d$ onto the parabola $\nu^2 = 4d^2(\mu + d^2)$. These parabolas intersect at $(c^2 - d^2, \pm 2|cd|)$. It is relevant to point out that as $c \to 0$ the parabola $\nu^2 = -4c^2(\mu - c^2)$ gets closer and closer to the negative real axis. This corresponds to the fact that the function $z^{\frac{1}{2}}$ maps $G = \mathbb{C} - \{z: z \le 0\}$ onto $\{z: \operatorname{Re} z > 0\}$. Notice also that $x = c$ and $x = -c$ (and $y = d$, $y = -d$) are mapped onto the same parabolas.

What happens to a circle centered at the origin? If $z = re^{i\theta}$ then $f(z) = r^2 e^{2i\theta}$; thus, the circle of radius r about the origin is mapped onto the circle of radius r^2 in a two to one fashion.

Finally, what happens to the sector $S(\alpha, \beta) = \{z: \alpha < \arg z < \beta\}$, for $\alpha < \beta$? It is easily seen that the image of $S(\alpha, \beta)$ is the sector $S(2\alpha, 2\beta)$. The restriction of f to $S(\alpha, \beta)$ will be one-one exactly when $\beta - \alpha < \pi$.

The above discussion sheds some light on the nature of $f(z) = z^2$ and, likewise, it is useful to study the mapping properties' other analytic functions. In the theory of analytic functions the following problem holds a paramount position: given two open connected sets G and Ω, is there an analytic function f defined on G such that $f(G) = \Omega$? Besides being intrinsically interesting, the solution (or rather, the information about the existence of a solution) of this problem is very useful.

3.1 Definition. A *path* in a region $G \subset \mathbb{C}$ is a continuous function $\gamma: [a,b] \to G$ for some interval $[a,b]$ in R. If $\gamma'(t)$ exists for each t in $[a,b]$ and $\gamma': [a,b] \to \mathbb{C}$ is continuous then γ is a *smooth path*. Also γ is *piecewise smooth* if there is a partition of $[a,b]$, $a = t_0 < t_1 < \ldots < t_n = b$, such that γ is smooth on each subinterval $[t_{j-1}, t_j]$, $1 \le j \le n$.

To say that a function $\gamma: [a,b] \to \mathbb{C}$ has a derivative $\gamma'(t)$ for each point t in $[a,b]$ means that

$$\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = \gamma'(t)$$

exists for $a < t < b$ and that the right and left sided limits exist for $t = a$ and $t = b$, respectively. This is, of course, equivalent to saying that $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ have a derivative (see Appendix A).

Suppose $\gamma: [a,b] \to G$ is a smooth path and that for some t_0 in (a,b), $\gamma'(t_0) \ne 0$. Then γ has a tangent line at the point $z_0 = \gamma(t_0)$. This line goes through the point z_0 in the direction of (the vector) $\gamma'(t_0)$; or, the slope of the line is $\tan(\arg\gamma'(t_0))$. If γ_1 and γ_2 are two smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \ne 0$, $\gamma_2'(t_2) \ne 0$, then define the *angle between the paths* γ_1 and γ_2 at z_0 to be

$$\arg\gamma_2'(t_2) - \arg\gamma_1'(t_1).$$

Suppose γ is a smooth path in G and $f: G \to \mathbb{C}$ is analytic. Then $\sigma = f \circ \gamma$ is also a smooth path and $\sigma'(t) = f'(\gamma(t))\gamma'(t)$. Let $z_0 = \gamma(t_0)$, and suppose that $\gamma'(t_0) \neq 0$ and $f'(z_0) \neq 0$; then $\sigma'(t_0) \neq 0$ and $\arg \sigma'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0)$. That is,

3.2 $$\arg \sigma'(t_0) - \arg \gamma'(t_0) = \arg f'(z_0).$$

Now let γ_1 and γ_2 be smooth paths with $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0 \neq \gamma_2'(t_2)$; let $\sigma_1 = f \circ \gamma_1$ and $\sigma_2 = f \circ \gamma_2$. Also, suppose that the paths γ_1 and γ_2 are not tangent to each other at z_0; that is, suppose $\gamma_1'(t_1) \neq \gamma_2'(t_2)$. Equation (3.2) gives

3.3 $$\arg \gamma_2'(t_2) - \arg \gamma_1'(t_1) = \arg \sigma_2'(t_2) - \arg \sigma_1'(t_1).$$

This says that given any two paths through z_0, f maps these paths onto two paths through $\omega_0 = f(z_0)$ and, when $f'(z_0) \neq 0$, the angles between the curves are preserved both in *magnitude and direction.* This summarizes as follows.

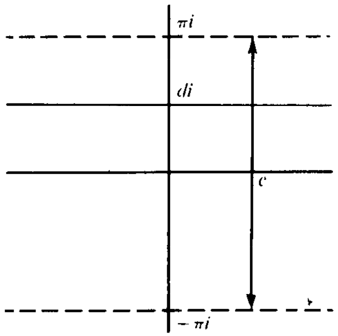
3.4 Theorem. *If f: $G \to \mathbb{C}$ is analytic then f preserves angles at each point z_0 of G where $f'(z_0) \neq 0$.*

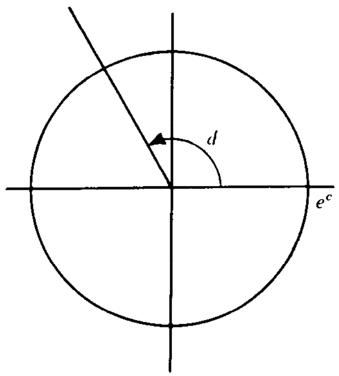
A function f: $G \to \mathbb{C}$ which has the angle preserving property and also has

$$\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|}$$

existing is called a *conformal map.* If f is analytic and $f'(z) \neq 0$ for any z then f is conformal. The converse of this statement is also true.

If $f(z) = e^z$ then f is conformal throughout $\mathbb{C}$; let us look at the exponential function more closely. If $z = c + iy$ where c is fixed then $f(z) = re^{iy}$ for $r = e^c$. That is, f maps the line $x = c$ onto the circle with center at the origin and of radius e^c. Also, f maps the line $y = d$ onto the infinite ray $\{re^{id}: 0 < r < \infty\}$.





We have already seen that e^z is one-one on any horizontal strip of width $< 2\pi$. Let $G = \{z: -\pi < \text{Im } z < \pi\}$. Then $f(G) = \Omega = \mathbb{C} - \{z: z \leq 0\}$; also f maps the vertical segments $\{z = c + iy, -\pi < y < \pi\}$ onto the part of the circle $\{e^c e^{i\theta}: -\pi < \theta < \pi\}$, and the horizontal line $y = d$, $-\pi < d < \pi$, goes onto the ray making an angle d with the positive real axis.

Notice that $\log z$, the principal branch of the logarithm, does the opposite. It maps Ω onto the strip G, circles onto vertical segments in G, rays onto horizontal lines in G.

The exploration of the mapping properties of $\cos z$, $\sin z$, and other analytic functions will be done in the exercises. We now proceed to an amazing class of mappings, the Möbius transformations.

3.5 Definition. A mapping of the form $S(z) = \dfrac{az+b}{cz+d}$ is called a *linear fractional transformation.* If a, b, c, and d also satisfy $ad - bc \neq 0$ then $S(z)$ is called a *Möbius transformation.*

If S is a Möbius transformation then $S^{-1}(z) = \dfrac{dz-b}{-cz+a}$ satisfies $S(S^{-1}(z)) = S^{-1}(S(z)) = z$; that is, S^{-1} is the inverse mapping of S. If S and T are both linear fractional transformations then it follows that $S \circ T$ is also. Hence, the set of Möbius maps forms a group under composition. Unless otherwise stated, the only linear fractional transformations we will consider are Möbius transformations.

Let $S(z) = \dfrac{az+b}{cz+d}$; if λ is any non-zero complex number, then

$$S(z) = \frac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)}.$$

That is, the coefficients a, b, c, d are not unique (see Exercise 20).

We may also consider S as defined on $\mathbb{C}_\infty$ with $S(\infty) = a/c$ and $S(-d/c) = \infty$. (Notice that we cannot have $a = 0 = c$ or $d = 0 = c$ since either situation would contradict $ad - bc \neq 0$.) Since S has an inverse it maps $\mathbb{C}_\infty$ onto $\mathbb{C}_\infty$.

If $S(z) = z + a$ then S is called a *translation*; if $S(z) = az$ with $a \neq 0$ then S is a *dilation*; if $S(z) = e^{i\theta}z$ then it is a *rotation*; finally, if $S(z) = 1/z$ it is the *inversion.*

3.6 Proposition. *If S is a Möbius transformation then S is the composition of translations, dilations, and the inversion.* (Of course, some of these may be missing.)

Proof. First, suppose $c = 0$. Hence $S(z) = (a/d)z + (b/d)$ so if $S_1(z) = (a/d)z$, $S_2(z) = z + (b/d)$, then $S_2 \circ S_1 = S$ and we are done.

Now let $c \neq 0$ and put $S_1(z) = z + d/c$, $S_2(z) = 1/z$, $S_3(z) = \dfrac{(bc-ad)}{c^2} z$, $S_4(z) = z + a/c$. Then $S = S_4 \circ S_3 \circ S_2 \circ S_1$. ■

What are the fixed points of S? That is, what are the points z satisfying $S(z) = z$. If z satisfies this condition then

$$cz^2 + (d-a)z - b = 0.$$

Hence, a Möbius transformation can have at most two fixed points unless $S(z) = z$ for all z.

Now let S be a Möbius transformation and let a, b, c be distinct points in $\mathbb{C}_\infty$ with $\alpha = S(a)$, $\beta = S(b)$, $\gamma = S(c)$. Suppose that T is another map with this property. Then $T^{-1} \circ S$ has a, b, and c as fixed points and, therefore, $T^{-1} \circ S = I =$ the identity. That is, $S = T$. Hence, a Möbius map is uniquely determined by its action on any three given points in $\mathbb{C}_\infty$.

Let z_2, z_3, z_4 be points in $\mathbb{C}_\infty$. Define $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \left(\frac{z-z_3}{z-z_4}\right)\Big/\left(\frac{z_2-z_3}{z_2-z_4}\right) \quad \text{if} \quad z_2, z_3, z_4 \in \mathbb{C};$$

$$S(z) = \frac{z-z_3}{z-z_4} \quad \text{if} \quad z_2 = \infty;$$

$$S(z) = \frac{z_2-z_4}{z-z_4} \quad \text{if} \quad z_3 = \infty;$$

$$S(z) = \frac{z-z_3}{z_2-z_3} \quad \text{if} \quad z_4 = \infty.$$

In any case $S(z_2) = 1$, $S(z_3) = 0$, $S(z_4) = \infty$ and S is the only transformation having this property.

3.7 Definition. If $z_1 \in \mathbb{C}_\infty$ then (z_1, z_2, z_3, z_4). (The *cross ratio* of z_1, z_2, z_3, and z_4) is the image of z_1 under the unique Möbius transformation which takes z_2 to 1, z_3 to 0, and z_4 to ∞.

For example: $(z_2, z_2, z_3, z_4) = 1$ and $(z, 1, 0, \infty) = z$. Also, if M is any Möbius map and w_2, w_3, w_4 are the points such that $Mw_2 = 1$, $Mw_3 = 0$, $Mw_4 = \infty$ then $Mz = (z, w_2, w_3, w_4)$.

3.8 Proposition. *If z_2, z_3, z_4 are distinct points and T is any Möbius transformation then*

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$$

for any point z_1.

Proof. Let $Sz = (z, z_2, z_3, z_4)$; then S is a Möbius map. If $M = ST^{-1}$ then $M(Tz_2) = 1$, $M(Tz_3) = 0$, $M(Tz_4) = \infty$; hence, $ST^{-1}z = (z, Tz_2, Tz_3, Tz_4)$ for all z in $\mathbb{C}_\infty$. In particular, if $z = Tz_1$ the desired result follows. ■

3.9 Proposition. *If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and $\omega_2, \omega_3, \omega_4$ are also distinct points of $\mathbb{C}_\infty$, then there is one and only one Möbius transformation S such that $Sz_2 = \omega_2$, $Sz_3 = \omega_3$, $Sz_4 = \omega_4$*

Proof. Let $Tz = (z, z_2, z_3, z_4)$, $Mz = (z, \omega_2, \omega_3, \omega_4)$ and put $S = M^{-1}T$. Clearly S has the desired property. If R is another Möbius map with $Rz_j = \omega_j$ for $j = 2, 3, 4$ then $R^{-1} \circ S$ has three fixed points (z_2, z_3, and z_4). Hence $R^{-1} \circ S = I$, or $S = R$. ■

It is well known from high school geometry that three points in the plane determine a circle. (Recall that a circle in $\mathbb{C}_\infty$ passing through ∞ corresponds to a straight line in $\mathbb{C}$. Hence there is no need to inject in the previous statement the word "non-colinear.) A straight line in the plane will be called a circle.) The next result explains when four points lie on a circle.

3.10 Proposition. *Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Then (z_1, z_2, z_3, z_4) is a real number iff all four points lie on a circle.*

Proof. Let $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by $Sz = (z, z_2, z_3, z_4)$; then $S^{-1}(\mathbb{R}) =$ the set of z such that (z, z_2, z_3, z_4) is real. Hence, we will be finished if we can show that the image of $\mathbb{R}_\infty$ under a Möbius transformation is a circle.

Let $Sz = \dfrac{az+b}{cz+d}$; if $z = x \in \mathbb{R}$ and $\omega = S^{-1}(x)$ then $x = S\omega$ implies that $S(\omega) = \overline{S(\omega)}$. That is,

$$\frac{a\omega+b}{c\omega+d} = \frac{\bar{a}\bar{\omega}+\bar{b}}{\bar{c}\bar{\omega}+\bar{d}}$$

Cross multiplying this gives

3.11 $$(a\bar{c}-\bar{a}c)\,|\omega|^2+(a\bar{d}-\bar{b}c)\omega+(b\bar{c}-d\bar{a})\bar{\omega}+(b\bar{d}-\bar{b}d) = 0.$$

If $a\bar{c}$ is real then $a\bar{c}-\bar{a}c = 0$; putting $\alpha = 2(a\bar{d}-\bar{b}c)$, $\beta = i(b\bar{d}-\bar{b}d)$ and multiplying (3.11) by i gives

3.12 $$0 = \operatorname{Im}(\alpha\omega)-\beta = \operatorname{Im}(\alpha\omega-\beta)$$

since β is real. That is, ω lies on the line determined by (3.12) for fixed α and β. If $a\bar{c}$ is not real then (3.11) becomes

$$|\omega|^2+\bar{\gamma}\omega+\gamma\bar{\omega}-\delta = 0$$

for some constants γ in $\mathbb{C}$, δ in $\mathbb{R}$. Hence,

3.13 $$|\omega+\gamma| = \lambda$$

where

$$\lambda = (|\gamma|^2+\delta)^{\frac{1}{2}} = \left|\frac{ad-bc}{\bar{a}c-a\bar{c}}\right| > 0.$$

Since γ and λ are independent of x and since (3.13) is the equation of a circle, the proof is finished. ■

3.14 Theorem. *A Möbius transformation takes circles onto circles.*

Proof. Let Γ be any circle in $\mathbb{C}_\infty$ and let S be any Möbius transformation. Let z_2, z_3, z_4 be three distinct points on Γ and put $\omega_j = Sz_j$ for $j = 2, 3, 4$. Then $\omega_2, \omega_3, \omega_4$ determine a circle Γ'. We claim that $S(\Gamma) = \Gamma'$. In fact, for any z in $\mathbb{C}_\infty$

3.15 $$(z, z_2, z_3, z_4) = (Sz, \omega_2, \omega_3, \omega_4)$$

by Proposition 3.8. By the preceding proposition, if z is on Γ then both sides of (3.15) are real. But this says that $Sz \in \Gamma'$. ■

Now let Γ and Γ' be two circles in $\mathbb{C}_\infty$ and let $z_2, z_3, z_4 \in \Gamma$; $\omega_2, \omega_3, \omega_4 \in \Gamma'$. Put $Rz = (z, z_2, z_3, z_4)$, $Sz = (z, \omega_2, \omega_3, \omega_4)$. Then $T = S^{-1} \circ R$ maps

Γ onto Γ'. In fact, $Tz_j = \omega_j$ for $j = 2, 3, 4$ and, as in the above proof, it follows that $T(\Gamma) = \Gamma'$.

3.16 Proposition. *For any given circles Γ and Γ' in $\mathbb{C}_\infty$ there is a Möbius transformation T such that $T(\Gamma) = \Gamma'$. Furthermore we can specify that T take any three points on Γ onto any three points of Γ'. If we do specify Tz_j for $j = 2, 3, 4$ (distinct z_j in Γ) then T is unique.*

Proof. The proof, except for the uniqueness statement, is given in the previous paragraph. The uniqueness part is a trivial exercise for the reader. ■

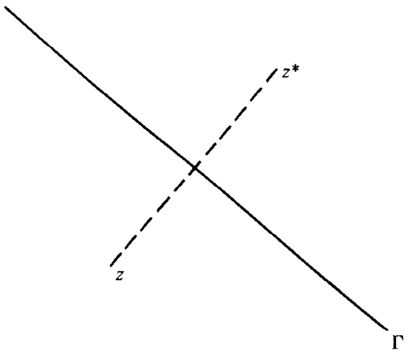
Now that we know that a Möbius map takes circles to circles, the next question is: What happens to the inside and the outside of these circles? To answer this we introduce some new concepts.

3.17 Definition. Let Γ be a circle through points z_2, z_3, z_4. The points z, z^* in $\mathbb{C}_\infty$ are said to be *symmetric* with respect to Γ if

3.18 $$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

As it stands, this definition not only depends on the circle but also on the points z_2, z_3, z_4. It is left as an exercise for the reader to show that symmetry is independent of the points chosen (Exercise 11).

Also, by Proposition 3.10 z is symmetric to itself with respect to Γ if and only if $z \in \Gamma$.



Let us investigate what it means for z and z^* to be symmetric. If Γ is a straight line then our linguistic prejudices lead us to believe that z and z^* are symmetric with respect to Γ if the line through z and z^* is perpendicular to Γ and z and z^* are the same distance from Γ but on opposite sides of Γ. This is indeed the case.

If Γ is a straight line then, choosing $z_4 = \infty$, equation (3.18) becomes

$$\frac{z^*-z_3}{z_2-z_3} = \frac{\bar z-\bar z_3}{\bar z_2-\bar z_3}.$$

This gives $|z^*-z_3| = |z-z_3|$; since z_3 was not specified, we have that z

and z^* are equidistant from each point on Γ. Also

$$\begin{aligned}\operatorname{Im}\frac{z^*-z_3}{z_2-z_3} &= \operatorname{Im}\frac{\bar z-\bar z_3}{\bar z_2-\bar z_3}\\ &= -\operatorname{Im}\frac{z-z_3}{z_2-z_3}\end{aligned}$$

Hence, we have (unless $z \in \Gamma$) that z and z^* lie in different half planes determined by Γ. It now follows that $[z, z^*]$ is perpendicular to Γ.

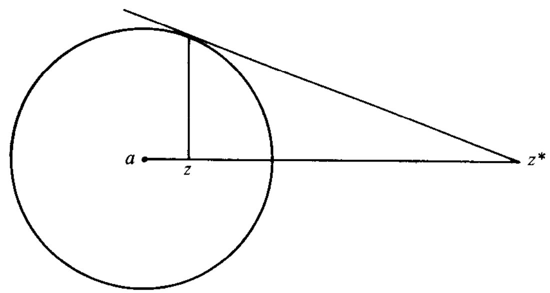
Now suppose that $\Gamma = \{z\colon |z-a| = R\}$ $(0 < R < \infty)$. Let z_2, z_3, z_4 be points in Γ; using (3.18) and Proposition 3.8 for a number of Möbius transformations gives

$$\begin{aligned}(z^*, z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)}\\ &= \overline{(z-a, z_2-a, z_3-a, z_4-a)}\\ &= \left(\bar z-\bar a, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}, \frac{R^2}{z_4-a}\right)\\ &= \left(\frac{R^2}{\bar z-\bar a}, z_2-a, z_3-a, z_4-a\right)\\ &= \left(\frac{R^2}{\bar z-\bar a}+a, z_2, z_3, z_4\right)\end{aligned}$$

Hence, $z^* = a+R^2(\bar z-\bar a)^{-1}$ or $(z^*-a)(\bar z-\bar a) = R^2$. From this it follows that

$$\frac{z^*-a}{z-a} = \frac{R^2}{|z-a|^2} > 0,$$

so that z^* lies on the ray $\{a+t(z-a)\colon 0 < t < \infty\}$ from a through z. Using the fact that $|z-a|\,|z^*-a| = R^2$ we can obtain z^* from z (if z lies inside Γ) as in the figure below. That is: Let L be the ray from a through z. Construct



a line P perpendicular to L at z and at the point where P intersects Γ construct the tangent to Γ. The point of intersection of this tangent with L is the point z^*. Thus, the points a and ∞ are symmetric with respect to Γ.

3.19 Symmetry Principle. *If a Möbius transformation T takes a circle Γ_1 onto the circle Γ_2 then any pair of points symmetric with respect to Γ_1 are mapped by T onto a pair of points symmetric with respect to Γ_2.*

Proof. Let $z_2, z_3, z_4 \in \Gamma_1$; it follows that if z and z^* are symmetric with respect to Γ_1 then

$$\begin{aligned}(Tz^*, Tz_2, Tz_3, Tz_4) &= (z^*, z_2, z_3, z_4)\\ &= \overline{(z, z_2, z_3, z_4)}\\ &= \overline{(Tz, Tz_2, Tz_3, Tz_4)}\end{aligned}$$

by Proposition 3.8. Hence Tz^* and Tz are symmetric with respect to Γ_2. ■

Now we will discuss orientation for circles in $\mathbb{C}_\infty$; this will enable us to distinguish between the "inside" and "outside" of a circle in $\mathbb{C}_\infty$. Notice that on $\mathbb{C}_\infty$ (the sphere) there is no obvious choice for the inside and outside of a circle.

3.20 Definition. If Γ is a circle then an *orientation* for Γ is an ordered triple of points (z_1, z_2, z_3) such that each z_j is in Γ.

Intuitively, these three points give a direction to Γ. That is we "go" from z_1 to z_2 to z_3. If only two points were given, this would, of course, be ambiguous.

Let $\Gamma = \mathbb{R}$ and let $z_1, z_2, z_3 \in \mathbb{R}$; also, put $Tz = (z, z_1, z_2, z_3) = \dfrac{az+b}{cz+d}$. Since $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ it follows that a, b, c, d can be chosen to be real numbers (see Exercise 8). Hence,

$$\begin{aligned}Tz &= \frac{az+b}{cz+d}\\ &= \frac{az+b}{|cz+d|^2}\cdot(c\bar{z}+d)\\ &= \frac{1}{|cz+d|^2}\,[ac|z|^2+bd+bc\bar{z}+adz]\end{aligned}$$

Hence,

$$\text{Im}\,(z, z_1, z_2, z_3) = \frac{(ad-bc)}{|cz+d|^2}\,\text{Im}\, z.$$

Thus, $\{z: \text{Im}\,(z, z_1, z_2, z_3) < 0\}$ is either the upper or lower half plane depending on whether $(ad-bc) > 0$ or $(ad-bc) < 0$. (Note that $ad-bc$ is the "determinant" of T.)

Now let Γ be arbitrary, and suppose that z_1, z_2, z_3 are on Γ; for any Möbius transformation S we have (by Proposition 3.8)

$$\begin{aligned}\{z: \text{Im}\,(z, z_1, z_2, z_3) > 0\} &= \{z: \text{Im}\,(Sz, Sz_1, Sz_2, Sz_3) > 0\}\\ &= S^{-1}\{z: \text{Im}\,(z, Sz_1, Sz_2, Sz_3) > 0\}\end{aligned}$$

In particular, if S is chosen so that S maps Γ onto $\mathbb{R}_\infty$, then $\{z: \text{Im}\,(z, z_1, z_2, z_3) > 0\}$ is equal to S^{-1} of either the upper or lower half plane.

If (z_1, z_2, z_3) is an orientation of Γ then we define the *right side of* Γ (with respect to (z_1, z_2, z_3)) to be

$$\{z: \text{Im}\,(z, z_1, z_2, z_3) > 0\}.$$

Similarly, we define the *left side of* Γ to be

$$\{z: \text{Im}\,(z, z_1, z_2, z_3) < 0\}.$$

The proof of the following theorem is left as an exercise.

3.21 Orientation Principle. *Let Γ_1 and Γ_2 be two circles in $\mathbb{C}_\infty$ and let T be a Möbius transformation such that $T(\Gamma_1) = \Gamma_2$. Let (z_1, z_2, z_3) be an orientation for Γ_1. Then T takes the right side and the left side of Γ_1 onto the right side and left side of Γ_2 with respect to the orientation (Tz_1, Tz_2, Tz_3).*

Consider the orientation $(1, 0, \infty)$ of $\mathbb{R}$. By the definition of the cross ratio, $(z, 1, 0, \infty) = z$. Hence, the right side of $\mathbb{R}$ with respect to $(1, 0, \infty)$ is the upper half plane. This fits our intuition that the right side lies on our right as we walk along $\mathbb{R}$ from 1 to 0 to ∞.

As an example consider the following problem: Find an analytic function $f: G \to \mathbb{C}$, where $G = \{z: \text{Re}\, z > 0\}$, such that $f(G) = D = \{z: |z| < 1\}$. We solve this problem by finding a Möbius transformation which takes the imaginary axis onto the unit circle and, by the Orientation Principle, takes G onto D (that is, we must choose this map carefully in order that it does not send G onto $\{z: |z| > 1\}$).

If we give the imaginary axis the orientation $(-i, 0, i)$ then $\{z: \text{Re}\, z > 0\}$ is on the right of this axis. In fact,

$$\begin{aligned}(z, -i, 0, i) &= \frac{2z}{z-i}\\ &= \frac{2z}{z-i}\cdot\frac{\bar{z}+i}{\bar{z}+i}\\ &= \frac{2}{|z-i|^2}\cdot(|z|^2+iz)\end{aligned}$$

Hence, $\{z: \text{Im}\,(z, -i, 0, i) > 0\} = \{z: \text{Im}\,(iz) > 0\} = \{z: \text{Re}\, z > 0\}$. Giving Γ the orientation $(-i, -1, i)$ we have that D lies on the right of Γ. Also,

$$(z, -i, -1, i) = \frac{2i}{i-1}\cdot\frac{z+1}{z-i}.$$

If

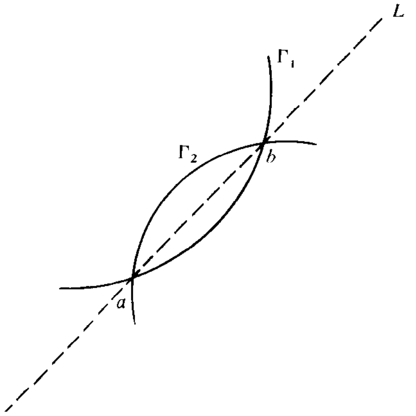
$$Sz = \frac{2z}{z-i} \quad\text{and}\quad Rz = \left(\frac{2i}{i-1}\right)\left(\frac{z+1}{z-i}\right)$$

then $T = R^{-1}S$ maps G onto D (and the imaginary axis onto Γ). By algebraic manipulations we have

$$Tz = \frac{z-1}{z+1}$$

Combining this with previous results we have that $g(z) = \dfrac{e^z-1}{e^z+1}$ maps the infinite strip $\{z: |\text{Im}\, z| < \pi/2\}$ onto the open unit disk D. (It is worth mentioning that $\dfrac{e^z-1}{e^z+1} = \tanh(z/2)$.)

Let G_1, G_2 be open connected sets; to try to find an analytic function f such that $f(G_1) = G_2$ we try to map both G_1 and G_2 onto the open unit disk. If this can be done, f can be obtained by taking the composition of one function with the inverse of the other.



As an example, let G be the open set inside two circles Γ_1 and Γ_2, intersecting at points a and b ($a \neq b$). Let L be the line passing through a and b and give L the orientation (∞, a, b). Then $Tz = (z, \infty, a, b) = \left(\frac{z-a}{z-b}\right)$ maps L onto the real axis ($T\infty = 1$, $Ta = 0$, $Tb = \infty$). Since T must map circles onto circles, T maps Γ_1 and Γ_2 onto circles through 0 and ∞. That is, $T(\Gamma_1)$ and $T(\Gamma_2)$ are straight lines. By the use of orientation we have that $T(G) = \{\omega - \alpha < \arg \omega < \alpha\}$ for some $\alpha > 0$, or the complement of some such closed sector. By the use of an appropriate power of z and possibly a rotation we can map this wedge onto the right half plane. Now, composing with the map $(z-1)(z+1)^{-1}$ gives a map of G onto $D = \{z: |z| < 1\}$.

Exercises

1. Find the image of $\{z: \text{Re } z < 0, |\text{Im } z| < \pi\}$ under the exponential function.
2. Do exercise 1 for the set $\{z: |\text{Im } z| < \pi/2\}$.
3. Discuss the mapping properties of $\cos z$ and $\sin z$.
4. Discuss the mapping properties of z^n and $z^{1/n}$ for $n \geq 2$. (Hint: use polar coordinates.)
5. Find the fixed points of a dilation, a translation and the inversion on $\mathbb{C}_\infty$.
6. Evaluate the following cross ratios: (a) $(7+i, 1, 0, \infty)$ (b) $(2, 1-i, 1, 1+i)$ (c) $(0, 1, i, -1)$ (d) $(i-1, \infty, 1+i, 0)$.
7. If $Tz = \dfrac{az+b}{cz+d}$ find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.

8. If $Tz = \dfrac{az+b}{cz+d}$ show that $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ iff we can choose a, b, c, d to be real numbers.
9. If $Tz = \dfrac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z: |z| = 1\}$.
10. Let $D = \{z: |z| < 1\}$ and find all Möbius transformations T such that $T(D) = D$.
11. Show that the definition of symmetry (3.17) does not depend on the choice of points z_2, z_3, z_4. That is, show that if $\omega_2, \omega_3, \omega_4$ are also in Γ then equation (3.18) is satisfied iff $(z^*, \omega_2, \omega_3, \omega_4) = \overline{(z, \omega_2, \omega_3, \omega_4)}$. (Hint: Use Exercise 8.)
12. Prove Theorem 3.4.
13. Give a discussion of the mapping $f(z) = \frac{1}{2}(z+1/z)$.
14. Suppose that one circle is contained inside another and that they are tangent at the point a. Let G be the region between the two circles and map G conformally onto the open unit disk. (Hint: first try $(z-a)^{-1}$.)
15. Can you map the open unit disk conformally onto $\{z: 0 < |z| < 1\}$?
16. Map $G = \mathbb{C} - \{z: -1 \leq z \leq 1\}$ onto the open unit disk oy an analytic function f. Can f be one-one?
17. Let G be a region and suppose that $f: G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.
18. Let $-\infty < a < b < \infty$ and put $Mz = \dfrac{z-ia}{z-ib}$. Define the lines $L_1 = \{z: \text{Im } z = b\}$, $L_2 = \{z: \text{Im } z = a\}$ and $L_3 = \{z: \text{Re } z = 0\}$. Determine which of the regions A, B, C, D, E, F in Figure 1, are mapped by M onto the regions U, V, W, X, Y, Z in Figure 2.

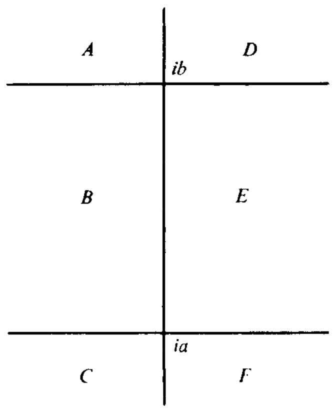


Figure 1

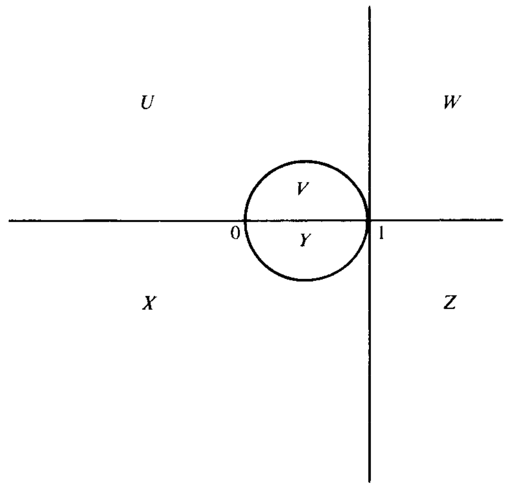


Figure 2

19. Let a, b, and M be as in Exercise 18 and let log be the principal branch of the logarithm.

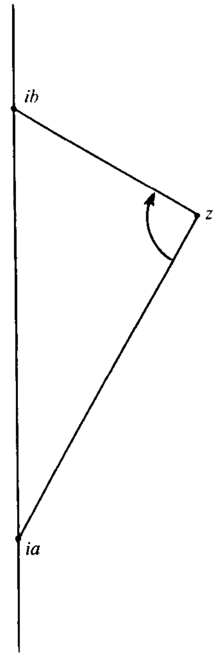
(a) Show that $\log(Mz)$ is defined for all z except $z = ic$, $a \le c \le b$; and if $h(z) = \operatorname{Im}[\log Mz]$ then $0 < h(z) < \pi$ for $\operatorname{Re} z > 0$.
(b) Show that $\log(z-ic)$ is defined for $\operatorname{Re} z > 0$ and any real number c; also prove that $|\operatorname{Im}\log(z-ic)| < \dfrac{\pi}{2}$ if $\operatorname{Re} z > 0$.
(c) Let h be as in (a) and prove that $h(z) = \operatorname{Im}[\log(z-ia)-\log(z-ib)]$.
(d) Show that

$$\int_a^b \frac{dt}{z-it} = i[\log(z-ib)-\log(z-ia)]$$

(Hint: Use the Fundamental Theorem of Calculus.)
(e) Combine (c) and (d) to get that

$$h(x+iy) = \int_a^b \frac{x}{x^2+(y-t)^2}\,dt = \arctan\left(\frac{y-a}{x}\right) - \arctan\left(\frac{y-b}{x}\right)$$

(f) Interpret part (e) geometrically and show that for $\operatorname{Re} z > 0$ $h(z)$ is the angle depicted in the figure.



20. Let $Sz = \dfrac{az+b}{cz+d}$ and $Tz = \dfrac{\alpha z+\beta}{\gamma z+\delta}$; show that $S = T$ iff there is a non zero complex number λ such that $\alpha = \lambda a$, $\beta = \lambda b$, $\gamma = \lambda c$, $\delta = \lambda d$.
21. Let T be a Möbius transformation with fixed points z_1 and z_2. If S is a Möbius transformation show that $S^{-1}TS$ has fixed points $S^{-1}z_1$ and $S^{-1}z_2$.
22. (a) Show that a Möbius transformation has 0 and ∞ as its only fixed points iff it is a dilation.

(b) Show that a Möbius transformation has ∞ as its only fixed point iff it is a translation.
23. Show that a Möbius transformation T satisfies $T(0) = \infty$ and $T(\infty) = 0$ iff $Tz = az^{-1}$ for some a in $\mathbb{C}$.
24. Let T be a Möbius transformation, $T \neq$ the identity. Show that a Möbius transformation S commutes with T if S and T have the same fixed points. (Hint: Use Exercises 21 and 22.)
25. Find all the abelian subgroups of the group of Möbius transformations.
26. (a) Let $GL_2(\mathbb{C})$ = all invertible 2×2 matrices with entries in $\mathbb{C}$ and let $\mathscr{M}$ be the group of Möbius transformations. Define $\varphi: GL_2(\mathbb{C}) \to \mathscr{M}$ by $\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dfrac{az+b}{cz+d}$. Show that φ is a group homomorphism of $GL_2(\mathbb{C})$ onto $\mathscr{M}$. Find the kernel of φ.
(b) Let $SL_2(\mathbb{C})$ be the subgroup of $GL_2(\mathbb{C})$ consisting of all matrices of determinant 1. Show that the image of $SL_2(\mathbb{C})$ under φ is all of $\mathscr{M}$. What part of the kernel of φ is in $SL_2(\mathbb{C})$?
27. If $\mathscr{G}$ is a group and $\mathscr{N}$ is a subgroup then $\mathscr{N}$ is said to be a *normal subgroup* of $\mathscr{G}$ if $S^{-1}TS \in \mathscr{N}$ whenever $T \in \mathscr{N}$ and $S \in \mathscr{G}$. $\mathscr{G}$ is a *simple group* if the only normal subgroups of $\mathscr{G}$ are $\{I\}$ (I = the identity of $\mathscr{G}$) and $\mathscr{G}$ itself. Prove that the group $\mathscr{M}$ of Möbius transformations is a simple group.
28. Discuss the mapping properties of $(1-z)^i$.
29. For complex numbers α and β with $|\alpha|^2+|\beta|^2=1$

$$u_{\alpha,\beta}(z)=\frac{\alpha z-\bar{\beta}}{\beta z-\bar{\alpha}} \quad \text{and let} \quad U=\left\{u_{\alpha,\beta}:|\alpha|^2+|\beta|^2=1\right\}.$$

(a) Show that U is a group under composition.
(b) If SU_2 is the set of all unitary matrices with determinant 1, show that SU_2 is a group under matrix multiplication and that for each A in SU_2 there are unique complex numbers α and β with $|\alpha|^2+|\beta|^2=1$ and $A=\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$.
(c) Show that $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto U_{\alpha,\beta}$ is an isomorphism of the group SU_2 onto U.
(d) If $l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ let H_l = all the polynomials of degree $\le 2l$. For $u_{\alpha,\beta} = u$ in U define $T_u^{(l)}: H_l \to H_l$ by $(T_u^{(l)}f)(z) = (\beta z+\bar{\alpha})^{2l} f(u(z))$. Show that $T_u^{(l)}$ is an invertible linear transformation on H_l and $u \mapsto T_u^{(l)}$ is an injective homomorphism of U into the group of invertible linear transformations of H_l onto H_l.
30. For $|z|<1$ define $f(z)$ by

$$f(z)=\exp\left\{-i\log\left[i\left(\frac{1+z}{1-z}\right)\right]^{1/2}\right\}.$$

(a) Show that f maps $D=\{z:|z|<1\}$ conformally onto an annulus G.
(b) Find all Mobius transformations $S(z)$ that map D onto D and such that $f(S(z))=f(z)$ when $|z|<1$.