But using (8.6) this gives

$$\left| \int_T f \right| \le 4^n \epsilon d \ell (\tfrac{1}{4})^n = \epsilon d \ell.$$

Since ϵ was arbitrary and d and ℓ are fixed, $\int_T f = 0$. ∎

Chapter V

Singularities

In this chapter functions which are analytic in a punctured disk (an open disk with the center removed) are examined. From information about the behavior of the function near the center of the disk, a number of interesting and useful results will be derived. In particular, we will use these results to evaluate certain definite integrals over the real line which cannot be evaluated by the methods of calculus.

§1. Classification of singularities

This section begins by studying the best behaved singularity—the removable kind.

1.1 Definition. A function f has an *isolated singularity* at $z = a$ if there is an $R > 0$ such that f is defined and analytic in $B(a;R) - \{a\}$ but not in $B(a;R)$. The point a is called a *removable singularity* if there is an analytic function $g : B(a;R) \to \mathbb{C}$ such that $g(z) = f(z)$ for $0 < |z - a| < R$.

The functions $\dfrac{\sin z}{z}$, $\dfrac{1}{z}$, and $\exp \dfrac{1}{z}$ all have isolated singularities at $z = 0$.

However, only $\dfrac{\sin z}{z}$ has a removable singularity (see Exercise 1). It is left to the reader to see that the other two functions do not have removable singularities.
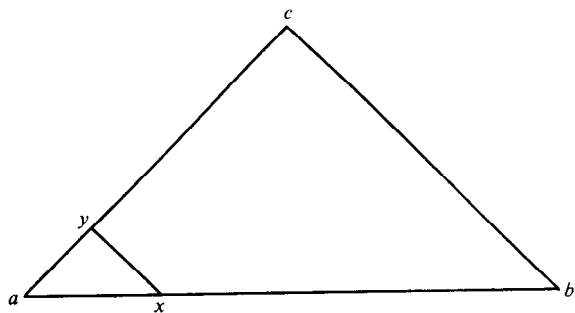
How can we determine when a singularity is removable? Since the function has an analytic extension to $B(a; R)$, $\int_\gamma f = 0$ for any closed curve in the punctured disk; but this may be difficult to apply. Also it must happen that $\lim_{z \to a} f(z)$ exists. This is easier to verify, but a much weaker criterion is available.

1.2 Theorem. *If f has an isolated singularity at a then the point $z = a$ is a removable singularity iff*

$$\lim_{z \to a} (z - a) f(z) = 0$$

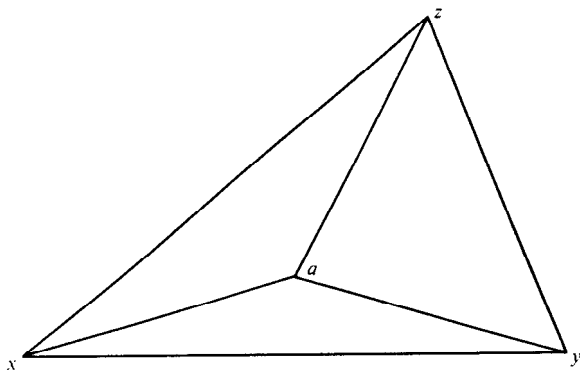*Proof.* Suppose f is analytic in $\{z: 0 < |z - a| < R\}$, and define $g(z) = (z - a) f(z)$ for $z \ne a$ and $g(a) = 0$. Suppose $\lim_{z \to a} (z - a) f(z) = 0$; then g is clearly a continuous function. If we can show that g is analytic then it follows that a is a removable singularity. In fact, if g is analytic we have $g(z) = (z - a) h(z)$ for some analytic function defined on $B(a; R)$ because $g(a) = 0$ (IV. 3.9). But then $h(z)$ and $f(z)$ must agree for $0 < |z - a| < R$, so that a is, by definition, a removable singularity.

To show that g is analytic we apply Morera's Theorem. Let T be a triangle in $B(a; R)$ and let Δ be the inside of T together with T. If $a \notin \Delta$ then $T \sim 0$ in $\{z: 0 < |z-a| < R\}$ and so, $\int_T g = 0$ by Cauchy's Theorem. If a is a vertex of T then we have $T = [a, b, c, a]$. Let $x \in [a, b]$ and $y \in [c, a]$ and

form the triangle $T_1 = [a, x, y, a]$. If P is the polygon $[x, b, c, y, x]$ then $\int_T g = \int_{T_1} g + \int_P g = \int_{T_1} g$ since $P \sim 0$ in the punctured disk. Since g is continuous and $g(a) = 0$, for any $\epsilon > 0$ x and y can be chosen such that $|g(z)| \leq \epsilon/\ell$ for any z on T_1, where ℓ is the length of T. Hence $|\int_T g| = |\int_{T_1} g| \leq \epsilon$; since ϵ was arbitrary we have $\int_T g = 0$.

If $a \in \Delta$ and $T = [x, y, z, x]$ then consider the triangles $T_1 = [x, y, a, x]$, $T_2 = [y, z, a, y]$, $T_3 = [z, x, a, z]$. From the preceding paragraph $\int_{T_j} g = 0$

for $j = 1, 2, 3$ and so, $\int_T g = \int_{T_1} g + \int_{T_2} g + \int_{T_3} g = 0$. Since this exhausts all possibilities, g must be analytic by Morera's Theorem. Since the converse is obvious, the proof of the theorem is complete. ∎

The preceding theorem points out another stark difference between functions of a real variable and functions of a complex variable. The function $f(x) = |x|$, $x \in \mathbb{R}$, is not differentiable because it has a "corner" at $x = 0$. Such a situation does not occur in the complex case. For a function to have an honest singularity (i.e., a non-removable one) the function must behave badly in the vicinity of the point. That is, either $|f(z)|$ becomes infinite as z nears the point (and does so at least as quickly as $(z-a)^{-1}$), or $|f(z)|$ doesn't have any limit as $z \to a$.

1.3 Definition. If $z = a$ is an isolated singularity of f then a is a *pole* of f if $\lim_{z \to a} |f(z)| = \infty$. That is, for any $M > 0$ there is a number $\epsilon > 0$ such that $|f(z)| \geq M$ whenever $0 < |z-a| < \epsilon$. If an isolated singularity is neither a pole nor a removable singularity it is called an *essential singularity*.

It is easy to see that $(z-a)^{-m}$ has a pole at $z = a$ for $m \geq 1$. Also, it is not difficult to see that although $z = 0$ is an isolated singularity of $\exp(z^{-1})$, it is neither a pole nor a removable singularity; hence it is an essential singularity.

Suppose that f has a pole at $z = a$; it follows that $[f(z)]^{-1}$ has a removable singularity at $z = a$. Hence, $h(z) = [f(z)]^{-1}$ for $z \neq a$ and $h(a) = 0$ is analytic in $B(a; R)$ for some $R > 0$. However, since $h(a) = 0$ it follows by Corollary IV. 3.9 that $h(z) = (z-a)^m h_1(z)$ for some analytic function h_1 with $h_1(a) \neq 0$ and some integer $m \geq 1$. But this gives that $(z-a)^m f(z) = [h_1(z)]^{-1}$ has a removable singularity at $z = a$. This is summarized as follows.

1.4 Proposition. *If G is a region with a in G and if f is analytic on $G - \{a\}$ with a pole at $z = a$ then there is a positive integer m and an analytic function $g: G \to \mathbb{C}$ such that*

1.5
$$f(z) = \frac{g(z)}{(z-a)^m}.$$

1.6 Definition. If f has a pole at $z = a$ and m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$ then f has a *pole of order m* at $z = a$.

Notice that if m is the order of the pole at $z = a$ and g is chosen to satisfy (1.5) then $g(a) \neq 0$. (Why?)

Let f have a pole of order m at $z = a$ and put $f(z) = g(z)(z-a)^{-m}$. Since g is analytic in a disk $B(a; R)$ it has a power series expansion about a. Let

$$g(z) = A_m + A_{m-1}(z-a) + \cdots + A_1(z-a)^{m-1} + (z-a)^m \sum_{k=0}^{\infty} a_k(z-a)^k.$$

Hence

1.7
$$f(z) = \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{(z-a)} + g_1(z)$$

where g_1 is analytic in $B(a; R)$ and $A_m \neq 0$.

1.8 Definition. If f has a pole of order m at $z = a$ and f satisfies (1.7) then $A_m(z-a)^{-m} + \cdots + A_1(z-a)^{-1}$ is called the *singular part* of f at $z = a$.

As an example consider a rational function $r(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials without common factors. That is, they have no common zeros; and consequently the poles of $r(z)$ are exactly the zeros of $q(z)$. The order of each pole of $r(z)$ is the order of the zero of $q(z)$. Suppose $q(a) = 0$ and let $S(z)$ be the singular part of $r(z)$ at a. Then $r(z) - S(z) = r_1(z)$ and $r_1(z)$ is a rational function whose poles are also poles of $r(z)$. More-

over, it is not difficult to see that the singular part of $r_1(z)$ at any of its poles is also the singular part of $r(z)$ at that pole. Using induction we arrive at the following: if $a_1, \cdots, a_n$ are the poles of $r(z)$ and $S_j(z)$ is the singular part of $r(z)$ at $z = a_j$ then

1.9
$$r(z) = \sum_{j=1}^{n} S_j(z) + P(z)$$

where $P(z)$ is a rational function without poles. But, by the Fundamental Theorem of Algebra, a rational function without poles is a polynomial! So $P(z)$ is a polynomial and (1.9) is nothing else but the expansion of a rational function by *partial fractions*.

Is this expansion by partial fractions (1.9) peculiar only to rational functions? Certainly it is if we require $P(z)$ in (1.9) to be a polynomial. But if we allow $P(z)$ to be any analytic function in a region G, then (1.9) is valid for any function $r(z)$ analytic in G except for a finite number of poles. Suppose we have a function f analytic in G except for infinitely many poles (e.g., $f(z) = (\cos z)^{-1}$); can we get an analogue of (1.9) where we replace the finite sum by an infinite sum? The answer to this is yes and is contained in Mittag-Leffler's Theorem which will be proved in Chapter VII.

There is an analogue of the singular part which is valid for essential singularities. Actually we will do more than this as we will investigate functions which are analytic in an annulus. But first, a few definitions.

1.10 Definition. If $\{z_n : n = 0, \pm 1, \pm 2, \ldots\}$ is a doubly infinite sequence of complex numbers, $\sum_{n=-\infty}^{\infty} z_n$ is *absolutely convergent* if both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ are absolutely convergent. In this case $\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n$. If u_n is a function on a set S for $n = 0, \pm 1, \ldots$ and $\sum_{-\infty}^{\infty} u_n(s)$ is absolutely convergent for each $s \in S$, then the convergence is *uniform* over S if both $\sum_{n=0}^{\infty} u_n$ and $\sum_{n=1}^{\infty} u_{-n}$ converge uniformly on S.

The reason we are limiting ourselves to absolute convergence is that this is the type of convergence we will be most concerned with. One can define convergence of $\sum_{-\infty}^{\infty} z_n$, but the definition is not that the partial sums $\sum_{n=-m}^{m} z_n$ converge. In fact, the series $\sum_{n \neq 0} \frac{1}{n}$ satisfies this criterion but it is clearly not a series we wish to have convergent. On the other hand, if $\sum_{-\infty}^{\infty} z_n$ is absolutely convergent with sum z then it readily follows that $z = \lim \sum_{n=-m}^{m} z_n$.

If $0 \le R_1 < R_2 \le \infty$ and a is any complex number, define ann $(a; R_1, R_2) = \{z : R_1 < |z - a| < R_2\}$. Notice that ann $(a; 0, R_2)$ is a punctured disk.

1.11 Laurent Series Development. *Let f be analytic in the annulus* ann $(a; R_1, R_2)$. *Then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$$

where the convergence is absolute and uniform over ann $(a; r_1, r_2)^-$ *if* $R_1 < r_1 < r_2 < R_2$. *Also the coefficients a_n are given by the formula*

1.12
$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz$$

where γ is the circle $|z - a| = r$ for any r, $R_1 < r < R_2$. Moreover, this series is unique.

Proof. If $R_1 < r_1 < r_2 < R_2$ and γ_1, γ_2 are the circles $|z - a| = r_1$, $|z - a| = r_2$ respectively, then $\gamma_1 \sim \gamma_2$ in ann $(a; R_1, R_2)$. By Cauchy's Theorem we have that for any function g analytic in ann $(a; R_1, R_2)$, $\int_{\gamma_1} g = \int_{\gamma_2} g$. In particular the integral appearing in (1.12) is independent of r so that for each integer n, a_n is a constant. Moreover, $f_2 : B(a; R_2) \to \mathbb{C}$ given by the formula
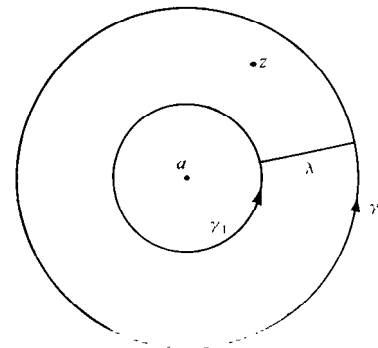
1.13
$$f_2(z) = \frac{1}{2\pi i} \int_{|w-a|=r_2} \frac{f(w)}{w-z} \, dw,$$

where $|z - a| < r_2$, $R_1 < r_2 < R_2$, is a well defined function. Also, by Lemma IV.5.1 f_2 is analytic in $B(a; R_2)$. Similarly, if $G = \{z : |z - a| > R_1\}$ then $f_1 : G \to \mathbb{C}$ defined by

1.14
$$f_1(z) = -\frac{1}{2\pi i} \int_{|w-a|=r_1} \frac{f(w)}{w-z} \, dw,$$

where $|z - a| > r_1$ and $R_1 < r_1 < R_2$, is analytic in G.

If $R_1 < |z - a| < R_2$ let r_1 and r_2 be chosen so that $R_1 < r_1 < |z - a| < r_2 < R_2$. Let $\gamma_1(t) = a + r_1 e^{it}$ and $\gamma_2(t) = a + r_2 e^{it}$, $0 \le t \le 2\pi$. Also choose a straight line segment λ going from a point on γ_1 radially to γ_2 which misses z. Since $\gamma_1 \sim \gamma_2$ in ann $(a; R_1, R_2)$ we have that the closed curve

$\gamma = \gamma_2 - \lambda - \gamma_1 + \lambda$ is homotopic to zero. Also $n(\gamma_2, z) = 1$ and $n(\gamma_1, z) = 0$ gives, by Cauchy's Integral Formula, that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z} \, dw - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} \, dw$$

$$= f_2(z) + f_1(z).$$

The plan now is to expand f_1 and f_2 in power series (f_1 having negative powers of $(z-a)$); then adding them together will give the Laurent series development of $f(z)$. Since f_2 is analytic in the disk $B(a; R_2)$ it has a power series expansion about a. Using Lemma IV. 5.1 to calculate $f_2^{(n)}(a)$,

1.15 $$f_2(z) = \sum_{n=0}^\infty a_n(z-a)^n$$

where the coefficients a_n are given by (1.12).

Now define $g(z)$ for

$$0 < |z| < \frac{1}{R_1} \text{ by } g(z) = f_1\left(a + \frac{1}{z}\right);$$

so $z = 0$ is an isolated singularity. We claim that $z = 0$ is a removable singularity. In fact, if $r > R_1$ then let $\rho(z) = d(z, C)$ where C is the circle $\{w: |w-a| = r\}$; also put $M = \max \{|f(w)|: w \in C\}$. Then for $|z-a| > r$

$$|f_1(z)| \le \frac{Mr}{\rho(z)}.$$

But $\lim_{z \to \infty} \rho(z) = \infty$; so that

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} f_1\left(a + \frac{1}{z}\right) = 0.$$

Hence, if we define $g(0) = 0$ then g is analytic in $B(0; 1/R_1)$. Let

1.16 $$g(z) = \sum_{n=1}^\infty B_n z^n$$

be its power series expansion about 0. It is easy to show that this gives

1.17 $$f_1(z) = \sum_{n=1}^\infty a_{-n}(z-a)^{-n}$$

where a_{-n} is defined by (1.12) (the details are to be furnished by the reader in Exercise 3). Also, by the convergence properties of (1.15) and (1.17), $\sum_{-\infty}^\infty a_n(z-a)^n$ converges absolutely and uniformly on properly smaller annuli.

The uniqueness of this expansion can be demonstrated by showing that if $f(z) = \sum_{n=-\infty}^\infty a_n(z-a)^n$ converges absolutely and uniformly on proper annuli then the coefficients a_n must be given by the formula (1.12). ∎

We now use the Laurent Expansion to classify isolated singularities.

1.18 Corollary. *Let $z = a$ be an isolated singularity of f and let $f(z) = \sum_{-\infty}^\infty a_n(z-a)^n$ be its Laurent Expansion in ann $(a; 0, R)$. Then:*

(a) *$z = a$ is a removable singularity iff $a_n = 0$ for $n \le -1$;*

(b) *$z = a$ is a pole of order m iff $a_{-m} \ne 0$ and $a_n = 0$ for $n \le -(m+1)$;*

(c) *$z = a$ is an essential singularity iff $a_n \ne 0$ for infinitely many negative integers n.*

Proof. (a) If $a_n = 0$ for $n \le -1$ then let $g(z)$ be defined in $B(a; R)$ by $g(z) = \sum_{n=0}^\infty a_n(z-a)^n$; thus, g must be analytic and agrees with f in the punctured disk. The converse is equally as easy.

(b) Suppose $a_n = 0$ for $n \le -(m+1)$; then $(z-a)^m f(z)$ has a Laurent Expansion which has no negative powers of $(z-a)$. By part (a), $(z-a)^m f(z)$ has a removable singularity at $z-a$. Thus f has a pole of order m at $z = a$. The converse follows by retracing the steps in the preceding argument.

(c) Since f has an essential singularity at $z = a$ when it has neither a removable singularity nor a pole, part (c) follows from parts (a) and (b). ∎

One can also classify isolated singularities by examining the equations

1.19 $$\lim_{z \to a} |z-a|^s |f(z)| = 0$$

1.20 $$\lim_{z \to a} |z-a|^s |f(z)| = \infty$$

where s is some real number. This is outlined in Exercises 7, 8, and 9; the reader is strongly encouraged to work through these exercises.

The following gives the best information which can be proved at this time concerning essential singularities. We know that f has an essential singularity at $z = a$ when $\lim_{z \to a} |f(z)|$ fails to exist ("existing" includes the possibility that the limit is infinity). This means that as z approaches a the values of $f(z)$ must wander through $\mathbb{C}$. The next theorem says that not only do they wander, but, as z approaches a, $f(z)$ comes arbitrarily close to every complex number. Actually, there is a result due to Picard that says that $f(z)$ assumes each complex value with at most one exception. However, this is not proved until Chapter XII.

1.21 Casorati–Weierstrass Theorem. *If f has an essential singularity at $z = a$ then for every $\delta > 0$, $\{f[\text{ann } (a; 0, \delta)]\}^- = \mathbb{C}$.*

Proof. Suppose that f is analytic in ann $(a; 0, R)$; it must be shown that if c and $\epsilon > 0$ are given then for each $\delta > 0$ we can find a z with $|z-a| < \delta$ and $|f(z)-c| < \epsilon$. Assume this to be false; that is, assume there is a c in $\mathbb{C}$ and $\epsilon > 0$ such that $|f(z)-c| \ge \epsilon$ for all z in $G = \text{ann } (a; 0, \delta)$. Thus $\lim_{z \to a} |z - a|^{-1}|f(z) - c| = \infty$, which implies that $(z-a)^{-1}(f(z)-c)$ has a pole at $z = a$. If m is the order of this pole then $\lim_{z \to a} |z - a|^{m+1}|f(z) - c| = 0$. Hence $|z - a|^{m+1}|f(z)| \le |z - a|^{m+1}|f(z) - c| + |z - a|^{m+1}|c|$ gives that

$\lim_{z \to a} |z-a|^{m+1}|f(z)| = 0$ since $m \geq 1$. But, according to Theorem 1.2, this gives that $f(z)(z-a)^m$ has a removable singularity at $z = a$. This contradicts the hypothesis and completes the proof of the theorem. ∎

Exercises

1. Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity determine $f(\{z: 0 < |z| < \delta\})$ for arbitrarily small values of δ.

(a) $f(z) = \dfrac{\sin z}{z}$;

(b) $f(z) = \dfrac{\cos z}{z}$;

(c) $f(z) = \dfrac{\cos z - 1}{z}$;

(d) $f(z) = \exp(z^{-1})$;

(e) $f(z) = \dfrac{\log(z+1)}{z^2}$;

(f) $f(z) = \dfrac{\cos(z^{-1})}{z^{-1}}$;

(g) $f(z) = \dfrac{z^2+1}{z(z-1)}$;

(h) $f(z) = (1-e^z)^{-1}$;

(i) $f(z) = z \sin \dfrac{1}{z}$;

(j) $f(z) = z^n \sin \dfrac{1}{z}$.

2. Give the partial fraction expansion of $r(z) = \dfrac{z^2+1}{(z^2+z+1)(z-1)^2}$.

3. Give the details of the derivation of (1.17) from (1.16).

4. Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$; give the Laurent Expansion of $f(z)$ in each of the following annuli: (a) ann $(0; 0, 1)$; (b) ann $(0; 1, 2)$; (c) ann $(0; 2, \infty)$.

5. Show that $f(z) = \tan z$ is analytic in $\mathbb{C}$ except for simple poles at $z = \dfrac{\pi}{2} + n\pi$, for each integer n. Determine the singular part of f at each of these poles.

6. If $f: G \to \mathbb{C}$ is analytic except for poles show that the poles of f cannot have a limit point in G.

7. Let f have an isolated singularity at $z = a$ and suppose $f \not\equiv 0$. Show that if either (1.19) or (1.20) holds for some s in $\mathbb{R}$ then there is an integer m such that (1.19) holds if $s > m$ and (1.20) holds if $s < m$.

8. Let f, a, and m be as in Exercise 7. Show: (a) $m = 0$ iff $z = a$ is a removable singularity and $f(a) \neq 0$; (b) $m < 0$ iff $z = a$ is a removable singularity and f has a zero at $z = a$ of order $-m$; (c) $m > 0$ iff $z = a$ is a pole of f of order m.

9. A function f has an essential singularity at $z = a$ iff neither (1.19) nor (1.20) holds for any real number s.

10. Suppose that f has an essential singularity at $z = a$. Prove the following strengthened version of the Casorati–Weierstrass Theorem. If $c \in \mathbb{C}$ and $\epsilon > 0$ are given then for each $\delta > 0$ there is a number α, $|c - \alpha| < \epsilon$, such that $f(z) = \alpha$ has infinitely many solutions in $B(a; \delta)$.

11. Give the Laurent series development of $f(z) = \exp\left(\dfrac{1}{z}\right)$. Can you generalize this result?

12. (a) Let $\lambda \in \mathbb{C}$ and show that

$$\exp\left\{\tfrac{1}{2}\lambda\left(z + \frac{1}{z}\right)\right\} = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

for $0 < |z| < \infty$, where for $n \geq 0$

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda \cos t} \cos nt \, dt$$

(b) Similarly, show

$$\exp\left\{\tfrac{1}{2}\lambda\left(z - \frac{1}{z}\right)\right\} = b_0 + \sum_{n=1}^{\infty} b_n\left(z^n + \frac{(-1)^n}{z^n}\right)$$

for $0 < |z| < \infty$, where

$$b_n = \frac{1}{\pi}\int_0^{\pi} \cos(nt - \lambda \sin t) \, dt.$$

13. Let $R > 0$ and $G = \{z: |z| > R\}$; a function $f: G \to \mathbb{C}$ has a *removable singularity, a pole, or an essential singularity at infinity* if $f(z^{-1})$ has, respectively, a removable singularity, a pole, or an essential singularity at $z = 0$. If f has a pole at ∞ then the order of the pole is the order of the pole of $f(z^{-1})$ at $z = 0$.
(a) Prove that an entire function has a removable singularity at infinity iff it is a constant.
(b) Prove that an entire function has a pole at infinity of order m iff it is a polynomial of degree m.
(c) Characterize those rational functions which have a removable singularity at infinity.
(d) Characterize those rational functions which have a pole of order m at infinity.
14. Let $G = \{z: 0 < |z| < 1\}$ and let $f: G \to \mathbb{C}$ be analytic. Suppose that γ is a closed rectifiable curve in G such that $n(\gamma; a) = 0$ for all a in $\mathbb{C} - G$. What is $\int_\gamma f$? Why?
15. Let f be analytic in $G = \{z: 0 < |z-a| < r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \to a$. Show that for any ω in $\mathbb{C}$ there is a sequence $\{z_n\}$ in G with $a = \lim z_n$ and $\omega = \lim f(z_n)$.

16. Determine the regions in which the functions $f(z)=(\sin\frac{1}{z})^{-1}$ and $g(z)=\int_0^1 (t-z)^{-1} dt$ are analytic. Do they have any isolated singularities? Do they have any singularities that are not isolated?

17. Let f be analytic in the region $G=\text{ann}(a;0,R)$. Show that if $\iint_G |f(x+iy)|^2\,dx\,dy<\infty$ then f has a removable singularity at $z=a$. Suppose that $p>0$ and $\iint_G |f(x+iy)|^p\,dx\,dy<\infty$; what can be said about the nature of the singularity at $z=a$?

§2. Residues

The inspiration behind this section is the desire for an answer to the following question: If f has an isolated singularity at $z=a$ what are the possible values for $\int_\gamma f$ when γ is a closed curve homologous to zero and not passing through a? If the singularity is removable then clearly the integral will be zero. If $z=a$ is a pole or an essential singularity the answer is not always zero but can be found with little difficulty. In fact, for some curves γ, the answer is given by equation (1.12) with $n=-1$.

2.1 Definition. Let f have an isolated singularity at $z=a$ and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n$$

be its Laurent Expansion about $z=a$. Then the *residue* of f at $z=a$ is the coefficient a_{-1}. Denote this by $\text{Res}\,(f;a)=a_{-1}$. The following is a generalization of formula (1.12) for $n=-1$.

2.2 Residue Theorem. *Let f be analytic in the region G except for the isolated singularities $a_1, a_2, \ldots, a_n$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G then*

$$\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} n(\gamma;\,a_k)\,\text{Res}\,(f;\,a_k).$$

Proof. Let $m_k=n(\gamma;a_k)$ for $1\le k\le m$, and choose positive numbers $r_1,\ldots,r_m$ such that no two disks $\bar{B}(a_k;r_k)$ intersect, none of them intersects $\{\gamma\}$, and each disk is contained in G. (This can be done by induction and by using the fact that γ does not pass through any of the singularities.) Let $\gamma_k(t)=a_k+r_k\exp(-2\pi i m_k t)$ for $0\le t\le 1$. Then for $1\le j\le m$

$$n(\gamma;a_j)+\sum_{k=1}^{m} n(\gamma_k;a_j)=0.$$

Since $\gamma\approx 0(G)$ and $\bar{B}(a_k;r_k)\subset G$,

$$n(\gamma;a)+\sum_{k=1}^{m} n(\gamma_k;a)=0$$

for all a not in $G-\{a_1,\ldots,a_m\}$. Since f is analytic in $G-\{a_1,\ldots,a_m\}$ Theorem IV.5.7 gives

$$2.3 \qquad\qquad 0=\int_\gamma f+\sum_{k=1}^{m}\int_{\gamma_k} f.$$

If $f(z)=\sum_{-\infty}^{\infty} b_n(z-a_k)^n$ is the Laurent expansion about $z=a_k$ then this series converges uniformly on $\bar{B}(a_k;r_k)$. Hence $\int_{\gamma_k} f=\sum_{-\infty}^{\infty} b_n\int_{\gamma_k}(z-a_k)^n$. But $\int_{\gamma_k}(z-a_k)^n=0$ if $n\ne-1$ since $(z-a_k)^n$ has a primitive. Also $\int_{\gamma_k}(z-a_k)^{-1}=2\pi i n(\gamma_k;a_k)\text{Res}(f;a_k)$. Hence (2.3) implies the desired result. ∎

Remark. The condition in the Residue Theorem that f have only a finite number of isolated singularities was made to simplify the statement of the theorem and not because the theorem is invalid when f has infinitely many isolated singularities. In fact, if f has infinitely many singularities they can only accumulate on ∂G. (Why?) If $r=d(\{\gamma\},\partial G)$ then the fact that $\gamma\approx 0$ gives that $n(\gamma;a)=0$ whenever $d(a;\partial G)<\frac{1}{2}r$. (See Exercise IV.7.2.)

The Residue Theorem is a two edged sword; if you can calculate the residues of a function you can calculate certain line integrals and vice versa. Most often, however, it is used as a means to calculate line integrals. To use it in this way we will need a method of computing the residue of a function at a pole.

Suppose f has a pole of order $m\ge 1$ at $z=a$. Then $g(z)=(z-a)^m f(z)$ has a removable singularity at $z=a$ and $g(a)\ne 0$. Let $g(z)=b_0+b_1(z-a)+\cdots$ be the power series expansion of g about $z=a$. It follows that for z near but not equal to a,

$$f(z)=\frac{b_0}{(z-a)^m}+\cdots+\frac{b_{m-1}}{(z-a)}+\sum_{k=0}^{\infty} b_{m+k}(z-a)^k.$$

This equation gives the Laurent Expansion of f in a punctured disk about $z=a$. But then $\text{Res}\,(f;a)=b_{m-1}$; in particular, if $z=a$ is a simple pole $\text{Res}\,(f;a)=g(a)=\lim_{z\to a}(z-a)f(z)$. This is summarized as follows.

2.4 Proposition. *Suppose f has a pole of order m at $z=a$ and put $g(z)=(z-a)^m f(z)$; then*

$$\text{Res}\,(f;a)=\frac{1}{(m-1)!}g^{(m-1)}(a).$$

The remainder of this section will be devoted to calculating certain integrals by means of the Residue Theorem

2.5 Example. Show

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}$$

If $f(z) = \dfrac{z^2}{1+z^4}$ then f has as its poles the fourth roots of -1. These are exactly the numbers $e^{i\theta}$ where

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}.$$

Let

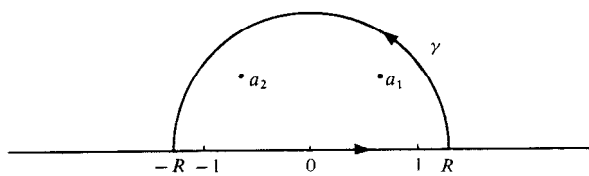$$a_n = \exp\left(i\left[\frac{\pi}{4} + (n-1)\frac{\pi}{2}\right]\right)$$

for $n = 1, 2, 3, 4$; then it is easily seen that each a_n is a simple pole of f. Consequently,

$$\text{Res}\,(f; a_1) = \lim_{z \to a_1} (z - a_1)f(z) = a_1^2(a_1 - a_2)^{-1}(a_1 - a_3)^{-1}(a_1 - a_4)^{-1}$$

$$= \frac{1 - i}{4\sqrt{2}} = \tfrac{1}{4}\exp\left(-\frac{\pi i}{4}\right).$$

Similarly

$$\text{Res}\,(f; a_2) = \frac{-1 - i}{4\sqrt{2}} = \tfrac{1}{4}\exp\left(\frac{-3\pi i}{4}\right).$$

Now let $R > 1$ and let γ be the closed path which is the boundary of the upper half of the disk of radius R with center zero, traversed in the counter-clockwise direction. The Residue Theorem gives

$$\frac{1}{2\pi i}\int_\gamma f = \text{Res}\,(f; a_1) + \text{Res}\,(f; a_2)$$

$$= \frac{-i}{2\sqrt{2}}.$$

But, applying the definition of line integral,

$$\frac{1}{2\pi i}\int_\gamma f = \frac{1}{2\pi i}\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx + \frac{1}{2\pi}\int_0^\pi \frac{R^3 e^{3it}}{1+R^4 e^{4it}}\,dt.$$

This gives

2.6
$$\int_{-R}^{R} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}} - iR^3\int_0^\pi \frac{e^{3it}}{1+R^4 e^{4it}}\,dt.$$

For $0 \leq t \leq \pi$, $1 + R^4 e^{4it}$ lies on the circle centered at 1 of radius R^4; hence $|1 + R^4 e^{4it}| \geq R^4 - 1$. Therefore

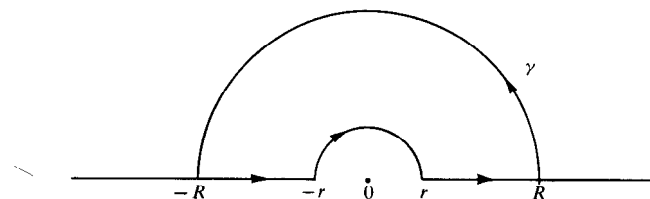$$\left| iR^3 \int_0^\pi \frac{e^{3it}}{1+R^4 e^{4it}}\,dt \right| \leq \frac{\pi R^3}{R^4 - 1};$$

and since $\dfrac{x^2}{1+x^4} \geq 0$ for all x in $\mathbb{R}$, it follows from (2.6) that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx$$

$$= \frac{\pi}{\sqrt{2}}$$

2.7 Example. Show

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

The function $f(z) = \dfrac{e^{iz}}{z}$ has a simple pole at $z = 0$. If $0 < r < R$ let γ be the closed curve that is depicted in the adjoining figure. It follows from Cauchy's

Theorem that $0 = \int_\gamma f$. Breaking γ into its pieces,

2.8
$$0 = \int_r^R \frac{e^{ix}}{x}\,dx + \int_{\gamma_R} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx + \int_{\gamma_r} \frac{e^{iz}}{z}\,dz$$

where γ_R and γ_r are the semicircles from R to $-R$ and $-r$ to r respectively. But

$$\int_r^R \frac{\sin x}{x} = \frac{1}{2i}\int_r^R \frac{e^{ix} - e^{-ix}}{x}\,dx$$

$$\frac{1}{2i}\int_r^R \frac{e^{ix}}{x}\,dx + \frac{1}{2i}\int_R^{-r} \frac{e^{ix}}{x}\,dx$$

Also

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}\,dz\right| = \left|i\int_0^\pi \exp\left(i\,R\,e^{i\theta}\right)d\theta\right|$$

$$\leq \int_0^\pi \left|\exp\left(i\,R\,e^{i\theta}\right)\right|d\theta$$

$$= \int_0^\pi \exp\left(-R\sin\theta\right)d\theta$$

By the methods of calculus we see that, for $\delta > 0$ sufficiently small, the largest possible value of $\exp\left(-R\sin\theta\right)$, with $\delta \leq \theta \leq \pi-\delta$, is $\exp\left(-R\sin\delta\right)$. (Note that δ does not depend on R if R is larger than 1.) This gives that

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}\,dz\right| \leq 2\delta + \int_\delta^{\pi-\delta}\exp\left(-R\sin\theta\right)d\theta$$

$$< 2\delta + \pi\exp\left(-R\sin\delta\right).$$

If $\epsilon > 0$ is given then, choosing $\delta < \frac{1}{3}\epsilon$, there is an R_0 such that $\exp\left(-R\sin\delta\right) < \frac{\epsilon}{3\pi}$ for all $R > R_0$. Hence

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z}\,dz = 0.$$

Since $\dfrac{e^{iz}-1}{z}$ has a removable singularity at $z = 0$, there is a constant $M > 0$ such that $\left|\dfrac{e^{iz}-1}{z}\right| \leq M$ for $|z| \leq 1$. Hence,

$$\left|\int_{\gamma_r}\frac{e^{iz}-1}{z}\,dz\right| \leq \pi r M;$$

that is,

$$0 = \lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}-1}{z}\,dz.$$

But $\displaystyle\int_{\gamma_r}\frac{1}{z}\,dz = -\pi i$ for each r so that

$$-\pi i = \lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}}{z}\,dz$$

So, if we let $r \to 0$ and $R \to \infty$ in (2.8)

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Notice that this example did not use the Residue Theorem. In fact, it could have been presented after Cauchy's Theorem. It was saved until now because the methods used to evaluate this integral are the same as the methods used in applying the Residue Theorem.

2.9 Example. Show that for $a > 1$,

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

If $z = e^{i\theta}$ then $\bar{z} = \dfrac{1}{z}$ and so

$$a+\cos\theta = a+\tfrac{1}{2}(z+\bar{z}) = a+\tfrac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+2az+1}{2z}.$$

Hence

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \tfrac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta}$$

$$= -i\int_\gamma \frac{dz}{z^2+2az+1}$$

where γ is the circle $|z| = 1$. But $z^2+2az+1 = (z-\alpha)(z-\beta)$ where $\alpha = -a+(a^2-1)^{\frac{1}{2}}$, $\beta = -a-(a^2-1)^{\frac{1}{2}}$. Since $a > 1$ it follows that $|\alpha| < 1$ and $|\beta| > 1$. By the Residue Theorem

$$\int_\gamma \frac{dz}{z^2+2az+1} = \frac{\pi i}{\sqrt{a^2-1}};$$

by combining this with the above equation we arrive at

$$\int_0^\pi \frac{d\theta}{a+\cos\theta} = \frac{\pi}{\sqrt{a^2-1}}.$$

2.10 Example. Show that

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0.$$

To solve this problem we *do not* use the principal branch of the logarithm. Instead define $\log z$ for z belonging to the region

$$G = \left\{z \in \mathbb{C} : z \neq 0 \text{ and } -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right\};$$

if $z = |z| e^{i\theta} \neq 0$ with $-\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$, let $\ell(z) = \log|z| + i\theta$. Let $0 < r < R$

and let γ be the same curve as in Example 2.7. Notice that $\ell(x) = \log x$ for $x > 0$, and $\ell(x) = \log|x| + \pi i$ for $x < 0$. Hence,

2.11
$$\int_\gamma \frac{\ell(z)}{1+z^2}\,dz = \int_r^R \frac{\log x}{1+x^2}\,dx + iR \int_0^\pi \frac{[\log R + i\theta]}{1+R^2 e^{2i\theta}} e^{i\theta}\,d\theta$$
$$+ \int_{-R}^{-r} \frac{\log|x| + \pi i}{1+x^2}\,dx + ir \int_\pi^0 \frac{[\log r + i\theta]}{1+r^2 e^{2i\theta}} e^{i\theta}\,d\theta$$

Now the only pole of $\ell(z)(1+z^2)^{-1}$ inside γ is at $z = i$; furthermore, this is a simple pole. By Proposition 2.4 the residue of $\ell(z)(1+z^2)^{-1}$ is $\dfrac{1}{2i}$

$[\log|i| + \tfrac{1}{2}\pi i] = \dfrac{\pi}{4}$. So,

$$\int_\gamma \frac{\ell(z)}{1+z^2}\,dz = \frac{\pi^2 i}{2}.$$

Also,

$$\int_r^R \frac{\log x}{1+x^2}\,dx + \int_{-R}^{-r} \frac{\log|x| + \pi i}{1+x^2}\,dx = 2\int_r^R \frac{\log x}{1+x^2}\,dx + \pi i \int_r^R \frac{dx}{1+x^2}$$

Letting $r \to 0+$ and $R \to \infty$, and using the fact that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

(Exercise 2(f)), it follows from (2.11) that

$$\int_0^\infty \frac{\log x}{1+x^2}\,dx = \tfrac{1}{2} \lim_{r \to 0+} ir \int_0^\pi \frac{[\log r + i\theta]}{1+r^2 e^{2i\theta}} e^{i\theta}\,d\theta$$
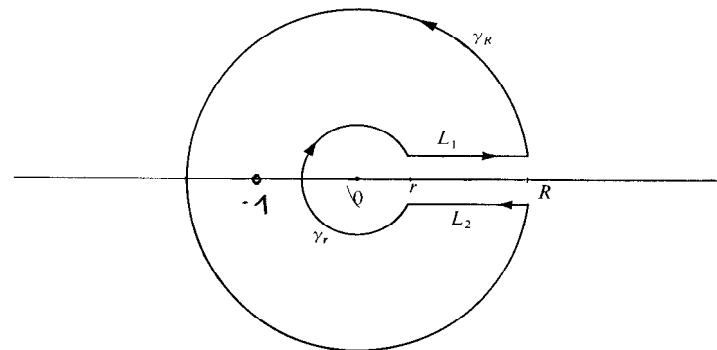$$- \tfrac{1}{2} \lim_{R \to \infty} iR \int_0^\pi \frac{[\log R + i\theta]}{1+R^2 e^{2i\theta}} e^{i\theta}\,d\theta.$$

We now show that both of these limits are zero. If $\rho > 0$ then

$$\left| \rho \int_0^\pi \frac{[\log \rho + i\theta]}{1+\rho^2 e^{i\theta}} e^{i\theta}\,d\theta \right| \leq \frac{\rho |\log \rho|}{|1-\rho^2|} \int_0^\pi d\theta + \frac{\rho}{|1-\rho^2|} \int_0^\pi \theta\,d\theta$$
$$= \frac{\pi\rho |\log \rho|}{|1-\rho^2|} + \frac{\rho\pi^2}{2|1-\rho^2|};$$

letting $\rho \to 0+$ or $\rho \to \infty$, the limit of this expression is zero.

2.12 Example. Show that $\displaystyle\int_0^\infty \frac{x^{-c}}{1+x}\,dx = \frac{\pi}{\sin \pi c}$ if $0 < c < 1$.

To evaluate this integral we must consider a branch of the function z^{-c}. The point $z = 0$ is called a branch point of z^{-c}, and the process used to

evaluate this integral is sometimes called integration around a branch point.

Let $G = \{z : z \neq 0 \text{ and } 0 < \arg z < 2\pi\}$; define a branch of the logarithm on G by putting $\ell(re^{i\theta}) = \log r + i\theta$ where $0 < \theta < 2\pi$. For z in G put $f(z) = \exp[-c\ell(z)]$; so f is a branch of z^{-c}. We now select an appropriate curve γ in G. Let $0 < r < 1 < R$ and let $\delta > 0$. Let L_1 be the line segment $[r+\delta i, R+\delta i]$; γ_R the part of the circle $|z| = R$ from $R+\delta i$ counterclockwise to $R-\delta i$; L_2 the line segment $[R-\delta i, r-\delta i]$; and γ_r the part of the circle $|z| = r$ from $r-\delta i$ clockwise to $r+\delta i$. Put $\gamma = L_1 + \gamma_R + L_2 + \gamma_r$.

Since $\gamma \sim 0$ in G and $\text{Res}(f(z)(1+z)^{-1}; -1) = f(-1) = e^{-i\pi c}$, the Residue Theorem gives

2.13
$$\int_\gamma \frac{f(z)}{1+z}\,dz = 2\pi i e^{-i\pi c}.$$

Using the definition of a line integral

$$\int_{L_1} \frac{f(z)}{1+z}\,dz = \int_r^R \frac{f(t+i\delta)}{1+t+i\delta}\,dt.$$

Let $g(t,\delta)$ be defined on the compact set $[r,R] \times [0, \tfrac{1}{2}\pi]$ by

$$g(t,\delta) = \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right|$$

when $\delta > 0$ and $g(t,0) = 0$. Then g is continuous and hence uniformly continuous. If $\epsilon > 0$ then there is a δ_0 such that if $(t-t')^2 + (\delta-\delta')^2 < \delta_0^2$ then $|g(t,\delta) - g(t',\delta')| < \epsilon/R$. In particular, $g(t,\delta) < \epsilon/R$ when $r \le t \le R$

and $\delta < \delta_0$. Thus

$$\int_r^R g(t, \delta)\, dt \leq \epsilon$$

for $\delta < \delta_0$. This implies that

2.14
$$\int_r^R \frac{t^{-c}}{1+t}\, dt = \lim_{\delta \to 0+} \int_{L_1} \frac{f(z)}{1+z}\, dz.$$

Similarly, using the fact that $l(\bar{z}) = \overline{l(z)} + 2\pi i$

2.15
$$-e^{-2\pi i c} \int_r^R \frac{t^{-c}}{1+t}\, dt = \lim_{\delta \to 0+} \int_{L_2} \frac{f(z)}{1+z}\, dz.$$

Now the value of the integral in (2.13) does not depend on δ. Therefore, letting $\delta \to 0+$ and using (2.14) and (2.15) gives

2.16
$$2\pi i e^{-i\pi c} - (1 - e^{-2\pi i c}) \int_r^R \frac{t^{-c}}{1+t}\, dt = \lim_{\delta \to 0+} \left[\int_{\gamma_r} \frac{f(z)}{1+z} + \int_{\gamma_R} \frac{f(z)}{1+z} \right].$$

Now if $\rho > 0$ and $\rho \neq 1$ and if γ_ρ is the part of the circle $|z| = \rho$ from $\sqrt{\rho^2 - \delta^2} + i\delta$ to $\sqrt{\rho^2 - \delta^2} - i\delta$ then

$$\left| \int_{\gamma_\rho} \frac{f(z)}{1+z}\, dz \right| \leq \frac{\rho^{-c}}{|1-\rho|} 2\pi\rho.$$

Since this estimate is independent of δ, (2.16) implies

$$\left| 2\pi i e^{-i\pi c} - (1 - e^{-2\pi i c}) \int_r^R \frac{t^{-c}}{1+t}\, dt \right| \leq \frac{r^{-c}}{|1-r|} 2\pi r + \frac{R^{-c}}{|1-R|} 2\pi R.$$

But as $r \to 0+$ and $R \to \infty$ the right-hand side of this last inequality converges to zero. Hence

$$2\pi i\, e^{-i\pi c} = (1 - e^{-2\pi i c}) \int_0^\infty \frac{t^{-c}}{1+t}\, dt;$$

or,

$$\int_0^\infty \frac{t^{-c}}{1+t}\, dt = \frac{2\pi i\, e^{-i\pi c}}{1 - e^{-2\pi i c}}$$

$$= \frac{2\pi i}{e^{\pi i c} - e^{-\pi i c}}$$

$$= \frac{\pi}{\sin \pi c}.$$

Exercises

1. Calculate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{x^2 dx}{x^4 + x^2 + 1}$

(b) $\displaystyle\int_0^\infty \frac{\cos x - 1}{x^2}\, dx$

(c) $\displaystyle\int_0^\pi \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$ where $a^2 < 1$

(d) $\displaystyle\int_0^\pi \frac{d\theta}{(a + \cos\theta)^2}$ where $a > 1$.

2. Verify the following equations:

(a) $\displaystyle\int_0^\infty \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}$, $a > 0$;

(b) $\displaystyle\int_0^\infty \frac{(\log x)^3}{1 + x^2}\, dx = 0$

(c) $\displaystyle\int_0^\infty \frac{\cos ax}{(1 + x^2)^2}\, dx = \frac{\pi(a + 1)\, e^{-a}}{4}$ if $a > 0$;

(d) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{a + \sin^2\theta} = \frac{\pi}{2[a(a+1)]^{\frac{1}{2}}}$, if $a > 0$;

(e) $\displaystyle\int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx = -\frac{\pi}{4}$;

(f) $\displaystyle\int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}$;

(g) $\displaystyle\int_{-\infty}^\infty \frac{e^{ax}}{1 + e^x}\, dx = \frac{\pi}{\sin a\pi}$ if $0 < a < 1$;

(h) $\displaystyle\int_0^{2\pi} \log \sin^2 2\theta\, d\theta = 4 \int_0^\pi \log \sin\theta\, d\theta = -4\pi \log 2$.

3. Find all possible values of $\int_\gamma \exp z^{-1}\, dz$ where γ is any closed curve not passing through $z = 0$.

4. Suppose that f has a simple pole at $z = a$ and let g be analytic in an open set containing a. Show that Res $(fg; a) = g(a)$ Res $(f; a)$.

5. Use Exercise 4 to show that if G is a region and f is analytic in G except for simple poles at $a_1, \ldots, a_n$; and if g is analytic in G then

$$\frac{1}{2\pi i} \int_\gamma fg = \sum_{k=1}^n n(\gamma; a_k)\, g(a_k)\, \text{Res}\, (f; a_k)$$

for any closed rectifiable curve γ not passing through $a_1, \ldots, a_n$ such that $\gamma \approx 0$ in G.

6. Let γ be the rectangular path $[n+\frac{1}{2}+ni, -n-\frac{1}{2}+ni, -n-\frac{1}{2}-ni, n+\frac{1}{2}-ni, n+\frac{1}{2}+ni]$ and evaluate the integral $\int_\gamma \pi(z+a)^{-2} \cot \pi z \, dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \int_\gamma \pi(z+a)^{-2} \cot \pi z \, dz = 0$ and, by using the first part, deduce that

$$\frac{\pi^2}{\sin^2 \pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}$$

(Hint: Use the fact that for $z = x + iy$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$ and $|\sin z|^2 = \sin^2 x + \sinh^2 y$ to show that $|\cot \pi z| \leq 2$ for z on γ if n is sufficiently large.)

7. Use Exercise 6 to deduce that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

8. Let γ be the polygonal path defined in Exercise 6 and evaluate $\int_\gamma \pi(z^2 - a^2)^{-1} \cot \pi z \, dz$ for $a \neq$ an integer. Show that $\lim_{n\to\infty} \int_\gamma \pi(z^2 - a^2)^{-1} \cot \pi z \, dz = 0$, and consequently

$$\pi \cot \pi a = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2a}{a^2 - n^2}$$

for $a \neq$ an integer.

9. Use methods similar to those of Exercises 6 and 8 to show that

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \sum_{n=1}^{\infty} \frac{2(-1)^n a}{a^2 - n^2}$$

for $a \neq$ an integer.

10. Let γ be the circle $|z| = 1$ and let m and n be non-negative integers. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{(z^2 \pm 1)^m dz}{z^{m+n+1}} = \begin{cases} \dfrac{(\pm 1)^p (n+2p)!}{p! \, (n+p)!}, & \text{if } m = 2p+n, \\ \qquad\qquad p \geq 0 \\ \\ \quad 0 & \text{otherwise} \end{cases}$$

11. In Exercise 1.12, consider a_n and b_n as functions of the parameter λ and use Exercise 10 to compute power series expansions for $a_n(\lambda)$ and $b_n(\lambda)$. ($b_n(\lambda)$ is called a Bessel function.)

12. Let f be analytic in the plane except for isolated singularities at $a_1, a_2, \ldots, a_m$. Show that

$$\text{Res}(f; \infty) = -\sum_{k=1}^{m} \text{Res}(f; a_k).$$

($\text{Res}(f; \infty)$ is defined as the residue of $-z^{-2}f(z^{-1})$ at $z = 0$. Equivalently, $\text{Res}(f; \infty) = -\frac{1}{2\pi i} \int f$ when $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, for sufficiently large R.) What can you say if f has infinitely many isolated singularities?

13. Let f be an entire function and let $a, b \in \mathbb{C}$ such that $|a| < R$ and $|b| < R$. If $\gamma(t) = Re^{it}$, $0 \leq t \leq 2\pi$, evaluate $\int_\gamma [(z-a)(z-b)]^{-1}f(z)dz$. Use this result to give another proof of Liouville's Theorem.

§3 The Argument Principle

Suppose that f is analytic and has a zero of order m at $z = a$. So $f(z) = (z-a)^m g(z)$ where $g(a) \neq 0$. Hence

3.1
$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

and g'/g is analytic near $z = a$ since $g(a) \neq 0$. Now suppose that f has a pole of order m at $z = a$; that is, $f(z) = (z-a)^{-m}g(z)$ where g is analytic and $g(a) \neq 0$. This gives

3.2
$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{g'(z)}{g(z)}$$

and again g'/g is analytic near $z = a$.

Also, to avoid the phrase "analytic except for poles" which may have already been used too frequently, we make the following standard definition.

3.3 Definition. If G is open and f is a function defined and analytic in G except for poles, then f is a *meromorphic function* on G.

Suppose that f is a meromorphic function on G and define $f: G \to \mathbb{C}_\infty$ by setting $f(z) = \infty$ whenever z is a pole of f. It easily follows that f is continuous from G into $\mathbb{C}_\infty$ (Exercise 4). This fact allows us to think of meromorphic functions as analytic functions with singularities for which we can remove the discontinuity of f, although we cannot remove the non-differentiability of f.

3.4 Argument Principle. *Let f be meromorphic in G with poles $p_1, p_2, \ldots, p_m$ and zeros $z_1, z_2, \ldots, z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1, \ldots, p_m$; $z_1, \ldots, z_n$; then*

3.5
$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

Proof. By a repeated application of (3.1) and (3.2)

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-a_k} - \sum_{j=1}^{m} \frac{1}{z-p_j} + \frac{g'(z)}{g(z)}$$

where g is analytic and never vanishes in G. Since this gives that g'/g is analytic, Cauchy's Theorem gives the result. ∎

Why is this called the "Argument Principle"? The answer to this is not completely obvious, but it is suggested by the fact that if we could define $\log f(z)$ then it would be a primitive for f'/f. Thus Theorem 3.4 would give that as z goes around γ, $\log f(z)$ would change by $2\pi i K$ where K is the integer on the right hand side of (3.5). Since $2\pi i K$ is purely imaginary this would give that $\operatorname{Im} \log f(z) = \arg f(z)$ changes by $2\pi K$.

Of course we can't define $\log f(z)$ (indeed, if we could then $\int_\gamma f'/f = 0$ since f'/f has a primitive). However, we can put the discussion in the above paragraph on a solid logical basis. Since no zero or pole of f lies on γ there is a disk $B(a; r)$, for each a in $\{\gamma\}$, such that a branch of $\log f(z)$ can be defined on $B(a; r)$ (simply select r sufficiently small that $f(z) \neq 0$ or ∞ in $B(a; r)$). The balls form an open cover of $\{\gamma\}$; and so, by Lebesgue's Covering Lemma, there is a positive number $\epsilon > 0$ such that for each a in $\{\gamma\}$ we can define a branch of $\log f(z)$ on $B(a; \epsilon)$. Using the uniform continuity of γ (suppose that γ is defined on $[0, 1]$), there is a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\gamma(t) \in B(\gamma(t_{j-1}); \epsilon)$ for $t_{j-1} \le t \le t_j$ and $1 \le j \le k$. Let ℓ_j be a branch of $\log f$ defined on $B(\gamma(t_{j-1}); \epsilon)$ for $1 \le j \le k$. Also, since the j-th and $(j+1)$-st sphere both contain $\gamma(t_j)$ we can choose $\ell_1, \ldots, \ell_k$ so that $\ell_1(\gamma(t_1)) = \ell_2(\gamma(t_1)); \ell_2(\gamma(t_2)) = \ell_3(\gamma t_2)); \ldots; \ell_{k-1}(\gamma(t_{k-1})) = \ell_k(\gamma(t_{k-1}))$.

If γ_j is the path γ restricted to $[t_{j-1}, t_j]$ then, since $\ell_j' = f'/f$,

$$\int_{\gamma_j} \frac{f'}{f} = \ell_j[\gamma(t_j)] - \ell_j[\gamma(t_{j-1})]$$

for $1 \le j \le k$. Summing both sides of this equation the right hand side "telescopes" and we arrive at

$$\int_\gamma \frac{f'}{f} = \ell_k(a) - \ell_1(a)$$

where $a = \gamma(0) = \gamma(1)$. That is, $\ell_k(a) - \ell_1(a) = 2\pi i K$. Because $2\pi i K$ is purely imaginary we get $\operatorname{Im} \ell_k(a) - \operatorname{Im} \ell_1(a) = 2\pi K$. This makes precise our contention that as z traces out γ, $\arg f(z)$ changes by $2\pi K$.

The proof of the following generalization is left to the reader (Exercise 1).

3.6 Theorem. *Let f be meromorphic in the region G with zeros $z_1, z_2, \ldots, z_n$ and poles $p_1, \ldots, p_m$ counted according to multiplicity. If g is analytic in G and γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through any z_i or p_j then*

$$\frac{1}{2\pi i} \int_\gamma g \frac{f'}{f} = \sum_{i=1}^n g(z_i) n(\gamma; z_i) - \sum_{j=1}^m g(p_j) n(\gamma; p_j).$$

We already know that a one-one analytic function f has an analytic inverse (IV. 7.6). It is a remarkable fact that Theorem 3.6 can be used to give

a formula for calculating this inverse. Suppose $R > 0$ and that f is analytic and one-one on $\bar{B}(a; R)$; let $\Omega = f[B(a; R)]$. If $|z - a| < R$ and $\xi = f(z) \in \Omega$ then $f(w) - \xi$ has one, and only one, zero in $B(a; R)$. If we choose $g(w) \equiv w$, Theorem 3.6 gives

$$z = \frac{1}{2\pi i} \int_\gamma \frac{w f'(w)}{f(w) - \xi} dw$$

where γ is the circle $|w - a| = R$. But $z = f^{-1}(\xi)$; this gives the following

3.7 Proposition. *Let f be analytic on an open set containing $\bar{B}(a; R)$ and suppose that f is one-one on $\bar{B}(a; R)$. If $\Omega = f[B(a; R)]$ and γ is the circle $|z - a| = R$ then $f^{-1}(\omega)$ is defined for each ω in Ω by the formula*

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)}{f(z) - \omega} dz.$$

This section closes with Rouché's Theorem.

3.8 Rouché's Theorem. *Suppose f and g are meromorphic in a neighborhood of $\bar{B}(a; R)$ with no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to their multiplicities and if*

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g.$$

Proof. From the hypothesis

$$\left| \frac{f(z)}{g(z)} + 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1$$

on γ. If $\lambda = f(z)/g(z)$ and if λ is a positive real number then this inequality becomes $\lambda + 1 < \lambda + 1$, a contradiction. Hence the meromorphic function f/g maps γ onto $\Omega = \mathbb{C} - [0, \infty)$. If l is a branch of the logarithm on Ω then $l(f(z)/g(z))$ is a well-defined primitive for $(f/g)'(f/g)^{-1}$ in a neighborhood of γ. Thus

$$0 = \frac{1}{2\pi i} \int_\gamma (f/g)'(f/g)^{-1}$$

$$= \frac{1}{2\pi i} \int_\gamma \left[\frac{f'}{f} - \frac{g'}{g} \right]$$

$$= (Z_f - P_f) - (Z_g - P_g). ∎$$

This statement of Rouché's Theorem was discovered by Irving Glicksberg (*Amer. Math. Monthly*, **83** (1976), 186-187). In the more classical statements of the theorem, f and g are assumed to satisfy the inequality

$|f+g|<|g|$ on γ. This weaker version often suffices in the applications as can be seen in the next paragraph.

Rouché's Theorem can be used to give another proof of the Fundamental Theorem of Algebra. If $p(z) = z^n+a_1z^{n-1}+\cdots+a_n$ then

$$\frac{p(z)}{z^n} = 1 + \frac{a_1}{z} + \cdots + \frac{a_n}{z^n}$$

and this approaches 1 as z goes to infinity. So there is a sufficiently large number R with

$$\left|\frac{p(z)}{z^n} - 1\right| < 1$$

for $|z| = R$; that is, $|p(z)-z^n| < |z|^n$ for $|z| = R$. Rouché's Theorem says that $p(z)$ must have n zeros inside $|z| = R$.

We also mention that the use of a circle in Rouché's Theorem was a convenience and not a necessity. Any closed rectifiable curve γ with $\gamma \sim 0$ in G could have been used, although the conclusion would have been modified by the introduction of winding numbers.

Exercises

1. Prove Theorem 3.6.

2. Suppose f is analytic on $\bar{B}(0; 1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicities) of the equation $f(z)=z^n$ where n is an integer larger than or equal to 1.

3. Let f be analytic in $B(0; R)$ with $f(0)=0$, $f'(0)\neq 0$ and $f(z)\neq 0$ for $0<|z| \le R$. Put $\rho=\min\{|f(z)|:|z|=R\}>0$. Define $g:B(0;\rho)\to\mathbb{C}$ by

$$g(\omega)= \frac{1}{2\pi i} \int_\gamma \frac{zf'(z)}{f(z)-\omega}$$

where γ is the circle $|z|=R$. Show that g is analytic and discuss the properties of g.

4. If f is meromorphic on G and $\tilde{f}: G\to\mathbb{C}_\infty$ is defined by $\tilde{f}(z)=\infty$ when z is a pole of f and $\tilde{f}(z)=f(z)$ otherwise, show that $\tilde{f}$ is continuous.

5. Let f be meromorphic on G; show that neither the poles nor the zeros of f have a limit point in G.

6. Let G be a region and let $H(G)$ denote the set of all analytic functions on G. (The letter "H" stands for *holomorphic*. Some authors call a differentiable function holomorphic and call functions analytic if they have a power series expansion about each point of their domain. Others reserve the term "analytic" for what many call the complete analytic function, which we will not describe here.) Show that $H(G)$ is an integral domain; that is, $H(G)$ is a commutative ring with no zero divisors. Show that $M(G)$, the meromorphic functions on G, is a field.

We have said that analytic functions are like polynomials; similarly, meromorphic functions are analogues of rational functions. The question

arises, is every meromorphic function on G the quotient of two analytic functions on G? Alternately, is $M(G)$ the quotient field of $H(G)$? The answer is yes but some additional theory will be required before this answer can be proved.

7. State and prove a more general version of Rouché's Theorem for curves other than circles in G.

8. Is a non-constant meromorphic function on a region G an open mapping of G into $\mathbb{C}$? Is it an open mapping of G into $\mathbb{C}_\infty$?

9. Let $\lambda > 1$ and show that the equation $\lambda-z-e^{-z} = 0$ has exactly one solution in the half plane $\{z: \text{Re } z > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?

10. Let f be analytic in a neighborhood of $D = \bar{B}(0; 1)$. If $|f(z)|<1$ for $|z|=1$, show that there is a unique z with $|z|<1$ and $f(z)=z$. If $|f(z)|\le 1$ for $|z|=1$, what can you say?