

# TRIGONOMETRIC SERIES AND FOURIER SERIES. AUXILIARY RESULTS

## 1. Trigonometric series

These are series of the form

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x). \quad (1.1)$$

Here  $x$  is a real variable and the *coefficients*  $a_0, a_1, b_1, \dots$  are independent of  $x$ . We may usually suppose, if we wish, that the coefficients are real; when they are complex the real and imaginary parts of (1.1) can be taken separately. The factor  $\frac{1}{2}$  in the constant term of (1.1) will be found to be a convenient convention.

Since the terms of (1.1) are all of period  $2\pi$ , it is sufficient to study trigonometric series in an interval of length  $2\pi$ , for example in  $(0, 2\pi)$  or  $(-\pi, \pi)$ .

Consider the power series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_{\nu} - ib_{\nu}) z^{\nu} \quad (1.2)$$

on the unit circle  $z = e^{ix}$ . The series (1.1) is the real part of (1.2). The series

$$\sum_{\nu=1}^{\infty} (a_{\nu} \sin \nu x - b_{\nu} \cos \nu x) \quad (1.3)$$

(with zero constant term), which is the imaginary part of (1.2), is called the series *conjugate* to (1.1). If  $S$  is the series (1.1), its conjugate will be denoted by  $\bar{S}$ . The conjugate of  $\bar{S}$  is, except for the constant term,  $-S$ .

A finite trigonometric sum

$$T(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^n (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x)$$

is called a *trigonometric polynomial* of order  $n$ . If  $|a_n| + |b_n| \neq 0$ ,  $T(x)$  is said to be strictly of order  $n$ . Every  $T(x)$  is the real part of an ordinary (power) polynomial  $P(z)$  of degree  $n$ , where  $z = e^{ix}$ .

We shall often use the term 'polynomial' instead of 'trigonometric polynomial'.

The fact that trigonometric series are real parts of power series often suggests a method of summing them. For example, the series

$$P(r, x) = \frac{1}{2} + \sum_{\nu=1}^{\infty} r^{\nu} \cos \nu x, \quad Q(r, x) = \sum_{\nu=1}^{\infty} r^{\nu} \sin \nu x \quad (0 \leq r < 1)$$

are respectively the real and imaginary parts of

$$\frac{1}{2} + z + z^2 + \dots = \frac{1}{2} \frac{1+z}{1-z},$$

where  $z = re^{ix}$ . This gives

$$P(r, x) = \frac{1}{2} \frac{1-r^2}{1-2r \cos x + r^2}, \quad Q(r, x) = \frac{r \sin x}{1-2r \cos x + r^2}.$$

Similarly, from the formula

$$\log \frac{1}{1-z} = z + \frac{1}{2}z^2 + \dots \quad (0 \leq r < 1),$$

we get 
$$\sum_{\nu=1}^{\infty} \frac{\cos \nu x}{r^{\nu}} = \frac{1}{2} \log \frac{1}{1-2r \cos x + r^2}, \quad \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{r^{\nu}} = \arctan \frac{r \sin x}{1-r \cos x}, \quad (1.4)$$

with  $\arctan 0 = 0$ .

Let us now consider the series

$$\frac{1}{2} - \sum_{\nu=1}^{\infty} \cos \nu x, \quad \sum_{\nu=1}^{\infty} \sin \nu x,$$

which are obtained by writing 1 for  $r$  in  $P(r, x)$  and  $Q(r, x)$ , and let us denote by  $D_n(x)$  and  $\bar{D}_n(x)$  the  $n$ th partial sums of these series. Arguing as before, we get

$$D_n(x) = \frac{1}{2} + \sum_{\nu=1}^n \cos \nu x = \frac{\sin(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}, \quad \bar{D}_n(x) = \sum_{\nu=1}^n \sin \nu x = \frac{\cos \frac{1}{2}x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}.$$

A slightly simpler method of proving the formula for  $D_n(x)$  is to multiply  $D_n(x)$  by  $2 \sin \frac{1}{2}x$  and replace the products  $2 \sin \frac{1}{2}x \cos \nu x$  by differences of sines. Then all the terms except the last cancel. Similarly for  $\bar{D}_n(x)$ .

These formulae show that  $D_n(x)$  and  $\bar{D}_n(x)$  are uniformly bounded indeed are absolutely less than  $\operatorname{cosec} \frac{1}{2}\epsilon$ , in each interval  $0 < \epsilon \leq x \leq 2\pi - \epsilon$ .

Many trigonometric expressions have a term  $2 \sin \frac{1}{2}x$  or  $2 \tan \frac{1}{2}x$  in the denominator, and in this connexion we often use the inequalities

$$\sin u \leq u, \quad \sin u \geq \frac{2}{\pi} u, \quad \tan u \geq u \quad (0 \leq u \leq \frac{1}{2}\pi).$$

Expressing the cosines and sines in terms of exponential functions, we write the  $n$ th partial sum  $s_n(x)$  of (1.1) in the form

$$\frac{1}{2}a_0 + \frac{1}{2} \sum_{\nu=1}^n \{(a_{\nu} - ib_{\nu}) e^{i\nu x} + (a_{\nu} + ib_{\nu}) e^{-i\nu x}\}.$$

If we define  $a_{-\nu}$  and  $b_{-\nu}$  for negative  $\nu$  by the conditions

$$a_{-\nu} = a_{\nu}, \quad b_{-\nu} = -b_{\nu}, \quad (\nu = 0, 1, 2, \dots)$$

(thus in particular  $b_0 = 0$ ),  $s_n$  is the  $n$ th *symmetric* partial sum, that is to say, the sum of the  $2n + 1$  central terms, of the Laurent series

$$\sum_{\nu=-\infty}^{+\infty} c_{\nu} e^{i\nu x} \quad (c_{\nu} = \frac{1}{2}(a_{\nu} - ib_{\nu})), \quad (1.5)$$

where, if the  $a_{\nu}$  and  $b_{\nu}$  are real,

$$c_{-\nu} = \bar{c}_{\nu} \quad (\nu = 0, 1, 2, \dots). \quad (1.6)$$

Conversely, any series (1.5) satisfying (1.6) may be written in the form (1.1) with  $a_{\nu}$  and  $b_{\nu}$  real. The series (1.5) satisfying (1.6) is a cosine series if and only if the  $c_{\nu}$  are real; it is a sine series if and only if the  $c_{\nu}$  are purely imaginary.

Whenever we speak of convergence or summability (see Chapter III) of a series (1.5), we are always concerned with the limit, ordinary or generalized, of the *symmetric* partial sums.

It is easily seen that the series conjugate to (1.1) is

$$-i \sum_{\nu=-\infty}^{+\infty} (\text{sign } \nu) c_{\nu} e^{i\nu x}, \quad (1.7)$$

where the symbol 'sign  $z$ ' is defined as follows:

$$\text{sign } 0 = 0, \quad \text{sign } z = z/|z| \quad (z \neq 0).$$

Each of the forms (1.1) and (1.5) of trigonometric series has its advantages. Where we are dealing with (1.1) we suppose, unless the contrary is stated, that the  $a$ 's and  $b$ 's are real. Where we are dealing with (1.5), on the other hand, it is convenient to leave the  $c$ 's unrestricted. The result is then that if (1.1) has complex coefficients and is of the form  $S_1 + iS_2$ , where  $S_1$  and  $S_2$  have real coefficients, then the series conjugate to (1.1) is  $\bar{S}_1 + i\bar{S}_2$ .

The following notation will also be used:

$$A_0(x) = \frac{1}{2}a_0, \quad A_n(x) = a_n \cos nx + b_n \sin nx, \quad B_n(x) = a_n \sin nx - b_n \cos nx \quad (n > 0),$$

so that (1.1) and (1.3) are respectively

$$\sum_{n=0}^{\infty} A_n(x), \quad \sum_{n=1}^{\infty} B_n(x).$$

We shall sometimes write (1.1) in the form

$$\sum_{n=0}^{\infty} \rho_n \cos(nx + \alpha_n), \quad \text{where } \rho_n = (a_n^2 + b_n^2)^{\frac{1}{2}} \geq 0$$

If  $c_{\nu} = 0$  for  $\nu < 0$ , (1.5) will be said to be of *power series type*. For such series,  $\bar{S}$  is except for the constant term,  $-iS$ . Obviously,  $S$  is the power series  $c_0 - c_1z + c_2z^2 - \dots$  on the unit circle  $|z| = 1$ .

In view of the periodicity of a trigonometric series it is often convenient to identify points  $x$  congruent mod  $2\pi$  and to accept all the implications of this convention. Thus, generally, we shall say that two points are distinct if they are not congruent mod  $2\pi$ ; a point  $x$  will be said to be outside a set  $E$  if it is outside every set congruent to  $E$  mod  $2\pi$ ; and so on. This convention amounts to considering points  $x$  as situated on the circumference of the unit circle. If on occasion the convention is not followed the position will be clear from the context.

## 2. Summation by parts

This is the name given to the formula

$$\sum_{\nu=1}^n u_{\nu} v_{\nu} = \sum_{\nu=1}^{n-1} U_{\nu}(v_{\nu} - v_{\nu+1}) + U_n v_n, \quad (2.1)$$

where  $U_k = u_1 + u_2 + \dots + u_k$  for  $k = 1, 2, \dots, n$ ; it is also called *Abel's transformation*.

(2.1) can be easily verified; it corresponds to integration by parts in the theory of integration. The following corollary is very useful.

(2.2) THEOREM. *If  $v_1, v_2, \dots, v_n$  are non-negative and non-increasing, then*

$$|u_1 v_1 + u_2 v_2 + \dots + u_n v_n| \leq v_1 \max_k |U_k|. \quad (2.3)$$

For the absolute value of the right-hand side of (2.1) does not exceed

$$\{(v_1 - v_2) + (v_2 - v_3) + \dots + (v_{n-1} - v_n) + v_n\} \max |U_k| = v_1 \max |U_k|.$$

The case when  $\{v_n\}$  is non-negative and non-decreasing can be reduced to the preceding one by reversing the sequence. The left-hand side of (2.3) then does not exceed

$$v_n \max |U_n - U_{k-1}| \leq 2v_n \max |U_k|.$$

A sequence  $v_0, v_1, \dots, v_n, \dots$  is said to be of *bounded variation* if the series

$$|v_1 - v_0| + |v_2 - v_1| + \dots + |v_n - v_{n-1}| + \dots$$

converges. This implies the convergence of  $(v_1 - v_0) + \dots + (v_n - v_{n-1}) + \dots = \lim (v_n - v_0)$ , and so every sequence of bounded variation is convergent.

The following result is an immediate consequence of (2.1).

**(2.4) THEOREM.** *If the series  $u_0(x) + u_1(x) + \dots$  converges uniformly and if the sequence  $\{v_n\}$  is of bounded variation, then the series  $u_0(x)v_0 + u_1(x)v_1 + \dots$  converges uniformly.*

*If the partial sums of  $u_0(x) + u_1(x) + \dots$  are uniformly bounded, and if the sequence  $\{v_n\}$  is of bounded variation and tends to 0, then the series  $u_0(x)v_0 + u_1(x)v_1 + \dots$  converges uniformly.*

The series (1.1) converges, and indeed uniformly, if  $\Sigma(|a_n| + |b_n|)$  converges. Apart from this trivial case the convergence of a trigonometric series is a delicate problem. Some special but none the less important results follow from the theorem just stated. Applying it to the series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu x, \quad \sum_{\nu=1}^{\infty} a_{\nu} \sin \nu x, \quad (2.5)$$

and taking into account the properties of  $D_n(x)$  and  $\tilde{D}_n(x)$  we have:

**(2.6) THEOREM.** *If  $\{a_n\}$  tends to 0 and is of bounded variation (in particular, if  $\{a_n\}$  tends monotonically to 0) both series (2.5), and so also the series  $\Sigma a_n e^{i\nu x}$ , converge uniformly in each interval  $\epsilon \leq x \leq 2\pi - \epsilon$  ( $\epsilon > 0$ ).*

As regards the neighbourhood of  $x=0$ , the behaviour of the cosine and sine series (2.5) may be totally different. The latter always converges at  $x=0$  (and so everywhere), while the convergence of the former is equivalent to that of  $a_1 + a_2 + \dots$ . If  $\{a_n\}$  is of bounded variation but does not tend to 0, the uniform convergence in Theorem (2.6) is replaced by uniform boundedness.

Transforming the variable  $x$  we may present (2.6) in different forms. For example, replacing  $x$  by  $x + \pi$  we have:

**(2.7) THEOREM.** *If  $\{a_n\}$  is of bounded variation and tends to 0, the series*

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu} a_{\nu} \cos \nu x, \quad \sum_{\nu=1}^{\infty} (-1)^{\nu} a_{\nu} \sin \nu x$$

*converge uniformly for  $|x| \leq \pi - \epsilon$  ( $\epsilon > 0$ ).*

By (2.6), the series  $\Sigma \nu^{-1} \cos \nu x$  and  $\Sigma \nu^{-1} \sin \nu x$  converge for  $x \neq 0$  (the latter indeed everywhere). Using the classical theorem of Abel which asserts that if  $\Sigma a_n$  converges

to  $s$  then  $\Sigma \alpha_r r^s \rightarrow s$  as  $r \rightarrow 1 - 0$  (see Chapter III, § 1, below), we deduce from (1.4) the formulae

$$\left. \begin{aligned} \sum_{\nu=1}^{\infty} \frac{\cos \nu x}{\nu} &= \log \left| \frac{1}{2 \sin \frac{1}{2} x} \right| \\ \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu} &= \frac{1}{2}(\pi - x) \end{aligned} \right\} \quad (0 < x < 2\pi). \quad (2.8)$$

### 3. Orthogonal series

A system of real- or complex-valued functions  $\phi_0(x), \phi_1(x), \phi_2(x), \dots$ , defined in an interval  $(a, b)$ , is said to be *orthogonal* over  $(a, b)$  if

$$\int_a^b \phi_m(x) \bar{\phi}_n(x) dx = \begin{cases} 0 & \text{for } m \neq n \\ \lambda_n > 0 & \text{for } m = n \end{cases} \quad (m, n = 0, 1, 2, \dots). \quad (3.1)$$

In particular,

(i) the functions  $|\phi_m(x)|^2$  are all integrable over  $(a, b)$ ;

(ii) no  $\phi_m(x)$  can vanish identically (for that would imply  $\lambda_m = 0$ ).

If in addition  $\lambda_0 = \lambda_1 = \lambda_2 = \dots = 1$ , the system is said to be *normal*. A system orthogonal and normal is called *orthonormal*. If  $\{\phi_n(x)\}$  is orthogonal,  $\{\phi_n(x)/\lambda_n^{1/2}\}$  is orthonormal.

The importance of orthogonal systems is based on the following fact. Suppose that

$$c_0 \phi_0(x) + c_1 \phi_1(x) + \dots + c_n \phi_n(x) + \dots$$

where  $c_0, c_1, \dots$  are constants, converges in  $(a, b)$  to a function  $f(x)$ . If we multiply both sides of the equation  $f = c_0 \phi_0 + c_1 \phi_1 + \dots$  by  $\bar{\phi}_n$  and integrate term by term over  $(a, b)$ , we have, after (3.1),

$$c_n = \frac{1}{\lambda_n} \int_a^b f(x) \bar{\phi}_n(x) dx \quad (n = 0, 1, 2, \dots). \quad (3.2)$$

The argument is purely formal, although in some cases easily justifiable, e.g. if the series defining  $f(x)$  converges almost everywhere, its partial sums are absolutely dominated by an integrable function, and each  $\phi_n$  is bounded. It suggests the following problem. Suppose that a function  $f(x)$  is defined in  $(a, b)$ . We compute the numbers  $c_n$  by means of (3.2), and we write

$$f(x) \sim c_0 \phi_0(x) + c_1 \phi_1(x) + \dots \quad (3.3)$$

We call the numbers  $c_n$  the *Fourier coefficients* of  $f$ , and the series in (3.3) the *Fourier series* of  $f$ , with respect to the system  $\{\phi_n\}$ . The sign ' $\sim$ ' in (3.3) only means that the  $c_n$  are connected with  $f$  by the formulae (3.2), and conveys no implication that the series is convergent, still less that it converges to  $f(x)$ . The problem is: *in what sense, and under what conditions, does the series (3.3) 'represent'  $f(x)$ ?*

This book is devoted to the study of a special but important orthogonal system, namely the trigonometric system (see § 4), and the theory of general orthogonal systems will be studied only in so far as it bears on this system. It may, however, be observed here that if an orthogonal system  $\{\phi_n\}$  is to be at all useful for developing functions it must be *complete*, that is to say, whenever a new function  $\psi$  is added to it the new system is no longer orthogonal. For otherwise there would exist a function

(namely, the function  $\psi$ ), not vanishing identically, whose Fourier series with respect to the system  $\{\phi_n\}$  would consist entirely of zeros.

If the functions  $\phi_n$  are real-valued, we may drop the bars in (3.1) and (3.2).

The following system  $\{\phi_n\}$ , orthonormal over  $(0, 1)$ , is instructive. Let  $\phi_0(x)$  be the function of period 1, equal to  $+1$  for  $0 < x < \frac{1}{2}$  and to  $-1$  for  $\frac{1}{2} < x < 1$ ; and let  $\phi_0(0) = \phi_0(\frac{1}{2}) = 0$ . Let

$$\phi_n(x) = \phi_0(2^n x) \quad (n = 0, 1, 2, \dots).$$

The function  $\phi_n(x)$  takes alternately the values  $\pm 1$  inside the intervals

$$(0, 2^{-n-1}), \quad (2^{-n-1}, 2 \cdot 2^{-n-1}), \quad (2 \cdot 2^{-n-1}, 3 \cdot 2^{-n-1}), \quad \dots$$

That  $\{\phi_n\}$  is orthogonal follows from the fact that if  $m > n$  the integral of  $\phi_m \phi_n$  over any of these intervals is 0. The system is obviously normal. It is not complete, since the function  $\psi(x) = 1$  may be added to it (see also Ex. 6 on p. 34). The functions  $\phi_n$  are called *Rademacher's functions*. Clearly,

$$\phi_n(x) = \text{sign} \sin 2^{n+1} \pi x. \dagger$$

For certain problems the following extension of the notion of orthogonality is useful. Let  $\omega(x)$  be a function non-decreasing over  $(a, b)$ , and let  $\phi_0, \phi_1, \phi_2, \dots$  be a system of functions in  $(a, b)$  such that

$$\int_a^b \phi_m(x) \bar{\phi}_n(x) d\omega(x) = \begin{cases} 0 & \text{for } m \neq n \\ \lambda_m > 0 & \text{for } m = n \end{cases} \quad (m, n = 0, 1, \dots), \quad (3.4)$$

where the integral is taken in the Stieltjes sense (Stieltjes-Riemann or Stieltjes-Lebesgue). The system  $\{\phi_n\}$  is then called *orthogonal over  $(a, b)$  with respect to  $d\omega(x)$* . If  $\lambda_0 = \lambda_1 = \dots = 1$ , the system is *orthonormal*. The Fourier coefficients of any function  $f$  with respect to  $\{\phi_n\}$  are

$$c_n = \frac{1}{\lambda_n} \int_a^b f(x) \bar{\phi}_n(x) d\omega(x), \quad (3.5)$$

and the series  $c_0 \phi_0 + c_1 \phi_1 + \dots$  is the Fourier series of  $f$ . If  $\omega(x) = x$ , this is the same as the old definition. If  $\omega(x)$  is absolutely continuous,  $d\omega(x)$  may be replaced by  $\omega'(x) dx$  and the functions  $\phi_n(x) \sqrt{\omega'(x)}$  are orthogonal in the old sense. The case when  $\omega(x)$  is a step function is important for trigonometric interpolation (see Chapter X).

#### 4. The trigonometric system

The system of functions  $e^{in\pi x} \quad (n = 0, \pm 1, \pm 2, \dots)$  (4.1)

is orthogonal over any interval of length  $2\pi$  since, for any real  $\alpha$ ,

$$\int_{\alpha}^{\alpha+2\pi} e^{imx} e^{-inx} dx = \begin{cases} 0 & (m \neq n), \\ 2\pi & (m = n). \end{cases}$$

With respect to (4.1) the Fourier series of any function  $f(x)$  defined, say, in the interval  $(-\pi, \pi)$  is

$$\sum_{-\infty}^{+\infty} c_n e^{in\pi x}. \quad (4.2)$$

† The values  $\phi_n(x)$  are closely related to the dyadic development of  $x$ . If  $0 < x < 1$ ,  $x$  is not a dyadic rational and has dyadic development  $\cdot d_1 d_2 \dots d_n \dots$  where the  $d_n$  are 0 or 1, then

$$d_n = d_n(x) = \frac{1}{2}(1 - \phi_{n-1}(x)).$$

where 
$$c_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-i\nu t} dt. \quad (4.3)$$

Let us set

$$a_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos \nu t dt, \quad b_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin \nu t dt \quad (\nu = 0, 1, \dots) \quad (4.4)$$

(thus  $b_0 = 0$ ), so that 
$$c_\nu = \frac{1}{2}(a_\nu - ib_\nu), \quad c_{-\nu} = \frac{1}{2}(a_\nu + ib_\nu) \quad (\nu \geq 0). \quad (4.5)$$

Bracketing together in (4.2) the terms with  $\pm \nu$ , we write the series in the form

$$c_0 + (c_1 e^{ix} + c_{-1} e^{-ix}) + \dots + (c_n e^{inx} + c_{-n} e^{-inx}) + \dots,$$

or, taking into consideration (4.5),

$$\frac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + \dots + (a_n \cos nx + b_n \sin nx) + \dots \quad (4.6)$$

Since the orthonormality of a pair of functions  $\phi_1, \phi_2$  implies the orthogonality of the pair  $\phi_1 \pm \phi_2$ , it is easily seen that the system

$$\frac{1}{2}, \quad \frac{e^{ix} + e^{-ix}}{2}, \quad \frac{e^{ix} - e^{-ix}}{2i}, \quad \dots, \quad \frac{e^{inx} + e^{-inx}}{2}, \quad \frac{e^{inx} - e^{-inx}}{2i}, \quad \dots,$$

or, what is the same thing, the system

$$\frac{1}{2}, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \dots, \quad (4.7)$$

is orthogonal over any interval of length  $2\pi$ .

The numbers  $\lambda$  (see § 3) for this system are  $\frac{1}{2}\pi, \pi, \pi, \dots$ , so that, in view of (4.4), (4.6) is the Fourier series of a function  $f(x)$ ,  $-\pi \leq x \leq \pi$ , with respect to the system (4.7).

If the function  $f(x)$  is even, that is, if  $f(-x) = f(x)$ , then

$$a_\nu = \frac{2}{\pi} \int_0^\pi f(t) \cos \nu t dt, \quad b_\nu = 0; \quad (4.8)$$

and if  $f(x)$  is odd, that is, if  $f(-x) = -f(x)$ , then

$$a_\nu = 0, \quad b_\nu = \frac{2}{\pi} \int_0^\pi f(t) \sin \nu t dt. \quad (4.9)$$

The set of functions (4.7) is called the *trigonometric system*, and (4.1) the *complex trigonometric system*. The numbers  $a_\nu, b_\nu$  will be called the *Fourier coefficients* (the adjective *trigonometric* being understood), and the numbers  $c_\nu$  the *complex Fourier coefficients*, of  $f$ . Finally, (4.6) is the *Fourier series* and (4.2) the *complex Fourier series*, of  $f$ . When no confusion can arise we shall simply speak of the *coefficients* of  $f$  and the *series* (or *development*) of  $f$ .

The *Fourier series* of  $f$  in either of the forms (4.2) and (4.6) will be denoted by

$$S[f],$$

and the series conjugate to  $S[f]$  by  $\bar{S}[f]$ .

The series (4.2) and (4.6) are merely variants of each other, and in particular the partial sums of the latter are symmetric partial sums of the former. For real-valued functions we shall in this book use the forms (4.2) and (4.6) interchangeably. For complex-valued functions, in principle, only the form (4.2) will be used. However, for many problems of Fourier series (e.g. the problem of the representation of  $f$  by  $S[f]$ ) the limitation to real-valued functions is no restriction of generality.

Since the terms of (4.2) and (4.6) have period  $2\pi$ , it is convenient to assume (as we shall always do in what follows) that the functions whose Fourier series we consider are defined not only in an interval of length  $2\pi$  but for all values of  $x$ , by the condition of periodicity

$$f(x + 2\pi) = f(x).$$

(This may necessitate a change in the value of  $f$  at one of the end-points of the interval, if initially the function had distinct values there.) In particular, when we speak of the Fourier series of a continuous function we shall always mean that the function is periodic and continuous in  $(-\infty, +\infty)$ . Similarly if we assert that a periodic  $f$  is integrable, of bounded variation, etc., we mean that  $f$  has these properties over a period. By *periodic functions* we shall always mean functions of period  $2\pi$ .

If  $\psi(x)$  is periodic, the integral of  $\psi(x)$  over any interval of length  $2\pi$  is always the same. In particular, since  $f(x)$  is now defined everywhere and is periodic, the interval of integration  $(-\pi, +\pi)$  in (4.3) and (4.4) may be replaced by any interval of length  $2\pi$ , for instance by  $(0, 2\pi)$ .

(4.10) THEOREM. *If (4.6) or (4.2) converges almost everywhere to  $f(x)$ , and its partial sums are absolutely dominated by an integrable function, the series is  $S[f]$  in particular the conclusion holds if the series converges uniformly.*

That  $a_n, b_n$  are given by (4.4) follows by the same argument which led to (3.2) and which is now justified.

A function  $f(x)$  defined in an interval of length  $2\pi$  (and continued periodically) has a uniquely defined  $S[f]$ . With a function  $f(x)$  defined in an interval  $(a, b)$  of length less than  $2\pi$  we can associate various Fourier series, for we may define  $f(x)$  arbitrarily in the remaining part of an interval of length  $2\pi$  containing  $(a, b)$ . The case  $(a, b) = (0, \pi)$  is of particular interest. If we define  $f(x)$  in  $(-\pi, 0)$  by the condition  $f(-x) = f(x)$ , so that the extended  $f$  is even, we get a cosine Fourier series. If the extended  $f$  is odd, we have a sine Fourier series. These two series are respectively called the *cosine* and *sine* Fourier series of the function  $f(x)$  defined in  $(0, \pi)$ .

By a linear change of variable we may transform the trigonometric system into a system orthogonal over any given finite interval  $(a, b)$ . For example, the functions

$$\exp\{2\pi i n x / (b - a)\} \quad (n = 0, \pm 1, \pm 2, \dots),$$

form an orthogonal system in  $(a, b)$ , and with any  $f(x)$  defined in that interval we may associate the Fourier series

$$\sum_{-\infty}^{+\infty} c_n \exp \frac{2\pi i n x}{\omega}, \quad \text{where } c_n = \frac{1}{\omega} \int_a^b f(t) \exp\left(-\frac{2\pi i n t}{\omega}\right) dt, \quad \omega = b - a.$$

By a change of variable the study of such series reduces to the study of ordinary Fourier series. The case of functions  $f(x)$  of period 1 is particularly important. Here

$$f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{2\pi i n x}, \quad \text{where } c_n = \int_0^1 f(t) e^{-2\pi i n t} dt.$$

The notion of Fourier coefficients  $a_n, b_n, c_n$  has a parallel notion, that of *Fourier transforms*

$$\alpha(\nu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x) \cos \nu x dx, \quad \beta(\nu) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x) \sin \nu x dx, \quad \gamma(\nu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-i\nu x} dx. \quad (4.11)$$

The function  $f$  here is defined over an infinite interval, and in general is not periodic;  $\nu$  is a continuous variable ranging from  $-\infty$  to  $+\infty$ . Unless  $f$  is a function absolutely integrable over  $(-\infty, \infty)$ , in which case the integrals (4.11) converge absolutely and uniformly for all  $\nu$ , one must specify the sense in which these integrals are taken. Fourier integrals occur sporadically in the theory of Fourier series, but a more detailed discussion of them is postponed to a later chapter (see Chapter XVI).

The problems of the theory of Fourier series are closely connected with the notion of integration. In the formulae (4.4) we tacitly assumed that the products  $f \cos \nu t$ ,  $f \sin \nu t$  were integrable. Thus we may consider Fourier-Riemann, Fourier-Lebesgue, Fourier-Denjoy, etc., series, according to the way in which the integrals are defined. In this book, except when otherwise stated, the integrals are always Lebesgue integrals. It is assumed that the reader knows the elements of the Lebesgue theory. Proofs of results of a special character will be given in the text, or the reader will be referred to standard text-books.

Every integrable function  $f(x)$ ,  $0 \leq x \leq 2\pi$ , has its Fourier series. It is even sufficient for  $f$  to be defined almost everywhere in  $(0, 2\pi)$ , that is to say, everywhere except in a set of measure zero. Functions  $f_1(x)$  and  $f_2(x)$  which are equal almost everywhere have the same Fourier series. Following the usage of Lebesgue, we call them equivalent (in symbols,  $f_1(x) \equiv f_2(x)$ ), and we do not distinguish between equivalent functions.

Throughout this book the following notations will consistently be used:

$$x \in A, \quad x \notin A, \quad A \subset B, \quad B \supset A.$$

The first means that  $x$  belongs to the set  $A$ ; the second that  $x$  does not belong to  $A$ ; the third and fourth that  $A$  is a subset of  $B$ .

The Lebesgue measure of a set (in particular, of an interval)  $E$  will be denoted by  $|E|$ . The sets and functions considered will always be measurable, even if this is not stated explicitly.

By a *denumerable* set we always mean a set which is either finite or denumerably infinite.

We list a few Fourier series which are useful in applications. Verifications are left to the reader.

(i) Let 
$$\phi(x) = \frac{1}{2}(\pi - x) \quad \text{for } 0 < x < 2\pi, \quad \phi(0) = \phi(2\pi) = 0.$$

Continued periodically  $\phi(x)$  is odd and

$$\phi(x) \sim \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu} = \frac{1}{2} \sum_{\nu=-\infty}^{+\infty} \frac{e^{i\nu x}}{i\nu} \quad (4.12)$$

(see also (2.8) above).

The function  $\phi(x)$  can be used to remove discontinuities of other functions. For it is continuous except at  $x = 0$ , where it has a jump  $\pi$ . Thus, if  $f(x)$  is periodic and at  $x = x_0$  has a jump  $d = f(x_0 + 0) - f(x_0 - 0)$ , the difference

$$\Delta(x) = f(x) - (d/\pi) \phi(x - x_0)$$

is continuous at  $x_0$ , or may be made so by changing the value of  $f(x_0)$ .

(ii) Let  $s(x) = +1$  for  $0 < x < \pi$  and  $s(x) = -1$  for  $-\pi < x < 0$ . Then

$$s(x) \sim \frac{4}{\pi} \sum_{\nu=1}^{\infty} \frac{\sin(2\nu - 1)x}{2\nu - 1}. \quad (4.13)$$

(iii) Given  $0 < h < \pi$ , let  $\chi(x) = 1$  in  $(-h, h)$ ,  $\chi(x) = 0$  at the remaining points of  $(-\pi, \pi)$ . The function  $\chi$  is even and

$$\chi(x) \sim \frac{2h}{\pi} \left[ \frac{1}{2} + \sum_{\nu=1}^{\infty} \frac{\sin \nu h}{\nu h} \cos \nu x \right] = \frac{h}{\pi} \sum_{\nu=-\infty}^{+\infty} \frac{\sin \nu h}{\nu h} e^{i\nu x}, \quad (4.14)$$

where the value of  $(\sin \nu h)/\nu h$  when  $\nu = 0$  is taken to be 1.

(iv) Let  $0 < k \leq h$ ,  $k + h \leq \pi$ , and let  $\mu(x) = \mu_{h,k}(x)$  be periodic, continuous, even, equal to 1 in  $(0, h - k)$ , 0 in  $(h + k, \pi)$ , linear in  $(h - k, h + k)$ . Then

$$\mu(x) \sim \frac{2h}{\pi} \left[ \frac{1}{2} + \sum_{\nu=1}^{\infty} \left( \frac{\sin \nu h}{\nu h} \right) \left( \frac{\sin \nu k}{\nu k} \right) \cos \nu x \right] = \frac{h}{\pi} \sum_{\nu=-\infty}^{+\infty} \left( \frac{\sin \nu h}{\nu h} \right) \left( \frac{\sin \nu k}{\nu k} \right) e^{i\nu x}. \quad (4.15)$$

The  $\nu$ th coefficient of  $\mu$  does not exceed a fixed multiple of  $\nu^{-2}$ , so that  $S[\mu]$  converges absolutely and uniformly. Using Theorem (6.3) below we see that the sign ' $\sim$ ' in (4.15) can be replaced by the sign of equality.

For  $k = 0$  the series (4.15) go into the series (4.14).

(v) The special case  $h = k$  of (4.15) deserves attention. Then

$$\lambda_h(x) = \mu_{h,h}(x) \sim \frac{2h}{\pi} \left[ \frac{1}{2} + \sum_{\nu=1}^{\infty} \left( \frac{\sin \nu h}{\nu h} \right)^2 \cos \nu x \right] = \frac{h}{\pi} \sum_{\nu=-\infty}^{+\infty} \left( \frac{\sin \nu h}{\nu h} \right)^2 e^{i\nu x}. \quad (4.16)$$

The function  $\lambda_h(x)$  is even, decreases linearly from 1 to 0 over the interval  $0 \leq x \leq 2h$ , and is zero in  $(2h, \pi)$ . It is useful to note that the coefficients of  $S[\lambda]$  are non-negative. Using the remark made above that  $S[\lambda]$  converges to  $\lambda$  and setting  $x = 0$ , we have the formula

$$\sum_{\nu=-\infty}^{+\infty} \left( \frac{\sin \nu h}{\nu h} \right)^2 = \frac{\pi}{h}, \quad (4.17)$$

which will be applied later.

The functions  $\mu$  can be expressed in terms of the  $\lambda$ 's:

$$\mu_{h,k} = \frac{1}{2} \left( \frac{h}{k} + 1 \right) \lambda_{\frac{1}{2}(h+k)} - \frac{1}{2} \left( \frac{h}{k} - 1 \right) \lambda_{\frac{1}{2}(h-k)}. \quad (4.18)$$

Since both sides here are even functions of  $x$  and represent polygonal lines, it is enough to check the formula for the values of  $x$  corresponding to the vertices, that is, for  $x = 0, h \pm k, \pi$ .

$\lambda$  is often called the *triangular* or *roof*, function and  $\mu$  the *trapezoidal* function.

(vi) Considering the Fourier series of the function  $e^{-i\alpha x}$ ,  $0 < x < 2\pi$ , where  $\alpha$  is any real or complex number, but not a real integer, we obtain the development

$$\frac{\sin \pi \alpha}{\alpha} e^{i(\pi-x)\alpha} \sim \sum_{n=-\infty}^{\infty} \frac{e^{in\pi}}{n+\alpha} \quad (4.19)$$

This degenerates to (4.12) when  $\alpha \rightarrow 0$ .

## 5. Fourier-Stieltjes series

Let  $F(x)$  be a function of bounded variation defined in the closed interval  $0 \leq x \leq 2\pi$ .

Let us consider the series  $\sum_{-\infty}^{+\infty} c_\nu e^{i\nu x}$  with coefficients given by the formula

$$c_\nu = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu t} dF(t) \quad (\nu = 0, \pm 1, \pm 2, \dots), \quad (5.1)$$

the integrals being Riemann-Stieltjes integrals. The numbers  $c_\nu$  will be called the *Fourier-Stieltjes coefficients* of  $F$ , or the *Fourier coefficients* of  $dF$ . We write

$$dF(x) \sim \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}$$

and call the series here the *Fourier-Stieltjes series* of  $F$  or the *Fourier series* of  $dF$ ; we denote it by  $S[dF]$ . If  $F(x)$  is absolutely continuous, then  $S[dF] = S[F']$ . We may also write  $S[dF]$  in the form (4.6) with

$$a_\nu = \frac{1}{\pi} \int_0^{2\pi} \cos \nu t dF(t), \quad b_\nu = \frac{1}{\pi} \int_0^{2\pi} \sin \nu t dF(t).$$

It is convenient to define  $F(x)$  for all  $x$  by the condition

$$F(x + 2\pi) - F(x) = F(2\pi) - F(0), \quad (5.2)$$

and this can be done without changing the values of  $F$  for  $0 \leq x \leq 2\pi$ . In the formulae for Fourier-Stieltjes coefficients we may then integrate over any interval of length  $2\pi$ .

If we change  $F(x)$  in a denumerable set, and if the new function is still of bounded variation, the numbers (5.1) remain unchanged. Thus we can assume once for all that the  $F$  we consider have no removable discontinuities.

The function  $F(x)$  defined for all  $x$  by (5.2) is periodic if and only if  $F(2\pi) - F(0)$  vanishes, i.e. if  $c_0 = 0$ . *The difference*

$$\Delta(x) = F(x) - c_0 x$$

is always periodic. For

$$\Delta(x + 2\pi) - \Delta(x) = F(x + 2\pi) - F(x) - 2\pi c_0 = 0.$$

A function  $F(x)$  of bounded variation satisfying (5.2) may be called a *mass distribution* (of positive and negative masses, in general) on the circumference of the unit circle. If  $(\alpha; \beta)$  is an arc on this circumference and  $0 < \beta - \alpha < 2\pi$ , then  $F(\beta) - F(\alpha)$  is, by definition, the mass situated on the semi-open arc  $\alpha < x \leq \beta$ . The series

$$\sum_{-\infty}^{+\infty} e^{i\nu x} = 2\left(\frac{1}{2} + \cos x + \cos 2x + \dots\right)$$

is the Fourier-Stieltjes series of a mass  $2\pi$  concentrated at the point  $x = 0$  of the circumference.

## 6. Completeness of the trigonometric system

This theorem is a simple corollary of results we shall obtain later, but the following elementary proof, due to Lebesgue, is of interest in itself.

Let  $f(x)$  be an integrable function whose coefficients  $a_0, a_1, b_1, \dots$  all vanish, so that

$$\int_{-\pi}^{\pi} f(x) T(x) dx = 0 \quad (6.1)$$

for any trigonometric polynomial  $T(x)$ . We have to show that  $f(x) \equiv 0$ . Let us assume first that  $f(x)$  is continuous and not identically zero. There is then a point  $x_0$  and two positive numbers  $\epsilon, \delta$  such that  $|f(x)| > \epsilon$ , say  $f(x) > \epsilon$ , in the interval  $I = (x_0 - \delta, x_0 + \delta)$ .

It will be enough to show that there is a sequence  $T_n(x)$  of trigonometric polynomials such that

- (i)  $T_n(x) \geq 0$  for  $x \in I$ ;
- (ii)  $T_n(x)$  tends uniformly to  $+\infty$  in every interval  $I'$  interior to  $I$ ;
- (iii) the  $T_n$  are uniformly bounded outside  $I$ .

For then the integral in (6.1), with  $T = T_n$ , may be split into two, extended respectively over  $I$  and over the rest of  $(-\pi, \pi)$ . By (i), the first integral exceeds

$$\epsilon |I'| \min_{x \in I'} T_n(x),$$

and so, by (ii), tends to  $+\infty$  with  $n$ . The second integral is bounded, in view of (iii). Thus (6.1) is impossible for  $T = T_n$  with  $n$  large.

If we set

$$T_n(x) = \{t(x)\}^n, \quad t(x) = 1 + \cos(x - x_0) - \cos \delta,$$

then  $t(x) \geq 1$  in  $I$ ,  $t(x) > 1$  in  $I'$ ,  $|t(x)| \leq 1$  outside  $I$ . Conditions (i), (ii) and (iii) being satisfied, the theorem is proved for  $f$  continuous.

Suppose now that  $f$  is merely integrable, and let  $F(x) = \int_{-\pi}^x f dt$ . The condition  $a_0 = 0$  implies that  $F(x + 2\pi) - F(x) = 0$ , so that  $F(x)$  is periodic. Let  $A_0, A_1, B_1, \dots$  be the coefficients of  $F$  and let us integrate by parts the integrals

$$\int_{-\pi}^{\pi} F(x) \cos \nu x dx, \quad \int_{-\pi}^{\pi} F(x) \sin \nu x dx$$

for  $\nu = 1, 2, \dots$ . Owing to the periodicity of  $F$ , the integrated terms vanish, and the hypothesis  $a_1 = b_1 = a_2 = \dots = 0$  implies that  $A_1 = B_1 = A_2 = \dots = 0$ . Let  $A'_0, A'_1, B'_1, \dots$  be the coefficients of  $F(x) - A_0$ . Obviously  $A'_0 = A'_1 = B'_1 = \dots = 0$ . Thus  $F(x) - A_0$ , being continuous, vanishes identically and  $f \equiv 0$ . This completes the proof. As corollaries we have:

(6.2) THEOREM. If  $f_1(x)$  and  $f_2(x)$  have the same Fourier series, then  $f_1 \equiv f_2$ .

(6.3) THEOREM. If  $f(x)$  is continuous, and  $S[f]$  converges uniformly, its sum is  $f(x)$ .

To prove (6.2) we observe that the coefficients of  $f_1 - f_2$  all vanish, so that  $f_1 - f_2 \equiv 0$ . To prove (6.3), let  $g(x)$  denote the sum of  $S[f]$ . Then the coefficients of  $S[f]$  are the Fourier coefficients of  $g$  (see (4.10)). Hence  $S[f] = S[g]$ , so that  $f \equiv g$  and,  $f$  and  $g$  being continuous,  $f = g$ . (For a more complete result see Chapter III, p. 89.)

## 7. Bessel's inequality and Parseval's formula

Let  $\phi_0, \phi_1, \dots$  be an orthonormal system of functions over  $(a, b)$  and let  $f(x)$  be a function such that  $|f(x)|^2$  is integrable over  $(a, b)$ . We fix an integer  $n \geq 0$ , set

$$\Phi = \gamma_0 \phi_0 + \gamma_1 \phi_1 + \dots + \gamma_n \phi_n$$

and seek the values of the constants  $\gamma_0, \gamma_1, \dots, \gamma_n$  which make the integral

$$J = \int_a^b |f - \Phi|^2 dx \tag{7.1}$$

a minimum.

If we observe that

$$\int_a^b |\Phi|^2 dx = \int_a^b (\Sigma \gamma_\mu \phi_\mu) (\Sigma \bar{\gamma}_\nu \bar{\phi}_\nu) dx = \Sigma |\gamma_\nu|^2,$$

$$\int_a^b f\Phi dx = \int_a^b f \cdot (\Sigma \bar{\gamma}_\nu \bar{\phi}_\nu) dx = \Sigma c_\nu \bar{\gamma}_\nu,$$

where  $c_0, c_1, \dots$  are the Fourier coefficients of  $f$  with respect to  $\{\phi_\nu\}$ , we have

$$J = \int_a^b (f - \Phi)(\bar{f} - \bar{\Phi}) dx = \int_a^b |f|^2 dx + \int_a^b |\Phi|^2 dx - 2\Re \int_a^b f\bar{\Phi} dx$$

$$= \int_a^b |f|^2 dx + \Sigma |\gamma_\nu|^2 - 2\Re \Sigma c_\nu \bar{\gamma}_\nu.$$

Adding and subtracting  $\Sigma |c_\nu|^2$  we get

$$J = \int_a^b |f - \Phi|^2 dx = \int_a^b |f|^2 dx - \sum_{\nu=0}^n |c_\nu|^2 + \sum_{\nu=0}^n |c_\nu - \gamma_\nu|^2. \quad (7.2)$$

It follows that  $J$  attains its minimum if  $\gamma_\nu = c_\nu$ , for  $\nu = 0, 1, \dots, n$ . Thus

(7.3) **THEOREM.** *If  $|f(x)|^2$  is integrable over  $(a, b)$  and if  $\Phi = \gamma_0 \phi_0 + \gamma_1 \phi_1 + \dots + \gamma_n \phi_n$ , where  $\phi_0, \phi_1, \dots$  form an orthonormal system over  $(a, b)$ , the integral (7.1) is a minimum when  $\Phi$  is the  $n$ -th partial sum of the Fourier series of  $f$  with respect to  $\{\phi_\nu\}$ .*

On account of (7.2) this minimum, necessarily non-negative, is

$$\int_a^b |f|^2 dx - \sum_{\nu=0}^n |c_\nu|^2. \quad (7.4)$$

Hence

$$\sum_{\nu=0}^n |c_\nu|^2 \leq \int_a^b |f|^2 dx.$$

This inequality is called *Bessel's inequality*. If  $\{\phi_\nu\}$  is infinite we may make  $n$  tend to infinity, when Bessel's inequality becomes

$$\sum_{\nu=0}^{\infty} |c_\nu|^2 \leq \int_a^b |f|^2 dx. \quad (7.5)$$

Since the system  $\{e^{i\nu x}/(2\pi)^{1/2}\}$  is orthonormal over  $(0, 2\pi)$ , we have

$$\sum_{\nu=-\infty}^{+\infty} |c_\nu|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f|^2 dx,$$

where the  $c_\nu$  are defined by (4.3). If  $f$  is real-valued this gives

$$\frac{1}{2}a_0^2 + \sum_{\nu=1}^{\infty} (a_\nu^2 + b_\nu^2) \leq \frac{1}{\pi} \int_0^{2\pi} f^2 dx.$$

It follows that the Fourier coefficients  $a_\nu, b_\nu, c_\nu$  tend to 0 with  $1/\nu$ , provided that  $|f|^2$  is integrable.

In some cases the sign ' $\leq$ ' in (7.5) can be replaced by '='. (From the preceding argument it follows that this is certainly the case if the Fourier series of  $f$  with respect to  $\{\phi_\nu\}$  converges uniformly to  $f$  and  $(a, b)$  is finite.) The equation we then get is called *Parseval's formula*. It will be shown in Chapter II, § 1, that Parseval's formula holds for the trigonometric system.

*Remark.* If the functions  $\phi_n$  form on  $(a, b)$  an orthonormal system with respect to a non-decreasing function  $\omega(x)$ , Theorem (7.3) remains valid provided we replace the integral (7.1) by

$$\int_a^b |f - \Phi|^2 d\omega(x).$$

This remark will be useful in trigonometric interpolation (see Chapter X).

### 8.† Remarks on series and integrals

Let  $f(x)$  and  $g(x)$  be two functions defined for  $x > x_0$ , and let  $g(x) \neq 0$  there. The symbols

$$f(x) = o(g(x)), \quad f(x) = O(g(x));$$

mean respectively that  $f(x)/g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ , and that  $f(x)/g(x)$  is bounded for  $x$  large enough. The same notation is used when  $x$  tends to a finite limit or to  $-\infty$ , or even when  $x$  tends to its limit through a discrete sequence of values. In particular, an expression is  $o(1)$  or  $O(1)$  if it tends to 0 or is bounded, respectively.

Two functions  $f(x)$  and  $g(x)$  defined in the neighbourhood of a point  $x_0$  (finite or infinite) are called *asymptotically equal* if  $f(x)/g(x) \rightarrow 1$  as  $x \rightarrow x_0$ . We write then

$$f(x) \sim g(x) \quad (x \rightarrow x_0).$$

If the ratios  $f(x)/g(x)$  and  $g(x)/f(x)$  are both bounded in the neighbourhood of  $x_0$ , we say that  $f(x)$  and  $g(x)$  are *of the same order* as  $x \rightarrow x_0$ , and write

$$f(x) \sim g(x) \quad (x \rightarrow x_0).$$

Let  $u_0, u_1, u_2, \dots$  be a sequence of numbers and let

$$U_n = u_0 + u_1 + \dots + u_n \quad (n = 0, 1, \dots).$$

A similar notation will be used with other letters. Let  $a$  be finite, and let  $f(x)$  be a function defined in a finite or infinite interval  $a \leq x < b$  and integrable over every interval  $(a, b')$ ,  $b' < b$ . We shall write

$$F(x) = \int_a^x f(t) dt \quad (a \leq x < b).$$

**(8.1) THEOREM.** Suppose that  $f(x)$  and  $g(x)$  are defined for  $a \leq x < b$  and integrable over each  $(a, b')$  ( $b' < b$ ), that  $g(x) \geq 0$ , and that  $G(x) \rightarrow +\infty$  as  $x \rightarrow b$ . Then, if  $f(x) = o(g(x))$  as  $x \rightarrow b$ , we have  $F(x) = o(G(x))$ .

Suppose that  $|f(x)/g(x)| < \frac{1}{2}\epsilon$  for  $x_0 < x < b$ . For such  $x$ ,

$$|F(x)| \leq \int_a^{x_0} |f| dt + \int_{x_0}^x |f| dt \leq \int_a^{x_0} |f| dt + \frac{1}{2}\epsilon G(x).$$

Since  $G(x) \rightarrow \infty$ , the last sum is less than  $\epsilon G(x)$  for  $x$  close enough to  $b$ , and since  $\epsilon$  is arbitrary, the result follows.

In this theorem the roles played by  $a$  and  $b$  can obviously be reversed. If  $a = 0$  and  $b = +\infty$ , it has the following analogue for series.

† The remainder of this chapter is not concerned with trigonometric series. It contains a concise presentation of various points from the theory of the real variable which will be frequently used later. Many of the results are familiar and we assemble them primarily for easy reference. We do not attempt to be complete. Some of the theorems, moreover, will be used later in a form more general than that in which they are proved here, but only when the general proof is essentially the same. (To give a typical example, the inequalities of Hölder and Minkowski will be applied to Stieltjes integrals although we prove them here only for ordinary Lebesgue integrals.) The material is not for detailed study, but only for consultation as required.

(8.2) THEOREM. Let  $\{u_n\}$  and  $\{v_n\}$  be two sequences of numbers, the latter positive. If  $u_n = o(v_n)$  and  $V_n \rightarrow +\infty$ , then  $U_n = o(V_n)$ .

The proof is the same as that of (8.1).

(8.3) THEOREM. Suppose that the series  $\Sigma v_n$  converges, that the  $v$ 's are positive, and that  $u_n = o(v_n)$ . Then

$$u_n + u_{n+1} + \dots = o(v_n + v_{n+1} + \dots).$$

This is obvious.

(8.4) THEOREM. Let  $f(x)$  be a positive, finite, and monotone function defined for  $x \geq 0$ , and let

$$F(x) = \int_0^x f dt, \quad F_n = f(0) + f(1) + \dots + f(n).$$

Then (i) if  $f(x)$  decreases,  $F(n) - F_n$  tends to a finite limit;

(ii) if  $f(x)$  increases,  $F(n) \leq F_n \leq F(n) + f(n)$ .

To prove (i) we note that  $f(k) \leq F(k) - F(k-1) \leq f(k-1)$  implies

$$0 \leq F(k) - F(k-1) - f(k) \leq f(k-1) - f(k) \quad (k=1, 2, \dots). \quad (8.5)$$

Since  $\Sigma \{f(k-1) - f(k)\}$  converges, so does the series  $\sum_1^{\infty} \{F(k) - F(k-1) - f(k)\}$ ; and it is enough to observe that its  $n$ th partial sum is  $F(n) - F_n + f(0)$ .

Case (ii) is proved by adding the obvious inequalities

$$f(k-1) \leq F(k) - F(k-1) \leq f(k) \quad (k=1, 2, \dots, n).$$

(8.6) THEOREM. Let  $f(x)$  be positive, finite and monotone for  $x \geq 0$ . If either (i)  $f(x)$  decreases and  $F(x) \rightarrow \infty$ , or (ii)  $f(x)$  increases and  $f(x) = o(F(x))$ , then

$$F_n \simeq F(n).$$

This follows from (8.4).

(8.7) THEOREM. Let  $f(x)$ ,  $x \geq 0$ , be positive, monotone decreasing and integrable over  $(0, +\infty)$ , and let

$$F^*(x) = \int_x^{\infty} f dt, \quad F_n^* = f(n) + f(n+1) + \dots$$

Then

$$F_{n+1}^* \leq F^*(n) \leq F_n^*.$$

If in addition  $f(x) = o(F^*(x))$ , then  $F_n^* \simeq F^*(n)$ .

It is enough to add the inequalities  $f(k+1) \leq F^*(k) - F^*(k+1) \leq f(k)$  for  $k=n, n+1, \dots$ .

Examples. From (8.6) and (8.7) it follows that

$$\sum_{k=1}^n k^\alpha \simeq \frac{n^{\alpha+1}}{\alpha+1}, \quad \sum_{k=n}^{\infty} k^{-\beta} \simeq \frac{n^{1-\beta}}{\beta-1} \quad (8.8)$$

for  $\alpha > -1$ ,  $\beta > 1$ .

Taking  $f(x) = 1/(1+x)$ ,  $n = m-1$ , we obtain from (8.4) that the difference

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} - \log m$$

tends to a finite limit  $C$  (Euler's constant) as  $m \rightarrow \infty$ .

A more precise formula is sometimes needed, namely,

$$1 + \frac{1}{2} + \dots + \frac{1}{m} = \log m + C + O\left(\frac{1}{m}\right). \quad (8.9)$$

To prove this, we observe that, for  $f(x) = 1/(1+x)$ , the right-hand side of (8.5) is  $1/k(k+1)$ . Hence the  $m$ th partial sum of the series with terms  $F(k) - F(k-1) - f(k)$  differs from the sum of the whole series by less than

$$\frac{1}{(m+1)(m+2)} + \frac{1}{(m+2)(m+3)} + \dots = \frac{1}{m+1},$$

and, arguing as in the proof of (8.4) (i), we get (8.9).

## 9. Inequalities

Let  $\phi(u)$  be a non-negative function defined for  $u \geq 0$ . We say that a function  $f(x)$  defined in an interval  $(a, b)$  belongs to the class  $L_\phi(a, b)$ , in symbols  $f \in L_\phi(a, b)$ , if  $\phi(|f(x)|)$  is integrable over  $(a, b)$ . If there is no danger of confusion, the class will be denoted simply by  $L_\phi$ . In particular, if  $f$  is periodic,  $f \in L_\phi$  will mean  $f \in L_\phi(0, 2\pi)$ . If  $\phi(u) = u^r$ ,  $r > 0$ ,  $L_\phi$  will be written  $L^r$ . More generally, we shall occasionally write  $\phi(L)$  for  $L_\phi$ ; thus, for example,  $L^\alpha(\log^+ L)^\beta$  will denote the class of functions  $f$  such that  $|f|^\alpha (\log^+ |f|)^\beta$  is integrable.†

We shall also systematically use the notation

$$\mathfrak{M}_r[f; a, b] = \left\{ \int_a^b |f(x)|^r dx \right\}^{1/r}, \quad \mathfrak{A}_r[f; a, b] = \left\{ \frac{1}{b-a} \int_a^b |f(x)|^r dx \right\}^{1/r}.$$

If  $(a, b)$  is fixed we may simply write  $\mathfrak{M}_r[f]$  and  $\mathfrak{A}_r[f]$ . Unlike  $\mathfrak{M}_r$ ,  $\mathfrak{A}_r$  is defined only if  $(a, b)$  is finite.

Similarly, given a finite or infinite sequence  $a = \{a_n\}$  and a finite sequence

$$b = \{b_1, b_2, \dots, b_N\},$$

we write

$$\mathfrak{S}_r[a] = \left\{ \sum |a_n|^r \right\}^{1/r},$$

Instead of  $L^1$ ,  $\mathfrak{M}_1$ ,  $\mathfrak{A}_1$ ,  $\mathfrak{S}_1$  we write  $L$ ,  $\mathfrak{M}$ ,  $\mathfrak{A}$ ,  $\mathfrak{S}$ .

Let  $\phi(u)$ ,  $u \geq 0$ , and  $\psi(v)$ ,  $v \geq 0$ , be two functions, continuous, vanishing at the origin, strictly increasing, and inverse to each other. Then for  $a, b \geq 0$  we have the following inequality, due to W. H. Young:

$$ab \leq \Phi(a) + \Psi(b), \quad \text{where} \quad \Phi(x) = \int_0^x \phi u du, \quad \Psi(y) = \int_0^y \psi v dv. \quad (9.1)$$

This is obvious geometrically, if we interpret the terms as areas. It is easy to see that we have equality in (9.1) if and only if  $b = \phi(a)$ . The functions  $\Phi$  and  $\Psi$  will be called *complementary functions* (in the sense of Young).

On setting  $\phi(u) = u^\alpha$ ,  $\psi(v) = v^{1/\alpha}$  ( $\alpha > 0$ ),  $r = 1 + \alpha$ ,  $r' = 1 + 1/\alpha$ , we get the inequality

$$ab \leq \frac{a^r}{r} + \frac{b^{r'}}{r'} \quad (a, b \geq 0), \quad (9.2)$$

where the 'complementary' exponents  $r$ ,  $r'$  both exceed 1 and are connected by the relation

$$1/r + 1/r' = 1.$$

† By  $\log^+ |f|$  we mean  $\log |f|$  wherever  $|f| \geq 1$ , and 0 otherwise.

This notation will be used systematically, so that, e.g.,  $p'$  will denote the number such that

$$1/p + 1/p' = 1.$$

If  $r = r' = 2$ , (9.2) reduces to the familiar inequality  $2ab \leq a^2 + b^2$ . Clearly, either  $r \leq 2 \leq r'$  or  $r' \leq 2 \leq r$ . If  $r \rightarrow 1$ , then  $r' \rightarrow \infty$ , and conversely. The connexion between  $r$  and  $r'$  may also be written

$$r' = \frac{r}{r-1}.$$

Integrating the inequality

$$|fg| \leq \Phi(|f|) + \Psi(|g|)$$

over  $a \leq x \leq b$ , we see that  $fg$  is integrable over  $(a, b)$  if  $f \in L_\Phi(a, b)$ ,  $g \in L_\Psi(a, b)$ .

In particular,  $fg$  is integrable if  $f \in L^r$ ,  $g \in L^{r'}$ .

Let us now consider (real or complex) sequences  $A = \{A_n\}$ ,  $B = \{B_n\}$ ,  $AB = \{A_n B_n\}$  and let us assume that  $\mathfrak{S}_r[A] = \mathfrak{S}_{r'}[B] = 1$ ,  $r > 1$ . If we sum the inequalities

$$|A_n B_n| \leq \frac{|A_n|^r}{r} + \frac{|B_n|^{r'}}{r'}$$

for  $n = 1, 2, \dots$ , we get  $\mathfrak{S}[AB] \leq 1$ .

Now let  $a = \{a_n\}$  and  $b = \{b_n\}$  be any two sequences such that  $\mathfrak{S}_r[a]$  and  $\mathfrak{S}_{r'}[b]$  are positive and finite, and let us set  $A_n = a_n/\mathfrak{S}_r[a]$ ,  $B_n = b_n/\mathfrak{S}_{r'}[b]$  for all  $n$ . Then  $\mathfrak{S}_r[A] = \mathfrak{S}_{r'}[B] = 1$ , so that  $\mathfrak{S}[AB] \leq 1$ . In other words,

$$\sum |a_n b_n| \leq \mathfrak{S}_r[a] \mathfrak{S}_{r'}[b], \quad (9.3)$$

and a fortiori

$$|\sum a_n b_n| \leq \mathfrak{S}_r[a] \mathfrak{S}_{r'}[b]. \quad (9.4)$$

These inequalities are called *Hölder's inequalities*. They are trivially true if  $\mathfrak{S}_r[a] = 0$  or  $\mathfrak{S}_{r'}[b] = 0$ .

*Hölder's inequality for integrals* is

$$\left| \int_a^b fg dx \right| \leq \mathfrak{M}_r[f] \mathfrak{M}_{r'}[g], \quad (9.5)$$

and its proof is similar to that of (9.4), summation being replaced by integration. If  $r = r' = 2$ , (9.4) and (9.5) reduce to the familiar Schwarz inequalities.

The remark concerning the sign of equality in (9.1) shows that we have equality in (9.2) if and only if  $a^r = b^r$ . Hence if we assume that  $\mathfrak{S}_r[a]$  and  $\mathfrak{S}_{r'}[b]$  are distinct from 0 the proof of (9.3) shows that the sign of equality holds there if and only if  $|A_n|^r = |B_n|^{r'}$  for all  $n$ ; or, again, if and only if  $|a_n|^r / |b_n|^{r'}$  is independent of  $n$ , with the understanding that a ratio 0/0 is to be disregarded. If  $\mathfrak{S}_r[a] = 0$ , or  $\mathfrak{S}_{r'}[b] = 0$ , we have automatic equality in (9.3), and at the same time  $|a_n|^r / |b_n|^{r'}$  is 'independent of  $n$ ', so that the rule is in this case also valid. Taking into account that the left hand sides of (9.3) and (9.4) are equal if and only if  $\arg(a_n b_n)$  is constant for all  $n$  for which  $a_n b_n \neq 0$ , we come to the following conclusion:

**(9.6) THEOREM.** *A necessary and sufficient condition for equality in (9.4) is that both sequences  $|a_n|^r / |b_n|^{r'}$  and  $\arg(a_n b_n)$  be independent of  $n$  (disregarding forms 0/0 and  $\arg 0$ ).*

An argument similar in principle shows that

(9.7) THEOREM. *The sign of equality holds in (9.5) if and only if (i) the ratio  $|f(x)|^r/|g(x)|^r$  is constant for almost all  $x$  for which it is not 0/0, (ii)  $\arg\{f(x)g(x)\}$  is constant for almost all  $x$  for which  $fg \neq 0$ .*

The inequality (9.5) (and similarly (9.4)) may be extended as follows:

(9.8) THEOREM. *If  $r_1, r_2, \dots, r_k$  are positive numbers such that  $1/r_1 + 1/r_2 + \dots + 1/r_k = 1$ , and if  $f_i \in L^{r_i}(a, b)$  for  $i = 1, 2, \dots, k$ , then*

$$\left| \int_a^b f_1 f_2 \dots f_k dx \right| \leq \mathfrak{M}_{r_1}[f_1] \mathfrak{M}_{r_2}[f_2] \dots \mathfrak{M}_{r_k}[f_k].$$

The proof (by induction) is left to the reader.

A number  $M$  is called the *essential upper bound* (sometimes the *least essential upper bound*) of the function  $f(x)$  in the interval  $(a, b)$  if (i) the set of points for which  $f(x) > M$  is of measure 0, (ii) for every  $M' < M$  the set of points for which  $f(x) > M'$  is of positive measure. Similarly we define the *essential lower bound*. If both bounds are finite,  $f(x)$  is said to be *essentially bounded*. (An equivalent definition is that  $f(x)$  is essentially bounded if it is bounded outside a set of measure 0, or, again, that  $f \equiv g$  where  $g$  is bounded.)

(9.9) THEOREM. *If  $M$  is the essential upper bound of  $|f(x)|$  in a finite interval  $(a, b)$ , then*

$$\mathfrak{M}_r[f; a, b] \rightarrow M \quad \text{as } r \rightarrow +\infty,$$

We may suppose that  $M > 0$ . Let  $0 < M' < M$ , and let  $E$  be the set of points where  $|f(x)| > M'$ . Then  $|E| > 0$ ,  $\mathfrak{M}_r[f] \geq M' |E|^{1/r}$ ,

so that  $\liminf_{r \rightarrow \infty} \mathfrak{M}_r[f] \geq M'$ . Hence  $\liminf_{r \rightarrow \infty} \mathfrak{M}_r[f] \geq M$ . In particular, the theorem is proved if  $M = +\infty$ . This part of the proof holds even if  $b - a = +\infty$ .

Suppose then that  $M < +\infty$ . Since  $\mathfrak{M}_r[f] \leq M(b-a)^{1/r}$ , we have  $\limsup_{r \rightarrow \infty} \mathfrak{M}_r[f] \leq M$ , and this, with the inequality  $\liminf_{r \rightarrow \infty} \mathfrak{M}_r[f] \geq M$  above, proves the theorem.

If  $b - a = +\infty$ , (9.9) is still true provided we assume that  $\mathfrak{M}_r[f]$  is finite for some  $r = r_0 > 0$ . (Otherwise the result is false: take, for instance,  $a = 2, b = +\infty, f(x) = 1/\log x$ .) We have to show that  $\limsup_{r \rightarrow \infty} \mathfrak{M}_r[f] \leq M < +\infty$ . Dividing by  $M$ , we may assume that  $M = 1$ . In order to show that  $\limsup_{r \rightarrow \infty} \mathfrak{M}_r[f] \leq 1$ , we write  $(a, b) = I + R$ , where  $I$  is a finite subinterval of  $(a, b)$  so large that  $\int_R |f|^r dx < 1$ . Since  $|f| \leq 1$  almost everywhere,

$$\int_a^b |f|^r dx = \int_I |f|^r dx + \int_R |f|^r dx \leq |I| + \int_R |f|^r dx \leq |I| + 1$$

for  $r \geq r_0$ . Hence  $\limsup_{r \rightarrow \infty} \mathfrak{M}_r[f] \leq 1$ .

Since any sequence  $a_0, a_1, \dots$  may be treated as a function  $f(x)$ , where  $f(x) = a_n$  for  $n \leq x < n+1$ , we see that  $\mathfrak{S}_r[a]$  tends to  $\max |a_n|$  as  $r \rightarrow \infty$ , provided that  $\mathfrak{S}_r[a]$  is finite for some  $r > 0$ .

In virtue of (9.9), it is natural to define  $\mathfrak{M}_\infty[f; a, b]$  as the essential upper bound of  $|f(x)|$  in  $(a, b)$ . By  $L^\infty$  we may denote the class of essentially bounded functions. The inequality (9.5) then remains meaningful and true for  $r = \infty, r' = 1$ .

Let  $a = \{a_n\}, b = \{b_n\}$  be two sequences of numbers, and let  $a + b = \{a_n + b_n\}$ . The inequality

$$\mathfrak{S}_r[a + b] \leq \mathfrak{S}_r[a] + \mathfrak{S}_r[b] \quad (r \geq 1) \quad (9.10)$$

is called *Minkowski's inequality*. To prove it for  $r > 1$  (it is obvious for  $r = 1$ ), we write

$$\Sigma |a_n + b_n|^r \leq \Sigma |a_n + b_n|^{r-1} |a_n| + \Sigma |a_n + b_n|^{r-1} |b_n|,$$

and apply Hölder's inequality, with exponents  $r'$  and  $r$ , to the sums on the right. We get

$$\mathfrak{S}_r[a + b] \leq \mathfrak{S}_{r'}[a + b] \mathfrak{S}_r[a] + \mathfrak{S}_{r'}[a + b] \mathfrak{S}_r[b],$$

from which (9.10) follows, provided  $\mathfrak{S}_r[a + b]$  is finite. Hence (9.10) holds when  $\{a_n\}$  and  $\{b_n\}$  are finite, and so also in the general case by passing to the limit.

A similar argument proves *Minkowski's inequality for integrals*

$$\mathfrak{M}_r[f + g] \leq \mathfrak{M}_r[f] + \mathfrak{M}_r[g] \quad (r \geq 1), \quad (9.11)$$

which implies that if  $f$  and  $g$  belong to  $L^r$  so does  $f + g$ .

Let  $h(x, y)$  be a function defined for  $a \leq x \leq b$ ,  $c \leq y \leq d$ . An argument similar to that which leads to (9.10) and (9.11) also gives the inequality

$$\left\{ \int_a^b \left| \int_c^d h(x, y) dy \right|^r dx \right\}^{1/r} \leq \int_c^d \left\{ \int_a^b |h(x, y)|^r dx \right\}^{1/r} dy \quad (r \geq 1) \quad (9.12)$$

which may be considered as a generalized form of Minkowski's inequality since it contains (9.10) and (9.11) as special cases. For if  $(c, d) = (0, 2)$ ,  $h(x, y) = f(x)$  for  $0 \leq y < 1$ ,  $h(x, y) = g(x)$  for  $1 \leq y \leq 2$ , (9.12) reduces to (9.11). If  $(c, d) = (0, 2)$ ,  $(a, b) = (0, +\infty)$ , and if for  $n \leq x < n + 1$  we set  $h(x, y) = a_n$  or  $h(x, y) = b_n$ , according as  $0 \leq y < 1$  or  $1 \leq y \leq 2$  ( $n = 0, 1, \dots$ ), (9.12) gives (9.10).

The inequality (9.12) can also be written slightly differently. Let

$$H(x) = \int_c^d h(x, y) dy.$$

Then

$$\mathfrak{M}_r^x[H(x)] \leq \int_c^d \mathfrak{M}_r^x[h(x, y)] dy,$$

where  $\mathfrak{M}_r^x$  means that integration is with respect to  $x$ .

If  $0 < r < 1$ , (9.10) and (9.11) cease to be true, but we have then the substitutes

$$\mathfrak{S}_r[a + b] \leq \mathfrak{S}_r[a] + \mathfrak{S}_r[b], \quad \mathfrak{M}_r^x[f + g] \leq \mathfrak{M}_r^x[f] + \mathfrak{M}_r^x[g] \quad (0 < r \leq 1). \quad (9.13)$$

These are corollaries of the inequality  $(x + y)^r \leq x^r + y^r$ , or, what is the same thing,

$$(1 + t)^r \leq 1 + t^r \quad (t \geq 0, 0 < r < 1).$$

To prove the latter we observe that  $(1 + t)^r - 1 - t^r$  vanishes for  $t = 0$  and has a negative derivative for  $t > 0$ .

In this connexion we may note in passing the inequality (a consequence of the last one)

$$|\Sigma a_n|^r \leq \Sigma |a_n|^r \quad (0 < r \leq 1).$$

(9.14) THEOREM. Given any function  $F(x)$ ,  $a \leq x \leq b$ , and a number  $1 \leq r < +\infty$ , we have

$$\mathfrak{M}_r[F; a, b] = \sup \left| \int_a^b FG dx \right|, \quad (9.15)$$

where the sup is taken over all  $G$  with  $\mathfrak{M}_r[G; a, b] \leq 1$ . The result holds if  $\mathfrak{M}_r[F] = +\infty$ .

We may suppose that  $\mathfrak{M}_r[F] > 0$ . Let  $I_G$  denote the integral on the right. By Hölder's inequality,

$$|I_G| \leq \mathfrak{M}_r[F] \mathfrak{M}_r[G] \leq \mathfrak{M}_r[F],$$

a result true even if  $\mathfrak{M}_r[F] = +\infty$ . On the other hand, if  $\mathfrak{M}_r[F] < +\infty$ , we set

$$G_0(x) = |F(x)|^{r-1} \operatorname{sign} \bar{F}(x) / \mathfrak{M}_r^{r-1}[F] \quad \text{for } r \geq 1.$$

We verify that  $\mathfrak{M}_r[G_0] = 1$ ,  $I_{G_0} = \mathfrak{M}_r[F]$ . This proves (9.15) if  $\mathfrak{M}_r[F]$  is finite.

If  $\mathfrak{M}_r[F] = +\infty$ , we have to show that there exist functions  $G$  with  $\mathfrak{M}_r[G] \leq 1$  and such that  $I_G$  exists and is arbitrarily large. Suppose first that  $b - a < +\infty$ , and let  $F^n(x)$  denote the function equal to  $F(x)$  where  $|F(x)| \leq n$ , and to 0 otherwise. Let  $G^n(x)$  be derived from  $F^n(x)$  in the same way as  $G_0$  was derived from  $F$ . If  $n$  is large enough,  $\mathfrak{M}_r[F^n]$  is positive (and finite), so that  $\mathfrak{M}_r[G^n] = 1$  but

$$I_{G^n} = \int_a^b F G^n dx = \int_a^b F^n G^n dx = \mathfrak{M}_r[F^n]$$

is arbitrarily large with  $n$ .

If  $(a, b)$  is infinite,  $(0, +\infty)$  for instance, we define  $F^n(x)$  as previously, but only in the interval  $0 \leq x \leq n$ , with  $F^n(x) = 0$  outside. This ensures the finiteness of  $\mathfrak{M}_r[F^n]$  for every  $n$ , and the rest of the argument is unchanged.

*Theorem (9.14) also holds for  $r = \infty$ .* The proof of  $|I_G| \leq \mathfrak{M}_r[F]$  remains unchanged in this case. On the other hand, if  $M$  is the essential upper bound of  $|F|$ , and if  $0 < M' < M$ , the set  $E$  of points where  $|F| \geq M'$  is of positive measure. If we choose a subset  $E_1$  of  $E$  with  $0 < |E_1| < \infty$ , then the function  $G(x)$ , equal to  $\operatorname{sign} \bar{F}(x) / |E_1|$  in  $E_1$  and to 0 elsewhere, has the property

$$\mathfrak{M}_r[G] = \int_a^b |G| dx = 1, \quad I_G = \frac{1}{|E_1|} \int_{E_1} |F| dx \geq M'$$

so that  $\sup |I_G| \geq M$ .

We conclude with the following theorem:

(9.16) THEOREM. Let  $f(x)$  be a non-negative function defined for  $x \geq 0$ , and let  $r > 1$ ,  $s < r - 1$ . Then if  $f^r(x) x^s$  is integrable over  $(0, \infty)$  so is  $\{x^{-1} F(x)\}^r x^s$ , where  $F(x) = \int_0^x f dt$ .

Moreover,

$$\int_0^\infty \left\{ \frac{F(x)}{x} \right\}^r x^s dx \leq \left( \frac{r}{r-s-1} \right)^r \int_0^\infty f^r(x) x^s dx. \quad (9.17)$$

We may suppose that  $f \neq 0$ . Hölder's inequality

$$\int_0^x f^{s(r-1)+r} dt \leq \left( \int_0^x f^r dt \right)^{1/r} \left( \int_0^x t^{-s(r-1)} dt \right)^{1/r'}$$

shows that  $f$  is integrable over any finite interval and that  $F(x) = o(x^{r-1-s/r})$  as  $x \rightarrow 0$ . The last estimate holds also as  $x \rightarrow \infty$ . For, applying the preceding argument to the integral defining  $F(x) - F(\xi)$ , we have

$$F(x) - F(\xi) < \epsilon x^{r-1-s/r}$$

if  $x > \xi$  and  $\xi = \xi(\epsilon)$  is large enough. Hence  $F(x) < 2\epsilon x^{r-1-s/r}$  for large  $x$ ; that is,

$$F(x) = o(x^{r-1-s/r}) \quad \text{as } x \rightarrow \infty$$

Now let  $0 < a < b < \infty$ . Integrating  $\int_a^b F^r x^{s-r} dx$  by parts and applying Hölder's inequality to  $\int_a^b F^{r-1} f x^{s-r+1} dx = \int_a^b (f x^{sr}) (F^{r-1} x^{s-r+1-s/r}) dx$ , we obtain

$$\int_a^b \left(\frac{F}{x}\right)^r x^s dx \leq \left[\frac{F^r x^{s-r+1}}{r-s-1}\right]_a^b + \frac{r}{r-s-1} \left(\int_a^b f^r x^s dx\right)^{1/r} \left(\int_a^b \left(\frac{F}{x}\right)^r x^s dx\right)^{1/r}$$

Divide both sides by the last factor on the right, which is positive if  $a$  and  $1/b$  are small enough. Since the integrated term tends to 0 as  $a \rightarrow 0$ ,  $b \rightarrow \infty$ , we are led to (9.17).

The cases  $s = 0$  and  $s = r - 2$  are the most interesting in application.

## 10. Convex functions

A function  $\phi(x)$  defined in an open or closed interval  $(a, b)$  is said to be *convex* if for every pair of points  $P_1, P_2$  on the curve  $y = \phi(x)$  the points of the arc  $P_1 P_2$  are below or on the chord  $P_1 P_2$ . For example,  $x^r$ , with  $r \geq 1$ , is convex in  $(0, +\infty)$ .

Jensen's inequality states that for any system of positive numbers  $p_1, p_2, \dots, p_n$  and for any system of points  $x_1, x_2, \dots, x_n$  in  $(a, b)$ ,

$$\phi\left(\frac{p_1 x_1 + p_2 x_2 + \dots + p_n x_n}{p_1 + p_2 + \dots + p_n}\right) \leq \frac{p_1 \phi(x_1) + \dots + p_n \phi(x_n)}{p_1 + \dots + p_n} \quad (10.1)$$

For  $n = 2$  this is just the definition of convexity, and for  $n > 2$  it follows by induction. Zero values of the  $p$ 's may be allowed provided that  $\Sigma p_i \neq 0$ . If  $-\phi(x)$  is convex,  $\phi(x)$  is called *concave*. Linear functions are the only ones which are both convex and concave. Concave functions satisfy the inequality opposite to (10.1).

Let  $P_1, P_2, P_3$  be three points on the convex curve  $y = \phi(x)$ , in the order indicated. Since  $P_2$  is below or on the chord  $P_1 P_3$ , the slope of  $P_1 P_2$  does not exceed that of  $P_1 P_3$ . Hence, if a point  $P$  approaches  $P_1$  from the right the slope of  $P_1 P$  is non-increasing. Thus the *right-hand side derivative*  $D^+ \phi(x)$  exists for every  $a \leq x < b$  and is less than  $+\infty$ . Similarly, the *left-hand side derivative*  $D^- \phi(x)$  exists for every  $a < x \leq b$  and is greater than  $-\infty$ .

If  $P_1, P, P_2$  are points on the curve, in this order, the slope of  $P_1 P$  does not exceed that of  $PP_2$ . Making  $P_1 \rightarrow P$ ,  $P_2 \rightarrow P$ , we have

$$-\infty < D^- \phi(x) \leq D^+ \phi(x) < +\infty \quad (a < x < b). \quad (10.2)$$

In particular,  $\phi(x)$  is continuous in the interior of  $(a, b)$ . The function  $\phi$  may, however, be discontinuous at the end-points  $a, b$  (take the example  $\phi(x) = 0$  for  $0 < x < 1$ ,  $\phi(0) = \phi(1) = 1$ ).

From the proof of the existence of  $D^+ \phi(x_0)$  and  $D^- \phi(x_0)$ , and from (10.2), we see that every straight line  $l$  passing through the point  $(x_0, \phi(x_0))$  and having a slope  $k$  satisfying  $D^- \phi(x_0) \leq k \leq D^+ \phi(x_0)$  has at least one point in common with the curve  $y = \phi(x)$ , and that the curve is above or on  $l$ . Such a straight line is called a *supporting line* for the curve  $y = \phi(x)$ .

Let  $x_1 < x < x_2$  be the abscissae of  $P_1, P, P_2$ . The slope of  $P_1 P$  does not exceed that of  $PP_2$ . The former is at least  $D^+ \phi(x_1)$ , the latter at most  $D^- \phi(x_2) \leq D^+ \phi(x_2)$ ; thus

$$D^+ \phi(x_1) \leq D^- \phi(x_2), \quad D^+ \phi(x_1) \leq D^+ \phi(x_2) \quad (x_1 < x_2). \quad (10.3)$$

The second inequality shows that  $D^+\phi(x)$  is a non-decreasing function of  $x$ . The same holds for  $D^-\phi(x)$ . Since  $D^+\phi(x)$  is non-decreasing, it is continuous except possibly at a denumerable set of points. Let  $x$  be a point of continuity of  $D^+\phi$  and let  $x_1 < x$ . Then, by (10.3) and (10.2),

$$D^+\phi(x_1) \leq D^-\phi(x) \leq D^+\phi(x).$$

Since  $D^+\phi(x_1) \rightarrow D^+\phi(x)$  as  $x_1 \rightarrow x$ , we see that  $D^-\phi'(x) = D^+\phi(x)$  and so  $\phi'(x)$  exists and is finite. Summing up we have:

(10.4) THEOREM. A function  $\phi(x)$  convex in an interval  $(a, b)$  is continuous for  $a < x < b$ . The one-sided derivatives of  $\phi$  exist, are non-decreasing and satisfy (10.2). The derivative  $\phi'(x)$  exists except possibly at a denumerable set of points.

We have seen that the continuity of  $\phi$  is a consequence of convexity. If, however, we assume that  $\phi(x)$  is continuous, we may modify the definition of convexity slightly. A continuous function  $\phi(x)$  is convex if and only if, given any arc  $P_1P_2$  of the curve, there is a subarc  $P'_1P'_2$  lying below or on the chord  $P_1P_2$ . The condition is obviously necessary. Suppose that it is satisfied, but  $\phi(x)$  is not convex. The curve would then contain an arc  $P_1P_2$  for which a certain subarc  $P'_1P'_2$  would be everywhere above the chord  $P_1P_2$ . Moving  $P'_1$  to the left,  $P'_2$  to the right, we may suppose that  $P'_1$  and  $P'_2$  are on the chord  $P_1P_2$  and the rest of the arc  $P'_1P'_2$  is above that chord. But then no subarc of  $P'_1P'_2$  is below the chord  $P'_1P'_2$ , contrary to hypothesis.

A convex function has no proper maximum† in the interior of the interval of definition. For if  $x_0$  were such a maximum, the arc  $y = \phi(x)$ ,  $|x - x_0| \leq \delta$  would, for  $\delta$  small enough, be partly above the chord.

(10.5) THEOREM. A necessary and sufficient condition that a function  $\phi(x)$ , continuous in  $(a, b)$ , should be convex is that for no pair of values  $\alpha, \beta$  should the sum  $\phi(x) + \alpha x + \beta$  have a proper maximum in the interior of  $(a, b)$ .

A sum of two convex functions being convex, the necessity of the condition is evident. To prove its sufficiency, suppose that  $\phi(x)$  is not convex. There is then an arc  $P_1P_2$  of the curve with all points above or on the chord  $P_1P_2$ . Let  $x_1, x_2$  be the abscissae of  $P_1, P_2$ , and let  $y = -\alpha x - \beta$  be the equation of the chord. Then  $\phi(x) + \alpha x + \beta$  vanishes at the end-points of  $(x_1, x_2)$  and takes some positive values inside; it has therefore a proper maximum inside  $(x_1, x_2)$  and so also inside  $(a, b)$ .

The following generalizations of ordinary first and second derivatives are useful. Given an  $F(x)$  defined in the neighbourhood of  $x_0$ , let us consider the ratios

$$\left. \begin{aligned} & \frac{F(x_0 + h) - F(x_0 - h)}{2h}, \\ & \frac{F(x_0 + h) + F(x_0 - h) - 2F(x_0)}{h^2} \end{aligned} \right\} \quad (10.6)$$

The limits (if they exist) of these expressions, as  $h \rightarrow 0$ , will be called respectively the *first* and *second symmetric derivatives* of  $F$  at the point  $x_0$ , and will be denoted by  $D_1F(x_0)$  and  $D_2F(x_0)$ . The limit superior and the limit inferior of the ratios (10.6) are called the

† We say that  $\phi(x)$  has a *proper maximum* at  $x_0$  if  $\phi(x_0) > \phi(x)$  in a neighbourhood of  $x_0$ , but  $\phi$  is not constant in any neighbourhood of  $x_0$ .

upper and the lower (first or second) symmetric derivatives and will be denoted by  $\bar{D}_1 F(x_0)$ ,  $\underline{D}_1 F(x_0)$ ,  $\bar{D}_2 F(x_0)$ ,  $\underline{D}_2 F(x_0)$ . The second symmetric derivative is often called the Schwarz (or Riemann) derivative.

If  $F'(x_0)$  exists, so does  $D_1 F(x_0)$  and both have the same value. For the first ratio (10.6) is half the sum of the ratios  $\{F(x_0 + h) - F(x_0)\}/h$  and  $\{F(x_0) - F(x_0 - h)\}/h$ , which tend to  $F'(x_0)$ .

If  $F''(x_0)$  exists, so does  $D_2 F(x_0)$  and both have the same value. For Cauchy's mean-value theorem applied to the second ratio (10.6),  $h$  being the variable, shows that it can be written as  $\{F'(x_0 + k) - F'(x_0 - k)\}/2k$  for some  $0 < k < h$ , and the last ratio tends to  $F''(x_0)$  as  $k \rightarrow 0$ .

These proofs actually give slightly more, namely: (i) both  $\bar{D}_1 F(x_0)$  and  $\underline{D}_1 F(x_0)$  are contained between the least and the greatest of the four Dini numbers of  $F$  at  $x_0$ ; (ii) if  $F'(x)$  exists in the neighbourhood of  $x_0$ , then both  $\bar{D}_2 F(x_0)$  and  $\underline{D}_2 F(x_0)$  are contained between the least and the greatest of the four Dini numbers of  $F'$  at  $x_0$ .

(10.7) THEOREM. A necessary and sufficient condition for a continuous  $\phi(x)$  to be convex in the interior of  $(a, b)$  is that  $\bar{D}_2 \phi(x) \geq 0$  there.

We may suppose that  $(a, b)$  is finite. Since

$$\phi(x+h) + \phi(x-h) - 2\phi(x) = \{\phi(x+h) - \phi(x)\} - \{\phi(x) - \phi(x-h)\} \geq 0$$

for a convex  $\phi$ , the necessity of the condition (even in the stronger form  $\underline{D}_2 \phi \geq 0$ ) follows. To prove the sufficiency, let us first assume slightly more, namely, that  $\bar{D}_2 \phi > 0$  in  $(a, b)$ . If  $\phi$  were not convex, the function  $\psi(x) = \phi(x) + ax + \beta$  would, for suitable  $a, \beta$ , have a maximum at a point  $x_0$  inside  $(a, b)$ , so that  $\psi(x_0 + h) + \psi(x_0 - h) - 2\psi(x_0)$  would be non-positive for small  $h$ . Since this expression equals

$$\phi(x_0 + h) + \phi(x_0 - h) - 2\phi(x_0),$$

it follows that  $\bar{D}_2 \phi(x_0) \leq 0$ , contrary to hypothesis.

Returning to the general case, consider the functions  $\phi_n(x) = \phi(x) + x^2/n$ . We have

$$\bar{D}_2 \phi_n(x) = \bar{D}_2 \phi(x) + 2/n > 0,$$

so that  $\phi_n$  is convex. The limit of a convergent sequence of convex functions is convex (applying (10.1) with  $n = 2$ ); and since  $\phi_n \rightarrow \phi$ ,  $\phi$  is convex.

A necessary and sufficient condition for a function  $\phi$  twice differentiable to be convex is that  $\phi'' \geq 0$ . This follows from (10.7i).

Suppose that  $\phi(u)$  is convex for  $u \geq 0$ , and that  $u_0$  is a minimum of  $\phi(u)$ . If  $\phi(u)$  is not constant for  $u \geq u_0$ , then it must tend to  $+\infty$  with  $u$  at least as rapidly as a fixed positive multiple of  $u$ . For let  $u_1 > u_0$ ,  $\phi(u_1) \neq \phi(u_0)$ . Clearly  $\phi(u_1) > \phi(u_0)$ . If  $P_0, P_1, P$  are points of the curve with abscissae  $u_0, u_1, u$ , where  $u > u_1$ , the slope of  $P_0P$  is not less than that of  $P_0P_1$ . This proves the assertion.

If  $\phi(u)$  is non-negative convex and non-decreasing in  $(0, +\infty)$ , but not constant, the relation  $f \in L_p(a, b)$ ,  $b - a < \infty$ , implies  $f \in L(a, b)$ . For then there is a  $k > 0$  such that  $\phi(|f(x)|) \geq k|f(x)|$ , if  $|f(x)|$  is large enough.

Jensen's inequality for integrals is

$$\phi \left\{ \frac{\int_a^b f(x) p(x) dx}{\int_a^b p(x) dx} \right\} \leq \frac{\int_a^b \phi\{f(x)\} p(x) dx}{\int_a^b p(x) dx}, \quad (10.8)$$

the hypotheses being that  $\phi(u)$  is convex in an interval  $\alpha \leq u \leq \beta$ , that  $\alpha \leq f(x) \leq \beta$  in  $a \leq x \leq b$ , that  $p(x)$  is non-negative and  $\neq 0$ , and that all the integrals in question exist.

Let

$$\gamma = \int_a^b f p dx / \int_a^b p dx, \quad (10.9)$$

so that  $\alpha \leq \gamma \leq \beta$ . Suppose first that  $\alpha < \gamma < \beta$ , and let  $k$  be the slope of a supporting line of  $\phi$  through the point  $(\gamma, \phi(\gamma))$ . Then

$$\phi(u) - \phi(\gamma) \geq k(u - \gamma) \quad (\alpha \leq u \leq \beta).$$

Replacing here  $u$  by  $f(x)$ , multiplying both sides by  $p(x)$ , and integrating over  $a \leq x \leq b$ , we get

$$\int_a^b \phi\{f(x)\} p(x) dx - \phi(\gamma) \int_a^b p(x) dx \geq k \left( \int_a^b f(x) p(x) dx - \gamma \int_a^b p(x) dx \right) = 0,$$

by (10.9). This gives (10.8). If  $\gamma = \beta$ , (10.9) can be written  $\int_a^b (f - \beta) p dx = 0$ , which shows that  $f(x) = \beta$  at almost all points at which  $p > 0$ . But then both sides of (10.8) reduce to  $\phi(\beta)$ . Similarly if  $\gamma = \alpha$ .

Jensen's inequality for Stieltjes integrals is

$$\phi \left\{ \frac{\int_a^b f(x) d\omega(x)}{\int_a^b d\omega(x)} \right\} \leq \frac{\int_a^b \phi\{f(x)\} d\omega(x)}{\int_a^b d\omega(x)}, \quad (10.10)$$

where  $\omega(x)$  is non-decreasing but not constant. The proof is similar to the one above.

(10.11) THEOREM. A necessary and sufficient condition that  $\phi(x)$  ( $a < x < b$ ) should be convex is that it should be the integral of a non-decreasing function.

If  $\phi(x)$  is convex, then, as is easily seen, the ratio  $\{\phi(x+h) - \phi(x)\}/h$  is uniformly bounded for  $x, x+h$  belonging to any interval  $(a', b')$  interior to  $(a, b)$ . Thus  $\phi(x)$  is absolutely continuous, and therefore is the integral of  $\phi'$ . The latter exists outside a denumerable set and is non-decreasing on the set where it exists. Completing  $\phi'$  at the exceptional points so that the new function is still non-decreasing, we see that  $\phi$  is the integral of a non-decreasing function. Conversely, suppose that

$\phi(x) = C + \int_c^x \psi(t) dt$ , where  $a < c < b$  and  $\psi(t)$  is non-decreasing in  $(a, b)$ . Let  $(a', b')$

be any subinterval of  $(a, b)$ , and let  $y = l(x)$  be the equation of the chord through  $(a', \phi(a'))$  and  $(b', \phi(b'))$ . We have to show that  $\phi(x) - \phi(a') \leq l(x) - l(a')$  for  $a' < x < b'$ , or, what is the same thing, that

$$\frac{1}{x - a'} \int_{a'}^x \psi dt \leq \frac{1}{b' - a'} \int_{a'}^{b'} \psi dt = \frac{\int_{a'}^x + \int_x^{b'}}{(x - a') + (b' - x)}.$$

Since the last expression is contained between  $\int_{a'}^x / (x - a')$  and  $\int_x^{b'} / (b' - x)$ , of which the latter is not less than the former (since  $\psi$  does not decrease), the proof of (10.11) is completed.

Let now  $\phi(x)$ ,  $x \geq 0$ , be an arbitrary function, non-negative, non-decreasing, vanishing at  $x=0$  and tending to  $+\infty$  with  $x$ . The curve  $y = \phi(x)$  may possess discontinuities and stretches of constancy. If at each point  $x_0$  of discontinuity of  $\phi$  we adjoin to the curve  $y = \phi(x)$  the vertical segment  $x = x_0$ ,  $\phi(x_0 - 0) \leq y \leq \phi(x_0 + 0)$ , we obtain a continuous curve, and we may define a function  $\psi(y)$  inverse to  $\phi(x)$  by defining  $\psi(y_0)$  ( $0 \leq y_0 < \infty$ ) to be any  $x_0$  such that the point  $(x_0, y_0)$  is on the continuous curve. The stretches of constancy of  $\phi$  then correspond to discontinuities of  $\psi$ , and conversely. The function  $\psi(y)$  is defined uniquely except for the  $y$ 's which correspond to the stretches of constancy of  $\phi$ , but since the set of such stretches is denumerable, our choice of  $\psi(y)$  has no influence upon the integral  $\Psi(y)$  of  $\psi(y)$ , and it is easy to see that the inequality (9.1) is valid in this slightly more general case.

From (10.11) it follows that every function  $\Phi(x)$ ,  $x \geq 0$ , which is non-negative, convex, and satisfies the relation  $\Phi(0) = 0$  and  $\Phi(x)/x \rightarrow \infty$ , may be considered as a Young's function (see p. 16). More precisely, to every such function corresponds another function  $\Psi(x)$  with similar properties, such that

$$ab \leq \Phi(a) + \Psi(b)$$

for every  $a \geq 0$ ,  $b \geq 0$ . It is sufficient to take for  $\Psi(y)$  the integral over  $(0, y)$  of the function  $\psi(x)$  inverse to  $\phi(x) = D^+ \Phi(x)$ . Since  $\Phi(x)/x$  tends to  $+\infty$  with  $x$ , it is easy to see that  $\phi(x)$  and  $\psi(x)$  also tend to  $+\infty$  with  $x$ . We have  $ab = \Phi(a) + \Psi(b)$  if and only if the point  $(a, b)$  is on the continuous curve obtained from the function  $y = \phi(x)$ .

A non-negative function  $\psi(u)$  ( $a \leq u \leq b$ ) will be called *logarithmically convex* if

$$\psi(t_1 u_1 + t_2 u_2) \leq \psi^{t_1}(u_1) \psi^{t_2}(u_2)$$

for  $u_1$  and  $u_2$  in  $(a, b)$ ,  $t_1$  and  $t_2$  positive and of sum 1. It is immediate that then either  $\psi$  is identically zero, or else  $\psi$  is strictly positive and  $\log \psi$  is convex.

(10.12) THEOREM. For any given function  $f$ , and for  $a > 0$ ,

- (i)  $\mathfrak{A}_a[f]$  is a non-decreasing function of  $\alpha$ ;
- (ii)  $\mathfrak{M}_a^\alpha[f]$  and  $\mathfrak{A}_a^\alpha[f]$  are logarithmically convex functions of  $\alpha$ ;
- (iii)  $\mathfrak{M}_{1/a}[f]$  and  $\mathfrak{A}_{1/a}[f]$  are logarithmically convex functions of  $\alpha$ .

If we substitute  $|f|^\alpha$  for  $f$  and 1 for  $g$  in (9.5), and divide both sides by  $b - a$ , we have  $\mathfrak{A}_a[f] \leq \mathfrak{A}_{a/r}[f]$  for  $r > 1$ . This proves (i). The result is not true for  $\mathfrak{M}_a$ , as may be seen from the example  $a = 0$ ,  $b = 2$ ,  $f(x) = 1$ .

Let now  $\alpha = \alpha_1 t_1 + \alpha_2 t_2$  with  $\alpha_i, t_i > 0$ ,  $t_1 + t_2 = 1$ , and suppose that  $f$  belongs to both  $L^{\alpha_1}$  and  $L^{\alpha_2}$ . Replacing the integrand  $|f|^\alpha$  in  $\mathfrak{M}_a^\alpha$  by  $|f|^{\alpha_1 t_1} |f|^{\alpha_2 t_2}$ , and applying Hölder's inequality with  $r = 1/t_1$ ,  $r' = 1/t_2$ , we find

$$\mathfrak{M}_a^\alpha \leq \mathfrak{M}_{a_1}^{\alpha_1 t_1} \mathfrak{M}_{a_2}^{\alpha_2 t_2},$$

which expresses the logarithmic convexity of  $\mathfrak{M}_a^\alpha[f]$ . Dividing both sides by  $b - a$ , we have the result for  $\mathfrak{A}_a^\alpha$ . Thus (ii) is proved.

To prove (iii) we apply Hölder's inequality with the exponents  $r = \alpha/\alpha_1 t_1 > 1$ ,  $r' = \alpha/\alpha_2 t_2$ . We get

$$\begin{aligned} \mathfrak{M}_{\alpha, \alpha} [f] &= \left\{ \int_a^b |f|^{1/\alpha} dx \right\}^\alpha = \left\{ \int_a^b |f|^{t_1/\alpha} |f|^{t_2/\alpha} dx \right\}^\alpha \\ &\leq \left( \int_a^b |f|^{1/\alpha_1} dx \right)^{\alpha_1 t_1} \left( \int_a^b |f|^{1/\alpha_2} dx \right)^{\alpha_2 t_2} = \mathfrak{M}_{1/\alpha_1}^{t_1} \mathfrak{M}_{1/\alpha_2}^{t_2}. \end{aligned}$$

Dividing both sides by  $(b-a)^\alpha$  we have the result for  $\mathfrak{M}_{1/\alpha}$ .

## 11. Convergence in $L^r$

Let  $f_1(x), f_2(x), \dots$  be a sequence of functions belonging to  $L^r(a, b)$ ,  $r > 0$ . If there is a function  $f \in L^r(a, b)$  such that  $\mathfrak{M}_r [f - f_n; a, b] \rightarrow 0$  as  $n \rightarrow \infty$ , we say that  $\{f_n(x)\}$  converges in  $L^r(a, b)$  (or, simply, in  $L^r$ ) to  $f(x)$ .

(11.1) THEOREM. A necessary and sufficient condition that a sequence of functions  $f_n(x) \in L^r(a, b)$ ,  $r > 0$ , should converge in  $L^r$  to some  $f(x)$  is that  $\mathfrak{M}_r [f_m - f_n]$  should tend to 0 as  $m, n \rightarrow \infty$ .

If  $r \geq 1$  the necessity of the condition follows from Minkowski's inequality, since, if  $\mathfrak{M}_r [f - f_m] \rightarrow 0$ ,  $\mathfrak{M}_r [f - f_n] \rightarrow 0$ , then

$$\mathfrak{M}_r [f_m - f_n] \leq \mathfrak{M}_r [f - f_m] + \mathfrak{M}_r [f - f_n] \rightarrow 0.$$

For  $0 < r < 1$ , we use instead the second inequality (9.13).

The proof of sufficiency depends on the following further theorems, themselves important.

(11.2) FATOU'S LEMMA. Let  $g_1(x), g_2(x), \dots$  be non-negative functions, integrable over  $(a, b)$  and satisfying

$$\int_a^b g_k dx \leq A < +\infty \quad (k = 1, 2, \dots). \quad (11.3)$$

If  $g(x) = \lim g_k(x)$  exists almost everywhere, then  $g$  is integrable and

$$\int_a^b g dx \leq A. \quad (11.4)$$

Let  $h_k(x) = \inf \{g_k(x), g_{k+1}(x), \dots\}$ . The function  $h_k$  is measurable and majorized by  $g_k$ , and so integrable. Since  $h_k \leq h_{k+1}$  and  $h_k \rightarrow g$ , (11.4) follows from (11.3) by Lebesgue's theorem on the integration of monotone sequences.

(11.5) THEOREM. Let  $\{u_n(x)\}$  be a sequence of non-negative functions, and write  $I_n = \int_a^b u_n dx$ . If  $I_1 + I_2 + \dots < \infty$ , then  $u_1(x) + u_2(x) + \dots$  converges almost everywhere in  $(a, b)$  to a finite sum. In particular  $u_n(x) \rightarrow 0$  almost everywhere in  $(a, b)$ .

For if  $u_1 + u_2 + \dots$  diverged to  $+\infty$  in a set of positive measure, Lebesgue's theorem mentioned above would imply that  $I_1 + I_2 + \dots = \infty$ .

(11.6) THEOREM. If  $\mathfrak{M}_r [f_m - f_n; a, b] \rightarrow 0$  as  $m, n \rightarrow \infty$ , we can find a subsequence  $\{f_{r_k}\}$  of  $\{f_n\}$  which converges almost everywhere in  $(a, b)$ .

Suppose first that  $r \geq 1$ . Let  $\epsilon_i = \sup \mathfrak{M}_r[f_m - f_n]$  for  $m, n \geq i$ . Since  $\epsilon_i \rightarrow 0$ , we have  $\epsilon_{n_1} + \epsilon_{n_2} + \dots < +\infty$  if  $n_k$  increases sufficiently rapidly. By Hölder's inequality,

$$\int_a^b |f_{n_k} - f_{n_{k+1}}| dx \leq (b-a)^{1/r} \mathfrak{M}_r[f_{n_k} - f_{n_{k+1}}] \leq (b-a)^{1/r} \epsilon_{n_k}, \quad (11.7)$$

and so, by (11.5), the series  $|f_{n_1}| + |f_{n_2} - f_{n_1}| + |f_{n_3} - f_{n_2}| + \dots$  converges almost everywhere in  $(a, b)$ . The function  $f(x) = f_{n_1} + (f_{n_2} - f_{n_1}) + \dots = \lim f_{n_k}$  thus exists almost everywhere.

To establish the existence of  $f(x)$  for  $0 < r < 1$ , we note that

$$(|f_{n_1}| + |f_{n_2} - f_{n_1}| + \dots)^r \leq |f_{n_1}|^r + |f_{n_2} - f_{n_1}|^r + \dots;$$

when we integrate the right-hand side of this inequality we obtain a finite number, provided  $n_k$  increases so fast that  $\epsilon_{n_1}^r + \epsilon_{n_2}^r + \dots < \infty$ .

In this proof we tacitly assumed that  $(a, b)$  was finite, but the argument holds even if  $b - a = \infty$ , since (11.7) remains valid if  $(a, b)$  is replaced by any of its finite sub-intervals.

Returning to (11.1), let  $\{n_k\}$  and  $\{\epsilon_i\}$  be the sequences of Theorem (11.6) and  $f = \lim f_{n_k}$ . We have  $\mathfrak{M}_r[f - f_{n_k}] \leq \epsilon_m$  for  $n_k \geq m$ . By (11.2),  $\mathfrak{M}_r[f - f] \leq \epsilon_m$ , and this completes the proof.

It is important to observe that the function  $f$  satisfying  $\mathfrak{M}_r[f - f_m] \rightarrow 0$  is unique. Suppose that  $\mathfrak{M}_r[f - f_m] \rightarrow 0$ ,  $\mathfrak{M}_r[g - f_m] \rightarrow 0$ . If  $r \geq 1$ , by Minkowski's inequality,

$$\mathfrak{M}_r[f - g] \leq \mathfrak{M}_r[f - f_m] + \mathfrak{M}_r[g - f_m] \rightarrow 0, \quad \text{so that } \mathfrak{M}_r[f - g] = 0, \quad f \equiv g.$$

If  $0 < r < 1$ , we use instead the inequality

$$\mathfrak{M}_r^r[f - g] \leq \mathfrak{M}_r^r[f - f_m] + \mathfrak{M}_r^r[g - f_m].$$

(11.8) THEOREM. Suppose that  $f \in L^r(a, b)$ ,  $0 < r < +\infty$ . Then, given any  $\epsilon > 0$ , there is a continuous function  $\phi(x)$  such that  $\mathfrak{M}_r[f - \phi] < \epsilon$ .

Suppose first that  $r \geq 1$ ,  $b - a < +\infty$ . There is a bounded function  $\psi(x)$  such that  $\mathfrak{M}_r[f - \psi] < \frac{1}{2}\epsilon$ ; for, taking  $N$  large enough, we may define  $\psi(x)$  as equal to  $f$  wherever  $|f| \leq N$ , and equal to 0 elsewhere. If we can find a continuous  $\phi(x)$  such that  $\mathfrak{M}_r[\psi - \phi] < \frac{1}{2}\epsilon$ , the result will follow by Minkowski's inequality. Let us set  $\psi(x) = 0$  outside  $(a, b)$ , and let  $\Psi(x)$  be the indefinite integral of  $\psi(x)$ . The functions

$$\psi_n(x) = n[\Psi(x + 1/n) - \Psi(x)]$$

are continuous and uniformly bounded, and, by Lebesgue's theorem on the differentiation of the indefinite integral, tend to  $\psi(x)$  almost everywhere in  $(a, b)$ . Thus  $\mathfrak{M}_r[\psi - \psi_n] \rightarrow 0$ , and it is enough to set  $\phi = \psi_n$ , with  $n$  large enough. The modifications in the case  $0 < r < 1$  are obvious.

The above argument holds if  $b - a = \infty$ , provided that  $f(x) = 0$  for  $|x|$  large enough. The general case can be reduced to this one; for if we modify  $f$  by setting it equal to 0 outside a sufficiently large interval, we get a function  $f_1$  with  $\mathfrak{M}_r[f - f_1]$  arbitrarily small.

(11.9) THEOREM. Suppose that a sequence of functions  $f_n(x)$  converges almost everywhere in a finite interval  $(a, b)$  to a limit  $f(x)$ , and that  $\mathfrak{M}_r[f_n; a, b] \leq M < +\infty$  for a fixed  $r > 0$  and all  $n$ . Then  $\mathfrak{M}_s[f_n - f] \rightarrow 0$  as  $n \rightarrow \infty$ , for  $0 < s < r$ .

Obviously,  $\mathfrak{M}_r[f] \leq M$ . Let  $E$  be a set of points on which  $\{f_n(x)\}$  converges uniformly to  $f(x)$ , and let  $D = (a, b) - E$ ;  $|D|$  can be arbitrarily small. Clearly

$$\int_a^b |f_n - f|^s dx = \int_E + \int_D \leq o(1) + \left( \int_D |f_n - f|^r dx \right)^{s/r} \cdot |D|^{1-s/r},$$

by Hölder's inequality. By Minkowski's inequality, if  $r \geq 1$ , the last term is not greater than  $(2M)^s |D|^{1-s/r}$ , and so is arbitrarily small with  $|D|$ . Hence  $\mathfrak{M}_s[f_n - f] \rightarrow 0$ . The proof is similar for  $0 < r < 1$  except that we use (9.13) instead of Minkowski's inequality.

## 12. Sets of the first and second categories

Let  $A$  be a linear point-set. By a *portion* of  $A$  we shall mean any *non-empty* intersection  $AI$  of  $A$  with an *open* interval  $I$ .

Let  $B$  be a subset of  $A$ .  $B$  is said to be *dense* in  $A$  if every portion of  $A$  contains points of  $B$ .  $B$  is said to be *non-dense* in  $A$  if every portion of  $A$  contains a portion (*subportion* of  $A$ ) without points in common with  $B$ . A set dense in  $(-\infty, +\infty)$  will be called *everywhere dense*.

Let  $B \subset A$ . If  $B$  can be decomposed into a denumerable sum of subsets (not necessarily disjoint) non-dense in  $A$ ,  $B$  will be said to be *of the first category* on  $A$ . Otherwise  $B$  will be called *of the second category* on  $A$ . When  $B = A$ , we say that  $A$  is of the first or second category (as the case may be) *on itself*.

If  $A = (-\infty, +\infty)$ , we shall simply say that  $B$  is of the first or second category, as the case may be.

The following fact is important:

(12.1) THEOREM. *A closed set  $A$  (in particular, an interval) is of the second category on itself.*

For suppose that  $A = A_1 + A_2 + \dots$ , where the  $A_i$  are non-dense on  $A$ . In particular, there is a portion  $I_1 A$  of  $A$  containing no points of  $A_1$ . In that portion we choose a subportion  $I_2 A$  containing no points of  $A_2$ . In  $I_2 A$  we choose a subportion  $I_3 A$  containing no points of  $A_3$ , and so on. We may suppose that  $I_{n+1}$  is strictly interior to  $I_n$ , and that  $|I_n| \rightarrow 0$ . The intervals  $I_1, I_2, \dots$  have a point  $x$  in common, and since all of them contain points of the closed set  $A$ ,  $x$  must belong to  $A$ . Since  $x \in I_n A$ ,  $x$  cannot belong to any of  $A_1, A_2, \dots, A_n$ . This being true for all  $n$ , we obtain a contradiction with the relation  $A = A_1 + A_2 + \dots$ .

If  $B_1, B_2, \dots$  are all of the first category on  $A$ , so is  $B_1 + B_2 + \dots$ ; thus a subset  $B$  of a closed set  $A$  and the complementary set  $A - B$  cannot both be of the first category on  $A$ .

An everywhere dense set may be of the first category (for example, any denumerable dense set). However, if a set  $E$  is both dense in an interval  $I$ , and a denumerable product of open sets, then  $E$  is of the second category on  $I$ . For the complementary set  $I - E$  is then a denumerable sum of closed sets. These closed sets cannot contain intervals, since that would contradict the assumption that  $E$  is dense in  $I$ ; so they are non-dense in  $I$ . Hence  $I - E$  is of the first category, and  $E$  is of the second category, on  $I$ .

(12.2) THEOREM. *Let  $f_1(x), f_2(x), \dots$  be a sequence of functions continuous in  $a \leq x \leq b$ . If the set  $E$  of points  $x$  at which the sequence  $\{f_n(x)\}$  is unbounded is dense in  $(a, b)$ ,*

$E$  is of the second category on  $(a, b)$ . (More precisely, the complement of  $E$  is of the first category on  $(a, b)$ .)

It is enough to show that  $E$  is a denumerable product of open sets. But if  $E_N$  is the set of points at which at least one of the inequalities  $|f_n(x)| > N$  is satisfied, then  $E_N$  is open, and  $E = E_1 E_2 \dots$ .

A set  $A \subset (0, 1)$  can be of measure 1 and of the first category, or of measure 0 and of the second category. Thus though we may think of the second category as 'richer' in points than the first category, the new classification cannot be compared with the one based on measure.

(12.3) THEOREM. Let  $f_1(x), f_2(x), \dots$  be continuous on a closed set  $E$ ; then

(i) if  $\limsup_{n \rightarrow \infty} f_n(x) < +\infty$  at each point of  $E$ , then there is a portion  $P$  of  $E$  and a number  $M$  such that  $f_n(x) \leq M$  for all  $n$  and all  $x \in P$ ;

(ii) if  $f_n(x)$  converges on  $E$  to  $f(x)$ , then for any  $\epsilon > 0$  there is a portion  $P$  of  $E$  and a number  $n_0$  such that  $|f(x) - f_n(x)| \leq \epsilon$  for  $x \in P, n \geq n_0$ . (12.4)

(iii) If  $E$  is, in addition, non-denumerable (in particular, if  $E$  is perfect), then the conclusions of (i) and (ii) hold even if the hypotheses fail to be satisfied in a denumerable subset  $D$  of  $E$ .

(i) Let  $E_M$  be the set of  $x$  such that  $f_n(x) \leq M$  for all  $n$ . Each  $E_M$  is closed and  $E = E_1 + E_2 + \dots$ . By (12.1), some  $E_M$  is not non-dense on  $E$  and so, being closed, must contain a portion  $P$  of  $E$ . This proves (i).

(ii) For every  $k = 1, 2, \dots$ , let  $E_k$  be the set of points  $x \in E$  such that  $|f_m(x) - f_n(x)| \leq \epsilon$  for  $m, n \geq k$ . The sets  $E_k$  are closed and  $E = E_1 + E_2 + \dots$ . As in (i), some  $E_{n_0}$  contains a portion  $P$  of  $E$ . We have  $|f_m(x) - f_n(x)| \leq \epsilon$  for  $x \in P$  and  $m, n \geq n_0$ ; this implies (12.4).

(iii) We begin with the extension of (i). Let  $x_1, x_2, \dots$  be the elements of  $D$ , and let  $E'_n$  be the set  $E_n$  in the proof of (i) augmented by the points  $x_1, x_2, \dots, x_n$ .  $E'_n$  is closed and  $E = E'_1 + E'_2 + \dots$ . Hence a certain  $E'_{m_0}$  contains a portion of  $E$ . If we take  $m_0$  so large that  $E'_{m_0}$  is infinite (observe that  $E'_1 \subset E'_2 \subset \dots$ ),  $E'_{m_0}$  will also contain a portion of  $E$ .

The extension of (ii) is proved similarly.

### 13. Rearrangements of functions. Maximal theorems of Hardy and Littlewood

In this section, unless otherwise stated, we shall consider only functions  $f(x)$ , defined in a fixed finite interval, which are non-negative and almost everywhere finite. We may suppose that the interval is of the form  $(0, a)$ .

For any  $f(x)$ , we shall denote by  $E(f > y)$  the set of points  $x$  such that  $f(x) > y$ . The measure  $|E(f > y)| = m(y)$  of this set will be called the *distribution function* of  $f$ . Two functions  $f$  and  $g$  will be called *equidistributed* if they have the same distribution functions. It is then clear that if  $f$  is integrable over  $(0, a)$ , so is  $g$ , and the integrals are equal. If  $f$  and  $g$  are equidistributed, so are  $\chi(f)$  and  $\chi(g)$  for any non-negative and, say, non-decreasing  $\chi(u)$ .

(13.1) THEOREM. For any  $f(x)$ , there exist functions  $f^*(x)$  and  $f_*(x)$  ( $0 < x < a$ ) equidistributed with  $f$  and respectively non-increasing and non-decreasing.

The function  $m(y) = |E(f > y)|$  is non-increasing and continuous to the right. Clearly  $m(y) = a$  for  $y$  negative, and  $m(+\infty) = 0$ . If  $m(y)$  is continuous and strictly decreasing for  $y \geq 0$ , then its inverse function, which we shall denote by  $f^*(x)$ , is decreasing and equidistributed with  $f(x)$ .

The definition of  $f^*$  just given holds, suitably modified, in the general case. Let us consider the curve  $x = m(y)$  and a point  $y_0$  of discontinuity of it. We adjoin to the curve the whole segment of points  $(x, y_0)$  with  $m(y_0 + 0) < x \leq m(y_0 - 0)$  (noting that the point  $x = m(y_0) = m(y_0 + 0)$  belongs to the initial curve) and we do this for every  $y_0$ . Every line  $x = x_0$ ,  $0 < x_0 \leq a$ , intersects the new curve in at least one point, whose ordinate we denote by  $f^*(x_0)$ . The function  $f^*(x)$  is defined uniquely for  $0 < x \leq a$ , except at those points which correspond to the stretches of constancy of  $m(y)$ . Such  $x$  are denumerable and for them we take for  $f^*(x)$  any value that preserves the monotonicity. Taking into account the discontinuities and the stretches of constancy of  $m(y)$ , we may verify geometrically that, for each  $y_0$ , the set of points  $x$  such that  $f^*(x) > y_0$  is a segment, with or without end-points, of length  $m(y_0)$ . Thus  $|E(f^* > y_0)| = |E(f > y_0)|$ .

We define  $f_*(x) = f^*(a - x)$ ; the properties of  $f_*$  then follow trivially from those of  $f^*$ .

Suppose that  $f(x)$  is integrable over  $(0, a)$ . For every  $x$ ,  $0 < x \leq a$ , we set

$$\theta(x) = \theta_f(x) = \sup_{\xi} \frac{1}{x - \xi} \int_{\xi}^x f(t) dt, \quad \text{where } 0 \leq \xi < x. \quad (13.2)$$

Clearly  $\theta(x)$  is finite at every point at which the integral of  $f$  is differentiable. If  $f$  is non-increasing, then

$$\theta_f(x) = \frac{1}{x} \int_0^x f dt. \quad (13.3)$$

In particular, this formula applies to the function  $f^*(x)$  introduced above.

(13.4) **THEOREM OF HARDY AND LITTLEWOOD.** For any non-decreasing and non-negative function  $\chi(t)$ ,  $t \geq 0$ , we have

$$\int_0^a \chi\{\theta_f(x)\} dx \leq \int_0^a \chi\{\theta_{f^*}(x)\} dx = \int_0^a \chi\left\{\frac{1}{x} \int_0^x f^* dt\right\} dx. \quad (13.5)$$

First of all we observe that for any  $g(x) \geq 0$  we have

$$\int_0^a g(x) dx = - \int_0^{\infty} y dm(y) = \int_0^{\infty} m(y) dy, \quad (13.6)$$

where  $m(y) = |E(g > y)|$ . For, if  $g$  is bounded, the first equation follows from the fact that the approximating Lebesgue sums for the first integral coincide with the approximating Riemann-Stieltjes sums for the second. In the general case, for  $u > 0$ ,

$$- \int_0^u y dm(y) = \int_{E(g < u)} g(x) dx,$$

and the result follows by making  $u \rightarrow \infty$ . Finally, the second equality in (13.6) follows from integration by parts, if we observe that

$$ym(y) \rightarrow 0 \text{ as } y \rightarrow \infty \quad \left( \text{since } ym(y) \leq \int_{E(g > y)} g dx \right).$$

Comparing the extreme terms of (13.6) we see that if we have another function  $g_1(x) \geq 0$  and the corresponding  $m_1(y)$ , then the inequality  $m_1(y) \geq m(y)$  for all  $y$

implies that the integral of  $g_1$  is not less than that of  $g$ . Hence,  $\chi(t)$  being monotone, the inequality (13.5) will follow if we show that

$$|E(\theta_f > y_0)| \leq |E(\theta_{f^*} > y_0)| \quad \text{for all } y_0. \quad (13.7)$$

We break up the proof of this inequality into three stages.

(13.8) LEMMA. Given a continuous  $F(x)$ ,  $0 \leq x \leq a$ , let  $H$  denote the set of points  $x$  for which there is a point  $\xi$  in  $0 \leq \xi \leq x$  such that  $F(\xi) < F(x)$ . The set  $H$  consists of a denumerable system of disjoint intervals  $(\alpha_k, \beta_k)$  such that  $F(\alpha_k) \leq F(\beta_k)$ . All these intervals are open except possibly one terminating at  $a$ .

Since small changes of  $x$  do not impair the inequality  $F(\xi) < F(x)$ , the set  $H$  is open, except possibly for the point  $a$ . Let  $(\alpha_k, \beta_k)$  be any one of the disjoint intervals (open, except when  $\beta_k = a$ ) constituting  $H$ . Assuming that  $F(\alpha_k) > F(\beta_k)$ , denote by  $x_0$  the smallest number in  $(\alpha_k, \beta_k)$  such that  $F(x_0) = \frac{1}{2}\{F(\alpha_k) + F(\beta_k)\}$ . Thus no  $\xi$  corresponding to  $x_0$  can belong to  $(\alpha_k, x_0)$ , since the points of this interval satisfy the inequality  $F(x) \geq F(x_0)$ . Hence  $\xi < \alpha_k$  and the inequalities  $F(\xi) < F(x_0) < F(\alpha_k)$  imply that  $\alpha_k \in H$ , which is false. It follows that  $F(\alpha_k) \leq F(\beta_k)$ .

Remark. We actually have  $F(\alpha_k) = F(\beta_k)$ , unless  $\beta_k = a$ . For no  $\beta_k < a$  belongs to  $H$ , so that  $F(\alpha_k) \geq F(\beta_k)$ .

$$(13.9) \text{ LEMMA. If } E \text{ is any set in } (0, a), \text{ then } \int_E f dx \leq \int_0^{|E|} f^* dx.$$

Let  $g(x)$  be equal to  $f(x)$  in  $E$  and to 0 elsewhere. Since  $g \leq f$ , we also have  $g^* \leq f^*$  and

$$\int_E f dx = \int_0^a g dx = \int_0^a g^* dx = \int_0^{|E|} g^* dx \leq \int_0^{|E|} f^* dx,$$

which proves (13.9).

Let  $E(y_0)$  and  $E^*(y_0)$  denote the sets in (13.7), and let  $E_1^*(y_0) = E(\theta_{f^*} \geq y_0)$ , with equality this time included. Having fixed  $y_0$  we drop it as an argument and write  $E, E^*, E_1^*$ . If we set  $F(x) = \int_0^x f dt - y_0 x$ , the set  $E$  becomes the set  $H$  of Lemma (13.8).

We show that

$$\int_0^{|E|} f^* dx \geq y_0 |E|. \quad (13.10)$$

In fact, if  $(\alpha_k, \beta_k)$  are the intervals making up  $E$ , then

$$\int_{\alpha_k}^{\beta_k} f dx \geq y_0(\beta_k - \alpha_k),$$

by (13.8), and summing over all  $k$  we get the inequality

$$\int_E f dx \geq y_0 |E|, \quad (13.11)$$

from which (13.10) follows, by (13.9).

Return now to (13.3), with  $f$  replaced by  $f^*$ . Since the right-hand side is a continuous and non-increasing function,  $|E_1^*|$  is the greatest number  $x \leq a$  such that  $x^{-1} \int_0^x f^* dt \geq y_0$ . Hence, by (13.10),  $|E| \leq |E_1^*|$ ; in full,

$$|E(\theta_f > y_0)| \leq |E(\theta_{f^*} \geq y_0)|.$$

If we replace here  $y_0$  by  $y_0 + \epsilon$  and make  $\epsilon$  decrease to 0, we get (13.7), and the proof of (13.4) is completed.

In addition to  $\theta_r(x)$ , we define the functions

$$\begin{aligned}\theta'_r(x) &= \sup_{\xi} \frac{1}{\xi - x} \int_x^{\xi} f dt \quad (x < \xi \leq a), \\ \Theta_r(x) &= \max \{ \theta_r(x), \theta'_r(x) \} = \sup_{\xi} \frac{1}{\xi - x} \int_x^{\xi} f dt \quad (0 \leq \xi \leq a).\end{aligned}\quad (13.12)$$

The inequality in (13.5) holds if we replace  $\theta_r$  by  $\theta'_r$  and  $f^*$  by  $f_{\bullet}$ . Since

$$\Theta = \max(\theta, \theta'),$$

we have  $\chi(\Theta) \leq \chi(\theta) + \chi(\theta')$  and

$$\int_0^a \chi(\Theta_r) dx \leq \int_0^a \chi(\theta_{r,\bullet}) dx + \int_0^a \chi(\theta'_{r,\bullet}) dx = 2 \int_0^a \chi(\theta_{r,\bullet}) dx.$$

Hence:

(13.13) **THEOREM.** *If  $f \in L(0, a)$  and  $\Theta(x) = \Theta_r(x)$  is defined by (13.12), then for a non-negative and non-decreasing  $\chi(u)$ ,*

$$\int_0^a \chi\{\Theta(x)\} dx \leq 2 \int_0^a \chi\left(\frac{1}{x} \int_0^x f^* dt\right) dx. \quad (13.14)$$

From this we deduce, by specifying  $\chi$ , the following corollaries:

(13.15) **THEOREM.** (i) *If  $f \in L^r(0, a)$ ,  $r > 1$ , then  $\Theta(x) \in L^r$  and*

$$\int_0^a \Theta^r dx \leq 2 \left(\frac{r}{r-1}\right)^r \int_0^a f^r dx.$$

(ii) *If  $f \in L(0, a)$ , then  $\Theta(x) \in L^\alpha$  for every  $0 < \alpha < 1$ , and*

$$\int_0^a \Theta^\alpha dx \leq \frac{2\alpha^{1-\alpha}}{1-\alpha} \left(\int_0^a f dx\right)^\alpha.$$

(iii) *If  $f \log^+ f \in L(0, a)$ , then  $\Theta(x) \in L$  and*

$$\int_0^a \Theta dx \leq 4 \int_0^a f \log^+ f dx + A.$$

with  $A$  depending on  $a$  only.

We have to estimate the right-hand side of (13.14) and we may suppose from the start that  $f$  is non-increasing, so that we may replace  $f^*$  by  $f$  there.

Case (i) then follows from Theorem (9.16), with  $s = 0$ ; it is enough to set  $f(x) = 0$  for  $x > a$ . Case (ii) follows by an application of Hölder's inequality. For, with  $\chi(u) = u^\alpha$ ,

$$\begin{aligned}\int_0^a \chi\left(\frac{1}{x} \int_0^x f dt\right) dx &= \int_0^a \frac{dx}{x^{\alpha(1-\alpha)}} \left\{ \frac{1}{x^\alpha} \int_0^x f dt \right\}^\alpha \leq \left(\int_0^a \frac{dx}{x^\alpha}\right)^{1-\alpha} \left(\int_0^a \frac{dx}{x^\alpha} \int_0^x f dt\right)^\alpha \\ &= \frac{\alpha^{(1-\alpha)^2}}{(1-\alpha)^{1-\alpha}} \left\{ \int_0^a f dt \int_0^a \frac{dx}{x^\alpha} \right\}^\alpha \leq \frac{\alpha^{1-\alpha}}{1-\alpha} \left(\int_0^a f dt\right)^\alpha.\end{aligned}$$

In case (iii), the right-hand side of (13.14), with  $\chi(u) = u$ , is

$$2 \int_0^a \frac{dx}{x} \int_0^x f(t) dt = 2 \int_0^a f(t) \log_t^a dt.$$

Let  $E_1$  and  $E_2$  be the sets of points at which, respectively,  $f < (a/t)^{\frac{1}{2}}$  and  $f \geq (a/t)^{\frac{1}{2}}$ . Clearly the integral of  $f \log(a/t)$  extended over  $E_1$  does not exceed a finite constant depending only on  $a$ . In  $E_2$  we have  $1 \leq (a/t) \leq f^2$ , so that

$$\int_E f \log(a/t) dt \leq 2 \int_{E_1} f \log f dt \leq 2 \int_0^a f \log^+ f dt,$$

and (iii) follows by collecting the estimates.

The example  $f(x) = 1/(x \log^2 x)$ , considered in the interval  $(0, \frac{1}{2})$ , shows that if  $f \in L$  the function  $\Theta$  need not be integrable. In this case  $\Theta(x) = 1/(x |\log x|)$ .

For applications to Fourier series a slight modification of the function  $\Theta(x)$  is useful. Let  $f(x)$  be periodic and integrable, but not necessarily non-negative (or even real-valued). We set

$$M(x) = M_f(x) = \sup_{0 < |t| \leq \pi} \frac{1}{t} \int_0^t |f(x+u)| du = \sup_{0 < |t| \leq \pi} \frac{1}{t} \int_x^{x+t} |f(u)| du \quad (13.16)$$

for  $-\pi \leq x \leq \pi$ . Clearly  $M_f(x)$  does not exceed the function  $\Theta_{|f|}(x)$  formed for the interval  $(-2\pi, 2\pi)$ , so that

$$\int_{-\pi}^{\pi} \chi\{M_f(x)\} dx \leq \int_{-\pi}^{\pi} \chi\{\Theta_{|f|}(x)\} dx.$$

From this and (13.15) we easily get the following analogues of (i), (ii), (iii):

$$\left. \begin{aligned} \int_{-\pi}^{\pi} M^r(x) dx &\leq 4 \left( \frac{r}{r-1} \right)^r \int_{-\pi}^{\pi} |f|^r dx \quad (r > 1), \\ \int_{-\pi}^{\pi} M^\alpha(x) dx &\leq 4 \frac{1-\alpha}{1-\alpha} \left( \int_{-\pi}^{\pi} |f| dx \right)^\alpha \quad (0 < \alpha < 1), \\ \int_{-\pi}^{\pi} M(x) dx &\leq 8 \int_{-\pi}^{\pi} |f| \log^+ |f| dx + A. \end{aligned} \right\} \quad (13.17)$$

The following inequalities, implicitly contained in the preceding proofs, deserve separate mention. First, for  $f$  integrable and non-negative in  $(0, a)$ ,

$$|E(\theta_f > y)| \leq y^{-1} \int_0^a f dx, \quad |E(\theta_f > y)| \leq 2y^{-1} \int_0^a f dx. \quad (13.18)$$

The first of these inequalities is contained in (13.11), and the second follows from the first by (13.12). Finally, for an  $f$  periodic and integrable but not necessarily non-negative,

$$|E\{M_f(x) > y, 0 \leq x \leq 2\pi\}| \leq 4y^{-1} \int_0^{2\pi} |f(x)| dx. \quad (13.19)$$

*Remark.* While in parts (ii) and (iii) of (13.15) we must necessarily assume that  $a$  is finite, part (i) holds, for infinite intervals. Suppose, e.g., that  $f \in L^r(-\infty, +\infty)$ ,  $r > 1$ , and consider the analogue of (13.15) (i) for the interval  $(-a, a)$  and the function  $f_a$ , which is  $f$  restricted to  $(-a, a)$ . The passage  $a \rightarrow +\infty$  leads to

$$\int_{-\infty}^{+\infty} \Theta^r dx \leq 2 \left( \frac{r}{r-1} \right)^r \int_{-\infty}^{+\infty} f^r dx.$$

## MISCELLANEOUS THEOREMS AND EXAMPLES

1. A sequence  $\{u_n\}$  is of bounded variation if and only if it is a difference of two non-negative and non-increasing sequences.

2. Of the two series,

$$(i) \frac{1}{2} + \cos x + \cos 2x + \dots + \cos nx + \dots,$$

$$(ii) \sin x + \sin 2x + \dots + \sin nx + \dots,$$

the first diverges for all  $x$ , the second for all  $x \not\equiv 0 \pmod{\pi}$ .

[(i) For no  $x_0$  do we have  $\cos nx_0 \rightarrow 0$ . For otherwise,

$$\sin^2 nx_0 = 1 - \cos^2 nx_0 \rightarrow 1, \quad \sin^2 nx_0 = \frac{1}{2}(1 - \cos 2nx_0) \rightarrow \frac{1}{2},$$

a contradiction.

(ii) If  $\sin nx_0 \rightarrow 0$ , then

$$\sin(n+1)x_0 - \sin(n-1)x_0 = 2 \sin x_0 \cos nx_0 \rightarrow 0,$$

that is,  $\sin x_0 = 0$  by (i).]

3. Using  $S[|\sin x|]$ , prove

$$|\sin x| = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2 nx}{4n^2 - 1}.$$

4. Let  $c_m = \frac{1}{2}(a_m - ib_m)$  for  $m > 0$ , and let  $c_{-m} = \bar{c}_m$ . Show that a necessary and sufficient condition for the existence of  $\lim_{\substack{N \\ -M}} \sum_{-M}^N c_m e^{imx}$  as  $M$  and  $N$  tend to  $+\infty$  independently of each other, is the simultaneous convergence of both series

$$\sum_1^{\infty} (a_m \cos mx_0 + b_m \sin mx_0) \quad \text{and} \quad \sum_1^{\infty} (a_m \sin mx_0 - b_m \cos mx_0).$$

5. Each of the two systems

$$(i) 1, \cos x, \cos 2x, \dots, \cos nx, \dots$$

$$(ii) \sin x, \sin 2x, \dots, \sin nx, \dots$$

is orthogonal and complete over  $(0, \pi)$ .

6. Let  $\{\phi_n\}$  denote Rademacher's system (see §3) and let

$$\chi_N(t) = 1, \quad \chi_N(t) = \phi_{n_1}(t) \phi_{n_2}(t) \dots,$$

where  $2^{n_1} + 2^{n_2} + \dots, n_1 > n_2 > \dots$ , is the dyadic development of the positive integer  $N$ . Show that the system  $\{\chi_N\}$  is orthonormal over the interval  $(0, 1)$ .

[The system  $\chi_N$ , in a different form, was first considered by Walsh [1]. See also Paley [1], who gave the above definition, and Kaczmarz [1].]

7. Let  $s_n(x)$  be the sum of the first  $N$  terms of the Fourier series  $\sum a_n \chi_n(x)$  of  $f(x)$ ,  $0 \leq x \leq 1$ . Prove the formula

$$s_n(x_0) = \int_0^1 f(t) \prod_{k=0}^{n-1} (1 + \phi_k(x_0) \phi_k(t)) dt,$$

and show that  $s_n(x) \rightarrow f(x)$  almost everywhere as  $n \rightarrow \infty$ . This implies, in particular, that the system  $\{\chi_N\}$  is complete over  $(0, 1)$ .

[If  $x_0$  is not a binary rational, and  $I_{n-1}$  is the interval of constancy of  $\phi_{n-1}$  containing  $x_0$ , then  $|I_{n-1}| = 2^{-n}$ , all functions  $\phi_0, \phi_1, \dots, \phi_{n-1}$  are constant in  $I_{n-1}$ , and the integral above is  $|I_{n-1}|^{-1} \int_{I_{n-1}} f(t) dt$ . If, therefore,  $F(x) = \int_0^x f dt$  is differentiable at  $x_0$ , we have  $s_n(x_0) \rightarrow F'(x_0)$ .]

8. Orthogonal systems can be defined in spaces of any dimension, intervals of integration being replaced by any fixed measurable set of positive measure. Show that if  $\{\phi_m(x)\}$  and  $\{\psi_n(y)\}$  are orthonormal and complete in intervals  $a \leq x \leq b$ ,  $c \leq y \leq d$  respectively, then the doubly infinite system  $\{\phi_m(x) \psi_n(y)\}$  is orthonormal and complete in the rectangle

$$R: a \leq x \leq b, \quad c \leq y \leq d.$$

[If  $\iint_R f(x, y) \bar{\phi}_m(x) \bar{\psi}_n(y) dx dy = 0$  for all  $m, n$ , the functions  $f_m(y) = \int_a^b f(x, y) \bar{\phi}_m(x) dx$  vanish for almost all  $y$ , and hence  $f(x, y)$  vanishes almost everywhere on almost all lines  $y = \text{const.}$ ]

## CHAPTER II

FOURIER COEFFICIENTS. ELEMENTARY THEOREMS  
ON THE CONVERGENCE OF  $S[f]$  AND  $\bar{S}[f]$ 1. Formal operations on  $S[f]$ (1-1) THEOREM. Let  $n$  be an integer,  $u$  a real number and

$$f(x) \sim \sum_{\nu=-\infty}^{+\infty} c_{\nu} e^{i\nu x}. \quad (1-2)$$

Then

$$(i) \quad \overline{f(x)} \sim \sum_{\nu=-\infty}^{+\infty} \bar{c}_{\nu} e^{-i\nu x} = \sum_{\nu=-\infty}^{+\infty} \bar{c}_{-\nu} e^{i\nu x},$$

$$(ii) \quad f(nx) \sim \sum_{\nu=-\infty}^{+\infty} c_{\nu} e^{i\nu nx} \quad (n \neq 0),$$

$$(iii) \quad e^{i\nu x} f(x) \sim \sum_{\nu=-\infty}^{+\infty} c_{\nu} e^{i(\nu+n)x} = \sum_{\nu=-\infty}^{+\infty} c_{\nu-n} e^{i\nu x},$$

$$(iv) \quad f(x+u) \sim \sum_{\nu=-\infty}^{+\infty} c_{\nu} e^{i\nu u} e^{i\nu x},$$

$$(v) \quad \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{x}{n} + \frac{2\pi k}{n}\right) \sim \sum_{\nu=-\infty}^{+\infty} c_{\nu n} e^{i\nu x} \quad (n > 0).$$

The proofs are simple:

$$(i) \quad \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-i\nu t} dt = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{i\nu t} dt = \bar{c}_{-\nu}.$$

(ii) Suppose first that  $n > 0$ . We observe that

$$\frac{1}{n} \sum_{k=0}^{n-1} \exp(2\pi\mu k i/n) = \begin{cases} 1 & (\mu = 0, \pm 1, \pm 2, \dots), \\ 0 & \end{cases} \quad (1-3)$$

according as  $\mu$  is or is not a multiple of  $n$ . Now

$$\int_0^{2\pi} f(nt) e^{-i\nu t} dt = \frac{1}{n} \int_0^{2\pi n} f(t) e^{-i\nu t/n} dt = \int_0^{2\pi} f(t) e^{-i\nu t/n} \left( \frac{1}{n} \sum_{k=0}^{n-1} e^{-2\pi i k \mu/n} \right) dt,$$

and this is  $2\pi c_{\nu}$ , or 0, according as  $\mu/n = \nu$  is an integer or not. The case  $n < 0$  reduces to  $n > 0$ , since, as we easily see,

$$f(-x) \sim \sum c_{\nu} e^{-i\nu x}.$$

$$(iii) \quad \int_0^{2\pi} f(t) e^{i\nu t} e^{-i\nu t} dt = 2\pi c_{\nu-n}.$$

$$(iv) \quad \int_0^{2\pi} f(t+u) e^{-i\nu t} dt = e^{i\nu u} \int_0^{2\pi} f(t+u) e^{-i\nu(t+u)} dt = 2\pi e^{i\nu u} c_{\nu}.$$

(v) This follows from (iv) and (1-3).

$$\text{If} \quad f(x) \sim \frac{1}{2} a_0 + \sum_1^{\infty} (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x), \quad (1-4)$$

(iv) can be written

$$f(x+u) \sim \sum_{\nu=0}^{\infty} \{A_{\nu}(u) \cos \nu x - B_{\nu}(u) \sin \nu x\}.$$

(1.5) THEOREM. If  $f(x)$  and  $g(x)$  are integrable and periodic, so is the function

$$h(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)g(t) dt. \quad (1.6)$$

$$\text{If } f \sim \sum c_{\nu} e^{i\nu x}, g \sim \sum d_{\nu} e^{i\nu x}, \text{ then } h(x) \sim \sum_{-\infty}^{+\infty} c_{\nu} d_{\nu} e^{i\nu x}. \quad (1.7)$$

We show first that the integral (1.6) exists for almost all  $x$ . We may assume that  $f$  and  $g$  are real-valued, and this case may, in turn, be reduced to  $f, g \geq 0$ . Then

$$\int_0^{2\pi} dx \int_0^{2\pi} f(x-t)g(t) dt = \int_0^{2\pi} g(t) \left( \int_0^{2\pi} f(x-t) dx \right) dt = \int_0^{2\pi} g(t) dt \int_0^{2\pi} f(x) dx. \quad (1.8)$$

The operations performed here are justified since  $f(x-t)g(t)$  is measurable in the  $(x, t)$  plane (being a product of measurable functions) and since (the integrand being non-negative) the order of integration is irrelevant. Thus  $h(x)$  is integrable and, in particular, finite almost everywhere. It is clearly periodic.

The function  $f(x-t)g(t)$  is integrable over the square  $0 \leq x \leq 2\pi, 0 \leq t \leq 2\pi$ . Thus for general  $f$  and  $g$ ,  $|f(x-t)g(t)e^{-i\nu x}|$  is integrable over the square and the following argument is also legitimate:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-i\nu x} dx &= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{2\pi} \int_0^{2\pi} f(x-t) e^{-i\nu(x-t)} g(t) e^{-i\nu t} dt \right\} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-i\nu t} \left\{ \frac{1}{2\pi} \int_0^{2\pi} f(x-t) e^{-i\nu(x-t)} dx \right\} dt = c_{\nu} d_{\nu}, \end{aligned}$$

and the proof of (1.5) is completed.

It is useful to observe that (1.7) is obtained by the formal process of multiplying  $S[f(x-t)] = \sum c_{\nu} e^{i\nu x} e^{-i\nu t}$  and  $S[g(t)] = \sum d_{\nu} e^{i\nu t}$  termwise and integrating each product term over  $0 \leq t \leq 2\pi$ .

The function

$$h(x) = I(f, g) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t)g(t) dt$$

of (1.6) is often called the *convolution*, or *composition*, of the functions  $f$  and  $g$ . Obviously  $I(f, g) = I(g, f)$ .

For  $f$  in (1.4) and  $g \sim \frac{1}{2}a'_0 + \sum (a'_{\nu} \cos \nu x + b'_{\nu} \sin \nu x)$ , (1.7) can also be written

$$\frac{1}{\pi} \int_0^{2\pi} f(x-t)g(t) dt \sim \frac{1}{2}a_0 a'_0 + \sum_1^{\infty} \{(a_{\nu} a'_{\nu} - b_{\nu} b'_{\nu}) \cos \nu x + (a_{\nu} b'_{\nu} + a'_{\nu} b_{\nu}) \sin \nu x\}. \quad (1.9)$$

Set  $g(t) = \bar{f}(-t)$  in (1.6) and replace  $t$  by  $-t$ . We obtain the special but interesting case

$$\frac{1}{2\pi} \int_0^{2\pi} f(x+t)\bar{f}(t) dt \sim \sum_{-\infty}^{+\infty} |c_{\nu}|^2 e^{i\nu x}. \quad (1.10)$$

Suppose that the  $f$  and  $g$  in (1.6) are in  $L^2$ . Then  $\sum |c_{\nu}|^2$  and  $\sum |d_{\nu}|^2$  converge. If we can show that the integral (1.6), which by Schwarz's inequality exists for every  $x$ , is a continuous function of  $x$ , then we can replace the sign ' $\sim$ ' in (1.7) by '=' (Chapter I, (6.3)). For this purpose we need the case  $p=2$  of the following lemma:

(1.11) LEMMA. If  $f$  is periodic and in  $L^p$ ,  $1 \leq p < \infty$ , the expression

$$J_p(t; f) = \mathfrak{M}_p[f(x+t) - f(x)] = \left\{ \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right\}^{1/p}$$

tends to 0 with  $t$ .

This is immediate for  $f$  continuous. Using Theorem (11.8) of Chapter I and its notation, we get, applying Minkowski's inequality twice,

$$J_p(t; f) \leq J_p(t; \phi) + J_p(t; f - \phi) \leq J_p(t; \phi) + 2\mathfrak{M}_p[f - \phi] < o(1) + 2\epsilon.$$

Hence  $J_p(t; f) < 3\epsilon$  for  $|t|$  small enough, and (1.11) follows.

Return to (1.6). If  $f$  and  $g$  are in  $L^2$ , then

$$|h(x+u) - h(x)| \leq \int_0^{2\pi} |f(x+u-t) - f(x-t)| |g(t)| dt \leq J_2(u; f) \mathfrak{M}_2[g] \rightarrow 0$$

as  $u \rightarrow 0$ , which shows that  $h$  is continuous. Hence:

(1.12) THEOREM. Suppose that  $f$  and  $g$  are in  $L^2$  and have coefficients  $c_\nu$  and  $d_\nu$  respectively. Then

$$\frac{1}{2\pi} \int_0^{2\pi} f(x-t)g(t) dt = \sum_{-\infty}^{+\infty} c_\nu d_\nu e^{i\nu x}$$

for all  $x$ , and the series on the right converges absolutely and uniformly. In particular

$$\left. \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f(x+t)\bar{f}(t) dt &= \sum_{-\infty}^{+\infty} |c_\nu|^2 e^{i\nu x}, \\ \frac{1}{2\pi} \int_0^{2\pi} f(t)g(t) dt &= \sum_{-\infty}^{+\infty} c_\nu d_{-\nu}, \\ \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt &= \sum_{-\infty}^{+\infty} |c_\nu|^2. \end{aligned} \right\} \quad (1.13)$$

The last equation is Parseval's formula for the trigonometric system. The name *Parseval's formula* is often given to the first two more general relations (1.13).

If  $f$  is real-valued and has coefficients  $a_\nu, b_\nu$ , we may write the last equation in the form

$$\frac{1}{\pi} \int_0^{2\pi} f^2(t) dt = \frac{1}{2} a_0^2 + \sum_1^\infty (a_\nu^2 + b_\nu^2). \quad (1.14)$$

Return to (1.5). If  $f$  and  $g$  are integrable, so is  $h$ . The following generalization of this result is of importance.

(1.15) THEOREM. Let  $f$  and  $g$  be periodic and in  $L^p$  and  $L^q$  respectively, where  $p \geq 1, q \geq 1, 1/p + 1/q > 1$ . Let

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1; \quad (1.16)$$

then the function  $h(x) = I(f, g)$  defined by (1.6) belongs to  $L^r$ , and

$$\mathfrak{A}_r[h] \leq \mathfrak{A}_p[f] \mathfrak{A}_q[g],$$

where  $\mathfrak{A}_r[h]$  stands for  $\mathfrak{A}_r[h; 0, 2\pi]$ , and similarly for  $\mathfrak{A}_p[f], \mathfrak{A}_q[g]$ .

Since  $|I(f, g)| \leq I(|f|, |g|)$ , we may suppose that  $f \geq 0, g \geq 0$ . Let  $\lambda, \mu, \nu$  be positive numbers satisfying  $1/\lambda + 1/\mu + 1/\nu = 1$ . Writing

$$f(x-t)g(t) = f^{p/\lambda} g^{q/\lambda} \cdot f^{p(\mu/\nu - 1/\lambda)} \cdot g^{q(\mu/\nu - 1/\lambda)},$$

and applying Hölder's inequality with the exponents  $\lambda, \mu, \nu$  (Chapter I, (9.3)), we see that  $h(x)$  does not exceed

$$\left[ \frac{1}{2\pi} \int_0^{2\pi} f^p(x-t) g^q(t) dt \right]^{1/\lambda} \left[ \frac{1}{2\pi} \int_0^{2\pi} f^{p\mu(1/\nu-1/\lambda)}(x-t) dt \right]^{1/\mu} \left[ \frac{1}{2\pi} \int_0^{2\pi} g^{q\mu(1/q-1/\lambda)}(t) dt \right]^{1/\nu}.$$

We use this with  $\lambda = r, 1/p - 1/\lambda = 1/\mu, 1/q - 1/\lambda = 1/\nu,$  (1.17)

so that  $\lambda, \mu, \nu$  are positive numbers and satisfy  $1/\lambda + 1/\mu + 1/\nu = 1$  by (1.16). The last two factors in the product are just  $\mathfrak{A}_p^{p/\mu}[f]$  and  $\mathfrak{A}_q^{q/\nu}[g]$ . Hence

$$\mathfrak{A}_r[h] = \mathfrak{A}_\lambda[h] \leq \mathfrak{A}_p^{p/\mu}[f] \mathfrak{A}_q^{q/\nu}[g] \left\{ \frac{1}{4\pi^2} \int_0^{2\pi} dx \int_0^{2\pi} f^\nu(x-t) g^q(t) dt \right\}^{1/r}.$$

The expression in curly brackets is  $\mathfrak{A}[f^p] \mathfrak{A}[g^q] = \mathfrak{A}_p^{p/\mu}[f] \mathfrak{A}_q^{q/\nu}[g]$ , and the right-hand side is therefore

$$\mathfrak{A}_p^{p(1/\mu+1/r)}[f] \mathfrak{A}_q^{q(1/\nu+1/r)}[g] = \mathfrak{A}_p[f] \mathfrak{A}_q[g],$$

by (1.17). This completes the proof.

The theorem holds when  $1/p + 1/q = 1$ . Moreover, by an argument similar to that preceding Theorem (1.12),  $h(x)$  is then continuous.

Let  $f_1, f_2, \dots, f_k$  be periodic integrable functions having respectively Fourier coefficients  $\{c_n^{(1)}\}, \{c_n^{(2)}\}, \dots, \{c_n^{(k)}\}$ . We define the convolution  $h(x)$ , or  $I(f_1, f_2, \dots, f_k)$ , of  $f_1, \dots, f_k$  by the induction formula

$$I(f_1, f_2, \dots, f_k) = I(I(f_1, \dots, f_{k-1}), f_k).$$

Then  $h(x)$  is a periodic integrable function, and obviously

$$h(x) = I(f_1, f_2, \dots, f_k) \sim \sum c_n^{(1)} c_n^{(2)} \dots c_n^{(k)} e^{inx}. \quad (1.18)$$

It follows that the operation of convolution is commutative and associative. Commutativity is anyway an immediate consequence of the definition of convolution, while associativity can also be derived directly from the formula

$$h(x) = (2\pi)^{-k} \int_0^{2\pi} \dots \int_0^{2\pi} f_1(x-t_1-\dots-t_k) f_2(t_1) \dots f_k(t_k) dt_1 \dots dt_k.$$

(1.19) THEOREM. Let  $f_1, f_2, \dots, f_k$  be periodic and of the classes  $L^{r_1}, L^{r_2}, \dots, L^{r_k}$  respectively. Suppose that  $r_j \geq 1$  for all  $j$  and that the number

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \dots + \frac{1}{r_k} - (k-1) \quad (1.20)$$

is positive. Then the convolution  $h(x) = I(f_1, \dots, f_k)$  is of class  $L^r$ . If the right-hand side of (1.20) is zero, then  $h$  is continuous. Moreover

$$\mathfrak{A}_r[h] \leq \mathfrak{A}_{r_1}[f_1] \dots \mathfrak{A}_{r_k}[f_k]. \quad (1.21)$$

This follows by induction from (1.15).

Let  $F(x)$  be a function satisfying the condition

$$F(x+2\pi) - F(x) = \text{const.} \quad (-\infty < x < +\infty) \quad (1.22)$$

and of bounded variation in  $(0, 2\pi)$ . Let  $G(x)$  be a similar function and write

$$H(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x-t) dG(t), \quad (1.23)$$

the convolution of  $F$  and  $dG$ . The integral here is taken in the Riemann-Stieltjes sense, and so exists for every  $x$  such that  $F(x-t)$ , qua function of  $t$ , and  $G(t)$  have no discontinuity in common. In other words, it exists for every  $x$  which does not belong to the denumerable set  $D$  of numbers  $\xi_j + \eta_k$ , where  $\{\xi_j\}$  and  $\{\eta_k\}$  are the discontinuities

of  $F(t)$  and  $G(t)$  respectively. Let  $E$  be the set complementary to  $D$ . The function  $H$  satisfies on  $E$  a condition analogous to (1.22). If  $\{(a_i, b_i)\}$  is a finite system of non-overlapping intervals with end-points in  $E$ , then

$$\sum |H(b_i) - H(a_i)| \leq \frac{1}{2\pi} \int_0^{2\pi} \{ \sum |F(b_i - t) - F(a_i - t)| \} |dG(t)|, \tag{1.24}$$

which shows that  $H$  is of bounded variation over any finite portion of  $E$ . It is therefore a difference of two functions monotone on  $E$ , and so can be extended to all  $x$ , e.g. by the condition

$$H(x) = \lim H(x') \quad (x \in D, x' \in E, x' \rightarrow x + 0). \tag{1.25}$$

Such an extension does not increase the variation of monotone functions, nor, therefore, the total variation of  $H$ . Let  $V_F$  denote the total variation of  $F$  over  $(0, 2\pi)$ . Since, as we see from (1.24), the total variation of  $H$  over  $(0, 2\pi) \setminus E$  does not exceed  $V_F V_G / 2\pi$ , it follows that

$$V_H \leq V_F V_G / 2\pi. \tag{1.26}$$

Summarizing, the integral (1.23) exists for all  $x$  outside a certain denumerable set  $D$  and can, by (1.25), be extended to all  $x$  as a function satisfying a condition similar to (1.22), of bounded variation over  $(0, 2\pi)$ , and also satisfying (1.26). Clearly if  $F$  and  $G$  are monotone so is  $H$ . We may add that if we used the Lebesgue-Stieltjes integral,  $H(x)$  could be defined by (1.23) for all  $x$ , and would have the same properties as the  $H(x)$  above.

Let  $c_n$  and  $c'_n$  be the Fourier coefficients of  $dF$  and  $dG$ . We shall show that

$$dH(x) \sim \sum c_n c'_n e^{inx}. \tag{1.27}$$

For let  $x_0 < x_1 < \dots < x_k = x_0 + 2\pi$  be points of  $E$ . The  $n$ th Fourier coefficient of  $dH$  is the limit of the sum

$$\frac{1}{2\pi} \sum_{j=1}^k e^{-inx_j} [H(x_j) - H(x_{j-1})] = \frac{1}{4\pi^2} \sum_{j=1}^k \int_0^{2\pi} \left\{ \int_{x_{j-1}-t}^{x_j-t} e^{-in(x_j-u)} dF(u) \right\} e^{-int} dG(t)$$

as  $\rho = \max(x_j - x_{j-1}) \rightarrow 0$ . Suppose  $\rho$  so small that the oscillation of  $e^{-inu}$  over every interval of length  $\leq \rho$  is less than  $\epsilon$ . Then on replacing  $e^{-in(x_j-u)}$  in the last integral by  $e^{-inu}$  we introduce an error at most

$$\frac{\epsilon}{4\pi^2} \sum_{j=1}^k \int_0^{2\pi} \left\{ \int_{x_{j-1}-t}^{x_j-t} |dF(u)| \right\} |dG(t)| = \frac{\epsilon}{4\pi^2} \int_0^{2\pi} \left\{ \int_{x_0-t}^{x_0-t} |dF(u)| \right\} |dG(t)| = \frac{\epsilon}{4\pi^2} V_F V_G.$$

But  $\int_0^{2\pi} \left\{ \int_{x_0-t}^{x_0-t} e^{-inu} dF(u) \right\} e^{-int} dG(t) = \int_0^{2\pi} e^{-inu} dF(u) \int_0^{2\pi} e^{-int} dG(t) = 4\pi^2 c_n c'_n.$

This completes the proof. As we see from (1.27), by interchanging the roles of  $F$  and  $G$  in (1.23) we modify  $H$  only by an additive constant,† a result which can also be obtained directly from (1.23) by integration by parts.

If  $F$  (or  $G$ ) is continuous, the integral (1.23) exists for all  $x$ , and since

$$2\pi |H(x+h) - H(x)| \leq \max_t |F(x+h-t) - F(x-t)| \int_0^{2\pi} |dG(t)|,$$

$H(x)$  is also continuous.

† See the last remark on p. 41.

A special case of  $H(x)$  is the function

$$F^*(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x+t) d\bar{F}(t). \quad (1.28)$$

$F^*(-x)$  is the convolution of  $F(-t)$  and  $d\bar{F}(t)$ . By (1.27) (if  $dF(x) \sim \sum c_n e^{inx}$ , then  $\bar{d}F(-x) \sim -\sum c_{-n} e^{inx}$ ),

$$dF^*(x) \sim \sum |c_n|^2 e^{inx}. \quad (1.29)$$

We shall show that the absence of a jump of  $F^*(x)$  at  $x=0$  is equivalent to the continuity of  $F$  at every point. More precisely:

(1.30) THEOREM. Let  $x_1, x_2, \dots$  be all the discontinuities of  $F$  in the interval  $0 \leq x < 2\pi$ . and let  $d_j = F(x_j+0) - F(x_j-0)$ . Then

$$F^*(+0) - F^*(-0) = (2\pi)^{-1} \sum |d_j|^2.$$

For let  $S_k(x)$  be a step function having jumps  $d_1, d_2, \dots, d_k$  at the points  $x_1, x_2, \dots, x_k$ , continuous elsewhere, and satisfying a condition analogous to (1.22). The difference  $F_k(x) = F(x) - S_k(x)$  is continuous at  $x_1, x_2, \dots, x_k$ , and has jumps  $d_{k+1}, d_{k+2}, \dots$  at the points  $x_{k+1}, x_{k+2}, \dots$ . The function  $F^*(x)$  equals

$$\frac{1}{2\pi} \int_0^{2\pi} F_k(x+t) d\bar{F}(t) + \frac{1}{2\pi} \int_0^{2\pi} S_k(x+t) d\bar{F}(t) = H_1(x) + H_2(x).$$

For  $\pm \delta \in E$ ,  $2\pi |H_1(+\delta) - H_1(-\delta)| \leq V_F \sup_t |F_k(t+\delta) - F_k(t-\delta)|$ ,

$$2\pi [H_2(+\delta) - H_2(-\delta)] = \int_0^{2\pi} [S_k(t+\delta) - S_k(t-\delta)] d\bar{F}(t). \quad (1.31)$$

The first inequality shows that by taking  $k$  large enough (i.e. by removing the 'heavier' discontinuities from  $F$ ) we can make  $H_1(+0) - H_1(-0)$  arbitrarily small. For  $\delta$  small enough,  $S_k(t+\delta) - S_k(t-\delta)$  is  $d_j$  in the  $\delta$  neighbourhood of  $x_j$ ,  $j=1, 2, \dots, k$ , and is zero elsewhere. This shows that the integral in (1.31) tends to  $|d_1|^2 + \dots + |d_k|^2$  as  $\delta \rightarrow 0$ . From these facts (1.30) follows.

## 2. Differentiation and integration of $S[f]$

Suppose that a periodic function  $f(x)$  is an integral, i.e. is absolutely continuous. Integration by parts gives

$$c_\nu = \frac{1}{2\pi} \int_0^{2\pi} f e^{-i\nu x} dx = \frac{1}{2\pi i \nu} \int_0^{2\pi} f' e^{-i\nu x} dx = \frac{c'_\nu}{i\nu} \quad (\nu \neq 0),$$

so that  $c'_\nu = i\nu c_\nu$ , the  $c'_\nu$  being the coefficients of  $f'$ . Since  $f$  is periodic,  $c'_0 = 0$ . Thus if  $S'[f]$  denotes the result of differentiating  $S[f]$  term by term, we have  $S'[f] = S[f']$ , or

$$f' \sim i \sum_{\nu=-\infty}^{+\infty} \nu c_\nu e^{i\nu x} = \sum_{\nu=1}^{\infty} \nu (b_\nu \cos \nu x - a_\nu \sin \nu x).$$

From this follows the general result:

(2.1) THEOREM. If  $f(x)$  is a  $k$ -th integral ( $k=1, 2, \dots$ ), then  $S^{(k)}[f] = S[f^{(k)}]$ .

The following result shows what happens when  $f$  has discontinuities, for simplicity a finite number of them:

(2.2) THEOREM. Suppose that  $f(x)$  has discontinuities of the first kind (jumps) at the points  $x_1 < x_2 < \dots < x_k < x_{k+1} = x_1 + 2\pi$ , and that  $f(x)$  is absolutely continuous in each of the intervals  $(x_i, x_{i+1})$ , if completed by continuity at the end-points  $x_i, x_{i+1}$ . Let

$$d_i = [f(x_i + 0) - f(x_i - 0)]/\pi, \quad D(x) = \frac{1}{2} + \sum_{\nu=1}^{\infty} \cos \nu x.$$

Then  $S[f] - S[f'] = d_1 D(x - x_1) + d_2 D(x - x_2) + \dots + d_k D(x - x_k)$ . (2.3)

The series  $D(x)$  diverges everywhere (see p. 34, Ex. 2), but is summable by various methods to 0 if  $x \neq 0$  (see Chapter III, §§ 1, 2). The statement (2.3) is, of course, to be interpreted *formally*: corresponding coefficients of the series on the two sides are equal.

We may suppose that  $f(x_i) = \frac{1}{2}[f(x_i + 0) + f(x_i - 0)]$  for all  $i$ . Let  $\phi(x)$  be the function defined in Chapter I, (4.12). Then  $S[\phi] = D(x) - \frac{1}{2}$ . The function

$$\Phi(x) = d_1 \phi(x - x_1) + \dots + d_k \phi(x - x_k)$$

has the same points of discontinuity, and the same jumps, as  $f$ . The difference  $g = f - \Phi$  is therefore continuous, indeed absolutely continuous. Moreover,

$$\Phi'(x) = -\frac{1}{2}(d_1 + \dots + d_k) = C,$$

say, except at the points  $x_i$ , so that  $g' = f' - C$  almost everywhere. Now

$$\begin{aligned} S[f] &= S[g + \Phi] = S[g] + S[\Phi] = S[g'] + S[\Phi] \\ &= S[f'] - C + \sum_i d_i \{D(x - x_i) - \frac{1}{2}\} = S[f'] + \sum_i d_i D(x - x_i). \end{aligned}$$

This completes the proof of (2.3).

Let  $F(x)$ ,  $0 \leq x \leq 2\pi$ , be a function of bounded variation, and let  $c_\nu$  be the Fourier coefficients of  $dF$ . The difference  $F(x) - c_0 x$  is periodic (Chapter I, § 5), and its Fourier coefficient  $C_\nu$ ,  $\nu \neq 0$ , is

$$\frac{1}{2\pi} \int_0^{2\pi} (F - c_0 x) e^{-i\nu x} dx = \frac{1}{2\pi i \nu} \int_0^{2\pi} e^{-i\nu x} d(F - c_0 x) = \frac{1}{2\pi i \nu} \int_0^{2\pi} e^{-i\nu x} dF = \frac{C_\nu}{i\nu}.$$

Let us agree to write  $F(x) \sim c_0 x + C_0 + \sum'_{\nu=-\infty}^{+\infty} \frac{C_\nu}{i\nu} e^{i\nu x}$

instead of  $F(x) - c_0 x \sim C_0 + \sum'_{\nu=-\infty}^{+\infty} \frac{C_\nu}{i\nu} e^{i\nu x}$ ,

where the dash signifies that the term  $\nu = 0$  is omitted in summation. Then  $S[dF]$  is obtained by formal differentiation of the first of these series, and we have:

(2.4) THEOREM. With the convention just stated, the class of Fourier-Stieltjes series coincides with the class of formally differentiated Fourier series of functions of bounded variation.

If  $S[dF]$  vanishes identically,  $S[F]$  consists of a constant term  $C$ . Thus  $F(x) \equiv C$ , and  $F$  is equal to  $C$  at every point of continuity, that is, outside a certain denumerable set. Hence, if two functions  $F_1$  and  $F_2$  with regular discontinuities have the same Fourier-Stieltjes coefficients, then  $F_1(x) - F_2(x) = C$ .

Let  $f$  be periodic and  $F$  the indefinite integral of  $f$ . Since  $F(x+2\pi) - F(x)$  is equal to the integral of  $f$  over  $(x, x+2\pi)$  or, what is the same thing, over  $(0, 2\pi)$ , a necessary and sufficient condition for the periodicity of  $F$  is that the constant term  $c_0$  of  $S[f]$  is zero. Suppose this condition satisfied. Then by (2.1)  $S[f] = S'[F]$ , so that  $S[F]$  is obtained by formal integration of  $S[f]$ . In other words,

$$F(x) \sim C + \sum_{\nu=-\infty}^{+\infty} \frac{c_\nu}{i\nu} e^{i\nu x} = C + \sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x)/\nu,$$

where  $C$  is the constant of integration.

If  $c_0 \neq 0$ , we apply the result to the function  $f - c_0$ , whose integral  $F - c_0 x$  is periodic. Hence we have:

(2.5) THEOREM. If  $f \sim \sum c_\nu e^{i\nu x}$ , and  $F$  is the indefinite integral of  $f$ , then

$$F(x) - c_0 x \sim C + \sum_{\nu=-\infty}^{+\infty} c_\nu e^{i\nu x}/i\nu = C + \sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x)/\nu.$$

Example. Let  $B_0(x)$ ,  $B_1(x)$ ,  $B_2(x)$ , ... be the periodic functions defined by the conditions

- (i)  $B_0(x) = -1$ ;
- (ii)  $B'_k(x) = B_{k-1}(x)$  for  $k = 1, 2, \dots$ ;
- (iii) the integral of  $B_k$  over  $(0, 2\pi)$  is zero for  $k = 1, 2, \dots$

Using Chapter I, (4.12), one verifies by induction that

$$B_k(x) \sim \sum_{\nu=-\infty}^{+\infty} \frac{e^{i\nu x}}{(i\nu)^k} \quad (k = 1, 2, \dots), \quad (2.6)$$

where the dash indicates that the term  $\nu = 0$  is omitted in summation. Inside  $(0, 2\pi)$ ,  $B_k(x)$  is a polynomial of degree  $k$  (Bernoulli's polynomial, except for a numerical factor). According as  $k$  is even or odd,

$$B_k(x) \sim 2(-1)^{k/2} \sum_{\nu=1}^{\infty} \frac{\cos \nu x}{\nu^k}, \quad B_k(x) \sim 2(-1)^{(k-1)/2} \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu^k}.$$

Suppose that  $f$  is a  $k$ th integral ( $k = 1, 2, \dots$ ). Replacing in (1.5)  $f$  by  $f^{(k)}$ ,  $g$  by  $B_k$ , we have the useful formula

$$f(x) - c_0 = \frac{1}{2\pi} \int_0^{2\pi} f^{(k)}(x-t) B_k(t) dt. \quad (2.7)$$

### 3. Modulus of continuity. Smooth functions

Let  $f(x)$  be defined in a closed interval  $I$ , and let

$$\omega(\delta) = \omega(\delta; f) = \sup |f(x_2) - f(x_1)| \quad \text{for } x_1 \in I, x_2 \in I, |x_2 - x_1| \leq \delta.$$

The function  $\omega(\delta)$  is called the *modulus of continuity* of  $f$ . If  $I$  is finite, then  $f$  is continuous in  $I$  if and only if  $\omega(\delta) \rightarrow 0$  with  $\delta$ . If for some  $\alpha > 0$  we have  $\omega(\delta) \leq C\delta^\alpha$ , with  $C$  independent of  $\delta$ , we shall say that  $f$  satisfies a *Lipschitz condition of order  $\alpha$  in  $(a, b)$* . We shall also say that  $f$  belongs to the class  $\Lambda_\alpha$ ; in symbols,

$$f \in \Lambda_\alpha.$$

Only the case  $0 < \alpha \leq 1$  is interesting: if  $\alpha > 1$ , then  $\omega(\delta)/\delta$  tends to zero with  $\delta$ ,  $f'(x)$  exists and is zero everywhere, and  $f$  is a constant.

The function  $f$  belongs to  $\Lambda_1$  if and only if  $f$  is the integral of a bounded function.

It is sometimes convenient to consider the classes  $\lambda_\alpha$  defined for  $0 \leq \alpha < 1$  by the condition  $\omega(\delta) = o(\delta^\alpha)$ , so that if  $I$  is finite  $\lambda_0$  is the class of continuous functions. By  $\lambda_1$  we mean the class of functions having a continuous derivative.

A function  $F(x)$  is said to be *smooth* at the point  $x_0$  if

$$\{F(x_0 + h) + F(x_0 - h) - 2F(x_0)\}/h = o(1) \quad \text{as } h \rightarrow 0. \quad (3.1)$$

This relation may also be written

$$\{F(x_0 + h) - F(x_0)\}/h - \{F(x_0) - F(x_0 - h)\}/h = o(1). \quad (3.2)$$

It follows immediately that if  $F'(x_0)$  exists and is finite then  $F$  is smooth at  $x_0$ . The converse is obviously false, but (as we see from (3.2)) if  $F$  is smooth at  $x_0$  and if a one-sided derivative of  $F$  at  $x_0$  exists, the derivative on the other side also exists and both are equal. The curve  $y = F(x)$  has then no angular points, and this is the reason for the terminology.

If  $F$  is smooth at every point of an interval  $I$ , we say that  $F$  is smooth in  $I$ . (If  $I$  is closed, this presupposes that  $F$  is defined in a larger interval containing  $I$ .) If  $F$  is continuous and satisfies (3.1) uniformly in  $x_0 \in I$  we shall say that  $F$  is *uniformly smooth*, and also that  $F$  belongs to the class  $\lambda_*$ . The class  $\Lambda_*$  is defined by the condition that  $F$  is continuous and that the left-hand side of (3.1) is  $O(1)$  uniformly in  $x_0$ .

If  $F \in \lambda_1$ , then  $F \in \lambda_*$ ; similarly, if  $F \in \Lambda_1$ , then  $F \in \Lambda_*$ . Thus  $\lambda_*$  and  $\Lambda_*$  are respectively generalizations of  $\lambda_1$  and  $\Lambda_1$ . They are sometimes important for trigonometric series as being more natural than  $\lambda_1$  and  $\Lambda_1$ . On the other hand, basic properties of  $\lambda_1$  and  $\Lambda_1$  do not hold for  $\lambda_*$  and  $\Lambda_*$ . Thus there exist functions  $F \in \Lambda_*$  which are nowhere differentiable and functions  $F \in \lambda_*$  differentiable in a set of measure zero only (p. 48). However, we do have:

(3.3) THEOREM. *If  $F(x)$  is real-valued, continuous and smooth in an interval  $I$ , the set  $E$  of points where  $F'(x)$  exists and is finite is of the power of the continuum in every subinterval  $I'$  of  $I$ .*

We may suppose that  $I' = I$ . Let  $L(x) = mx + n$  be the linear function coinciding with  $F(x)$  at the end-points  $a, b$  of  $I$ . Then  $G(x) = F(x) - L(x)$  is continuous and smooth, and vanishes for  $x = a, b$ . If  $x_0$  is a point inside  $I$  where  $G$  attains its maximum or minimum, the two terms on the left in

$$\{G(x_0 + h) - G(x_0)\}/h + \{G(x_0 - h) - G(x_0)\}/h \rightarrow 0$$

are of the same sign for  $|h|$  small. Thus the right- and left-hand derivatives of  $G$  at the point  $x_0$  exist and are zero, so that  $G'(x_0) = 0$ ,  $F'(x_0) = m = [F(b) - F(a)]/(b - a)$ .

Hence  $E$  is dense in  $I$ . Let now  $a < c < b$ . The above proof shows that there is a point  $x_1$  inside  $(a, c)$  such that  $F'(x_1)$  exists and equals the slope of the chord through  $(a, F(a))$  and  $(c, F(c))$ . Hence, if the slopes corresponding to two different  $c$ 's are different, the corresponding points  $x_1$  must also be different. But unless  $F(x)$  is a linear function, in which case (3.3) is obvious, the set of the different slopes, and so also of the points  $x_1$ , is of the power of the continuum.

It is well known that a function  $f(x)$  may be non-measurable and yet satisfy the condition

$$f(x + h) + f(x - h) - 2f(x) = 0$$

for all  $x$  and  $h$ . This is the reason why in the definition of classes  $\lambda_*$  and  $\Lambda_*$  we assumed the continuity of  $f$ . It turns out that the functions of  $\lambda_*$  and  $\Lambda_*$  have 'a considerable degree of continuity'.

(3.4) THEOREM. Let  $f(x)$  be defined in a finite interval  $(a, b)$ . If  $f \in \Lambda_*$ , then

$$\omega(\delta; f) = O(\delta \log \delta)$$

and in particular  $f \in \Lambda_x$  for every  $\alpha < 1$ . If  $f \in \lambda_*$ , then  $\omega(\delta; f) = o(\delta \log \delta)$ .

It is enough to prove the part concerning  $\Lambda_*$ . Let  $M = \max |f(x)|$ . The hypothesis  $f \in \Lambda_*$  implies

$$|f(x + \tau) - 2f(x + \frac{1}{2}\tau) + f(x)| \leq A\tau.$$

for  $x \in (a, b)$  and  $\tau$  small enough,  $0 < \tau \leq \gamma$ . Let us fix  $x$  and set  $f(x + \tau) - f(x) = g(\tau)$ . The left-hand side of the inequality above is  $|g(\tau) - 2g(\frac{1}{2}\tau)|$ . Replacing here  $\tau$  successively by  $\tau/2, \tau/2^2, \dots$  we get

$$|g(\tau) - 2g(\tau/2)| \leq A\tau, \quad |2g(\tau/2) - 2^2g(\tau/2^2)| \leq A\tau, \quad \dots, \\ |2^{n-1}g(\tau/2^{n-1}) - 2^n g(\tau/2^n)| \leq A\tau,$$

where  $n$  will be defined in a moment. By termwise addition,

$$|g(\tau) - 2^n g(\tau/2^n)| \leq An\tau. \quad (3.5)$$

Suppose now that  $h$  tends to 0 through positive values. Let  $0 < h \leq \frac{1}{2}\gamma$  and let  $n$  be a positive integer such that  $2^n h$  is in the interval  $(\frac{1}{2}\gamma, \gamma)$ . The inequality  $2^n h \leq \gamma$  implies that  $n = O(\log h)$ . From (3.5), with  $\tau = 2^n h$ , we get

$$|g(h)| \leq \frac{2M}{2^n} + \frac{An\tau}{2^n} = \frac{2Mh}{2^n h} + Anh \leq \frac{2Mh}{\frac{1}{2}\gamma} + O(h \log h) = O(h \log h),$$

or  $f(x+h) - f(x) = O(h \log h)$ , which proves the theorem.†

A function  $f(x)$  defined on a set  $E$  will be said to have property D if, given any two points  $\alpha, \beta$  in  $E$ , the function  $f$  takes on the product set  $(\alpha, \beta) \times E$  all values between  $f(\alpha)$  and  $f(\beta)$ . Property D may be considered as a (rather weak) substitute for continuity. A classical result of Darboux asserts that any exact derivative has property D in an interval where it exists.

(3.6) THEOREM. Under the hypothesis of (3.3),  $F'(x)$  has property D on  $E$ .

For let  $\alpha < \beta$ ,  $\alpha \in E$ ,  $\beta \in E$ .  $F'(\alpha) = A$ ,  $F'(\beta) = B$ .

Let  $C$  be any number between  $A$  and  $B$ , say  $A < C < B$ . We have to show the existence in  $(\alpha, \beta)$  of a point  $\gamma$  such that  $F'(\gamma) = C$ . By subtracting  $Cx$  from  $F$ , we may suppose that  $C = 0$ . Then  $A < 0 < B$ . Consider the function  $G(x) = \{F(x+h) - F(x)\}/h$ , where  $h < \beta - \alpha$  is fixed, positive, and so small that

$$G(\alpha) < 0, \quad G(\beta - h) = \{F(\beta) - F(\beta - h)\}/h > 0. \quad (3.7)$$

Since  $G(x)$  is continuous, there is a point  $x_0$  inside  $(\alpha, \beta - h)$  such that  $G(x_0) = 0$ , that is,  $F(x_0 + h) = F(x_0)$ . If  $\gamma$  is a point inside  $(x_0, x_0 + h)$  at which  $F$  attains its maximum or minimum, then  $F'(\gamma) = 0 = C$ . Since  $(x_0, x_0 + h) \subset (\alpha, \beta)$ , the theorem follows.

Remark. The argument even shows that if  $A < C < B$ , and

$$\liminf_{h \rightarrow 0} \{F(\alpha + h) - F(\alpha)\}/h \leq A, \quad \limsup_{h \rightarrow 0} \{F(\beta) - F(\beta - h)\}/h \geq B,$$

then there is a point  $\gamma$  between  $\alpha$  and  $\beta$  such that  $F'(\gamma) = C$ .

† The same proof shows that if  $f(x+h) + f(x-h) - 2f(x) = O(h^\alpha)$ ,  $0 < \alpha < 1$ , then  $f \in \Lambda_x$  (see also Remark (d) on p. 120).

Let us now confine our attention to periodic functions. Given an  $f \in L^p$ ,  $p \geq 1$ , the expression

$$\omega_p(\delta) = \omega_p(\delta; f) = \sup_{0 < h < \delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p dx \right\}^{1/p}$$

will be called the *integral modulus of continuity* (in  $L^p$ ) of  $f$ . Theorem (1.11) implies that  $\omega_p(\delta) \rightarrow 0$  with  $\delta$ , for every  $f \in L^p$ . Obviously  $\omega_p(\delta)$  is a non-decreasing function of  $\delta$  and  $p$ . If  $f$  is continuous, then  $\omega_p(\delta) \rightarrow \omega(\delta)$  as  $p \rightarrow \infty$ . Unlike  $\omega(\delta)$ ,  $\omega_p(\delta)$  is not affected by a change of  $f$  in a set of measure 0.

If  $\omega_p(\delta) = O(\delta^\alpha)$ , we write  $f \in \Lambda_p^\alpha$ ; and if  $\omega_p(\delta) = o(\delta^\alpha)$ , then  $f \in \lambda_p^\alpha$ . Here again the case  $\alpha > 1$  is of no interest: if  $\omega_p(\delta) = o(\delta)$ , then  $f \equiv \text{const}$ . Since  $\omega_p(\delta) \geq \omega_1(\delta)$ , it is enough to take  $p = 1$ . Let

$$F(x) = \int_0^x f dt, \quad 0 < x_2 - x_1 < 2\pi, \quad \delta > 0.$$

Then 
$$\left| \delta^{-1} \int_{x_1}^{x_2} \{f(x+\delta) - f(x)\} dx \right| = \left| \delta^{-1} \int_{x_1}^{x_2+\delta} f(u) du - \delta^{-1} \int_{x_1}^{x_2} f(u) du \right|.$$

The left-hand side here is not greater than  $2\pi \delta^{-1} \omega_1(\delta) = o(1)$ , as  $\delta \rightarrow 0$ . The right-hand side tends to  $|F'(x_2) - F'(x_1)|$ , provided that  $F'(x_1)$ ,  $F'(x_2)$  exist. Hence  $F'(x)$  is constant outside a set of measure 0, which means that  $f \equiv \text{const}$ .

We may also consider the class  $\Lambda_p^\alpha$  of periodic functions  $F \in L^p$ ,  $p \geq 1$ , such that

$$\left\{ \int_0^{2\pi} |F(x+h) + F(x-h) - 2F(x)|^p dx \right\}^{1/p} = O(h).$$

Replacing  $O(h)$  by  $o(h)$  we define the class  $\lambda_p^\alpha$ ; and for  $p = \infty$  and  $F$  continuous we get the classes  $\Lambda_*$  and  $\lambda_*$  respectively.

#### 4. Order of magnitude of Fourier coefficients

The Fourier coefficients  $c_\nu$  of a function  $f$  satisfy the inequalities

$$|c_\nu| \leq \frac{1}{2} \omega(\pi/|\nu|), \quad |c_\nu| \leq \frac{1}{2} \omega_1(\pi/|\nu|) \quad (\nu \neq 0), \tag{4.1}$$

where  $\omega$  and  $\omega_1$  denote the moduli of continuity of  $f$  (see § 3). For, replacing  $x$  by  $x + \pi/\nu$  in the integral defining  $c_\nu$ , and taking the mean value of the new and old integrals, we find that  $2\pi c_\nu$  is

$$\int_0^{2\pi} f(x) e^{-i\nu x} dx = - \int_0^{2\pi} f\left(x + \frac{\pi}{\nu}\right) e^{-i\nu x} dx = \frac{1}{2} \int_0^{2\pi} \left\{ f(x) - f\left(x + \frac{\pi}{\nu}\right) \right\} e^{-i\nu x} dx.$$

Hence 
$$|c_\nu| \leq \frac{1}{4\pi} \int_0^{2\pi} \left| f(x) - f\left(x + \frac{\pi}{\nu}\right) \right| dx, \tag{4.2}$$

and the right-hand side here exceeds neither  $\frac{1}{2} \omega(\pi/|\nu|)$  nor  $\frac{1}{2} \omega_1(\pi/|\nu|)$ . If  $f \in L^p$ ,  $p \geq 1$ , (4.2) implies

$$|c_\nu| \leq \frac{1}{2} \omega_p(\pi/|\nu|) \tag{4.3}$$

(see Chapter I, (10.12) (i)).

From the second inequality (4.1), and the fact that  $\omega_1(\delta) \rightarrow 0$  with  $\delta$ , we obtain the following important theorem:

(4.4) **THEOREM OF RIEMANN-LEBESGUE.** *The Fourier coefficients  $c_\nu$  of an integrable  $f$  tend to 0 as  $|\nu| \rightarrow \infty$ .*

The same result holds of course for the coefficients  $a_\nu, b_\nu$ , since  $c_\nu = \frac{1}{2}(a_\nu - ib_\nu)$  for  $\nu > 0$ .

A slightly different proof of (4.4) runs as follows. We set  $f = f_1 + f_2$ , where  $f_1$  is bounded and  $\int_0^{2\pi} |f_2| dx < \epsilon$ . Correspondingly  $c_r = c'_r + c''_r$ . Here  $f_1 \in L^2$ , so that  $c'_r \rightarrow 0$  (a consequence of  $\sum |c'_r|^2 < \infty$ ). Since

$$|c''_r| \leq \frac{1}{2\pi} \int_0^{2\pi} |f_2| dx < \epsilon/2\pi,$$

$|c_r|$  is less than  $\epsilon$  for  $|\nu|$  large enough. This concludes the proof. The reader will notice that it proves (4.4) for the general uniformly bounded orthonormal system.

The following corollary of (4.4) is useful:

(4.5) THEOREM. Let  $E$  be a measurable set in  $(0, 2\pi)$ , and let  $\xi_1, \xi_2, \dots$  be any sequence of real numbers. Then

$$\int_E \cos^2(nx + \xi_n) dx \rightarrow \frac{1}{2} |E| \quad (n \rightarrow \infty).$$

For the integrand here is  $\frac{1}{2} + \frac{1}{2} \cos 2nx \cos 2\xi_n - \frac{1}{2} \sin 2nx \sin 2\xi_n$ , and the integrals of  $\cos 2nx$  and  $\sin 2nx$  over  $E$  tend to 0 since they are the Fourier coefficients (with a factor  $\pi$ ) of the characteristic function of the set  $E$ .

The following is a slightly more general form of (4.4):

(4.6) THEOREM. Let  $f \in L(a, b)$ , where  $(a, b)$  is finite or infinite, and let  $\lambda$  be a real variable. Let  $a \leq a' < b' \leq b$ . The integral

$$\gamma_\lambda = \gamma_\lambda(f) = \gamma_\lambda(f; a', b') = \int_{a'}^{b'} f(x) e^{i\lambda x} dx$$

tends to 0 as  $\lambda \rightarrow \pm \infty$ , and the convergence is uniform in  $a'$  and  $b'$ .

Suppose first that  $b - a < \infty$ . If  $f = C$  the result is obvious, since then  $|\gamma_\lambda| \leq 2|C|/|\lambda|$ . Hence the result holds if  $f$  is a step-function (that is, if  $(a, b)$  can be broken up into a finite number of subintervals in each of which  $f$  is constant). Since a continuous  $f$  may be uniformly approximated by step-functions, (4.6) is valid for continuous functions. Applying Theorem (11.8) of Chapter I with  $r = 1$  and writing  $f$  as  $\phi + (f - \phi)$ , we find that

$$|\gamma_\lambda(f)| \leq |\gamma_\lambda(\phi)| + |\gamma_\lambda(f - \phi)| \leq |\gamma_\lambda(\phi)| + \epsilon < 2\epsilon$$

for  $|\lambda|$  large enough.

If  $b - a = \infty$ , for example if  $(a, b) = (-\infty, +\infty)$ , we write  $f = f_1 + f_2$ , where  $f_1 = f$  in the interval  $(-N, +N)$  and  $f_1 = 0$  elsewhere. If  $N$  is large enough, then

$$\int_a^b |f_2| dx < \epsilon, \quad |\gamma_\lambda(f)| \leq |\gamma_\lambda(f_1)| + |\gamma_\lambda(f_2)| \leq o(1) + \epsilon,$$

and the result follows.

(4.7) THEOREM. (i) If  $f \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , or if only  $f \in \Lambda_\alpha^p$ , then  $c_r = O(|\nu|^{-\alpha})$ ;  
 (ii) If  $f \in \Lambda_\alpha$ , or if only  $f \in \Lambda_\alpha^p$ , then  $c_r = O(\nu^{-1})$ .

Case (i) follows from (4.1) and (4.3). Here 'O' cannot be replaced by 'o' (see below), except in the extreme case  $\alpha = 1, f \in \Lambda_1$ . In this latter case  $f$  is an integral,  $S[f]$  is still a Fourier series, and  $\nu_r \rightarrow 0$ .

To prove (ii) we replace  $x$  by  $x \pm \pi/\nu$  in the integrals defining  $c_\nu$ . Then

$$\begin{aligned} 2\pi c_\nu &= \int_0^{2\pi} f(x) e^{-i\nu x} dx = - \int_0^{2\pi} f\left(x \pm \frac{\pi}{\nu}\right) e^{-i\nu x} dx \\ &= -\frac{1}{4} \int_0^{2\pi} \left[ f\left(x + \frac{\pi}{\nu}\right) + f\left(x - \frac{\pi}{\nu}\right) - 2f(x) \right] e^{-i\nu x} dx, \\ 8\pi |c_\nu| &\leq \int_0^{2\pi} \left| f\left(x + \frac{\pi}{\nu}\right) + f\left(x - \frac{\pi}{\nu}\right) - 2f(x) \right| dx = O\left(\frac{1}{\nu}\right). \end{aligned}$$

For  $f \in \lambda_\alpha^*$ , we have  $c_\nu = o(1/\nu)$ .

A good illustration of (4.7) is the Weierstrass function

$$f(x) = f_\alpha(x) = \sum_1^\infty b^{-n\alpha} \cos b^n x, \tag{4.8}$$

where  $b > 1$  is an integer and  $\alpha$  is a positive number. The series here converges absolutely and uniformly. The results which follow hold also for  $\sum f^{-n\alpha} \sin f^n x$ .

(4.9) THEOREM. If  $0 < \alpha < 1$ , then  $f_\alpha \in \Lambda_\alpha$ . The function  $f_1$  belongs to  $\Lambda_\alpha$  but not to  $\Lambda_1$ . Let  $0 < \alpha < 1$ ,  $h > 0$ . Then

$$\begin{aligned} f(x+h) - f(x) &= \sum_1^N b^{-n\alpha} \sin b^n \left(x + \frac{1}{2}h\right) 2 \sin \frac{1}{2}b^n h \\ &= - \sum_1^N - \sum_{N+1}^\infty = P + Q, \end{aligned}$$

where  $N = N(h)$  is the largest integer satisfying  $b^N h \leq 1$ , so that  $b^{N+1} h > 1$ . Now

$$\begin{aligned} |P| &\leq \sum_1^N b^{-n\alpha} \cdot 1 \cdot b^n h = O\{h \cdot (b^N)^{1-\alpha}\} = O\{h \cdot h^{\alpha-1}\} = O(h^\alpha), \\ |Q| &\leq \sum_{N+1}^\infty b^{-n\alpha} \cdot 1 \cdot 2 = O(b^{-(N+1)\alpha}) = O(h^\alpha). \end{aligned}$$

Hence  $P + Q = O(h^\alpha)$  uniformly in  $x$ , and  $f \in \Lambda_\alpha$ .

In order to show that  $f_1 \in \Lambda_\alpha$ , we write

$$\begin{aligned} f_1(x+h) + f_1(x-h) - 2f_1(x) &= - \sum b^{-n} \cos b^n x (2 \sin \frac{1}{2}b^n h)^2 \\ &= - \sum_1^N - \sum_{N+1}^\infty = R + T, \end{aligned}$$

with the same  $N$  as before. Then

$$\begin{aligned} |R| &\leq h^2 \sum_1^N b^n = h^2 O(b^N) = h^2 O(h^{-1}) = O(h), \\ |T| &\leq \sum_{N+1}^\infty b^{-n} \leq b^{-N} = O(h), \end{aligned}$$

so that  $R + T = O(h)$  and  $f_1 \in \Lambda_\alpha$ . That  $f_1 \notin \Lambda_1$  follows from the fact that otherwise  $S'[f_1]$  would be a Fourier series and the coefficients of  $S'[f_1]$  would tend to zero, which is not the case.

Minor changes in the preceding argument give the following result:

(4.10) THEOREM. Let  $\epsilon_n \rightarrow 0$  and

$$g(x) = g_\alpha(x) = \sum_1^\infty \epsilon_n b^{-n\alpha} \cos b^n x. \tag{4.11}$$

Then  $g_\alpha \in \lambda_\alpha$  for  $0 < \alpha < 1$ , and  $g_1 \in \lambda_\alpha$ .

Weierstrass showed that for  $\alpha$  small enough the function  $f_\alpha(x)$  is nowhere differentiable. The extension to  $\alpha \leq 1$  was first proved by Hardy. (For  $\alpha > 1$ ,  $f'(x)$  clearly exists and is continuous, since  $S'[f]$  then converges absolutely and uniformly.)

$f_1$  is an example of a function of class  $\Lambda_*$  which is nowhere differentiable. On account of (4.10) and (3.3),  $g_1(x)$  is differentiable in a set of the power of the continuum in every interval. As we shall see in Chapter V, p. 206, if  $\sum \epsilon_n^2 = \infty$  (for example if  $\epsilon_n = n^{-1}$ ), then  $g_1$  is differentiable in a set of measure zero only. Thus, *smooth functions may be non-differentiable almost everywhere.*

If we write (4.8) in the form  $\sum a_k \cos kx$  then  $a_k = O(k^{-\alpha})$ , and for  $k = b^n$  this is the exact estimate. This shows that the results of (4.7) (i) cannot be improved.

(4.12) THEOREM. Let  $F(x)$  be a function of bounded variation over  $0 \leq x \leq 2\pi$ , and let  $C_\nu$  and  $c_\nu$  be the coefficients of  $F$  and  $dF$  respectively. If  $V$  is the total variation of  $F$  over  $(0, 2\pi)$ , then

$$|C_\nu| \leq \frac{V}{\pi |\nu|} \quad (\nu \neq 0), \quad |c_\nu| \leq \frac{V}{2\pi}. \quad (4.13)$$

The second inequality follows from the formula

$$2\pi |c_\nu| = \left| \int_0^{2\pi} e^{-i\nu x} dF(x) \right| \leq \int_0^{2\pi} |dF(x)| = V.$$

Integrating by parts, we see that

$$2\pi C_\nu = \int_0^{2\pi} e^{-i\nu x} F(x) dx = \frac{F(2\pi) - F(0)}{-i\nu} + \frac{2\pi c_\nu}{i\nu}$$

for  $\nu \neq 0$ , and the last sum is absolutely  $\leq 2V/|\nu|$ .

Thus the coefficients of a function of bounded variation are  $O(1/\nu)$ . The example of the series  $\sum \nu^{-1} \sin \nu x$  (see Chapter I, (4.12)) shows that we cannot replace 'O' by 'o' here. The function in this example is, however, discontinuous; examples of *continuous* functions of bounded variation with coefficients not  $o(1/\nu)$  are much less obvious and will be given later (see, for example, Chapter V, §§ 3 and 7).

Consider the Fourier sine series  $\sum b_\nu \sin \nu x$  of a function  $f(x)$  defined in  $(0, \pi)$ . For the existence of the coefficients

$$b_\nu = \frac{2}{\pi} \int_0^\pi f \sin \nu x dx,$$

it is not necessary to suppose that  $f$  is integrable over  $(0, \pi)$ ; it is enough to assume the integrability of  $f \sin x$ , for then  $f \sin \nu x$  is also integrable. In this case we shall call our series a *generalized Fourier sine series*. For example, we have, in this sense,

$$\frac{1}{2} \cot \frac{1}{2} x \sim \sin x + \sin 2x + \dots + \sin nx + \dots, \quad (4.14)$$

a relation suggested by making  $r \rightarrow 1$  in the formula for  $\sum r^\nu \sin \nu x$  (see Chapter I, § 1). We have only to verify that the numbers

$$\beta_\nu = \frac{2}{\pi} \int_0^\pi \frac{1}{2} \cot \frac{1}{2} x \sin \nu x dx$$

satisfy the relations  $\beta_1 = 1$ ,  $\beta_\nu - \beta_{\nu+1} = 0$  for  $\nu = 1, 2, \dots$ , so that  $\beta_1 = \beta_2 = \dots = 1$ .

This example shows that the *generalized Fourier sine coefficients need not tend to 0.*

They are, however,  $o(\nu)$ . For  $b_{\nu+1} - b_{\nu-1}$  is the  $\nu$ th cosine coefficient of the integrable function  $2f(x) \sin x$ , and so tends to 0. Hence if, for example,  $\nu$  is odd,

$$b_\nu = b_1 + (b_3 - b_1) + \dots + (b_\nu - b_{\nu-2}) = o(\nu);$$

and the same argument holds for  $\nu$  even.

The following result both generalizes and illuminates the Riemann-Lebesgue theorem.

(4.15) THEOREM. Let  $\alpha(x)$  be integrable,  $\beta(x)$  bounded, both periodic. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha(x) \beta(nx) dx \rightarrow \frac{1}{2\pi} \int_0^{2\pi} \alpha(x) dx \frac{1}{2n} \int_0^{2n} \beta(x) dx \quad \text{as } n \rightarrow \infty. \quad (4.16)$$

Observe that if, for every  $\epsilon > 0$ , we have  $\alpha = \alpha_1 + \alpha_2$  with  $\mathfrak{M}[\alpha_1] < \epsilon$  and with the relation (4.16) holding for  $\alpha_2$  and each bounded  $\beta$ , then (4.16) is true. Now (4.16) is certainly true if  $\alpha$  is the characteristic function of an interval and so, more generally, a step function. If  $\alpha$  is integrable, we set  $\alpha = \alpha_1 + \alpha_2$ , where  $\alpha_2$  is a step function and  $\mathfrak{M}[\alpha_1]$  small.

The Riemann-Lebesgue theorem is the special cases  $\beta = e^{-iz}$ . As the above proof shows, (4.16) holds if we replace  $\beta(nx)$  by  $\beta(nx + \theta_n)$ , where  $\theta_n$  are arbitrary numbers. In this, moreover,  $n$  may tend to infinity by continuous values.

### 5. Formulae for partial sums of $S[f]$ and $\tilde{S}[f]$

Given an integrable and periodic  $f$ , let

$$a_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos \nu t dt, \quad b_\nu = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin \nu t dt, \quad (5.1)$$

so that  $\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$ ,  $\sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x)$

are  $S[f]$  and  $\tilde{S}[f]$  respectively. The partial sums of  $S[f]$  will be denoted by  $S_n[f]$ , or by  $\mathcal{S}_n(x; f)$ , or simply by  $S_n(x)$ ; those of  $\tilde{S}[f]$  by  $\tilde{S}_n[f]$ ,  $\tilde{\mathcal{S}}_n(x; f)$ , or  $\tilde{S}_n(x)$ . Using (5.1), we have

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{\nu=1}^n \cos \nu(t-x) \right\} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t-x) dt,$$

$$\tilde{S}_n(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \sum_{\nu=1}^n \sin \nu(t-x) \right\} dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \tilde{D}_n(t-x) dt,$$

where  $D_n(v) = \frac{1}{2} + \sum_{\nu=1}^n \cos \nu v = \sin(n + \frac{1}{2})v / 2 \sin \frac{1}{2}v$ ,

$$\tilde{D}_n(v) = \sum_{\nu=1}^n \sin \nu v = \{\cos \frac{1}{2}v - \cos(n + \frac{1}{2})v\} / 2 \sin \frac{1}{2}v$$

(cf. Chapter I, § 1). The polynomials  $D_n(v)$  and  $\tilde{D}_n(v)$  are called *Dirichlet's kernel* and *Dirichlet's conjugate kernel* respectively. The formulae for  $S_n$  and  $\tilde{S}_n$  may also be written

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) D_n(u) du, \quad \tilde{S}_n(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+u) \tilde{D}_n(u) du.$$

Sometimes there is a slight advantage in taking the last term in  $S_n$  or  $\tilde{S}_n$  with a

factor  $\frac{1}{2}$ . The new expressions will be called the *modified partial sums*, and will be denoted by  $S_n^*$  and  $\bar{S}_n^*$  respectively. Thus

$$S_n^*(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n-1} (a_\nu \cos \nu x + b_\nu \sin \nu x) + \frac{1}{2}(a_n \cos nx + b_n \sin nx) = \frac{1}{2}(S_n(x) + S_{n-1}(x)),$$

and  $\bar{S}_n^*$  is defined similarly. If we set

$$\left. \begin{aligned} D_n^*(v) &= D_n(v) - \frac{1}{2} \cos nv = \sin nv / 2 \tan \frac{1}{2}v, \\ \bar{D}_n^*(v) &= \bar{D}_n(v) - \frac{1}{2} \sin nv = (1 - \cos nv) / 2 \tan \frac{1}{2}v, \end{aligned} \right\} \quad (5.2)$$

and proceed as before, we get

$$S_n^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n^*(t) dt, \quad \bar{S}_n^*(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \bar{D}_n^*(t) dt. \quad (5.3)$$

By (4.4),  $S_n - S_n^*$  tends uniformly to 0;  $S_n^*$  and  $S_n$  are equivalent with regard to convergence, and  $S_n^*$  is slightly the simpler. Similarly for  $\bar{S}_n, \bar{S}_n^*$ . We call  $D_n^*$  the *modified Dirichlet kernel*,  $\bar{D}_n^*$  the *modified conjugate Dirichlet kernel*.

With a fixed  $f$  and a fixed point  $x$  we set

$$\begin{aligned} \phi(t) &= \phi_x(t) = \phi_x(t; f) = \frac{1}{2}(f(x+t) + f(x-t) - 2f(x)), \\ \psi(t) &= \psi_x(t) = \psi_x(t; f) = \frac{1}{2}(f(x+t) - f(x-t)), \end{aligned}$$

and we shall adhere throughout the book to this notation.

The polynomial

$$D_n^*(u) = \frac{1}{2} + \cos u + \dots + \frac{1}{2} \cos nu$$

is even, and integrating it term by term we see that

$$\begin{aligned} \int_{-\pi}^{\pi} D_n^*(t) dt &= \pi. \\ \text{Hence} \quad S_n^*(x) - f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n^*(t) dt - \frac{f(x)}{\pi} \int_{-\pi}^{\pi} D_n^*(t) dt \\ &= \frac{2}{\pi} \int_0^{\pi} \phi_x(t) D_n^*(t) dt = \frac{2}{\pi} \int_0^{\pi} \frac{\phi_x(t)}{2 \tan \frac{1}{2}t} \sin nt dt. \end{aligned} \quad (5.4)$$

$\bar{D}_n^*(u)$  being odd, we similarly get

$$\bar{S}_n^*(x) = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \bar{D}_n^*(t) dt = -\frac{2}{\pi} \int_0^{\pi} \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} (1 - \cos nt) dt. \quad (5.5)$$

For future reference we also state the following formulae:

$$\left. \begin{aligned} S_n(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t) dt = \frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)] D_n(t) dt, \\ S_n(x) - f(x) &= \frac{2}{\pi} \int_0^{\pi} \phi_x(t) D_n(t) dt = \frac{2}{\pi} \int_0^{\pi} \phi_x(t) \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt, \\ \bar{S}_n(x) &= -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \bar{D}_n(t) dt = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt. \end{aligned} \right\} \quad (5.6)$$

Our main task in this chapter will be to show that, subject to suitable conditions on  $f$  at the point  $x$ ,  $S_n(x; f)$ , or, what amounts to the same thing,  $S_n^*(x; f)$ , tends to  $f(x)$

as  $n \rightarrow \infty$ . The summation problem for the conjugate series  $\tilde{S}[f]$  leads us to consider the expression

$$-\frac{2}{\pi} \int_0^\pi \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} dt = -\frac{1}{\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan \frac{1}{2}t} dt, \tag{5.7}$$

where the integral is meant as the limit (if it exists) of

$$-\frac{2}{\pi} \int_\epsilon^\pi \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} dt \tag{5.8}$$

for  $\epsilon \rightarrow +0$ . The value of the expression (5.7), wherever it exists, will be denoted by  $\tilde{f}(x)$ , and the function  $\tilde{f}(x)$  will be said to be *conjugate* to  $f(x)$ . The expression (5.8) will be denoted by  $\tilde{f}(x; \epsilon)$ . We show later (see Chapter IV, § 3 and Chapter VII, § 1) that for any integrable  $f$  the function  $\tilde{f}$  exists almost everywhere; but the proof of this is far from simple.

The expression (5.7) can also be written

$$-\frac{1}{\pi} \int_{-\pi}^\pi \frac{f(x+t)}{2 \tan \frac{1}{2}t} dt \quad \text{or} \quad -\frac{1}{\pi} \int_{-\pi}^\pi \frac{f(t)}{2 \tan \frac{1}{2}(t-x)} dt,$$

where the integrals are taken in the 'principal value' sense, that is are the limits, for  $\epsilon \rightarrow +0$ , of integrals taken over the complements of intervals of length  $2\epsilon$  around the point of non-integrability of the integrand ( $t=0$  in the first case and  $t=x$  in the second).

From (5.5) we get formally

$$\tilde{S}_n^*(x) - f(x) = \frac{2}{\pi} \int_0^\pi \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} \cos nt dt. \tag{5.9}$$

There is an analogy between this integral and the last integral in (5.4), though the latter always converges, even absolutely, whereas in (5.9) both  $\tilde{f}(x)$  and the right-hand side may not exist at some points. We shall see later that to a theorem on the convergence (or summability) of  $S[f]$  there usually corresponds an analogous theorem for  $\tilde{S}[f]$ .

We record some inequalities useful in 'convergence theory':

$$|D_n^*(t)| < n, \quad |D_n^*(t)| \leq 1/t \quad (0 < t \leq \pi; n = 1, 2, \dots). \tag{5.10}$$

For  $|D_n^*(t)| < \frac{1}{2} + 1 + \dots + \frac{1}{2} = n$ ; and the second estimate follows from (5.2), since  $2 \tan \frac{1}{2}t \geq t$ . The first inequality (5.10) is preferable for  $t$  not too large in comparison with  $1/n$ , for example for  $0 < t \leq \pi/n$ , the second for larger  $t$ 's. Similarly

$$|D_n^*(t)| < n, \quad |D_n^*(t)| \leq 2/t. \tag{5.11}$$

Analogous inequalities hold for  $D_n$  and  $\tilde{D}_n$ .

With the notation of § 1 of Chapter I we have easily

$$\begin{aligned} \frac{1}{2}[f(x_0+t) + f(x_0-t)] &\sim \sum_0^\infty A_n(x_0) \cos nt, \\ \frac{1}{2}[f(x_0+t) - f(x_0-t)] &\sim -\sum_1^\infty B_n(x_0) \sin nt. \end{aligned}$$

Thus  $S[f]$  at  $x=x_0$  is the same as the Fourier series at  $t=0$  of the even function  $\frac{1}{2}[f(x_0+t) + f(x_0-t)]$ ;  $\tilde{S}[f]$  at  $x=x_0$  is the series conjugate to the Fourier series at  $t=0$  of the odd function  $\frac{1}{2}[f(x_0+t) - f(x_0-t)]$ .

## 6. The Dini test and the principle of localization

(6.1) THEOREM. *If the first of the integrals*

$$\int_0^\pi \frac{|\phi_x(t)|}{2 \tan \frac{1}{2}t} dt, \quad \int_0^\pi \frac{|\psi_x(t)|}{2 \tan \frac{1}{2}t} dt \quad (6.2)$$

*is finite, then  $S[f]$  converges at the point  $x$  to sum  $f(x)$ . If the second integral is finite, then  $\tilde{f}(x)$  exists and  $\tilde{S}[f]$  at  $x$  converges to  $\tilde{f}(x)$ .*

The formulæ (5.4) and (5.9) display the fundamental fact that, formally at least,  $S_n^*(x) - f(x)$  and  $\tilde{S}_n^*(x) - \tilde{f}(x)$  are the sine and cosine coefficients of certain functions. In each of the cases the function concerned is, by hypothesis, integrable, and in the second case  $\tilde{f}(x)$  exists. Thus, by (4.4), we have respectively

$$S_n^*(x) - f(x) \rightarrow 0, \quad \tilde{S}_n^*(x) - \tilde{f}(x) \rightarrow 0.$$

The first part of (6.1) is called the *Dini test* for the convergence of  $S[f]$ . The second part is due to Pringsheim. Since  $2 \tan \frac{1}{2}t \simeq t$  as  $t \rightarrow 0$ , the finiteness of the integrals (6.2) is equivalent to that of

$$\int_0^\pi \frac{|\phi_x(t)|}{t} dt, \quad \int_0^\pi \frac{|\psi_x(t)|}{t} dt.$$

Both integrals are finite if, for example,

$$f(x+t) - f(x) = O(|t|^\alpha) \quad (\alpha > 0)$$

as  $t \rightarrow 0$ , and in particular if  $f'(x)$  exists and is finite. The first integral converges even if  $f$  is discontinuous at  $x$ , provided that  $\phi_x(t)$  tends to 0 sufficiently rapidly. The second integral diverges if  $f(x \pm 0)$  exist and are different, and we shall see later that  $\tilde{S}[f]$  always diverges at such points.

(6.3) THEOREM. *If  $f(x)$  vanishes in an interval  $I$ , then  $S[f]$  and  $\tilde{S}[f]$  converge uniformly in every interval  $I'$  interior to  $I$ , and the sum of  $S[f]$  there is 0.*

If the word 'uniformly' is omitted, (6.3) is a corollary of (6.1). For if  $x \in I'$ , both  $\phi_x(t)$  and  $\psi_x(t)$  vanish for small  $|t|$  and the integrals (6.2) converge. To prove the general result, we need the following lemma:

(6.4) LEMMA. *Let  $f$  be integrable,  $g$  bounded, and both periodic. Then the Fourier coefficients of the function  $\chi(t) = f(x+t)g(t)$  tend to 0 uniformly in the parameter  $x$ .*

By the second inequality in (4.1) it is enough to show that  $\omega_1(\delta; \chi) \rightarrow 0$  uniformly in  $x$ . Now

$$\begin{aligned} \int_{-\pi}^\pi |\chi(t+h) - \chi(t)| dt &\leq \int_{-\pi}^\pi |f(x+t+h) - f(x+t)| |g(t+h)| dt \\ &\quad + \int_{-\pi}^\pi |f(x+t)| |g(t+h) - g(t)| dt = P + Q, \end{aligned}$$

say. Suppose that  $|g| < M$ ,  $|h| \leq \delta$ . Then

$$P \leq M\omega_1(\delta; f) \rightarrow 0.$$

In order to show that  $Q \rightarrow 0$ , we set  $f = f_1 + f_2$ , where  $f_1$  is bounded, say  $|f_1| \leq B$ , and

$$\int_{-\pi}^{\pi} |f_2| dt < \epsilon/4M. \text{ Then}$$

$$Q = \int_{-\pi}^{\pi} |f_1(x+t)| |g(t+h) - g(t)| dt + \int_{-\pi}^{\pi} |f_2(x+t)| |g(t+h) - g(t)| dt \leq B\omega_1(\delta; g) + \frac{1}{2}\epsilon.$$

and so is less than  $\epsilon$  for  $\delta$  small enough. This proves (6.4).

Returning to (6.3), let  $x \in I'$ . Then  $f(x+t) = 0$  for  $|t| < \eta$ , say. Let  $\lambda(t)$  be the periodic function equal to 0 for  $|t| < \eta$  and to 1 elsewhere. Using (5.3) and (5.2), we write

$$S_n^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\lambda(t)}{2 \tan \frac{1}{2}t} \sin nt dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) g(t) \sin nt dt. \tag{6.5}$$

Here  $g(t) = \lambda(t)/2 \tan \frac{1}{2}t$  is bounded. By (6.4),  $S_n^*(x)$  tends uniformly to 0 for  $x \in I'$ .

Similarly  $S_n^*(x)$  tends uniformly to  $f(x)$  in  $I'$ ; for the difference  $S_n^*(x) - f(x)$  is represented by (6.5) with  $\sin nt$  replaced by  $\cos nt$ .

The result may be stated differently. Let us call two series  $u_0 + u_1 + \dots$  and  $v_0 + v_1 + \dots$  (convergent or not) *equiconvergent* if the difference  $(u_0 - v_0) + (u_1 - v_1) + \dots$  converges to 0. If this difference converges, but not necessarily to 0, the two series will be called *equiconvergent in the wider sense*. It is clear what 'uniform equiconvergence' means. The theorem that follows is a consequence of (6.3) when we set  $f = f_1 - f_2$ .

(6.6) THEOREM. *If two functions  $f_1$  and  $f_2$  are equal in an interval  $I$ , then  $S[f_1]$  and  $S[f_2]$  are uniformly equiconvergent in any interval  $I'$  interior to  $I$ ;  $\tilde{S}[f_1]$  and  $\tilde{S}[f_2]$  are uniformly equiconvergent in  $I'$  in the wider sense.*

Considering for simplicity convergence at a single point, we see that the convergence of  $S[f]$  and  $\tilde{S}[f]$ , and the sum of  $S[f]$  (but not that of  $\tilde{S}[f]$ ) at a point  $x$ , depend only on the behaviour of  $f$  in an arbitrarily small neighbourhood of  $x$ .

Theorems (6.3) and (6.6) express the *Riemann-Lebesgue localization principle*.

(6.7) THEOREM. (i) *Let  $f(x)$  be integrable,  $\rho(x)$  bounded, both periodic. If at a point  $x_0$  the Dini numbers of  $\rho$  are bounded, the series  $S[\rho f]$  and  $\rho(x_0) S[f]$  are equiconvergent for  $x = x_0$ . The series  $\tilde{S}[\rho f]$  and  $\rho(x_0) \tilde{S}[f]$  are equiconvergent at  $x_0$  in the wider sense.*

(ii) *If  $\rho(x) \in \Lambda_1$ , the equiconvergence of  $S[\rho f]$  and  $\rho(x_0) S[f]$ , and that (in the wider sense) of  $\tilde{S}[\rho f]$  and  $\rho(x_0) \tilde{S}[f]$ , is uniform in  $x_0$ .*

If  $\rho(x_0) = 1$ , case (i) may be interpreted as follows: 'slight' modifications of  $f$  in the neighbourhood of  $x_0$  which leave  $f(x_0)$  unaltered have no influence either upon the convergence of  $S[f]$  and  $\tilde{S}[f]$  at  $x_0$ , or on the sum of  $S[f]$  at that point (though they can influence the sum of  $\tilde{S}[f]$ ).

To prove (i), we observe that

$$S_n^*(x_0; \rho f) - \rho(x_0) S_n^*(x_0; f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+t) g(t) \sin nt dt.$$

where

$$g(t) = g_{x_0}(t) = [\rho(x_0+t) - \rho(x_0)]/2 \tan \frac{1}{2}t$$

is a bounded function. Hence the integral on the right, being the Fourier coefficient

of an integrable function, tends to 0 with  $1/n$ . For  $S_n^*(x_0; \rho f) - \rho(x_0) S_n^*(x_0; f)$  we have the value

$$\begin{aligned} -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\rho(x_0+t) - \rho(x_0)}{2 \tan \frac{1}{2}t} f(x_0+t) (1 - \cos nt) dt \\ = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+t) g(t) dt + \frac{1}{\pi} \int_{-\pi}^{\pi} f(x_0+t) g(t) \cos nt dt, \end{aligned}$$

and the last integral tends to 0. This proves (i).

Let us set

$$\chi(t) = \chi_x(t) = f(x+t) g_x(t).$$

The above argument and (4.1) will give (ii), if we can prove that  $\omega_1(\delta; \chi) \rightarrow 0$  uniformly in  $x$  as  $\delta \rightarrow 0$ . Arguing as in the proof of (6.4), and observing that  $|g_x(t)| < M$ , say, we have only to show that the integral  $\int_{-\pi}^{\pi} |g_x(t+h) - g_x(t)| dt$  tends to 0 with  $h$ , uniformly in  $x$ . We break up the interval of integration into two parts: the interval  $|t| \leq \epsilon/8M$ , and the remainder of  $(-\pi, \pi)$ . The first integral does not exceed  $2M \cdot 2\epsilon/8M = \frac{1}{2}\epsilon$ . Outside the first interval the function  $g_x(t)$  is continuous in  $t$ , uniformly in  $x$ , so that the second integral tends to 0 with  $h$ , uniformly in  $x$ . The whole is thus less than  $\epsilon$  for small  $|h|$ , and this completes the proof. For the conjugate series we argue similarly.

Theorem (6.7) includes (6.3). For let  $\rho(x)$  denote the continuous function which is equal to 0 in  $I'$ , is equal to 1 outside  $I$ , and is linear elsewhere. For  $x_0 \in I'$  we have  $\rho(x_0) S[f] = 0$ , and since  $S[\rho f] = S[f]$ , (6.7) implies that  $S[f]$  and  $\tilde{S}[f]$  converge uniformly in  $I'$ , the sum of  $S[f]$  being 0 there.

The analogue of (6.1) for uniform convergence is as follows.

**(6.8) THEOREM.** *Suppose that  $f$  is continuous in a closed interval  $I = (a, b)$  and let  $\omega(\delta)$  be its modulus of continuity there. If  $\omega(\delta)/\delta$  is integrable near  $\delta = 0$ , and if the integrals*

$$\int_0^{\pi} \frac{|f(a) - f(a-t)|}{t} dt, \quad \int_0^{\pi} \frac{|f(b+t) - f(b)|}{t} dt,$$

*are finite, then both  $S[f]$  and  $\tilde{S}[f]$  converge uniformly in  $I$ , to  $f$  and  $\bar{f}$  respectively.*

For let  $\xi(t)$  be the sum of the numbers

$$\omega(t), \quad |f(a) - f(a-t)|, \quad |f(b+t) - f(b)| \quad \text{for } 0 \leq t \leq h = b - a.$$

The function  $\xi(t)/t$  is integrable. Write

$$S_n^*(x) - f(x) = \frac{1}{\pi} \left\{ \int_{|t| < \sigma} + \int_{\sigma < |t| < \pi} \right\} [f(x+t) - f(x)] D_n^*(t) dt = P + Q, \quad (6.9)$$

say, where  $0 < \sigma \leq h$ , and consider first the term  $P$ . Let  $x \in I$ . If  $x+t$  is in  $I$ , then  $|f(x+t) - f(x)| \leq \omega(|t|)$ . If  $x+t$  is not in  $I$ , say  $x+t > b$ , then

$$|f(x+t) - f(x)| \leq |f(b) - f(x)| + |f(t+x) - f(b)| \leq \omega(t) + |f(t+x) - f(b)|,$$

and since  $|D_n^*(t)| \leq |t|^{-1}$  it is easy to see that

$$|P| \leq \frac{2}{\pi} \int_0^{\sigma} \frac{\xi(t)}{t} dt < \epsilon,$$

provided  $\sigma$  is small enough. Since  $Q$  is the Fourier coefficient of the function  $\{f(x+t) - f(x)\} g(t)$ , where  $g(t)$  is 0 in  $(-\sigma, \sigma)$  and  $\frac{1}{2} \cot \frac{1}{2}t$  outside, we see from (6.4) that  $Q \rightarrow 0$  uniformly in  $I$ . Hence  $S_n^*(x) \rightarrow f(x)$  uniformly in  $I$ .

With the hypotheses of (6.8), the integral defining  $f(x)$  converges absolutely and uniformly in  $I$ , since

$$\int_0^\sigma \frac{|\psi_x(t)|}{2 \tan \frac{1}{2}t} dt \leq \int_0^\sigma \frac{\xi(t)}{t} dt \quad (0 < \sigma \leq h).$$

In particular,  $f(x)$  is continuous in  $I$ . An argument similar to that above shows that  $S_n^*(x) - f(x) \rightarrow 0$  uniformly in  $I$ .

(6.10) THEOREM. *If  $f \in L$ ,  $\rho \in \Lambda_1$ , the integrals*

$$\int_{-\pi}^\pi \rho(t) f(t) \frac{1}{2} \cot \frac{1}{2}(t-x) dt, \quad \int_{-\pi}^\pi \rho(x) f(t) \frac{1}{2} \cot \frac{1}{2}(t-x) dt, \quad (6.11)$$

*taken in the 'principal value' sense, are uniformly equiconvergent in the wider sense.*

This is immediate, since  $[\rho(t) - \rho(x)] \frac{1}{2} \cot \frac{1}{2}(t-x)$  is bounded in  $x, t$ .

### 7. Some more formulae for partial sums

Let  $\epsilon$  be a fixed positive number less than  $\pi$ . It is sometimes convenient to use the formulae

$$\left. \begin{aligned} S_n(x) &= \frac{1}{\pi} \int_{-\epsilon}^\epsilon f(x+t) \frac{\sin nt}{t} dt + o(1), \\ S_n(x) - f(x) &= \frac{2}{\pi} \int_0^\epsilon \phi_x(t) \frac{\sin nt}{t} dt + o(1). \end{aligned} \right\} \quad (7.1)$$

*In the former the  $o(1)$  term tends to 0 uniformly in  $x$ ; in the latter it tends to 0 for every  $x$  and uniformly in every interval where  $f$  is bounded.*

To prove the first formula we note that the difference between the integral on the right and the integral defining  $S_n^*(x)$  ( $= S_n(x) + o(1)$ ) is the sine coefficient of the function  $f(x+t)g(t)$ , where  $g(t)$  is the function equal to  $1/t - \frac{1}{2} \cot \frac{1}{2}t = O(1)$  for  $|t| < \epsilon$  and to  $-\frac{1}{2} \cot \frac{1}{2}t$  at the remaining points of  $(-\pi, \pi)$ . Similarly, the difference between the second integral and the one defining  $S_n^*(x) - f(x)$  is the sine coefficient of

$$\{f(x+t) - f(x)\}g(t),$$

and the second formula (7.1) follows.

We note also the formula

$$S_n(x) = -\frac{1}{\pi} \int_{-\epsilon}^\epsilon f(x+t) \frac{1 - \cos nt}{t} dt + R_n(x),$$

where  $R_n(x)$  tends uniformly to a continuous function of  $x$ .

It is instructive to compare this and the first formula (7.1) with the exact formulae

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{\sin \omega t}{t} dt = \sum'_{\nu < \omega} A_\nu(x), \quad (7.2)$$

$$-\frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{1 - \cos \omega t}{t} dt = \sum'_{\nu < \omega} B_\nu(x), \quad (7.3)$$

where  $\omega$  is positive but not necessarily an integer, the integrals are defined as  $\lim_{T \rightarrow +\infty} \int_{-T}^T$ , and the dash indicates that if  $\omega$  is an integer then the last term of the sum is taken with a factor  $\frac{1}{2}$ .

We take the first formula only, the proof of the second being analogous. The familiar equation

$$\frac{1}{\pi} \int_{-\sigma}^{+\sigma} \frac{\sin t}{t} dt = 1$$

(see (3.4) below) shows that

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin \lambda t}{t} dt = \text{sign } \lambda \quad (-\infty < \lambda < +\infty). \quad (7.4)$$

Hence if  $f(x) = e^{i\nu x}$ , the left-hand side of (7.2) is

$$\pi^{-1} e^{i\nu x} \lim_{T \rightarrow \infty} \int_{-T}^{+T} e^{itx} \frac{\sin \omega t}{t} dt = (2\pi)^{-1} e^{i\nu x} \int_{-\infty}^{+\infty} \left[ \frac{\sin(\nu + \omega)t}{t} + \frac{\sin(\nu - \omega)t}{t} \right] dt,$$

and the last integral is  $2\pi$ ,  $\pi$ ,  $0$  according as  $|\nu| < \omega$ ,  $|\nu| = \omega$ ,  $|\nu| > \omega$ . This proves the formula if  $f$  is a trigonometric polynomial. Hence we may assume that  $c_n = 0$  for  $|n| \leq \omega$ .

We now use a result which will be established in Chapter IV, n. 160, and which asserts that a Fourier series can be integrated termwise over any finite interval after having been multiplied by any function of bounded variation. Thus if  $\Sigma f] = \Sigma c_n e^{inx}$  we have

$$\frac{1}{\pi} \int_{-T}^{+T} f(x+t) \frac{\sin \omega t}{t} dt = \Sigma c_n e^{inx} \frac{1}{\pi} \int_0^T \left[ \frac{\sin(\nu + \omega)t}{t} + \frac{\sin(\nu - \omega)t}{t} \right] dt \quad (7.5)$$

Integrating by parts twice we get

$$\int_u^\infty \frac{\sin t}{t} dt = \frac{\cos u}{u} + \frac{\sin u}{u^2} - 2 \int_u^\infty \frac{\sin t}{t^3} dt = \frac{\cos u}{u} + O\left(\frac{1}{u^2}\right) \quad (u > 0).$$

Since  $|\nu| > \omega$  and  $\int_0^{+T} = \int_0^\infty - \int_T^\infty$ , the sum in (7.5) is

$$\begin{aligned} & \frac{1}{\pi} \Sigma c_n e^{inx} \left[ \frac{\cos T(\nu - \omega)}{T(\nu - \omega)} - \frac{\cos T(\nu + \omega)}{T(\nu + \omega)} + O\left(\frac{1}{T^2 \nu^2}\right) \right] \\ &= \frac{1}{\pi T} \Sigma c_n e^{inx} \left[ \frac{\cos T(\nu - \omega)}{\nu} - \frac{\cos T(\nu + \omega)}{\nu} + O\left(\frac{1}{T^2}\right) \right] \\ &= \frac{2 \sin T\omega}{\pi T} \Sigma c_n \frac{e^{i\nu x}}{\nu} \sin \nu T + o(1) \end{aligned}$$

as  $T \rightarrow \infty$  (observe that  $\Sigma |c_n| \nu^{-2} < \infty$ ). If  $F$  denotes the integral of  $f$ ,  $F$  is bounded and periodic, and the penultimate term is  $(\pi T)^{-1} \sin T\omega [F(x+T) - F(x-T)] = o(1)$ . Hence (7.5) tends to 0 as  $T \rightarrow \infty$  and this completes the proof of (7.2).

The integral (7.4) converges uniformly in  $\lambda$  outside an arbitrarily small neighbourhood of  $\lambda = 0$ . From the preceding proof it follows that the integrals (7.2) and (7.3) converge uniformly over the set obtained by removing from any finite interval  $|\omega| \leq \Omega$  arbitrarily small neighbourhoods of the points  $0, \pm 1, \pm 2, \dots$  (The neighbourhoods must be removed, since the right-hand sides of the formulae are, in general, discontinuous at the points  $\nu$ .)

We have also the formula

$$f(x) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(x+t)}{t} dt = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \left\{ \lim_{T \rightarrow +\infty} \left( \int_{-T}^{-\epsilon} + \int_{\epsilon}^{+T} \right) \frac{f(x+t)}{t} dt \right\}. \quad (7.6)$$

valid at every point at which  $f(x)$  exists. (The internal limit always exists.) For if we subtract from  $f$  a constant, which changes nothing in (7.6), we may assume that the integral of  $f$  over a period is zero, and then the application of the second mean-value theorem gives the existence of each of the integrals

$$-\frac{1}{\pi} \int_{\pi}^{\infty} \frac{f(x+t)}{t} dt, \quad -\frac{1}{\pi} \int_{-\infty}^{-\pi} \frac{f(x+t)}{t} dt \tag{7.7}$$

separately. Their sum is

$$-\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left[ \sum_{k \neq 0} \frac{1}{t+2k\pi} \right] dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left[ \frac{1}{2} \cot \frac{1}{2}t - \frac{1}{t} \right] dt,$$

where the dash ' indicates the omission of  $k=0$  in the summation. This is

$$-\frac{1}{\pi} \left( \int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) \left[ \frac{1}{2} \cot \frac{1}{2}t - \frac{1}{t} \right] f(x+t) dt + o(1) = f(x; \epsilon) + \frac{1}{\pi} \int_{\epsilon < |t| < \pi} \frac{f(x+t)}{t} dt + o(1),$$

as  $\epsilon \rightarrow +0$ , and (7.6) easily follows.

### 8. The Dirichlet-Jordan test

This name is usually given to the following theorem (see also (5.14) below).

(8.1) THEOREM. Suppose that  $f(x)$  is of bounded variation over  $(0, 2\pi)$ . Then

(i) at every point  $x_0$ ,  $S_n[f]$  converges to the value  $\frac{1}{2}[f(x_0+0) + f(x_0-0)]$ ; in particular  $S_n[f]$  converges to  $f(x)$  at every point of continuity of  $f$ ;

(ii) if further  $f$  is continuous at every point of a closed interval  $I$ , then  $S_n[f]$  converges uniformly in  $I$ .

We prove first the following lemma, only part of which is needed here:

(8.2) LEMMA. The integrals

$$\frac{2}{\pi} \int_0^{\xi} D_n(t) dt, \quad \frac{2}{\pi} \int_0^{\xi} D_n^*(t) dt, \quad \frac{2}{\pi} \int_0^{\xi} \frac{\sin nt}{t} dt \quad (0 \leq \xi \leq \pi)$$

are all uniformly bounded in  $n$  and  $\xi$ . The difference between any two of these integrals tends to 0 with  $1/n$ , uniformly in  $\xi$ .

Let us denote these integrals respectively by  $\alpha_n(\xi)$ ,  $\beta_n(\xi)$ ,  $\gamma_n(\xi)$ . Plainly,  $\beta_n - \alpha_n$  is uniformly bounded and tends uniformly to 0 as  $n \rightarrow \infty$ . Furthermore,

$$\gamma_n - \beta_n = \frac{2}{\pi} \int_0^{\xi} \left\{ \frac{1}{t} - \frac{1}{2} \cot \frac{1}{2}t \right\} \sin nt dt = \frac{2}{\pi} \int_0^{\pi} w_{\xi}(t) \sin nt dt,$$

where  $w_{\xi}(t)$  is  $1/t - \frac{1}{2} \cot \frac{1}{2}t$  in  $(0, \xi)$  and 0 in  $(\xi, \pi)$ . Since the total variation of  $w_{\xi}(t)$  over  $(0, \pi)$  is uniformly bounded, the last integral is uniformly bounded and tends uniformly to 0 (see (4.13)). It is thus enough to show the boundedness of

$$\gamma_n(\xi) = \frac{2}{\pi} \int_0^{n\xi} \frac{\sin u}{u} du = \frac{2}{\pi} G(n\xi)$$

where

$$G(v) = \int_0^v \frac{\sin u}{u} du, \tag{8.3}$$

and this will follow if we show that  $G(v)$  tends to a limit as  $v \rightarrow +\infty$ . Since the integrand tends to 0 it is enough to prove the existence of  $\lim G(n\pi)$ . But  $\alpha_n(\pi) = 1$  and

$\alpha_n(\pi) - \gamma_n(\pi) \rightarrow 0$  together imply  $G(n\pi) \rightarrow \frac{1}{2}\pi$ , which proves (8.2). We have obtained incidentally the well-known formula

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{1}{2}\pi \quad (8.4)$$

Return now to (8.1), and apply the last remark of § 5. Replacing  $f(x)$  by

$$\frac{1}{2}[f(x_0 + x) + f(x_0 - x)],$$

we may assume that  $x_0 = 0$  and that  $f(x)$  is even. We have to show that  $S_n(0) \rightarrow f(+0)$ .

Suppose first that  $f(x)$  is non-negative and non-decreasing in  $(0, \pi)$ . Let  $C$  be a number greater than  $|\beta_n(\xi)|$  for all  $n$  and  $\xi$ . We write

$$S_n^*(0) - f(+0) = \frac{2}{\pi} \int_0^\pi [f(t) - f(+0)] D_n^*(t) dt = \frac{2}{\pi} \left( \int_0^\eta + \int_\eta^\pi \right) = A + B, \quad (8.5)$$

say, where  $\eta$  is so chosen that  $|f(\eta) - f(+0)| < \epsilon/4C$ . Since  $f(t) - f(+0)$  is non-negative and non-decreasing the second mean-value theorem gives

$$|A| = \left| \{f(\eta) - f(+0)\} \frac{2}{\pi} \int_\eta^\pi D_n^*(t) dt \right| \leq \frac{\epsilon}{4C} 2C = \frac{1}{2}\epsilon \quad (0 < \eta' < \eta).$$

For fixed  $\eta$ ,  $B$  is a sine coefficient of the function  $w_\eta(t)$  equal to 0 in  $(0, \eta)$  and to  $\{f(t) - f(+0)\} \frac{1}{2} \cot \frac{1}{2}t$  in  $(\eta, \pi)$ . Thus, by (4.13),

$$B \rightarrow 0, \quad |A + B| < \epsilon \quad \text{for } n > n_0, \quad S_n(0) \rightarrow f(+0).$$

In the general case  $f$  is, in  $(0, \pi)$ , the difference  $f_1 - f_2$  of two non-negative and non-decreasing functions (the positive and negative variations of  $f$ ). If we define  $f_1$  and  $f_2$  in  $(-\pi, 0)$  by the condition of evenness, the formula  $f = f_1 - f_2$  becomes valid in  $(-\pi, \pi)$  and the general result follows from the special case just proved.

Case (ii) follows from the argument just used if we note that the continuity of  $f$  in  $I$  implies the continuity of the positive and negative variations of  $f$  in that interval, and that all the estimates obtained above hold uniformly for  $x_0 \in I$ .

In Chapter III, § 3, we give a different proof of (8.1) that does not require the theorem on the continuity of the positive and negative variations.

A sequence of functions  $s_n(x)$  defined in the neighbourhood of  $x = x_0$  and converging for  $x = x_0$  (but not necessarily for  $x \neq x_0$ ) is said to converge uniformly at  $x_0$  to limit  $s$ , if to every  $\epsilon > 0$  there is a  $\delta = \delta(\epsilon)$  and a  $p = p(\epsilon)$  such that

$$|s_n(x) - s| < \epsilon \quad \text{for } |x - x_0| < \delta \quad \text{and } n > p.$$

An equivalent definition is that  $s_n(x_n) \rightarrow s$  for each sequence  $x_n \rightarrow x_0$ .

(8.6) THEOREM. *If  $f$  is of bounded variation,  $S[f]$  converges uniformly at every point of continuity of  $f$ .*

It is enough to consider the case when  $x_0 = 0$  and  $f$  is even and non-decreasing in  $(0, \pi)$ . A proof similar to that of (8.1)(i) shows that if  $n$  is large enough and  $x$  small enough and, e.g., positive then  $S_n(x)$  is arbitrarily close to  $f(+0) = f(0)$ . We omit the details since a simpler proof will be given in Chapter III, § 3.

(8.7) THEOREM. Let  $a_r, b_r$  be the coefficients of  $f$ , and let  $F(x)$  be the indefinite integral of  $f$ . Then

$$F(x) = \frac{1}{2}a_0x + C + \sum_{r=1}^{\infty} (a_r \sin vx - b_r \cos vx)/v, \tag{8.8}$$

the series on the right being uniformly convergent.

For the proof it is enough to observe that the right-hand side without the term  $\frac{1}{2}a_0x$  is  $S[F - \frac{1}{2}a_0x]$  (see (2.5)), and that  $F(x) - \frac{1}{2}a_0x$  is a continuous function of bounded variation.

It follows from (8.8) that

$$\int_{\alpha}^{\beta} f(x) dx = [\frac{1}{2}a_0x]_{\alpha}^{\beta} + \sum_{r=1}^{\infty} \left[ \frac{a_r \sin vx - b_r \cos vx}{v} \right]_{\alpha}^{\beta}$$

for every  $\alpha, \beta$ . Thus, if  $S[f]$  is integrated term by term over any interval  $(\alpha, \beta)$ , the resulting series converges to  $\int_{\alpha}^{\beta} f dx$ .

Putting  $x = 0$  in (8.8) we see that the series  $\sum b_r/v$  converges for any  $f$ . This may be false for  $\sum a_r/v$  (see Chapter V, (1.11)).

The following result is an analogue of (8.1)(i) for  $\tilde{S}[f]$ :

(8.9) THEOREM. If  $f(x)$  is of bounded variation, a necessary and sufficient condition for the convergence of  $\tilde{S}[f]$  at  $x$  is the existence of the integral

$$f(x) = -\frac{2}{\pi} \int_0^{\pi} \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} dt = \lim_{h \rightarrow +0} \left\{ -\frac{2}{\pi} \int_h^{\pi} \frac{\psi_x(t)}{2 \tan \frac{1}{2}t} dt \right\} = \lim_{h \rightarrow +0} f(x; h), \tag{8.10}$$

which represents then the sum of  $\tilde{S}[f]$ .

We first show that:

(8.11) LEMMA. If  $f$  is of bounded variation,  $S_n^*(x) - f(x; \pi/n)$  tends to 0 at every point of continuity of  $f$  and is bounded at every point of discontinuity.

Let  $\psi(t) = \frac{1}{2}[f(x_0 + t) - f(x_0 - t)]$ . Since  $\tilde{S}[f]$  at  $x_0$  is the same thing as  $\tilde{S}[\psi]$  at  $t = 0$  we may suppose that  $x_0 = 0$  and that  $f(x)$  is odd. Hence  $\psi_x(t) = f(t)$ . Let us also temporarily assume that  $f(x)$  is non-negative and non-decreasing in  $(0, \pi)$ . Then

$$S_n^*(0) - f(0; \pi/n) = -\frac{2}{\pi} \int_0^{\pi/n} f(t) D_n^*(t) dt + \frac{2}{\pi} \int_{\pi/n}^{\pi} \frac{f(t)}{2 \tan \frac{1}{2}t} \cos nt dt = A + B. \tag{8.12}$$

Suppose first that  $f$  is continuous at  $t = 0$ , i.e. that  $f(t) \rightarrow 0$  with  $t$ . Since  $|D_n^*| \leq n$ ,

$$|A| \leq \frac{2n}{\pi} \int_0^{\pi/n} f(t) dt \leq 2f(\pi/n) = o(1).$$

Given  $\epsilon > 0$ , we choose  $\eta$  such that  $f(\eta) < \epsilon$ , and write

$$B = \frac{2}{\pi} \left( \int_{\pi/n}^{\eta} + \int_{\eta}^{\pi} \right) = B^{\epsilon} + B^{\eta}.$$

say. Applying the second mean-value theorem twice, we get

$$|B^{\eta}| = \left| \frac{1}{\pi} \cot(\pi/2n) \int_{\pi/n}^{\eta} f(t) \cos nt dt \right| = \left| \frac{1}{\pi} \cot(\pi/2n) f(\eta') \int_{\pi/n}^{\eta} \cos nt dt \right| < \frac{1}{\pi} \frac{2n}{\pi} \epsilon \frac{2}{\pi}.$$

so that  $|B'| < \epsilon$ . Since  $B'$  is a Fourier coefficient, it tends to 0. Thus (8.12) tends to 0. We prove similarly that  $\lim_{h \rightarrow 0} f(+0) \neq 0$ , (8.12) is bounded.

When  $f$  is no longer assumed to be non-negative and non-decreasing in  $(0, \pi)$ , we decompose it into its positive and negative variations. These are continuous at  $t=0$  when  $f$  is, so that (8.11) is proved.

Now let  $\pi/(n+1) \leq h \leq \pi/n$ . In the general case of an  $f$  of bounded variation we have

$$\left| f\left(0; \frac{\pi}{n}\right) - f(0; h) \right| \leq \frac{2}{\pi} \int_{\pi/(n+1)}^{\pi/n} |f(t)| \frac{dt}{2 \tan \frac{1}{2}t} \leq \frac{2(n+1)}{\pi^2} \left( \frac{\pi}{n} - \frac{\pi}{n+1} \right) \sup |f(t)| = o(1)$$

as  $n \rightarrow \infty$ , and this, together with (8.11), proves (8.9).

If  $f$  has a jump at a point  $x$ , then obviously  $f(x; \pi/n) \rightarrow \pm \infty$ . Thus  $\tilde{S}[f]$  diverges at  $x$  to  $\pm \infty$ . This is also contained in the following more precise result, in which only the integrability of  $f$  is assumed.

(8.13) THEOREM. *If  $f(x_0 \pm 0)$  exist, and if  $f(x_0 + 0) - f(x_0 - 0) = l$ , then*

$$S_n(x_0)/\log n \rightarrow -l/\pi.$$

We may suppose that  $x_0 = 0$  and that  $f$  is odd. It is easy to verify the result for the function  $\phi(x)$  defined in Chapter I, (4.12), using the fact that the partial sums of the harmonic series are asymptotically equal to  $\log n$ . Subtracting  $(l/\pi)\phi(x)$  from  $f$  we obtain an odd function  $g$  continuous and vanishing at the origin, and it is enough to prove that  $S_n^*(0; g) = o(\log n)$ . For this purpose we write (cf. (5.11))

$$\begin{aligned} |S_n^*(0; g)| &\leq \frac{2}{\pi} \int_0^\pi |g(t)| |D_n^*(t)| dt = \frac{2}{\pi} \int_0^{\pi/n} + \frac{2}{\pi} \int_{\pi/n}^\pi \\ &\leq \frac{2n}{\pi} \int_0^{\pi/n} |g(t)| dt + \frac{4}{\pi} \int_{\pi/n}^\pi \frac{|g(t)|}{t} dt = o(1) + o(\log n) = o(\log n) \end{aligned}$$

(cf. Chapter I, (8.1)).

A corollary of (8.13) is that, if the Fourier coefficients  $a_n, b_n$  of  $f$  are  $o(1/n)$ ,  $f$  cannot have discontinuities of the first kind. For the hypothesis implies that

$$S_n(x) = o\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) = o(\log n).$$

In particular, if the Fourier coefficients of a function  $f$  of bounded variation are  $o(1/n)$ , the function  $f$  has only removable discontinuities. For  $f(x+0) = f(x-0)$  for every  $x$ , and by changing the values of  $f$  at the at most denumerable set of points where  $f$  is discontinuous, we can make  $f$  everywhere continuous.

(8.14) THEOREM. *Suppose that  $f$  is integrable and periodic, and of bounded variation in an interval  $I$ . Then  $S[f]$  converges to  $\frac{1}{2}[f(x+0) + f(x-0)]$  at every point  $x$  interior to  $I$ . If, in addition,  $f$  is continuous in  $I$ , the convergence is uniform in any interval interior to  $I$ . A necessary and sufficient condition for the convergence of  $\tilde{S}[f]$  at an  $x$  interior to  $I$  is the existence of the integral  $\int f(x)$ , which represents then the sum of  $\tilde{S}[f]$ .*

For we can modify  $f$  by making it equal to 0 outside  $I$ . The new function is of bounded variation, and it is enough to combine (8.1) and (8.9) with (8.6).

9. Gibbs's phenomenon

We now study the partial sums  $s_n(x)$  of the special series

$$\sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu} = \frac{1}{2}(\pi - x) = \phi(x) \quad (0 < x < 2\pi) \tag{9.1}$$

in the neighbourhood of  $x = 0$ . The series cannot converge uniformly there since  $\phi(x)$  is discontinuous at  $x = 0$ . Supposing that  $x > 0$ , and using (8.2), we see that

$$s_n(x) + \frac{1}{2}x = \int_0^x D_n(t) dt = \int_0^{nx} \frac{\sin t}{t} dt + o(1),$$

uniformly in  $0 \leq x \leq \pi$ . Hence the  $s_n(x)$  are uniformly bounded, and

$$s_n(x) = \int_0^{nx} \frac{\sin t}{t} dt + R_n(x), \tag{9.2}$$

where  $|R_n(x)| < \epsilon$  if  $x < \epsilon$ ,  $n > n_0(\epsilon)$ .

Consider the integral (8.3). The integrals of  $(\sin t)/t$  over the intervals  $(k\pi, (k+1)\pi)$  decrease in absolute value and are of alternating sign when  $k = 0, 1, 2, \dots$ . This shows that the curve  $y = G(x)$  has a wave-like shape with maxima  $M_1 > M_2 > M_3 > \dots$  at  $\pi, 3\pi, 5\pi, \dots$  and minima  $m_2 < m_4 < m_6 < \dots$  at  $2\pi, 4\pi, \dots$ . Substituting  $x = \pi/n$  in (9.2), we get

$$s_n(\pi/n) \rightarrow G(\pi) > G(\infty) = \frac{1}{2}\pi.$$

Thus, though  $s_n(x)$  tends to  $\phi(x)$  at every fixed  $x$ ,  $0 < x < 2\pi$ , the curves  $y = s_n(x)$ , which pass through the point  $(0, 0)$ , condense to the interval  $0 \leq y \leq G(\pi)$  of the  $y$ -axis (cf. also (9.4), below), the ratio of whose length to that of the interval  $0 \leq y \leq \phi(+0) = \frac{1}{2}\pi$  is

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt = 1.179\dots$$

Similarly, to the left of  $x = 0$  the curves  $y = s_n(x)$  condense to the interval  $-G(\pi) \leq y \leq 0$ . This behaviour is called *Gibbs's phenomenon*, and its generalized form may be described as follows. Suppose that a sequence  $\{f_n(x)\}$  converges for  $x_0 < x \leq x_0 + h$ , say, to limit  $f(x)$  and that  $f(x_0 + 0)$  exists. Suppose that, when  $n \rightarrow \infty$  and  $x \rightarrow x_0$  independently, we have

$$\limsup f_n(x) > f(x_0 + 0) \quad \text{or} \quad \liminf f_n(x) < f(x_0 + 0);$$

then we say that  $\{f_n(x)\}$  shows Gibbs's phenomenon in the right-hand neighbourhood of  $x = x_0$ . Similarly for the left-hand neighbourhood. If  $f(x) = \lim f_n(x)$  is defined and continuous at  $x_0$ , the absence of the phenomenon at the point  $x_0$  is equivalent to the uniform convergence of  $\{f_n(x)\}$  at  $x_0$ .

**(9.3) THEOREM.** *If  $f$  is of bounded variation† and has no removable discontinuities,  $S[f]$  shows Gibbs's phenomenon at every point of discontinuity of  $f$  and only there.*

We may suppose that  $f$  has only regular discontinuities, i.e.

$$f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$$

for each  $x$ . Suppose that  $f(\xi + 0) - f(\xi - 0) = l \neq 0$ . The function

$$\Delta(x) = f(x) - \frac{l}{\pi} \phi(x - \xi)$$

† It is enough to assume that the coefficients of  $f$  are  $O(1/n)$ ; see Theorem (3.8) of Chapter III.

is continuous at  $\xi$ . Hence  $S[\Delta]$  converges uniformly at  $\xi$  (see (8.6)). The behaviour of  $S_n(x; f)$  near  $x$  is then effectively dominated by that of  $S_n[\pi^{-1}\phi(x - \xi)]$  so that  $S[f]$  will show Gibbs's phenomenon at  $x = \xi$ . If  $f$  is continuous at  $\xi$ ,  $S[f]$  converges uniformly at  $\xi$  and the phenomenon is absent.

(9.4) THEOREM. *The partial sums  $s_n(x)$  of the series (9.1) are strictly positive for  $0 < x < \pi$ .*

The theorem is true for  $n = 1$ . Suppose that it is established for  $n - 1$  and that  $s_n(x)$  has a non-positive minimum at a point  $x_0$ ,  $0 < x_0 < \pi$ . Since

$$s'_n(x_0) = D_n(x_0) - \frac{1}{2} = \{\sin(n + \frac{1}{2})x_0 - \sin \frac{1}{2}x_0\} / 2 \sin \frac{1}{2}x_0 = 0,$$

we infer that  $\sin(n + \frac{1}{2})x_0 = \sin \frac{1}{2}x_0$  and so also that  $|\cos(n + \frac{1}{2})x_0| = \cos \frac{1}{2}x_0$ . Hence

$$\sin nx_0 = \sin(n + \frac{1}{2})x_0 \cos \frac{1}{2}x_0 - \cos(n + \frac{1}{2})x_0 \sin \frac{1}{2}x_0 \geq 0.$$

It follows that  $s_n(x_0) - s_{n-1}(x_0) \geq 0$  and so  $s_{n-1}(x_0) \leq s_n(x_0) \leq 0$ , contrary to hypothesis.

## 10. The Dini-Lipschitz test

We know that  $S_n^*(x) - f(x)$  is formally a Fourier sine coefficient (see (5.4)). We may therefore apply to it the device which led to the estimate (4.2) for Fourier coefficients.

We fix  $x$  and take  $\phi(t) = \phi_x(t)$ ,  $\chi(t) = \phi(t) \frac{1}{2} \cot \frac{1}{2}t$ ,  $\eta = \pi/n$ .

$$\begin{aligned} \text{Then } n[S_n^*(x) - f(x)] &= 2 \int_0^\pi \chi(t) \sin nt dt = -2 \int_{-\eta}^{\pi-\eta} \chi(t + \eta) \sin nt dt \\ &= \int_0^\pi \chi(t) \sin nt dt - \int_{-\eta}^{\pi-\eta} \chi(t + \eta) \sin nt dt \\ &= \int_{\eta}^{\pi-\eta} \{\chi(t) - \chi(t + \eta)\} \sin nt dt + \int_{\pi-\eta}^{\pi} \chi(t) \sin nt dt \\ &\quad + \int_0^{\eta} \chi(t) \sin nt dt - \int_{-\eta}^{\eta} \chi(t + \eta) \sin nt dt. \end{aligned}$$

Denote the last four integrals by  $I_1, I_2, I_3, I_4$  respectively. Since  $|\sin nt \frac{1}{2} \cot \frac{1}{2}t| \leq n$ , we have

$$|I_3| + |I_4| \leq n \int_0^{\eta} |\phi(t)| dt + n \int_{-\eta}^{\eta} |\phi(t + \eta)| dt \leq 2n \int_0^{2\eta} |\phi(t)| dt.$$

For  $n \geq 2$  and  $t \in (\pi - \eta, \pi)$ ,

$$|\chi(t) \sin nt| \leq |\phi(t)| \leq \frac{1}{2} \{ |f(x+t)| + |f(x-t)| + 2|f(x)| \},$$

and since the indefinite integral is a continuous function we have  $I_2 = o(1)$ , and uniformly in every interval where  $f$  is bounded. Finally,  $|I_1|$  does not exceed

$$\int_{\eta}^{\pi-\eta} |\phi(t)| \left\{ \frac{1}{2 \tan \frac{1}{2}t} - \frac{1}{2 \tan \frac{1}{2}(t + \eta)} \right\} dt + \int_{\eta}^{\pi-\eta} \frac{|\phi(t) - \phi(t + \eta)|}{2 \tan \frac{1}{2}t} dt$$

The difference inside curly brackets here is  $\frac{1}{2} \sin \frac{1}{2}\eta / \sin \frac{1}{2}t \sin \frac{1}{2}(t + \eta) < \pi^2 \eta / 4t^2$ . Collecting results and observing that  $2 \tan \frac{1}{2}t \geq t$  in  $(0, \pi)$  we have

(10.1) THEOREM. *Let  $\eta = \pi/n$ . For every  $x$ ,  $|S_n^*(x) - f(x)|$  is majorized by*

$$\frac{1}{\pi} \int_{\eta}^{\pi} \frac{|\phi(t) - \phi(t + \eta)|}{t} dt + \eta \int_{\eta}^{\pi} \frac{|\phi(t)|}{t^2} dt + 2\eta^{-1} \int_0^{2\eta} |\phi(t)| dt + o(1), \quad (10.2)$$

the  $o(1)$  being uniform in every interval where  $f$  is bounded.

As a corollary we obtain the following:

(10·3) THEOREM OF DINI-LIPSCHITZ. If  $f$  is continuous and its modulus of continuity  $\omega(\delta)$  satisfies the condition  $\omega(\delta) \log \delta \rightarrow 0$  with  $\delta$ , then  $S[f]$  converges uniformly.

For since

$$|\phi(t) - \phi(t + \eta)| \leq \frac{1}{2} |f(x+t) - f(x+t+\eta)| + \frac{1}{2} |f(x-t) - f(x-t-\eta)| \leq \omega(\eta), \quad (10\cdot4)$$

the first term in (10·2) does not exceed  $\omega(\eta) \log n = o(1)$ . Similarly, since  $\phi(t) \rightarrow 0$  uniformly in  $x$ , the remaining terms in (10·2) tend uniformly to 0 (Chapter I, (8·1)).

(10·5) THEOREM. If the modulus of continuity of  $f$  in an interval  $I$  is  $o(|\log \delta|^{-1})$ , then  $S[f]$  converges uniformly in every interval interior to  $I$ .

For the continuous function coinciding with  $f$  on  $I$  and, say, linear outside  $I$  satisfies the hypothesis of (10·3), and it is enough to apply (8·6).

We shall see in Chapter VIII, § 2, that the condition

$$f(x_0 \pm t) - f(x_0) = o\left(\frac{1}{|\log t|}\right) \quad (t \rightarrow +0) \quad (10\cdot6)$$

does not ensure the convergence of  $S[f]$  at  $x_0$ , so that (10·5) is primarily a result about uniform convergence. However:

(10·7) THEOREM.  $S[f]$  converges at  $x_0$  to sum  $f(x_0)$  provided the condition (10·6) is satisfied and the coefficients of  $f$  are  $O(n^{-\delta})$  for some  $\delta > 0$ .

Without loss of generality we may suppose that  $x_0 = 0$ ,  $f$  is even,  $f(0) = 0$ . It is also convenient to have  $a_0 = 0$ , which may be achieved by subtracting  $\frac{1}{2}a_0(1 - \cos x)$  from  $S[f]$ . Finally, suppose that  $|a_n| \leq n^{-\delta}$  ( $0 < \delta < 1$ ) for  $n = 1, 2, \dots$ . We set  $r = \frac{1}{2}\delta$  and write

$$S_n^*(0) = \frac{2}{\pi} \int_0^\pi f(t) \frac{\sin nt}{2 \tan \frac{1}{2}t} dt = \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-r}} + \int_{n^{-r}}^\pi = P + Q + R.$$

Here  $P \rightarrow 0$  as  $n \rightarrow \infty$ , since  $f$  is continuous at  $t = 0$  and  $|D_n^*| \leq n$ . If

$$\epsilon(t) = \sup |f(u) \log u| \quad \text{for } 0 < u \leq t,$$

then

$$Q \leq \epsilon(n^{-r}) \int_{n^{-1}}^{n^{-r}} \frac{dt}{t \log(1/t)} = \epsilon(n^{-r}) \log 1/r = o(1),$$

and it only remains to prove that  $R \rightarrow 0$ . To this end we shall take for granted a result which will be established later (Chapter IV, Theorem (8·16)), namely that Fourier series can be integrated term by term after multiplication by any function of bounded variation. Then

$$R = \sum_{\nu=1}^\infty a_\nu \frac{2}{\pi} \int_{n^{-r}}^\pi \frac{\sin nt \cos \nu t}{2 \tan \frac{1}{2}t} dt.$$

We replace the products  $\sin nt \cos \nu t$  by differences of sines and apply the second mean-value theorem to the factor  $\frac{1}{2} \cot \frac{1}{2}t$ . We find that for  $\nu \neq n$  the factor of  $a_\nu$  does not exceed  $4n^\nu/\pi |\nu - n|$  in absolute value. The factor of  $a_n$  is bounded. Hence

$$|R| \leq o(1) + \frac{4}{\pi} \sum_{\nu=1}^\infty \frac{\nu^\delta n^\nu}{|\nu - n|} = o(1) + \sum_{\nu=1}^{n-1} + \sum_{\nu=n+1}^\infty = o(1) + R_1 + R_2,$$

say, where the dash indicates the omission of the term  $\nu = n$ . Now

$$\frac{\pi}{4} R_1 < \frac{n^r}{\frac{1}{2}n} \sum_{\nu=1}^{[n]} \nu^{-\delta} + n^r (\frac{1}{2}n)^{-\delta} \sum_{[n]+1}^{n-1} \frac{1}{n-\nu} = O(n^{-\frac{1}{2}\delta}) + O(n^{-\frac{1}{2}\delta} \log n) = o(1),$$

$$\frac{\pi}{4} R_2 < n^r \sum_{\nu=n+1}^{2n} \frac{1}{\nu-n} + n^r \sum_{2n+1}^{\infty} \frac{\nu^{-\delta}}{\frac{1}{2}\nu} = O(n^{-\frac{1}{2}\delta} \log n) + O(n^{-\frac{1}{2}\delta}) = o(1),$$

so that  $R \rightarrow 0$ , and this completes the proof.

A similar argument shows that, under the hypotheses of (10.7),

$$\tilde{S}_n(x_0) - \tilde{f}(x_0; \pi/n) \rightarrow 0.$$

The proof of the following theorem is very similar to that of (10.3).

**(10.8) THEOREM.** *If  $f \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , then  $S_n(x; f) - f(x) = O(n^{-\alpha} \log n)$ , uniformly in  $x$ .*

For now  $\phi(t) = O(t^\alpha)$ ,  $\phi(t + \eta) - \phi(t) = O(\eta^\alpha)$  (cf. (10.4)). The first term in (10.2) is  $O(n^{-\alpha} \log n)$ ; the second is  $O(n^{-\alpha})$  or  $O(n^{-1} \log n)$ , according as  $\alpha < 1$  or  $\alpha = 1$ ; the third is  $O(n^{-\alpha})$ . A glance back at the source of the fourth term shows that it is  $O(\eta) = O(n^{-\alpha})$ .

It can be shown by examples that the factor  $\log n$  in (10.8) cannot, in general, be omitted (see p. 315, Example 10). Suppose, however, that there is a constant  $C$  such that the function  $f(x) + Cx$  is monotone for all  $x$ . (The function  $f$  itself, being periodic, cannot be monotone, unless it is constant.) Such functions  $f$  will be called of *monotonic type*. We have now:

**(10.9) THEOREM.** *If  $f$  is of monotonic type and of class  $\Lambda_\alpha$ ,  $0 < \alpha < 1$ , then*

$$S_n(x, f) - f(x) = O(n^{-\alpha}), \quad \tilde{S}_n(x, f) - \tilde{f}(x) = O(n^{-\alpha}), \quad (10.10)$$

uniformly in  $x$ .

Suppose that  $g(x) = f(x) + Cx$  is increasing. The difference  $S_n(x) - f(x)$  is given by the integral of  $\pi^{-1}\{f(x+t) - f(x)\} D_n(t)$  extended over  $(-\pi, \pi)$ . We can replace  $f$  by  $g$  in this, since the integral of  $t D_n(t)$  over  $(-\pi, \pi)$  is zero. It is enough to show that the integral over  $(0, \pi)$  is  $O(n^{-\alpha})$ , the proof for the remaining integral being similar. Let  $2^{k-1} \leq n < 2^k$ . Our integral is

$$\frac{1}{\pi} \int_0^\pi \{g(x+t) - g(x)\} D_n(t) dt = \int_0^{\pi 2^{-k}} + \sum_1^k \int_{\pi 2^{-j}}^{\pi 2^{-(j-1)}} = P + \sum_1^k Q_j.$$

Since  $g(x+t) - g(x) = O(t^\alpha)$  and  $D_n = O(n)$ , it follows that  $P = O(n 2^{-(1+\alpha)k}) = O(n^{-\alpha})$ . Also  $g(x+t) - g(x)$  is non-negative and increasing. Applying the second mean-value theorem twice, we get for  $Q_j$  the value

$$O(2^{-j\alpha}) \int_{\xi}^{\pi 2^{-(j-1)}} \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} dt = O(2^{-j\alpha}) O(2^j) \int_{\xi}^{\xi'} \sin(n + \frac{1}{2})t dt = O(2^{j(1-\alpha)} n^{-1}),$$

from which it follows that

$$\sum_1^k Q_j = \sum_1^k O(n^{-1}) 2^{j(1-\alpha)} = O(n^{-1} 2^{k(1-\alpha)}) = O(n^{-\alpha}).$$

Hence  $P + \sum Q_j = O(n^{-\alpha})$ , and the first estimate (10.10) follows. The proof of the second is similar.

## 11. Lebesgue's test

(11.1) THEOREM. If  $f$  is integrable, then, for almost all  $x$ ,

$$\int_0^h |f(x \pm t) - f(x)| dt = o(h) \quad \text{as } h \rightarrow +0. \quad (11.2)$$

This theorem is due to Lebesgue. It generalizes the familiar fact (to which it reduces if the sign of absolute value on the left of (11.2) is dropped) that the derivative of the indefinite integral of  $f(x)$  exists and equals  $f(x)$  for almost all  $x$ . The set of  $x$  for which we have (11.2) is sometimes called the *Lebesgue set* of  $f$ .

We shall prove the following slightly more general result.

(11.3) THEOREM. Suppose that  $f \in L^r$  ( $r \geq 1$ ). Then, for almost all  $x$ ,

$$\int_0^h |f(x \pm t) - f(x)|^r dt = o(h) \quad \text{as } h \rightarrow +0. \quad (11.4)$$

Let  $\alpha$  be any rational number. The function  $|f(x) - \alpha|^r$  is integrable and so

$$h^{-1} \int_0^h |f(x \pm t) - \alpha|^r dt \rightarrow |f(x) - \alpha|^r$$

for almost all  $x$ . Let  $E_\alpha$  be the set of the  $x$  for which this does not hold. Since  $|E_\alpha| = 0$ , the sum  $E$  of all the  $E_\alpha$  is of measure 0. We shall prove (11.4) for  $x$  not in  $E$ . Suppose that  $x_0$  is not in  $E$  and  $\epsilon > 0$  is given, and let  $\beta$  be rational and such that  $|f(x_0) - \beta| < \frac{1}{2}\epsilon$ . In the inequality

$$\left\{ \frac{1}{h} \int_0^h |f(x_0 \pm t) - f(x_0)|^r dt \right\}^{1/r} \leq \left\{ \frac{1}{h} \int_0^h |f(x_0 \pm t) - \beta|^r dt \right\}^{1/r} + \left\{ \frac{1}{h} \int_0^h |\beta - f(x_0)|^r dt \right\}^{1/r},$$

the first term on the right tends by hypothesis to  $|f(x_0) - \beta| < \frac{1}{2}\epsilon$ . The second term is  $|\beta - f(x_0)|$ . Thus the right-hand side is  $< \epsilon$  for  $h$  small enough, and (11.3) follows.

We shall systematically use the notation

$$\Phi(h) = \Phi_{x_0}(h) = \int_0^h |\phi_{x_0}(t)| dt, \quad \Psi(h) = \Psi_{x_0}(h) = \int_0^h |\psi_{x_0}(t)| dt.$$

It follows from (11.1) that  $\Phi_x(h) = o(h)$ ,  $\Psi_x(h) = o(h)$  for almost all  $x$ .

The following test for the convergence of  $S[f]$  is due to Lebesgue:

(11.5) THEOREM.  $S[f]$  converges to  $f(x)$  at every point  $x$  at which

$$\Phi(h) = o(h), \quad \int_\eta^\pi \left| \frac{\phi(t) - \phi(t+\eta)}{t} \right| dt \rightarrow 0 \quad \text{as } \eta = \pi/n \rightarrow 0, \quad (11.6)$$

and the convergence is uniform over any closed interval of continuity of  $f$  where the second condition (11.6) is satisfied uniformly.

We apply (10.1). The first term in (10.2) is  $o(1)$  by hypothesis. The third term there is  $2\eta^{-1}\Phi(2\eta) = o(1)$ . Integration by parts gives for the second term the value

$$\eta \left\{ [\Phi(t)t^{-2}]_\eta^\pi + 2 \int_\eta^\pi \Phi(t)t^{-3} dt \right\} = o(1),$$

since  $\Phi(t) = o(t)$  (see Chapter I, (8.1)). This completes the proof.

Using the analogue of (10·1) for conjugate series we find, by a similar argument, that the conditions

$$\Psi_x(h) = o(h), \quad \int_{\eta}^{\pi} \frac{|\psi(t) - \psi(t+\eta)|}{t} dt \rightarrow 0 \quad (11\cdot7)$$

together imply the relation  $S_n^*(x) - f(x; \pi/n) \rightarrow 0$ .

If we also observe that, for  $\pi/(n+1) \leq h \leq \pi/n$ , the first condition (11·7) gives

$$|f(x; h) - f(x; \pi/(n+1))| \leq \frac{2}{\pi} \int_{\pi/(n+1)}^{\pi/n} \frac{|\psi(t)|}{t} dt \leq 2\pi^{-2}(n+1) \Psi(\pi/n) = o(1) \quad (11\cdot8)$$

as  $n \rightarrow \infty$ , we deduce that under the conditions (11·7),  $\tilde{S}[f]$  converges at the point  $x$  if and only if  $f(x)$  exists.

The conditions (11·7) are certainly satisfied if  $f$  satisfies the Dini-Lipschitz condition in an interval containing  $x$ .

In Chapter VIII, § 4, we shall see that there exist integrable functions  $f$  such that  $S_n(x; f)$  is unbounded at every  $x$ . We now show, in the opposite direction, that  $S_n(x)$  and  $\tilde{S}_n(x)$  are  $o(\log n)$  at almost all  $x$ . More precisely,

(11·9) THEOREM. If  $\Phi_x(h) = o(h)$ , then  $S_n(x_0; f) = o(\log n)$ ; if  $\Psi_x(h) = o(h)$ , then  $\tilde{S}_n(x_0; f) = o(\log n)$ .

By (5·4) and (5·10),  $|S_n^*(x_0) - f(x_0)|$  does not exceed

$$n \int_0^{1/n} |\phi(t)| dt + \int_{1/n}^{\pi} t^{-1} |\phi(t)| dt = n\Phi(1/n) + [\Phi(t)t^{-1}]_{1/n}^{\pi} + \int_{1/n}^{\pi} t^{-2}\Phi(t) dt.$$

The sum of the first two terms on the right is  $\Phi(\pi)/\pi = O(1) = o(\log n)$ . Since  $\Phi(t) = o(t)$ , the remaining integral is  $o(\log n)$ . Thus  $S_n^*(x_0) = o(\log n)$ .

Similarly (cf. (5·5) and (5·11)),

$$|\tilde{S}_n^*(x_0)| \leq n \int_0^{1/n} |\psi(t)| dt + 2 \int_{1/n}^{\pi} \frac{|\psi(t)|}{t} dt = o(\log n).$$

By (11·9),  $S_n(x)$  and  $\tilde{S}_n(x)$  are  $o(\log n)$  at every point of continuity of  $f$ . Moreover, if  $f$  is continuous in an interval  $I$ , then  $S_n(x)$  and  $\tilde{S}_n(x)$  are  $o(\log n)$  uniformly in every interval interior to  $I$ . The proofs are slightly simpler than those of (11·9), no integration by parts being necessary.

The most important tests for the convergence of Fourier series are those of Dini, Dini-Lipschitz and Dirichlet-Jordan, each of which is based on a different idea. Lebesgue's test may be shown to include the other three, but in practice it is less convenient to use because the second condition (11·6) corresponds to no simple property of the function  $f$ . The following application of Lebesgue's test is, however, of interest.

(11·10) THEOREM. Suppose that  $f \in \lambda_{p'}^1$ ,  $p > 1$ . Then  $S[f]$  converges to  $f(x)$  at each point  $x$  of the Lebesgue set of  $f$ ; and the convergence is uniform over any closed arc of continuity of  $f$ . At each point of the Lebesgue set where  $f(x)$  exists,  $\tilde{S}[f]$  converges to  $f(x)$ .

It is enough to prove the part about  $S[f]$ . This will follow if we show that the second condition (11·6) is satisfied everywhere and uniformly in  $x$ . By Hölder's inequality, with  $p' = p/(p-1)$ ,

$$\int_{\eta}^{\pi} \frac{|\phi(t) - \phi(t+\eta)|}{t} dt \leq \left\{ \int_{\eta}^{\pi} |\phi(t) - \phi(t+\eta)|^p dt \right\}^{1/p} \left\{ \int_{\eta}^{\pi} \frac{dt}{t^{p'}} \right\}^{1/p'} = o(\eta^{1/p}) O(\eta^{-1/p}) = o(1),$$

uniformly in  $x$ .

12. Lebesgue constants

This name is given to the numbers

$$L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt = \frac{2}{\pi} \int_0^{\pi} \left| \frac{\sin(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \right| dt.$$

It is clear that if  $|f| \leq 1$  then

$$|S_n(x; f)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t)| |D_n(t)| dt \leq L_n$$

for all  $x$ ; and for  $f(t) = \text{sign } D_n(t)$ , we actually have  $S_n(0; f) = L_n$ . While the function  $\text{sign } D_n(t)$  is discontinuous at a finite number of points, given any  $\epsilon > 0$  we can, by smoothing this function slightly at the points of discontinuity, obtain a continuous  $f$  such that  $S_n(0; f) > L_n - \epsilon$ . Thus, for each  $n$ ,  $L_n$  is

- (i) the maximum of  $|S_n(x; f)|$  for all  $x$  and  $f$  satisfying  $|f| \leq 1$ ;
- (ii) the upper bound† of  $|S_n(x; f)|$  for all  $x$  and all continuous  $f$  satisfying  $|f| \leq 1$ .

We shall prove that  $L_n = 4\pi^{-2} \log n + O(1) \simeq 4\pi^{-2} \log n$  as  $n \rightarrow \infty$ . (12·1)

Since  $|D_n - D_n^*| \leq \frac{1}{2}$  and the function  $1/t - \frac{1}{2} \cot \frac{1}{2}t$  is bounded for  $|t| \leq \pi$ ,

$$\begin{aligned} L_n &= \frac{2}{\pi} \int_0^{\pi} |D_n^*(t)| dt + O(1) = \frac{2}{\pi} \int_0^{\pi} \frac{|\sin nt|}{t} dt + O(1) \\ &= \frac{2}{\pi} \sum_{k=0}^{n-1} \int_{k\pi/n}^{(k+1)\pi/n} \frac{|\sin nt|}{t} dt + O(1) = \frac{2}{\pi} \int_0^{\pi/n} \sin nt \left( \sum_{k=1}^{n-1} \frac{1}{t + k\pi/n} \right) dt + O(1) \end{aligned}$$

The sum in curly brackets lies between

$$n\pi^{-1}(1 + \frac{1}{2} + \dots + 1/(n-1)) \quad \text{and} \quad n\pi^{-1}(\frac{1}{2} + \dots + 1/n),$$

and so is equal to  $\pi^{-1}n[\log n + O(1)]$ . Since the integral of  $\sin nt$  over  $(0, \pi/n)$  is  $2/n$ , we obtain (12·1).

We may add that, since  $|D_n(t)|$  is uniformly bounded in any interval  $\epsilon \leq t \leq \pi$ ,  $0 < \epsilon < \pi$ , the formula (12·1) implies that

$$\frac{2}{\pi} \int_0^{\epsilon} |D_n(t)| dt = \frac{4}{\pi^2} \log n + O(1) \simeq \frac{4}{\pi^2} \log n \tag{12·2}$$

for any fixed  $\epsilon$  ( $0 < \epsilon \leq \pi$ ).

The formulae

$$L_n = \frac{2}{\pi} \int_0^{\pi} |\tilde{D}_n(t)| dt \simeq \frac{2}{\pi} \log n, \quad \frac{2}{\pi} \int_0^{\pi} \tilde{D}_n^*(t) dt \simeq \frac{2}{\pi} \log n \tag{12·3}$$

are also useful. They are equivalent, since  $|\tilde{D}_n - \tilde{D}_n^*| \leq \frac{1}{2}$  and  $\tilde{D}_n^*(t) \geq 0$  in  $(0, \pi)$ . The left-hand member of the second formula represents  $-S_n^*(0; f)$  where  $f(t) = \text{sign } t$  ( $-\pi < t < \pi$ ). Since  $f$  has jump 2 at  $t = 0$ , (8·13) gives  $-S_n^*(0; f) \simeq 2\pi^{-1} \log n$  and (12·3) follows.

The first integral (12·3) is an analogue of the Lebesgue constant  $L_n$ , and is the maximum of  $|S_n(0; f)|$  for all functions  $f$  with  $|f| \leq 1$ . This maximum is attained for  $f(t) = \text{sign } \tilde{D}_n(t)$ .

† By upper bound we always mean the least upper bound. Similarly, for the lower bound.

### 13. Poisson's summation formula

The notion of the Fourier transform

$$\gamma(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x) e^{-iux} dx \quad (13.1)$$

of a function  $g(x)$  defined in  $(-\infty, +\infty)$  (see Chapter I, § 4) is useful in the theory of Fourier series in connexion with the following simple fact.

Suppose first that  $g(x)$  is absolutely integrable over  $(-\infty, +\infty)$ . The series

$$\sum_{k=-\infty}^{+\infty} g(x + 2k\pi) \quad (13.2)$$

is then absolutely convergent at almost all  $x$  in  $(0, 2\pi)$ , as is seen from the inequality

$$\sum_{k=-\infty}^{+\infty} \int_0^{2\pi} |g(x + 2k\pi)| dx = \int_{-\infty}^{+\infty} |g(x)| dx < \infty$$

(cf. Chapter I, (11.5)). Let  $G_n(x)$  be the  $n$ th symmetric partial sum, and  $G(x) = \lim G_n(x)$  the sum, of (13.2). The function  $G(x)$  exists for almost all  $x$  and is periodic. Since the  $G_n(x)$  are majorized by an integrable function, the Fourier coefficient  $c_\nu$  of  $G$  is

$$\lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_0^{2\pi} G_n(x) e^{-i\nu x} dx = \lim_{n \rightarrow +\infty} \frac{1}{2\pi} \int_{-2n\pi}^{2(n+1)\pi} g(x) e^{-i\nu x} dx = \gamma(\nu). \quad (13.3)$$

Thus, with our hypothesis, the Fourier coefficient  $c_\nu$  of the sum  $G(x)$  of (13.2) is equal to the Fourier transform  $\gamma(\nu)$  of  $g(x)$ .

If, moreover, it happens that  $\sum c_\nu e^{i\nu x} = S[G]$  converges at  $x = 0$ , and to a value which is the sum of (13.2) at  $x = 0$ , we are led to the equation

$$\sum_{k=-\infty}^{+\infty} g(2k\pi) = \sum_{\nu=-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(x) e^{-i\nu x} dx. \quad (13.4)$$

This is called the *Poisson summation formula*, and it has many applications. The sum on the right is defined as the limit of the symmetric partial sums.

Suppose, for example, that  $g(x)$  is not only absolutely integrable over  $(-\infty, +\infty)$  but also of bounded variation, and that  $2g(x) = g(x+0) + g(x-0)$  for all  $x$ . We shall prove (13.4) under these hypotheses.

Let  $v_k$  be the total variation of  $\bar{g}$  over the interval  $I_k = (2k\pi, 2(k+1)\pi)$ ,  $k = 0, \pm 1, \dots$ . Since the series (13.2) converges absolutely at some point  $x_0$  in  $I_0$ , the inequalities  $|g(x + 2k\pi) - g(x_0 + 2k\pi)| \leq v_k$ , for  $x \in I_0$ , and  $\sum v_k < \infty$ , prove that (13.2) converges absolutely and uniformly in  $I_0$  to a sum  $G(x)$ , obviously of bounded variation and such that  $2G(x) = G(x+0) + G(x-0)$ . Formula (13.4) is then a consequence of Theorem (8.1).

The equation  $c_\nu = \gamma(\nu)$ , slightly modified, remains valid in cases when  $g(x)$  is not absolutely integrable over  $(-\infty, +\infty)$ . Suppose, namely, that  $g(x)$  is integrable over every finite interval and that

$$(i) \int_{2k\pi}^{2(k+1)\pi} |g(x)| dx \rightarrow 0 \text{ as } k \rightarrow \pm \infty;$$

(ii) the function  $g^*$  defined by the formula

$$g^*(x) = g(x) - \frac{1}{2\pi} \int_{2k\pi}^{2(k+1)\pi} g(t) dt \quad \text{for } 2k\pi \leq x < 2(k+1)\pi \quad (k=0, \pm 1, \pm 2, \dots),$$

is absolutely integrable over  $(-\infty, +\infty)$ .

Conditions (i) and (ii) are certainly satisfied if, for example,  $g(x)$  tends to 0 monotonically in the neighbourhoods of  $+\infty$  and of  $-\infty$ . We prove now the following theorem:

(13.5) THEOREM. Under conditions (i) and (ii) the integral (13.1), defined as  $\lim_{u \rightarrow \infty} \int_{-u}^{+u}$ , exists for  $u = \pm 1, \pm 2, \dots$ , and the Fourier coefficients  $c_\nu^*$  of the function

$$G^*(x) = \lim_{n \rightarrow +\infty} \left\{ \sum_{k=-n}^n g(x+2k\pi) - \frac{1}{2\pi} \int_{-2n\pi}^{2n\pi} g(x) dx \right\} \quad (13.6)$$

satisfy the equations  $c_0^* = 0, c_\nu^* = \gamma(\nu)$  for  $\nu = \pm 1, \pm 2, \dots$ .

The function  $G^*(x)$  can be written in the form  $\sum g^*(x+2k\pi)$ ; thus, by the case already dealt with, it is integrable over  $(0, 2\pi)$ , and  $c_\nu^*$  is given by (13.3), with  $G_n^*$  for  $G_n$  and  $g^*$  for  $g$ . Since the integral of  $e^{-i\nu x}$  over a period is zero for  $\nu = \pm 1, \pm 2, \dots$ , we have, by (i),

$$c_\nu^* = \lim_{u \rightarrow \infty} \frac{1}{2\pi} \int_{-u}^u g(x) e^{-i\nu x} dx = \gamma(\nu) \quad (\nu = \pm 1, \pm 2, \dots).$$

The integral of  $g^*(x)$  over any interval  $(2k\pi, 2(k+1)\pi), k=0, \pm 1, \dots$ , is evidently 0.

Hence  $\int_0^{2\pi} G^*(x) dx = 0$  and  $c_0^* = 0$ .

The following application of Theorem (13.5) will be useful in fractional integration (Chapter XII, § 8):

(13.7) THEOREM. Let  $0 < \alpha < 1$ , and let  $\Psi_\alpha(x)$  be the periodic function defined for  $0 < x < 2\pi$  by the formula

$$\Psi_\alpha(x) = \lim_{n \rightarrow \infty} \frac{2\pi}{\Gamma(\alpha)} \left\{ x^{\alpha-1} + (x+2\pi)^{\alpha-1} + \dots + (x+2n\pi)^{\alpha-1} - \frac{(2\pi)^\alpha}{\alpha} n^\alpha \right\}. \quad (13.8)$$

Then 
$$\Psi_\alpha(x) \sim \sum'_{\nu=-\infty}^{+\infty} \frac{e^{-\frac{1}{2}\pi i \alpha \text{sign } \nu}}{|\nu|^\alpha} e^{i\nu x}, \quad (13.9)$$

where the dash indicates that the term  $\nu = 0$  is omitted from the summation.

The function  $\Psi_\alpha(x)$  is the  $G^*(x)$  corresponding to a  $g(x)$  equal to  $2\pi x^{\alpha-1}/\Gamma(\alpha)$  for  $x > 0$  and to 0 elsewhere. The coefficients of  $\Psi_\alpha$  are

$$c_0^* = 0, \quad c_\nu^* = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-i\nu t} dt \quad \text{for } \nu \neq 0.$$

We now observe that†

$$e^{-\frac{1}{2}\pi i \alpha} \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad \text{for } 0 < \alpha < 1. \quad (13.10)$$

† This formula is easily obtainable from the classical definition  $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$  by applying Cauchy's theorem to the integral of the function  $z^{\alpha-1} e^{-z}$ , taken along the boundary of the domain limited by the arcs  $0 \leq \arg z \leq \frac{1}{2}\pi$  of the circles  $|z| = \epsilon$  and  $|z| = R$ , and by the rectilinear segments  $(\epsilon, R)$  and  $(i\epsilon, iR)$  of the real and imaginary axes; if  $0 < \alpha < 1$ , the integrals taken along the circular arcs tend to 0 as  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$ , and (13.10) follows.

Substituting  $t = \nu u$ , we see that  $c_\nu^* = \nu^{-\alpha} \exp(-\frac{1}{2}\pi i \alpha)$  for  $\nu > 0$ , and since  $c_{-\nu}^* = \bar{c}_\nu^*$ , we get (13.9).

By (2.6) of Chapter I, the series in (13.9) converges uniformly outside any neighbourhood of  $x = 0$ . If we omit the term  $x^{\alpha-1}$  in (13.8), the limit will exist uniformly in  $0 \leq x \leq 2\pi$  (this will be the function  $G^*$  corresponding to a  $g(x)$  equal to  $2\pi x^{\alpha-1}/\Gamma(\alpha)$  for  $x > 2\pi$  and to zero elsewhere). Hence, with an error uniformly  $O(1)$ , the periodic function  $\Psi_\alpha(x)$  is 0 for  $-\pi \leq x < 0$  and is  $2\pi x^{\alpha-1}/\Gamma(\alpha)$  for  $0 < x \leq \pi$ .

By considering the Fourier series of  $\Psi_\alpha(x) \pm \Psi_\alpha(-x)$ , and using the relation  $\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin \pi\alpha$ , we get the useful formulae

$$\left. \begin{aligned} \sum_{\nu=1}^{\infty} \frac{\cos \nu x}{\nu^\alpha} &= \Gamma(1-\alpha) \sin \frac{1}{2}\pi\alpha \cdot x^{\alpha-1} + O(1) \simeq \Gamma(1-\alpha) \sin \frac{1}{2}\pi\alpha \cdot x^{\alpha-1} \\ \sum_{\nu=1}^{\infty} \frac{\sin \nu x}{\nu^\alpha} &= \Gamma(1-\alpha) \cos \frac{1}{2}\pi\alpha \cdot x^{\alpha-1} + O(1) \simeq \Gamma(1-\alpha) \cos \frac{1}{2}\pi\alpha \cdot x^{\alpha-1} \end{aligned} \right\} (0 < x \leq \pi). \quad (13.11)$$

Poisson's summation formula may be written in a slightly different and more symmetric form. Let  $a > 0$  and let  $g(x) = h(ax/2\pi)$ . Then, by (13.4),

$$\sum_{k=-\infty}^{+\infty} h(ak) = \sum_{\nu=-\infty}^{+\infty} \frac{1}{a} \int_{-\infty}^{+\infty} h(y) e^{-2\pi i \nu y/a} dy. \quad (13.12)$$

If, as is often convenient, we modify the definition (13.1) of the Fourier transform by replacing the factor  $1/2\pi$  there by  $1/(2\pi)^{\frac{1}{2}}$ , and accordingly set

$$\chi(u) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} h(y) e^{-iuy} dy, \quad (13.13)$$

the terms on the right of (13.12) can be written  $(2\pi)^{\frac{1}{2}} a^{-1} \chi(2\pi\nu/a)$  and Poisson's formula becomes

$$\sqrt{a} \sum_{k=-\infty}^{+\infty} h(ak) = \sqrt{b} \sum_{k=-\infty}^{+\infty} \chi(bk), \quad (13.14)$$

where  $a, b$  are any two positive numbers satisfying the condition  $ab = 2\pi$ , and  $\chi(u)$  is the Fourier transform (13.13) of  $h$ .

### MISCELLANEOUS THEOREMS AND EXAMPLES

1. Given any sequence of positive numbers  $\epsilon_n$  tending to 0, there is always a continuous function  $f(x)$  whose Fourier coefficients satisfy the inequality  $|a_n| + |b_n| \geq \epsilon_n$  for infinitely many  $n$ . (Lebesgue [1], Hardy [1].)

[Take, for example,  $f(x) = \epsilon_{n_1} \cos n_1 x + \epsilon_{n_2} \cos n_2 x + \dots$ , where  $\{n_s\}$  increases so rapidly that  $\sum \epsilon_{n_s} < \infty$ .]

2. Suppose that  $f \in \Lambda_{p_1}^2$  and  $f \in \Lambda_{p_2}^2$ ,  $p_1 \geq 1$ ,  $p_2 \geq 1$ . Show that  $f \in \Lambda_p^2$  if the point with Cartesian co-ordinates  $(\alpha, 1/p)$  is on the segment joining  $(\alpha_1, 1/p_1)$ ,  $(\alpha_2, 1/p_2)$ . Hardy and Littlewood [5].

[Use Chapter I, (10.12) (iii).]

3. Using the equation

$$\sum_{\lambda}^k (a_n \sin n x - b_n \cos n x)/n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sum_{\lambda} \frac{\sin n(x-t)}{n} dt,$$

prove (8.7) and the formula  $\sum_1^{\infty} b_n/n = \frac{1}{\pi} \int_0^{2\pi} f(t) \frac{1}{2}(\pi-t) dt$ .

4. The number  $O_k = 1 + 2^{-2k} + 3^{-2k} + \dots$  ( $k = 1, 2, \dots$ ) is a rational multiple of  $\pi^{2k}$ .

[Integrate the series  $\sin x + \frac{1}{2} \sin 2x + \dots$  an odd number of times and set  $x = 0$ .]

5. If  $f(x)$  is periodic and has  $k$  continuous derivatives, the Fourier coefficients of  $f$  satisfy the inequality

$$|c_n| \leq \frac{\omega(\pi/n, f^{(k)})}{2n^k} \quad (n > 0).$$

If  $f^{(k)}$  is of bounded variation, then

$$|c_n| \leq V/\pi n^k \quad (n > 0),$$

where  $V$  is the total variation of  $f^{(k)}$  over  $0 \leq x \leq 2\pi$ .

6. Suppose that  $f(x)$ ,  $0 \leq x \leq 2\pi$ , has  $k$  continuous derivatives in the closed interval  $(0, 2\pi)$ , but is not necessarily continuous when continued periodically. Show that the Fourier coefficient  $c_n$  of  $f$  is

$$\frac{\alpha}{n} + \frac{\beta}{n^2} + \dots + \frac{\lambda + o(1)}{n^k} \quad \text{as } n \rightarrow \pm \infty.$$

7. Let  $f(x) \sim \sum c_n e^{inx}$ ,  $h > 0$ ,  $f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt$ . Show that

$$f_h(x) \sim c_n + \sum' o_n \left( \frac{\sin nh}{nh} \right) e^{inx}.$$

The dash indicates that the term  $n = 0$  is omitted in summation. The sign ' $\sim$ ' can also be replaced by '='.

[If  $F(x)$  is the integral of  $f$ , then  $f_h(x) = [F(x+h) - F(x-h)]/2h$ . Apply (8.7).]

8. Let  $0 \leq \alpha < 1$ . The system  $\{e^{i(n+\alpha)x}\}_{n=0, \pm 1, \pm 2, \dots}$  is orthogonal and complete over any interval of length  $2\pi$ .

Each of the systems  $\cos(n + \frac{1}{2})x$  and  $\sin(n + \frac{1}{2})x$ ,  $n = 0, 1, 2, \dots$ , is orthogonal and complete over  $(0, \pi)$ .

The joint system  $\cos(n + \frac{1}{2})x$ ,  $\sin(n + \frac{1}{2})x$  is orthogonal and complete over any interval of length  $2\pi$ .

Show that 
$$\frac{1}{2}(\pi - x) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{e^{i(n+\frac{1}{2})x}}{(2n+1)^2} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(n + \frac{1}{2})x}{(2n+1)^2} \quad (0 \leq x \leq 2\pi),$$

the series being absolutely and uniformly convergent.

9. Let  $f(x)$  be defined for  $0 \leq x \leq 2\pi$ . No matter how well behaved the function  $f(x)$ ,  $S[f]$  cannot converge uniformly if  $f(+0) \neq f(2\pi-0)$ . Suppose, however, that  $\alpha$  is such that  $g(x) = f(x)e^{-i\alpha x}$  takes the same values at the end-points  $0$  and  $2\pi$ . (Such an  $\alpha$  always exists, though it need not be real, provided  $f(+0) \neq 0$  and  $f(2\pi-0) \neq 0$ .) If  $S[g]$  converges uniformly to  $g(x)$ , we get the uniformly convergent representation

$$f(x) = \sum_{-\infty}^{+\infty} c_n e^{i(n+\alpha)x}, \quad \text{with } c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-i(n+\alpha)x} dx.$$

The series here is a Fourier series if  $\alpha$  is real; if  $f(+0) = -f(2\pi-0)$ , we have  $\alpha = \frac{1}{2}$  (compare Ex. 8).

10. If  $\alpha$  is not a real integer, then

$$\sum_{n=-\infty}^{+\infty} \frac{\sin(n+\alpha)x}{n+\alpha} = \pi, \quad \sum_{n=-\infty}^{+\infty} \frac{\cos(n+\alpha)x}{n+\alpha} = \pi \cot \pi \alpha$$

for  $0 < x < 2\pi$ .

[See Chapter I, (4.19).]

11. Let  $f(x)$  be periodic, non-negative and not identically 0. Prove that the Fourier coefficients of  $f$  satisfy the inequalities

$$|a_m| < a_0, \quad |b_m| < a_0, \quad |c_m| < a_0 \quad (m \neq 0). \quad (\text{Carathéodory [1].})$$

12. Let  $g(x)$  be periodic, odd, non-negative in  $(0, \pi)$  and not identically 0. The Fourier coefficients  $b_m$  of  $g$  satisfy the inequality

$$|b_m| < m b_1 \quad (m = 2, 3, \dots). \quad (\text{Rogosinski [1], Dieudonné [1].})$$

[Prove, by induction, that  $|\sin m\epsilon| \leq m |\sin \epsilon|$ ,  $m = 1, 2, \dots$ ]

13. Let  $0 < \alpha < 1$ . The (non-integrable) function  $x^{-\alpha-1}$ ,  $0 < x \leq \pi$ , has generalized sine coefficients  $b_n \approx Cn^\alpha$ , where  $C \neq 0$  is independent of  $n$ .

$$\left[ \left(\frac{1}{2}\pi\right) b_n = \int_0^\pi x^{-\alpha-1} \sin nx \, dx = n^\alpha \int_0^{n\pi} x^{-\alpha-1} \sin x \, dx. \right]$$

14. Show that

$$r \sin x + r^3 \sin 3x + \dots = \frac{1}{2} \frac{k \sin x}{1 - k^2 \cos^2 x}, \quad k = \frac{2r}{1+r^2} \quad (0 \leq r < 1),$$

$$\frac{1}{2 \sin x} \sim \sin x + \sin 3x + \sin 5x + \dots$$

The second formula is, formally, a limiting case of the first when  $r \rightarrow 1$ .

15. Let  $f(x)$ ,  $0 \leq x \leq \pi$ , (not necessarily integrable) be such that  $g(x) = f(x) \sin x$  is of bounded variation. Then  $b_n$  tends to a finite limit. If also  $g(+0) = g(\pi-0) = 0$ , then  $b_n = o(1)$ .

[Observe that

$$b_n = \frac{2}{\pi} \int_0^\pi g(x) \frac{\sin nx}{\sin x} \, dx = \frac{2}{\pi} \int_0^{2\pi} g\left(\frac{1}{2}x\right) \frac{\sin \frac{1}{2}nx}{2 \sin \frac{1}{2}x} \, dx.$$

If  $n = 2k + 1$ , the last integral is  $2S_k(0; g(\frac{1}{2}x))$ .

16. Let

$$(*) \sum c_n e^{i(n+\alpha)x}$$

be the Fourier series of any integrable  $f(x)$ ,  $0 \leq x \leq 2\pi$ , with respect to the orthogonal system  $e^{i(n+\alpha)x}$ ,  $0 \leq \alpha < 1$ . (Thus  $c_n = c_n^{(\alpha)}$ .) Show that all the series (\*) are uniformly equiconvergent in every interval  $(\epsilon, 2\pi - \epsilon)$ . The equiconvergence cannot, in general, hold in the whole interval  $(0, 2\pi)$  (if, for instance,  $\alpha = \frac{1}{2}$ , the sum  $S(x)$  of the series (\*) satisfies the condition  $S(x + 2\pi) = -S(x)$ ).

17. Let  $f(x)$  be periodic and integrable, and consider the functions

$$f_1(t) = \frac{f(x_0+t) + f(x_0-t) - 2f(x_0)}{4 \tan \frac{1}{2}t}, \quad f_2(t) = \frac{f(x_0+t) + f(x_0-t) - 2f(x_0)}{4 \sin \frac{1}{2}t},$$

which are periodic and odd. Show that the generalized sine coefficients of  $f_1(t)$  are  $S_n^\alpha(x_0; f) - f(x_0)$ , and that the coefficients of  $f_2(t)$  with respect to the system  $\sin(n + \frac{1}{2})t$  are  $S_n(x_0) - f(x_0)$ .

18. Let  $f(x)$  be periodic and integrable, and suppose that the even functions

$$f_3(t) = \frac{f(x_0+t) - f(x_0-t)}{4 \tan \frac{1}{2}t}, \quad f_4(t) = \frac{f(x_0+t) - f(x_0-t)}{4 \sin \frac{1}{2}t}$$

are integrable over  $(0, \pi)$ . (This amounts to the integrability of  $|f(x_0+t) - f(x_0-t)| t^{-1}$ .) Show that the cosine coefficients of  $f_3(t)$  and the coefficients of  $f_4(t)$  with respect to the system  $\cos(n + \frac{1}{2})t$  are respectively  $\tilde{S}_n^\alpha(x_0) - f(x_0)$  and  $\tilde{S}_n(x_0) - f(x_0)$ .

19. If  $f(x)$  is the characteristic function of the interval  $(-h, h)$ ,  $0 < h \leq \pi$ , then

$$\tilde{f}(x) = \frac{1}{\pi} \log \left| \frac{\sin \frac{1}{2}(x+h)}{\sin \frac{1}{2}(x-h)} \right|.$$

[Observe that  $\int_a^b \frac{1}{2} \cot \frac{1}{2}t \, dt = \log \left| \frac{\sin \frac{1}{2}b}{\sin \frac{1}{2}a} \right|$  for any subinterval  $(a, b)$  of  $(-2\pi, 2\pi)$ .]

20. Let  $g(x) = 0$  for  $|x| \leq h$ , and  $g(x) = \frac{1}{2} \cot \frac{1}{2}x$  elsewhere in  $(-\pi, \pi)$ ,  $0 < h \leq \pi$ . Show that

$$\tilde{g}(x) = \frac{1}{2} \left( 1 - \frac{h}{\pi} \right) - \frac{1}{\pi} \frac{1}{2} \cot \frac{1}{2}x \log \left| \frac{\sin \frac{1}{2}(x+h)}{\sin \frac{1}{2}(x-h)} \right|.$$

In particular,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \cot \frac{1}{2}x \log \left| \frac{\sin \frac{1}{2}(x+h)}{\sin \frac{1}{2}(x-h)} \right| dx = \pi - h,$$

the integrand on the left being non-negative. (M. Riesz [1].)

21. Considering  $S[\cos \alpha x]$  at the points  $x = 0, \pi$  show that

$$\frac{\pi}{\sin \alpha \pi} = \frac{1}{\alpha} + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^2 - k^2} = \lim_{N \rightarrow +\infty} \sum_{k=-N}^N \frac{(-1)^k}{\alpha + k},$$

$$\frac{\pi}{\tan \alpha \pi} = \frac{1}{\alpha} + 2\alpha \sum_{k=1}^{\infty} \frac{1}{\alpha^2 - k^2} = \lim_{N \rightarrow +\infty} \sum_{k=-N}^N \frac{1}{\alpha + k}.$$

22. If  $f(x_0 + t) + f(x_0 - t)$  increases monotonically to  $+\infty$  as  $t \rightarrow 0, 0 < t < t_0$ , then  $S[f]$  diverges to  $+\infty$  at the point  $x_0$ .

[Let  $[f(x_0 + t) + f(x_0 - t)]/t = \chi(t)$ . Then

$$\int_0^{\pi/n} \chi(t) \sin nt \, dt + o(1) \geq \pi S_n^*(x_0) \geq \int_0^{\pi/n} [\chi(t) - \chi(t + \pi/n)] \sin nt \, dt + o(1)$$

$$\geq \frac{1}{2} \int_0^{\pi/n} \chi(t) \sin nt \, dt + o(1).]$$

23. Using for the Lebesgue constants (see § 12) the formula

$$L_n = \frac{1}{\pi} \int_0^{2\pi} \text{sign} \sin(n + \frac{1}{2})x \left[ \frac{1}{2} + \sum_{k=1}^n \cos kx \right] dx,$$

and integrating termwise, prove that

$$L_n = \frac{1}{2n+1} + \frac{2}{\pi} \sum_{k=1}^n \frac{1}{k} \tan \frac{\pi k}{2n+1}. \quad (\text{Fejér [8].})$$

24. Using  $S[|\sin x|]$  (see p. 34, Example 3) and the formula

$$(\sin kx)^2 / \sin x = \sin x + \sin 3x + \dots + \sin(2k-1)x$$

prove

$$L_n = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2k(2n+1)-1}}{4k^2 - 1}.$$

This equation shows, in particular, that  $\{L_n\}$  is strictly increasing. (Szegő [1].)

25. Show that the conclusions of (6.7) (i) remain valid if the hypotheses are replaced by the following ones: (i)  $f(x)$  is bounded; (ii)  $\rho(x)$  is integrable and satisfies

$$\int_{-\pi}^{\pi} |\rho(x_0 + t) - \rho(x_0)| |t|^{-1} dt < \infty. \quad (\text{W. H. Young [11].})$$

26. Consider the periodic functions  $f_p(x)$  defined by the formulae

$$f_p(x) = |x|^{p'} \quad (1 < p < \infty),$$

$$f_1(x) = \log |x|,$$

for  $|x| \leq \pi, p' = p/(p-1)$ . Show that (i)  $f_p(x)$  belongs to  $\Lambda_p^*$ ,  $1 \leq p < \infty$ , but not to  $\Lambda_q^*$ ,  $q > p$ ; (ii)  $f_p(x)$  does not belong to  $\Lambda_1^*$ .

27. Let  $0 \leq \alpha < 1, -\infty < \beta < +\infty$ . The modulus of continuity of

$$f_{\alpha, \beta}(x) = \sum_{n=1}^{\infty} b^{-n\alpha} n^{-\beta} \cos b^n x \quad (b = 2, 3, \dots)$$

is  $O(\delta^\alpha \log^{-\beta} 1/\delta)$  if  $\alpha > 0$ , and is  $O(\log^{-\beta-1} 1/\delta)$  if  $\alpha = 0, \beta > 1$ .

## CHAPTER III

## SUMMABILITY OF FOURIER SERIES

## 1. Summability of numerical series

We consider a doubly infinite matrix

$$M = \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} & \dots \\ a_{10} & a_{11} & \dots & a_{1n} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n0} & a_{n1} & \dots & a_{nn} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

of numbers. With every sequence  $s_0, s_1, s_2, \dots$  we associate the sequence  $\{\sigma_n\}$  given by

$$\sigma_n = a_{n0}s_0 + a_{n1}s_1 + \dots + a_{nn}s_n + \dots \quad (n = 0, 1, 2, \dots), \quad (1.1)$$

provided the series on the right converges for all  $n$ . If  $\sigma_n$  tends to a limit  $s$  we shall say that the sequence  $\{s_n\}$ , or the series whose partial sums are  $s_n$ , is *summable M* to limit (sum)  $s$ . The  $\sigma_n$  are also called the *linear means* (determined by the matrix  $M$ ) of the  $\{s_n\}$ . Matrices  $M$  such that  $a_{n\nu} = 0$  for  $n < \nu$  are called *triangular*.

Let us suppose that the numbers

$$N_n = |a_{n0}| + |a_{n1}| + \dots, \quad A_n = a_{n0} + a_{n1} + \dots$$

exist (and are finite) for all  $n$ . The matrix will be called *regular* if the following conditions are satisfied:

- (i)  $\lim_{n \rightarrow \infty} a_{n\nu} = 0$  for  $\nu = 0, 1, \dots$
- (ii) the  $N_n$  are bounded;
- (iii)  $\lim_{n \rightarrow \infty} A_n = 1$ .

The finiteness of  $N_n$  implies the existence of  $A_n$ . It also implies the convergence of the series (1.1) for every bounded (in particular, convergent) sequence  $\{s_n\}$ .

(1.2) THEOREM. *If M is a regular matrix, and if  $s_n$  tends to a finite limit  $s$ , then  $\sigma_n \rightarrow s$ .*

For let  $s_n = s + \epsilon_n$ ,  $\epsilon_n \rightarrow 0$ . Correspondingly,  $\sigma_n = \sigma'_n + \sigma''_n$  where

$$\sigma'_n = sA_n, \quad \sigma''_n = \epsilon_0 a_{n0} + \epsilon_1 a_{n1} + \dots$$

Here  $\sigma'_n \rightarrow s$  by condition (iii). Let  $N$  be the upper bound of the  $N_n$ ; and let  $|\epsilon_n| < \eta/2N$  for  $\nu > \nu_0$ , where  $\eta$  is any given positive number. Then

$$|\sigma''_n| \leq (|\epsilon_0| |a_{n0}| + \dots + |\epsilon_{\nu_0}| |a_{n\nu_0}|) + (|a_{n,\nu_0+1}| + |a_{n,\nu_0+2}| + \dots) \eta \cdot 2N.$$

The first sum on the right tends to 0 as  $n \rightarrow \infty$  (condition (i)), and so is less than  $\frac{1}{2}\eta$  for  $n > n_0$ . The remainder does not exceed  $N\eta/2N = \frac{1}{2}\eta$ . Hence  $|\sigma''_n| < \eta$  for  $n > n_0$ ,  $\sigma''_n \rightarrow 0$ ,  $\sigma_n \rightarrow s$ , as desired.

We note that if  $s = 0$  condition (iii) is not needed in the above argument.

Condition (ii) by itself shows that the boundedness of  $\{s_\nu\}$  implies that of  $\{\sigma_n\}$ ; for if  $|s_\nu| \leq A$  for all  $\nu$ , then  $|\sigma_n| \leq A N$ .

It is interesting to observe that conditions (i), (ii), (iii) are also *necessary*, if  $\{\sigma_n\}$  is to tend to  $s$  for every  $\{s_\nu\} \rightarrow s$ . For consider the sequence  $s_\nu = 1$  for all  $\nu$ , and the sequence  $s_\nu = 0$  for all  $\nu \neq \mu$ ,  $s_\mu = 1$ . In the first case  $s = 1$ , in the second  $s = 0$ . Since  $\sigma_n = A_n$  in the first case and  $\sigma_n = a_{n\mu}$  in the second, the necessity of conditions (i) and (iii) is evident. The necessity of (ii) is less simple and will not be discussed here (see Chapter IV, p. 168).

Condition (ii) is a consequence of (iii) if the numbers  $a_{\nu\mu}$  are non-negative. Such matrices  $M$  are called *positive*.

(1.3) THEOREM. *If  $M$  is a positive regular matrix, then*

$$\liminf s_\nu \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_\nu \tag{1.4}$$

for any sequence  $\{s_\nu\}$  for which the  $\sigma_n$  are defined.

In particular, if  $M$  is positive Theorem (1.2) holds for  $s = +\infty$  and  $s = -\infty$ .

Let  $\liminf s_\nu = \underline{s}$ ,  $\limsup s_\nu = \bar{s}$ . To prove the last inequality in (1.4), we may suppose that  $\bar{s} < +\infty$ . Let  $\alpha$  be any number greater than  $\bar{s}$ . Then  $s_\nu < \alpha$  for  $\nu > \nu_0$  and, by (i),

$$\sigma_n \leq o(1) + \alpha(a_{n, \nu_0+1} + a_{n, \nu_0+2} + \dots) = o(1) + \alpha(A_n + o(1)).$$

Hence, by (iii),  $\limsup \sigma_n \leq \alpha$ , and so  $\limsup \sigma_n \leq \bar{s}$ . The first inequality (1.4) is proved similarly.

If  $M$  is not positive, (1.4) need not be true. However, we have:

(1.5) THEOREM. *Let  $M$  be a regular matrix, and let  $C = \limsup N_i$ . For any  $\{s_\nu\}$  for which  $s$  and  $\bar{s}$  are finite, the numbers  $\underline{\sigma} = \liminf \sigma_n$ ,  $\bar{\sigma} = \limsup \sigma_n$  are both contained in the interval whose end-points are*

$$\frac{1}{2}(s + \bar{s}) \pm C \cdot \frac{1}{2}(\bar{s} - s).$$

In other words,  $\underline{\sigma}$  and  $\bar{\sigma}$  lie in the interval concentric with  $(s, \bar{s})$  and  $C$  times as large.

Let us set  $s_\nu = s'_\nu + s''_\nu$ , where  $s'_\nu = \frac{1}{2}(s + \bar{s})$  for all  $\nu$ . Then  $\limsup |s'_\nu| = \frac{1}{2}(\bar{s} - s)$ . Correspondingly  $\sigma_n = \sigma'_n + \sigma''_n$ , where

$$\sigma'_n \rightarrow \frac{1}{2}(s + \bar{s}), \quad \limsup |\sigma''_n| \leq C \cdot \frac{1}{2}(\bar{s} - s).$$

This completes the proof.

(1.6) THEOREM. *Let  $p_0, p_1, p_2, \dots$  and  $q_0, q_1, q_2, \dots$  be two sequences and let*

$$P_n = p_0 + p_1 + \dots + p_n, \quad Q_n = q_0 + q_1 + \dots + q_n, \quad q_n > 0 \text{ for all } n, \quad Q_n \rightarrow \infty.$$

*Under these conditions, if  $p_n/q_n \rightarrow s$  then  $P_n/Q_n \rightarrow s$ .*

Set  $s_\nu = p_\nu/q_\nu$ ,  $\sigma_n = P_n/Q_n$ . Then

$$\sigma_n = (q_0 s_0 + q_1 s_1 + \dots + q_n s_n)/Q_n.$$

The  $\sigma$ 's here are linear means of the  $s$ 's, and we may verify that the matrix  $M$  is a positive regular matrix. It is therefore enough to apply (1.3).

In particular, taking  $q_\nu = 1$  for all  $\nu$ , we obtain the classical result of Cauchy: if  $s_\nu \rightarrow s$ , then  $(s_0 + s_1 + \dots + s_n)/(n+1) \rightarrow s$ .

Given a sequence  $s_0, s_1, s_2, \dots$  we define, for every  $k = 0, 1, \dots$ , the sequence  $S_0^k, S_1^k, S_2^k, \dots$  by the conditions

$$S_n^0 = s_n, \quad S_n^k = S_0^{k-1} + S_1^{k-1} + \dots + S_n^{k-1} \quad (k = 1, 2, \dots; n = 0, 1, \dots).$$

Similarly, for  $k = 0, 1, 2, \dots$  we define the sequence of numbers  $A_0^k, A_1^k, A_2^k, \dots$  by the conditions

$$A_n^0 = 1, \quad A_n^k = A_0^{k-1} + A_1^{k-1} + \dots + A_n^{k-1} \quad (k = 1, 2, \dots; n = 0, 1, \dots).$$

We say that the sequence  $s_0, s_1, s_2, \dots$  (or the series whose partial sums are  $s_n$ ) is *summable by the  $k$ -th arithmetic mean of Cesàro*, or, briefly, *summable (C,  $k$ )*, to limit (sum)  $s$ , if

$$\lim_{n \rightarrow \infty} S_n^k / A_n^k = s.$$

Summability (C, 0) is ordinary convergence. Summability (C,  $k$ ) of a sequence implies summability (C,  $k + 1$ ) to the same limit (take  $p_n = S_n^k, q_k = A_n^k$  in (1.6)). To find the numerical values of the  $A_n^k$  we use the following proposition: *If*

$$A_n = a_0 + a_1 + \dots + a_n$$

for all  $n$ , and if  $|x| < 1$ , then

$$\sum_{n=0}^{\infty} a_n x^n = (1-x) \sum_{n=0}^{\infty} A_n x^n, \quad (1.7)$$

provided either series converges. For if  $\sum A_n x^n$  converges and if we multiply out the right-hand side and collect similar terms, we obtain the series on the left. Conversely, if  $|x| < 1$  and  $\sum a_n x^n$  converges, then

$$(1-x)^{-1} \sum_0^{\infty} a_n x^n = \sum_0^{\infty} x^n \sum_0^{\infty} a_n x^n = \sum_0^{\infty} A_n x^n,$$

by Cauchy's rule of multiplication of power series, and  $\sum A_n x^n$  converges.

In particular,

$$\begin{aligned} \sum_{n=0}^{\infty} A_n^k x^n &= (1-x)^{-1} \sum_{n=0}^{\infty} A_n^{k-1} x^n = (1-x)^{-2} \sum_{n=0}^{\infty} A_n^{k-2} x^n = \dots = (1-x)^{-(k+1)}, \\ \sum_{n=0}^{\infty} S_n^k x^n &= (1-x)^{-1} \sum_{n=0}^{\infty} S_n^{k-1} x^n = (1-x)^{-2} \sum_{n=0}^{\infty} S_n^{k-2} x^n = \dots = (1-x)^{-k} \sum_{n=0}^{\infty} S_n^0 x^n. \end{aligned}$$

This permits us to restate our definition as follows: *A sequence  $s_0, s_1, \dots$  (or a series  $u_0 + u_1 + u_2 + \dots$  with partial sums  $s_0, s_1, s_2, \dots$ ) is summable (C,  $\alpha$ ) to limit (sum)  $s$ , if*

$$\sigma_n^\alpha = S_n^\alpha / A_n^\alpha \rightarrow s \quad \text{as } n \rightarrow \infty, \quad (1.8)$$

$S_n^\alpha$  and  $A_n^\alpha$  being given by the formulae

$$\left. \begin{aligned} \sum_{n=0}^{\infty} A_n^\alpha x^n &= (1-x)^{-\alpha-1}, \\ \sum_{n=0}^{\infty} S_n^\alpha x^n &= (1-x)^{-\alpha} \sum_{n=0}^{\infty} s_n x^n = (1-x)^{-\alpha-1} \sum_{n=0}^{\infty} u_n x^n. \end{aligned} \right\} \quad (1.9)$$

We may then also write (C,  $\alpha$ )  $\lim s_n = s$ , or (C,  $\alpha$ )  $\sum_0^{\infty} u_n = s$ , as the case may be.

If  $\{\sigma_n^\alpha\}$  is bounded, we say that  $\{s_n\}$  is *bounded (C,  $\alpha$ )*.

In these new definitions  $\alpha$  need no longer be a non-negative integer. The only restriction is that  $\alpha \neq -1, -2, -3, \dots$  (otherwise, as may be seen from the first formula (1.9),  $A_n^\alpha$  is zero for large enough  $n$ ). It turns out, however, that only the case  $\alpha > -1$  is of interest. The numbers  $S_n^\alpha$  and  $\sigma_n^\alpha$  will be called respectively the *Cesàro sums* and *Cesàro means* of order  $\alpha$  of the sequence  $\{s_n\}$  (series  $\Sigma u_n$ ). The  $A_n^\alpha$  are called the *Cesàro numbers*

of order  $\alpha$ . It is useful to remember that in the case of a series  $u_0 + u_1 + u_2 + \dots$  we have  $S_n^{-1} = u_n$ .

From the definitions of  $A_n^\alpha$  and  $S_n^\alpha$  it follows that

$$(i) A_n^{\alpha+\beta+1} = \sum_{\nu=0}^n A_\nu^\alpha A_{n-\nu}^\beta, \quad (ii) S_n^{\alpha+\beta+1} = \sum_{\nu=0}^n S_\nu^\alpha A_{n-\nu}^\beta. \quad (1.10)$$

for all  $\alpha$  and  $\beta$ . In particular, replacing  $\alpha$  by  $\alpha - 1$ ,  $\beta$  by 0,

$$A_n^\alpha = \sum_{\nu=0}^n A_\nu^{\alpha-1}, \quad S_n^\alpha = \sum_{\nu=0}^n S_\nu^{\alpha-1}. \quad (1.11)$$

Hence

$$A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}, \quad S_n^\alpha - S_{n-1}^\alpha = S_n^{\alpha-1}. \quad (1.12)$$

From (1.10) (ii) we get the fundamental formula

$$S_n^\beta = \sum_{\nu=0}^n A_{n-\nu}^{\beta-1} s_\nu = \sum_{\nu=0}^n A_{n-\nu}^\beta u_\nu, \quad (1.13)$$

which shows that

$$\sigma_n^\beta = \sum_{\nu=0}^n \frac{A_{n-\nu}^{\beta-1}}{A_n^\beta} s_\nu = \sum_{\nu=0}^n \frac{A_{n-\nu}^\beta}{A_n^\beta} u_\nu. \quad (1.14)$$

The first formula in (1.9) implies, first,

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} = \binom{n+\alpha}{n} \approx \frac{n^\alpha}{\Gamma(\alpha+1)} \quad (\alpha \neq -1, -2, \dots), \quad (1.15)$$

so that

$$\sum_{\nu=0}^{\infty} |A_\nu^\alpha| < +\infty \quad \text{for } \alpha < -1, \quad (1.16)$$

and secondly

(1.17) THEOREM.  $A_n^\alpha$  is positive for  $\alpha > -1$ , increasing (as a function of  $n$ ) for  $\alpha > 0$  and decreasing for  $-1 < \alpha < 0$ ; and  $A_n^0 = 1$  for all  $n$ . If  $\alpha < -1$ ,  $A_n^\alpha$  is of constant sign for  $n$  large enough.

The  $\Gamma$  in (1.15) is Euler's gamma function, and in fact the formula itself is just Gauss's definition of that function. Later (in Chapter V, § 2) we shall need a slightly more precise formula, namely

$$A_n^\alpha = \frac{n^\alpha}{\Gamma(\alpha+1)} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}. \quad (1.18)$$

To prove it we note that  $\log(1+u) = u + O(u^2)$  for small  $|u|$ , and hence

$$\log A_n^\alpha = \sum_{\nu=1}^n \log \left( 1 + \frac{\alpha}{\nu} \right) = \alpha \sum_{\nu=1}^n \frac{1}{\nu} + \sum_{\nu=1}^n O\left(\frac{1}{\nu^2}\right) = \alpha \log n + \text{const.} + O\left(\frac{1}{n}\right). \quad (1.19)$$

by formula (8.9) of Chapter I and the fact that the remainders of a series with terms  $O(1/\nu^2)$  are  $O(1/n)$ . Comparing the last formula with (1.15) we see that our constant is  $\log\{1/\Gamma(\alpha+1)\}$ , and (1.18) follows from (1.19).

It is useful to note that if  $\alpha$  is a positive integer, then

$$A_n^\alpha = \binom{n+\alpha}{n} = \binom{n+\alpha}{\alpha} = \frac{(n+1)(n+2)\dots(n+\alpha)}{\alpha!}. \quad (1.20)$$

(1.21) THEOREM. If a series is summable (C,  $\alpha$ ),  $\alpha > -1$ , to sum  $s$ , it is also summable (C,  $\alpha+h$ ) to  $s$ , for every  $h > 0$ .

(1·22) THEOREM. If a series  $u_0 + u_1 + \dots$  is summable  $(C, \alpha)$ ,  $\alpha > -1$ , to a finite sum then  $u_n = o(n^\alpha)$ .

Let  $\sigma_n^\alpha$  be the Cesàro means of the series. Then

$$\sigma_n^{\alpha+h} = \left( \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} S_\nu^\alpha \right) / A_n^{\alpha+h} = \left( \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} A_\nu^\alpha \sigma_\nu^\alpha \right) / A_n^{\alpha+h}.$$

Hence the  $\sigma_n^{\alpha+h}$  are linear means of the  $\sigma_n^\alpha$ , and using (1·10) and (1·15) we verify that conditions (i), (ii) and (iii) of regularity are satisfied for  $\alpha > -1$ ,  $h > 0$ . This proves (1·21). Moreover, since the matrix here is positive, the limits of indetermination of the sequence  $\{\sigma_n^{\alpha+h}\}$  are contained between those of  $\{\sigma_n^\alpha\}$ .

To prove (1·22) we write

$$u_n / A_n^\alpha = \left( \sum_{\nu=0}^n A_{n-\nu}^{-\alpha-2} S_\nu^\alpha \right) / A_n^\alpha = \left( \sum_{\nu=0}^n A_{n-\nu}^{-\alpha-2} A_\nu^\alpha \sigma_\nu^\alpha \right) / A_n^\alpha.$$

Suppose that  $\sigma_\nu^\alpha \rightarrow 0$  (by subtracting a constant from  $u_0$ , we may assume that  $u_0 + u_1 + \dots$  is summable  $(C, \alpha)$  to 0). We have to show that the coefficients of the  $\sigma_\nu^\alpha$  here satisfy conditions (i) and (ii) (condition (iii) is superfluous since  $\sigma_\nu^\alpha \rightarrow 0$ ). Condition (i) follows from (1·15), since  $\alpha > -1$ . To prove (ii) suppose first that  $\alpha \geq 0$ . Then, since  $A_n^\alpha \geq A_\nu^\alpha > 0$ , we have

$$N_n \leq \sum_{\nu=0}^n |A_{n-\nu}^{-\alpha-2}| = \sum_{\nu=0}^n |A_\nu^{-\alpha-2}| \leq \sum_{\nu=0}^{\infty} |A_\nu^{-\alpha-2}| < \infty$$

(cf. (1·16)). If  $-1 < \alpha < 0$ , then  $A_0^{-\alpha-2} = 1$ ,  $A_\nu^{-\alpha-2} < 0$  for  $\nu > 0$ , and using (1·11) we verify that  $N_n = 2$  for all  $n$ . This proves (1·22).

(1·23) THEOREM. Under the hypotheses of (1·22), if  $\gamma < \alpha$  then  $S_n^\gamma = o(n^\alpha)$ .

For  $\gamma = -1$ , this is Theorem (1·22). (1·23) follows from the preceding argument applied to the formula

$$S_n^\gamma / A_n^\alpha = \left( \sum_{\nu=0}^n A_{n-\nu}^{\gamma-\alpha-1} A_\nu^\alpha \sigma_\nu^\alpha \right) / A_n^\alpha.$$

For  $\gamma \neq -1, -2, \dots$  the conclusion may be written  $\sigma_n^\gamma = o(n^{\alpha-\gamma})$ .

Consider a series  $u_0 + u_1 + \dots$ . Its partial sum will be denoted by  $s_n$ , its first arithmetic means by  $\sigma_n$ . Thus

$$\sigma_n = \frac{s_0 + s_1 + \dots + s_n}{n+1} = \sum_{\nu=0}^n \left( 1 - \frac{\nu}{n+1} \right) u_\nu. \quad (1·24)$$

It is often useful to consider the difference

$$\Delta_n = s_n - \sigma_n = \frac{u_1 + 2u_2 + \dots + nu_n}{n+1}. \quad (1·25)$$

If  $\Delta_n \rightarrow 0$ , and if the series is summable  $(C, 1)$ , it is also convergent. In particular, a series  $u_0 + u_1 + \dots$  summable  $(C, 1)$  and having terms  $u_\nu = o(1/\nu)$  is convergent†. If  $u_\nu = O(1/\nu)$  and if the series is bounded  $(C, 1)$ , the partial sums of the series are bounded. Less obvious is the following:

(1·26) THEOREM OF HARDY. If  $u_0 + u_1 + \dots$  is summable  $(C, 1)$  and if  $u_\nu = O(1/\nu)$ , the series is convergent.

† Another useful corollary of (1·25) is: if  $\Sigma u_n$  converges then  $u_1 + 2u_2 + \dots + nu_n = o(n)$ .

We shall return to this presently. Meanwhile we observe that the condition  $\Delta_n \rightarrow 0$  may be satisfied in cases other than  $u_\nu = o(1/\nu)$ . We note two such cases:

(a)  $\sum \nu |u_\nu|^2 = M$  is finite;

(b)  $u_\nu \rightarrow 0, u_\nu = 0$  if  $\nu$  does not belong to a sequence  $n_1 < n_2 < n_3 < \dots$  of integers satisfying  $n_{k+1}/n_k > q$ , where  $q$  is fixed and greater than 1.

Suppose (a) is satisfied. By Schwarz's inequality,

$$|\Delta_n| \leq \frac{1}{n+1} \sum_{\nu=1}^n \nu^{\frac{1}{2}} |u_\nu| \nu^{\frac{1}{2}} \leq \frac{1}{n+1} \left( \sum_{\nu=1}^n \nu |u_\nu|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu=1}^n \nu \right)^{\frac{1}{2}} \leq \left( \sum_{\nu=1}^{\infty} \nu |u_\nu|^2 \right)^{\frac{1}{2}},$$

so that  $\limsup |\Delta_n| \leq M^{\frac{1}{2}}$ . But  $\limsup |\Delta_n|$  is not affected if we replace each of  $u_0, u_1, u_2, \dots, u_k$  by zeros. Taking  $k$  large enough we may make  $M$  arbitrarily small; thus  $\lim \Delta_n = 0$ .

We may replace (a) by (a')  $\sum \nu^p |u_\nu|^{p+1} < \infty$ ,

where  $p > 0$ . The conclusion and proof hold, if in the latter we use Hölder's inequality instead of Schwarz's.

In case (b), let  $|u_{n_\nu}| = \epsilon_\nu$  and let  $n_k \leq n < n_{k+1}$ . Then

$$|\Delta_n| \leq (n_k + 1)^{-1} \sum_{\nu=1}^k n_\nu \epsilon_\nu \leq \sum_{\nu=1}^k (n_\nu/n_k) \epsilon_\nu \leq \sum_{\nu=1}^k \epsilon_\nu q^{\nu-k},$$

and the sum on the right is a linear mean of  $\{\epsilon_\nu\}$ . Conditions (i) and (ii) of regularity are satisfied, so that  $\epsilon_\nu \rightarrow 0$  implies  $\Delta_n \rightarrow 0$ .

A series  $\sum u_\nu$  will be said to possess a gap  $(p, q)$  if  $c_\nu = 0$  for  $p < \nu \leq q$ . Case (b) may be generalized as follows.

(1.27) THEOREM. If a series  $\sum u_\nu$ , with partial sums  $s_n$ , has infinitely many gaps  $(m_k, m'_k)$  such that  $m'_k/m_k \geq q > 1$ , and is summable  $(C, 1)$  to sum  $s$ , then  $s_{m_k} \rightarrow s$  (and so also  $s_{m'_k} \rightarrow s$ ).

We may suppose that  $s = 0$ . Since  $s_0 + s_1 + \dots + s_n = (n+1)\sigma_n$ , we have

$$(m'_k - m_k) s_{m_k} = s_{m_k} + s_{m_k+1} + \dots + s_{m'_k-1} \\ = m'_k \sigma_{m'_k-1} - m_k \sigma_{m_k-1} = o(m'_k) + o(m_k) = o(m'_k) = o(m'_k - m_k), \quad (1.28)$$

whence  $s_{m_k} = o(1)$  and (1.27) follows.

If we assume nothing about the summability of  $\sum u_\nu$  and set

$$s^* = \sup_k |s_{m_k}|, \quad \sigma^* = \sup_n |\sigma_n|,$$

the identity  $(m'_k - m_k) s_{m_k} = m'_k \sigma_{m'_k-1} - m_k \sigma_{m_k-1}$  gives

$$|s_{m_k}| \leq \sigma^* \frac{m'_k + m_k}{m'_k - m_k} \leq \frac{q+1}{q-1} \sigma^*,$$

and so

$$s^* \leq A_q \sigma^*, \quad (1.29)$$

where  $A_q$  depends on  $q$  only.

Let  $k$  be a positive integer. Consider the expression

$$\sigma_{n,k} = \frac{s_n + s_{n+1} + \dots + s_{n+k-1}}{k} = \frac{(n+k)\sigma_{n+k-1} - n\sigma_{n-1}}{k} = \left(1 + \frac{n}{k}\right) \sigma_{n+k-1} - \frac{n}{k} \sigma_{n-1}. \quad (1.30)$$

We verify that 
$$\sigma_{n,k} = s_n + \sum_{\nu=n+1}^{n+k-1} \left(1 - \frac{\nu-n}{k}\right) u_\nu. \quad (1.31)$$

(If  $k=1$ , we interpret the sum  $\Sigma$  on the right as 0.) If  $k$  tends to  $\infty$  with  $n$  in such a way that  $n/k$  is bounded, then  $\sigma_{n,k}$  defines a method of summability which is at least as strong as the  $(C, 1)$  method: if  $\sigma_n \rightarrow s$ , then  $\sigma_{n,k} \rightarrow s$ . This follows from (1.30) if we set  $\sigma_\nu = s + \epsilon_\nu$ ,  $\epsilon_\nu \rightarrow 0$ .

The peculiarity of the method is that  $\sigma_{n,k}$  is obtained from  $s_n$  by adding to it a linear combination, with coefficients positive and less than 1, of the terms  $u_{n+1}, u_{n+2}, \dots, u_{n+k-1}$  (cf. (1.31)); it is useful in certain applications. The case

$$\sigma_{n,n} = 2\sigma_{2n-1} - \sigma_{n-1}$$

is particularly simple. We may call the  $\sigma_{n,k}$  the *delayed first arithmetic means*. We observe that  $\sigma_{n,1} = s_n$ ,  $\sigma_{0,n} = \sigma_{n-1}$ .

Returning to (1.26), suppose that  $\sigma_\nu \rightarrow s$ ,  $|u_\nu| < A/\nu$  for  $\nu = 1, 2, \dots$ . By the remark just made,

$$|\sigma_{n,k} - s_n| \leq \sum_{\nu=n+1}^{n+k-1} |u_\nu| \leq A \sum_{\nu=n+1}^{n+k-1} \frac{1}{\nu} < A \frac{k-1}{n}.$$

Let  $\epsilon$  be any positive number, and let  $k = [n\epsilon] + 1$ .† The last expression is then  $A[n\epsilon]/n \leq A\epsilon$ . Since  $n/k < n/n\epsilon = 1/\epsilon$  is bounded,  $\sigma_{n,k} \rightarrow s$ , so that  $\limsup |\sigma_n - s| \leq A\epsilon$ . This gives,  $\epsilon$  being arbitrary,  $\lim s_n = s$ .

The series  $u_0 + u_1 + \dots$  is said to be summable by *Abel's method* (sometimes called *Poisson's*), or *summable A*, to sum  $s$ , if the series  $u_0 + u_1 x + u_2 x^2 + \dots$  is convergent for  $|x| < 1$  and if

$$\lim_{x \rightarrow 1-0} \sum_{\nu=0}^{\infty} u_\nu x^\nu = s, \quad (1.32)$$

where  $x$  tends to 1 along the real axis. By (1.7), summability A of a sequence  $\{s_\nu\}$  may be defined as the existence of

$$\lim_{x \rightarrow 1-0} (1-x) \sum_{\nu=0}^{\infty} s_\nu x^\nu.$$

(1.33) THEOREM. *If the series  $u_0 + u_1 + \dots$  is summable  $(C, \alpha)$ ,  $\alpha > -1$ , to sum  $s$  (finite or not), it is also summable A to  $s$ .*

For let  $f(x) = u_0 + u_1 x + u_2 x^2 + \dots$ , and let  $\{x_n\}$  be any sequence of points on the real axis tending to 1 from the left. By (1.9),

$$f(x_n) = (1-x_n)^{\alpha+1} \sum_{\nu=0}^{\infty} S_\nu^\alpha x_n^\nu = (1-x_n)^{\alpha+1} \sum_{\nu=0}^{\infty} \sigma_\nu^\alpha A_\nu^\alpha x_n^\nu.$$

The expression on the right is a linear mean of the sequence  $\{\sigma_\nu^\alpha\}$ , and corresponds to the matrix M with  $a_{n\nu} = A_\nu^\alpha (1-x_n)^{\alpha+1} x_n^\nu$ . This matrix is positive and satisfies conditions (i) and (iii). Hence  $f(x_n) \rightarrow s$ , and (1.33) is proved.

The argument also shows that the *limits of indetermination by the method A* (i.e.  $\liminf_{x \rightarrow 1} f(x)$  and  $\limsup_{x \rightarrow 1} f(x)$ ) are contained between those by the method  $(C, \alpha)$ .

(1.34) THEOREM. *If  $u_0 + u_1 + \dots$  is summable  $(C, \alpha)$ ,  $\alpha > -1$ , to a finite sum  $s$ , then (1.32) holds as  $x$  tends to 1 along any path  $L$  lying between two chords of the unit circle which pass through  $x = 1$ .*

† By  $[x]$  we denote the integral part of  $x$ .

Such paths  $L$  will hereafter be spoken of as *non-tangential*. In the neighbourhood of the point 1 they are characterized by an inequality

$$|1 - x|/(1 - |x|) \leq \text{const.} \tag{1.35}$$

For the proof of (1.34) we observe that, if  $\{x_n\}$  is any sequence of points tending to 1 along  $L$ , then  $f(x_n)$  is as before a linear mean of the  $\sigma_r^\alpha$  generated by a matrix  $M$  satisfying conditions (i) and (iii).  $M$  is no longer positive, but

$$\sum_{\nu=0}^{\infty} |a_{n\nu}| = \sum_{\nu=0}^{\infty} A_{\nu}^{\alpha} |1 - x_n|^{\alpha+1} |x_n|^{\nu} = |1 - x_n|^{\alpha+1} / (1 - |x_n|)^{\alpha+1}$$

is bounded, by (1.35). This proves (1.34).

(1.36) **THEOREM OF TAUBER.** *Let  $s_n$  and  $f(x)$  denote the partial sums and the Abel means of a series  $\sum u_n$  with terms  $o(1/n)$ . Then, if  $N = \left[ \frac{1}{1-x} \right]$ , we have*

$$f(x) - s_N \rightarrow 0 \text{ as } x \rightarrow 1 - 0. \tag{1.37}$$

In particular, the series is Abel summable if and only if it converges.

The relation (1.37) still holds if the condition  $u_n = o(1/n)$  is replaced by

$$u_1 + 2u_2 + \dots + nu_n = o(n).$$

Suppose first that  $\eta_n = nu_n \rightarrow 0$ . The left-hand side in (1.37) is

$$\sum_1^N u_n(x^n - 1) + \sum_{N+1}^{\infty} u_n x^n = P + Q,$$

say. Observing that  $N \leq 1/(1-x) < N+1$  and  $1-x^n \leq n(1-x)$ , we have

$$|P| \leq (1-x) \sum_1^N |\eta_n| \leq N^{-1} \sum_1^N |\eta_n| \rightarrow 0,$$

$$|Q| \leq \sum_{N+1}^{\infty} \frac{|\eta_n|}{n} x^n \leq (N+1)^{-1} \max_{n>N} |\eta_n| \sum_{N+1}^{\infty} x^n \leq \frac{1}{(N+1)(1-x)} \max_{n>N} |\eta_n| \rightarrow 0.$$

Hence  $P + Q \rightarrow 0$ , and (1.37) follows.

Let now  $v_0 = 0$ ,  $v_n = u_1 + 2u_2 + \dots + nu_n$  for  $n > 0$ , and suppose that  $v_n = o(n)$ . Summation by parts gives

$$\sum_0^n u_k = u_0 + \sum_{k=1}^n \frac{v_k - v_{k-1}}{k} = u_0 + \sum_{k=1}^n \frac{v_k}{k(k+1)} + \frac{v_n}{n+1}.$$

Since  $v_n = o(n)$ , the series  $\sum_0^{\infty} u_k$  and  $u_0 + \sum_1^{\infty} v_k/k(k+1)$  are equiconvergent. If, therefore,  $t_n$  and  $g(x)$  are the partial sums and the Abel means of the second series, we have  $s_N - t_N \rightarrow 0$ ,  $f(x) - g(x) \rightarrow 0$ . But the terms of the second series are  $o(1/n)$ , so that, by the case already dealt with we have  $g(x) - t_N \rightarrow 0$ . This and the preceding two relations imply (1.37).

The following theorem partly generalizes (1.36).

(1.38) **THEOREM OF LITTLEWOOD.** *A series  $\sum_0^{\infty} u_n$  summable A and with terms  $O(1/n)$  converges.*

We may suppose that  $|u_n| \leq 1/n$  for  $n > 0$  and that  $f(x) = \sum u_n x^n \rightarrow 0$  as  $x \rightarrow 1$ . If we substitute here  $x^k$  for  $x$  ( $k=1, 2, \dots$ ), we see that for every power polynomial  $P(x)$  without constant term we have

$$\sum_0^{\infty} u_n P(x^n) \rightarrow 0$$

as  $x \rightarrow 1$ .

Suppose that given any two numbers  $0 < \xi' < \xi < 1$  and a positive  $\delta$  we can find a  $P(x)$  such that

- (i)  $0 \leq P(x) \leq 1$  in  $(0, 1)$ ,
- (ii)  $P(x) \leq \delta x$  in  $(0, \xi')$ ,
- (iii)  $1 - P(x) \leq \delta(1 - x)$  in  $(\xi, 1)$ .

We show that we can then prove the convergence of  $\sum u_n$ . Given any  $0 < x < 1$ , let  $N = N(x)$  be the greatest non-negative integer satisfying  $x^N \geq \xi$ , and let  $N' = N'(x)$  be the least positive integer satisfying  $x^{N'} \leq \xi'$ . Both  $N$  and  $N'$  are non-decreasing functions of  $x$  taking successively all values  $1, 2, 3, \dots$  as  $x \rightarrow 1$ . Clearly  $N < N'$  and

$$N \simeq \frac{\log(1/\xi)}{\log(1/x)}, \quad N' \simeq \frac{\log(1/\xi')}{\log(1/x)}.$$

For a  $P$  satisfying (i), (ii), (iii) we have,

$$\begin{aligned} \sum_0^{\infty} u_n P(x^n) - s_N &= \sum_1^N u_n \{P(x^n) - 1\} + \sum_{N+1}^{N'} u_n P(x^n) + \sum_{N'+1}^{\infty} u_n P(x^n) \\ &= A(x) + B(x) + C(x), \end{aligned}$$

say, and

$$|A(x)| \leq \delta \sum_1^N \frac{1}{n} (1 - x^n) \leq \delta \sum_1^N (1 - x) = \delta N(1 - x).$$

$$|C(x)| \leq \delta \sum_{N'+1}^{\infty} \frac{x^n}{n} < \frac{\delta}{N'(1-x)},$$

$$|B(x)| \leq \sum_{N+1}^{N'} \frac{1}{n} < \frac{N' - N}{N}.$$

Take any  $\epsilon > 0$ . From the asymptotic expressions for  $N$  and  $N'$  we see that if  $\xi'$  and  $\xi$  are sufficiently close to each other and both away from 0 and 1 (we may take  $\xi'$  and  $\xi$  symmetrically with respect to  $\frac{1}{2}$ ), we have  $\limsup |B(x)| \leq \epsilon$ . Having fixed  $\xi'$  and  $\xi$ , and observing that  $N(1-x)$  and  $N'(1-x)$  tend to finite non-zero limits, we obtain  $\limsup |A(x)| < \epsilon$ ,  $\limsup |C(x)| < \epsilon$ , if  $\delta$  is small enough. Since  $\sum u_n P(x^n) \rightarrow 0$ , we get  $\limsup |s_N| \leq 3\epsilon$ , that is  $s_N \rightarrow 0$ .

It remains to construct the required  $P$ , and it is enough to assume that  $\xi' = \frac{1}{2} - \eta$ ,  $\xi = \frac{1}{2} + \eta$ . Write

$$R_k(x) = \{4x(1-x)\}^k, \quad P(x) = \int_0^x R_k(t) dt \Big/ \int_0^1 R_k(t) dt,$$

where  $k$  is a positive integer. Clearly  $P$  satisfies (i). Since  $R_k(x) \leq (1 - 4\eta^2)^k$  in  $(0, 1)$  but outside  $(\frac{1}{2} - \eta, \frac{1}{2} + \eta)$ , and

$$\int_0^1 R_k(t) dt \geq \int_{\frac{1}{2}-\eta}^{\frac{1}{2}+\eta} \left(1 - \frac{1}{k^2}\right)^k dt \simeq \frac{1}{k},$$

we find that

$$P(x) \leq x \max_{0 \leq x \leq \frac{1}{2} - \eta} R_k(x) \int_0^1 R_k(t) dt \leq \delta x$$

for  $0 \leq x \leq \frac{1}{2} - \eta$  and  $k$  large enough. This is (ii), and (iii) is proved similarly.

Return for a moment to (1.2). If the sequence  $\sigma_n$  in (1.1) is of bounded variation, we say that  $\{s_n\}$  (or the series whose partial sums are  $s_n$ ) is *absolutely summable M*. Only absolute summability A, however, is of interest for trigonometric series. The parameter  $n$  in this case is a continuous variable and the definition must be modified in an obvious way: a series  $\Sigma u_n$  is absolutely summable A if the function  $f(r) = \Sigma u_n r^n$  is of bounded variation over  $0 \leq r < 1$ . Every absolutely convergent series with, say, real terms is a difference of two convergent series with non-negative terms, and, correspondingly,  $f(r)$  is a difference of two non-decreasing bounded functions. Hence every absolutely convergent series is absolutely summable A.

Parallel to the theory of divergent series, one can construct a theory of divergent integrals. We shall only consider the analogue of the method (C, 1). Suppose we have a function  $A(u)$  defined for  $u \geq 0$  and integrable over every finite interval  $0 \leq u \leq u_0$ . We say that the function  $A(u)$  tends to limit  $s$ , as  $u \rightarrow \infty$ , by the method of the first arithmetic mean, if

$$u^{-1} \int_0^u A(v) dv \rightarrow s \quad \text{as } u \rightarrow +\infty. \quad (1.39)$$

We shall then write (C, 1)  $\lim_{u \rightarrow \infty} A(u) = s$ . It is easily seen that if  $A(u)$  tends to  $s$  as  $u \rightarrow \infty$ , then we also have (1.39).

Consider an integral

$$\int_0^\infty a(v) dv, \quad (1.40)$$

where  $a(v)$  is integrable over every finite interval  $0 \leq v \leq v_0$ . We shall say that (1.40) is *summable (C, 1) to  $s$*  if the partial integrals  $A(u) = \int_0^u a(v) dv$  satisfy (1.39), and we shall write

$$(C, 1) \int_0^\infty a(v) dv = s.$$

The latter relation is satisfied if (1.40) converges to  $s$ , that is, if  $A(u) \rightarrow s$  as  $u \rightarrow +\infty$ . As an example one easily verifies that

$$\int_0^\infty e^{ixv} dv$$

is summable (C, 1) to sum  $ix^{-1}$  or  $+\infty$ , according as  $x \neq 0$  or  $x = 0$ .

Consider a series  $a_0 + a_1 + \dots$ . Generalizing the notion of the partial sum  $A_n = a_0 + \dots + a_n$ , we shall introduce that of *sum function*  $A(u)$ , defined by the formula

$$A(u) = \sum_{n < u} a_n \quad (u \geq 0).$$

Thus  $A(u)$  is a step function such that  $A(u) = A_n$  for  $n \leq u < n+1$ . At  $u = n$ ,  $A(u)$  has jump  $a_n$ .

(1.41) THEOREM. A necessary and sufficient condition for  $a_0 + a_1 + \dots$  to be summable (C, 1) to finite sum  $s$  is that (C, 1)  $\lim_{u \rightarrow \infty} A(u) = s$ .

For suppose that  $n \leq u < n+1$ . Then, denoting by  $s_n$  the partial sums of the series,

$$\frac{1}{u} \int_0^u A(t) dt = \frac{s_0 + \dots + s_{n-1} + (u-n)s_n}{u} = \frac{n}{u} \sigma_{n-1} + \frac{u-n}{u} s_n. \quad (1.42)$$

If  $\sigma_n \rightarrow s$ , then  $s_n = o(n)$ , and so the right-hand side of (1.42) tends to  $s$ . The converse follows by substituting  $u = n$  in (1.42).

The definition of summability  $(C, 1)$  of  $a_0 + a_1 + \dots$  in terms of the sum function  $A(u)$  is often useful. Expressing the second member of (1.42) in terms of the  $a$ 's, we get

$$\frac{1}{u} \int_0^u A(t) dt = \sum_{\nu < u} a_\nu \left(1 - \frac{\nu}{u}\right), \quad (1.43)$$

a formula analogous to (1.24). (1.43) may be called the *integral*  $(C, 1)$  *mean* of  $\Sigma a_n$ .

*Remark.* In the foregoing discussion of summability we confined ourselves to series of constants. Analogous results hold for series of functions and various modes of convergence or summability, bounded, uniform, etc. For example, if a series  $\Sigma u_\nu(t)$  is uniformly summable  $(C, \alpha)$ ,  $\alpha > -1$ , on a set  $E$  of points  $t$ , it is also uniformly summable  $(C, \beta)$ ,  $\beta > \alpha$ , and uniformly summable  $A$ , on  $E$ . We shall use such results in the sequel without stating them explicitly. Changes of argument that the extensions call for can easily be supplied by the reader.

## 2. General remarks about the summability of $S[f]$ and $\tilde{S}[f]$

Given a sequence  $s_0, s_1, \dots$ , consider the linear means

$$\sigma_n = a_{n0}s_0 + a_{n1}s_1 + \dots + a_{nk}s_k + \dots \quad (2.1)$$

generated by a matrix  $M$  satisfying conditions (i), (ii) and (iii) of regularity (see § 1). If the  $s_k$  are the partial sums of a series  $\Sigma u_k$ , and we write  $s_k = u_0 + u_1 + \dots + u_k$  in (2.1) we get

$$\sigma_n = \alpha_{n0}u_0 + \alpha_{n1}u_1 + \dots + \alpha_{nk}u_k + \dots, \quad (2.2)$$

where  $\alpha_{nk} = a_{nk} + a_{n, k+1} + \dots$ . On the right here we have linear means of the series  $\Sigma u_k$ .

The passage from (2.1) to (2.2) can be justified under very general conditions (it is trivially true if the matrix  $M$  is row-finite). We shall not do so here since in all the cases which interest us either the justification is immediate or, as happens for instance in the case of Abel summability, the form (2.2) is simpler and more natural than (2.1). We shall take (2.2) as a fresh starting-point and apply the idea to Fourier series.

We recall that the case when the parameter  $n$  is a continuous variable may be reduced to the standard case by considering discrete sequences of the parameter, and in what follows we shall not dwell on this point.

In all cases which interest us the  $\alpha_{nk}$  satisfy the condition

$$\sum_k |\alpha_{nk}| < \infty \quad (2.3)$$

for all  $n$ , and we assume this once for all for the sake of simplicity. It is automatically satisfied if the matrix is row-finite.

A very important role is played by the linear means of the two basic series

$$\frac{1}{2} + \cos t + \cos 2t + \dots, \quad (2.4)$$

$$0 + \sin t + \sin 2t + \dots \quad (2.5)$$

These means will be denoted by  $K_n(t)$  and  $\tilde{K}_n(t)$ :

$$K_n(t) = \frac{1}{2}\alpha_{n_0} + \sum_{k=1}^{\infty} \alpha_{nk} \cos kt, \tag{2.6}$$

$$\tilde{K}_n(t) = \sum_{k=1}^{\infty} \alpha_{nk} \sin kt. \tag{2.7}$$

They are both continuous and are even and odd functions respectively of  $t$ . If the linear forms are given in the form (2.1) then, clearly,

$$K_n(t) = \sum_{k=0}^{\infty} \alpha_{nk} D_k(t) = \frac{1}{2 \sin \frac{1}{2}t} \sum_{k=0}^{\infty} \alpha_{nk} \sin (k + \frac{1}{2})t, \tag{2.8}$$

$$\tilde{K}_n(t) = \sum_{k=0}^{\infty} \alpha_{nk} \tilde{D}_k(t) = A_n \cdot \frac{1}{2} \cot \frac{1}{2}t - \frac{1}{2 \sin \frac{1}{2}t} \sum_{k=0}^{\infty} \alpha_{nk} \cos (k + \frac{1}{2})t, \tag{2.9}$$

where  $D_k$  and  $\tilde{D}_k$  denote the Dirichlet and the conjugate Dirichlet kernel and  $A_n = \alpha_{n_0} + \alpha_{n_1} + \dots$ .

It is customary to call linear means of (2.4) *kernels* corresponding to method M. Linear means of (2.5) are called *conjugate kernels*.

Let  $a_k, b_k$  be the Fourier coefficients of an  $f$ . The linear means for  $S[f]$  and  $\tilde{S}[f]$  are

$$\sigma_n(x) = \sigma_n(x; f) = \frac{1}{2}\alpha_0 \alpha_{n_0} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \alpha_{nk}, \tag{2.10}$$

$$\tilde{\sigma}_n(x) = \tilde{\sigma}_n(x; f) = \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) \alpha_{nk}. \tag{2.11}$$

Under the hypothesis (2.3) we have

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n(t) dt, \tag{2.12}$$

$$\tilde{\sigma}_n(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \tilde{K}_n(t) dt. \tag{2.13}$$

For the left-hand side of (2.12) is

$$\begin{aligned} \alpha_{n_0} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{k=1}^{\infty} \alpha_{nk} \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos k(t-x) dt &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[ \frac{1}{2}\alpha_{n_0} + \sum_{k=1}^{\infty} \alpha_{nk} \cos k(t-x) \right] dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) K_n(t-x) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n(t) dt, \end{aligned}$$

the interchange of the order of integration and summation being justified by (2.3); and similarly we prove (2.13).

We shall always assume that

$$\alpha_{n_0} = 1 \quad \text{for } n = 0, 1, \dots \tag{2.14}$$

(Observe that in any case  $\alpha_{n_0} \rightarrow 1$ , if the linear means (2.2) of the series  $1 + 0 + 0 + \dots$ , converging to 1 are also to tend to 1.) We shall call (2.14) *condition (A)*. It can also be written

$$(A) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) dt = 1.$$

If it is satisfied, then 
$$\sigma_n(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \phi_x(t) K_n(t) dt, \tag{2.15}$$

since the left-hand side is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n(t) dt - \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) K_n(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \{f(x+t) - f(x)\} K_n(t) dt,$$

and the last integral equals the right-hand side of (2.15) since  $K_n(t)$  is even.

If  $K_n$  satisfies conditions (A) and (B)  $K_n \geq 0$

for all  $n$ , we shall call  $K_n$  a *positive kernel*. If we have merely

$$(B') \quad \frac{1}{\pi} \int_{-\pi}^{\pi} |K_n(t)| dt \leq C,$$

with  $C$  independent of  $n$ , we shall call the kernel  $K_n$  *quasi-positive*. Every positive kernel is quasi-positive, as we see from condition (A).

The following theorem is an immediate corollary of (2.12):

(2.16) THEOREM. *If  $K_n$  is a positive kernel, then for any  $f$  satisfying*

$$m \leq f \leq M$$

we have (2.17)

$$m \leq \sigma_n(x; f) \leq M.$$

If  $K_n$  is quasi-positive,  $|f| \leq M$  implies

$$|\sigma_n(x; f)| \leq CM, \quad (2.18)$$

with the same  $C$  as in condition (B').

We now introduce a condition (c) in addition to (A) and (B), or (B'), so far considered.

Let

$$\mu_n(\delta) = \max_{\delta < t \leq \pi} |K_n(t)| \quad (0 < \delta \leq \pi); \quad (2.19)$$

we shall say that the kernels  $K_n$  satisfy *condition (c)*, if

$$(c) \quad \mu_n(\delta) \rightarrow 0 \quad \text{for each fixed } \delta \quad (0 < \delta \leq \pi)$$

Condition (c) implies that (2.4) is uniformly summable  $M$  to zero outside an arbitrarily small neighbourhood of  $t=0$ .

If condition (c) is satisfied, then the decomposition (see (2.12))

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\delta}^{\delta} f(x+t) K_n(t) dt + \frac{1}{\pi} \int_{\delta < |t| \leq \pi} f(x+t) K_n(t) dt, \quad (2.20)$$

in which the last term is numerically majorized by

$$\mu_n(\delta) \frac{1}{\pi} \int_{\delta < |t| \leq \pi} |f(x+t)| dt \leq \mu_n(\delta) \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t)| dt,$$

shows that the behaviour of  $\sigma_n(x)$  at the point  $x$  depends solely on the values of  $f$  in an arbitrarily small neighbourhood  $(x-\delta, x+\delta)$ . We know, of course, that this result holds also for the Dirichlet kernel, which does not satisfy condition (c), but in the new case the last integral in (2.20) is uniformly small for all functions  $f$  such that  $\int_{-\pi}^{\pi} |f(t)| dt$  is bounded.

(2.21) THEOREM. *Suppose that the kernel  $K_n$  satisfies conditions (A), (B)—or merely (B')—and (c). For any integrable  $f$ , if the numbers  $f(x_0 \pm 0)$  exist and are finite, then we have*

$$\sigma_n(x_0) \rightarrow \frac{1}{2} \{f(x_0+0) + f(x_0-0)\}. \quad (2.22)$$

In particular, if  $f$  is continuous at  $x_0$ , we have

$$\sigma_n(x_0) \rightarrow f(x_0). \quad (2.23)$$

If  $f$  is continuous at every point of a closed interval  $I = (\alpha, \beta)$ , the relation (2.23) holds uniformly in  $x_0 \in I$ . In particular, if  $f$  is everywhere continuous we have (2.23) uniformly in  $x_0$ .

Suppose first that  $K_n$  is positive. By changing if necessary the value of  $f(x_0)$  we may suppose that  $f(x_0) = \frac{1}{2}[f(x_0 + 0) + f(x_0 - 0)]$ , so that

$$|\phi_{x_0}(t)| < \frac{1}{2}\epsilon \quad \text{for } 0 \leq t \leq \delta = \delta(\epsilon). \quad (2.24)$$

By (2.15),  $|\sigma_n(x_0; f) - f(x_0)|$  does not exceed

$$\begin{aligned} \frac{2}{\pi} \int_0^\pi |\phi(t)| K_n(t) dt &= \frac{2}{\pi} \left( \int_0^\delta + \int_\delta^\pi \right) \leq \frac{6}{\pi} \int_0^\delta K_n(t) dt + \frac{2\mu_n(\delta)}{\pi} \int_\delta^\pi |\phi| dt \\ &< \frac{6}{\pi} \int_0^\pi K_n(t) dt + \frac{2\mu_n(\delta)}{\pi} \int_0^\pi |\phi(t)| dt = P + Q, \end{aligned}$$

say. Here

$$P = \frac{1}{2}\epsilon \quad (2.25)$$

by condition (A), and

$$Q \rightarrow 0 \quad (2.26)$$

by condition (C), so that

$$P + Q < \epsilon \quad \text{for } n > n_0, \quad (2.27)$$

which proves the first part of (2.21).

If  $f$  is continuous at every point of  $I$  (which is understood to mean that  $f$  is also continuous to the left at  $x = \alpha$  and to the right at  $x = \beta$ ), we can find a  $\delta$  independent of  $x_0$  such that (2.24) holds for all  $x_0 \in I$ . As before we have (2.25). The integral in  $Q$  does not exceed

$$\int_0^\pi (|f(x_0 + t)| + |f(x_0 - t)| + 2|f(x_0)|) dt = \int_{-\pi}^\pi |f(t)| dt + 2\pi |f(x_0)|.$$

(2.26) holds uniformly in  $I$ , and (2.27) holds for all  $x_0 \in I$ .

Only minor modifications are needed in the preceding argument if the kernel  $K_n$  is quasi-positive. In the inequality for  $|\sigma_n - f|$  we have to replace  $K_n$  by  $|K_n|$ , which gives, with the previous notation,

$$P \leq \frac{1}{2}C\epsilon, \quad Q \rightarrow 0,$$

and the conclusion follows as before.

(2.28) THEOREM. Suppose that a positive kernel  $K_n$  satisfies condition (c), and that  $m \leq f(x) \leq M$  for  $x \in I = (a, b)$ . Then for any  $\epsilon > 0$  and  $0 < \delta < \frac{1}{2}(b - a)$  there is an  $n_0$  such that

$$m - \epsilon \leq \sigma_n(x) \leq M + \epsilon \quad \text{for } x \in I_\delta = (a + \delta, b - \delta), \quad n > n_0. \quad (2.29)$$

It is enough to prove the second inequality. In view of the remark concerning the decomposition (2.20), we have

$$\sigma_n(x) = \frac{1}{\pi} \int_{-\delta}^\delta f(x+t) K_n(t) dt + o(1),$$

where the  $o(1)$  is uniform in  $x$ . Suppose that  $x \in I_\delta$ . Then  $x + t \in I$  for  $|t| < \delta$ , the integral last written does not exceed

$$M \frac{1}{\pi} \int_{-\delta}^\delta K_n(t) dt \leq M \frac{1}{\pi} \int_{-\pi}^\pi K_n(t) dt = M,$$

and the result follows.

Given a function  $f(x)$ , let  $M(a, b)$  and  $m(a, b)$  be the upper and lower bounds of  $f$  in the interval  $a < x < b$ . For every  $x$  we set

$$M(x) = \lim_{h \rightarrow +0} M(x-h, x+h), \quad m(x) = \lim_{h \rightarrow +0} m(x-h, x+h).$$

These limits exist, since the expressions under the limits  $h \rightarrow +0$  are monotone functions of  $h$ . We shall call  $M(x)$  and  $m(x)$  the *maximum* and *minimum* of  $f$  at the point  $x$ .

**(2.30) THEOREM.** *If a positive kernel  $K_n$  satisfies condition (c), then for any sequence  $\{x_n\} \rightarrow x_0$  we have*

$$m(x_0) \leq \liminf \sigma_n(x_n) \leq \limsup \sigma_n(x_n) \leq M(x_0).$$

*In particular,  $\{\sigma_n(x)\}$  converges uniformly at every point of continuity of  $f$ .*

For if  $h$  is small enough the values of  $f$  in the interval  $(x_0 - h, x_0 + h)$  will be contained between  $m(x_0) - \epsilon$  and  $M(x_0) + \epsilon$ , and so for  $n$  large enough  $\sigma_n(x_n)$  is, by (2.28), contained between  $m(x_0) - 2\epsilon$  and  $M(x_0) + 2\epsilon$ .

A special case of (2.30) may be stated separately: *if  $f(x)$  tends to  $+\infty$  as  $x \rightarrow x_0$  (that is, if  $M(x_0) = m(x_0) = +\infty$ ) then  $\sigma_n(x_0) \rightarrow +\infty$ .*

As regards the convergence of  $\{\tilde{\sigma}_n(x)\}$ , there are no results as simple as (2.21). Let us suppose that  $A_n = 1$  in (2.9). (This holds for all important methods of summability in any case,  $A_n \rightarrow 1$ , by condition (iii) of regularity.) Then the difference

$$H_n(t) = \tilde{K}_n(t) - \frac{1}{2} \cot \frac{1}{2}t$$

has some resemblance to  $K_n(t)$  (see (2.8)), which suggests that to results about  $\sigma_n(x)$  there correspond results about

$$\tilde{\sigma}_n(x) - \left\{ -\frac{1}{\pi} \int_{1/n}^{\pi} \{f(x+t) - f(x-t)\} \frac{1}{2} \cot \frac{1}{2}t dt \right\} = \tilde{\sigma}_n(x) - f(x; 1/n).$$

Without aiming unduly at generality we shall find that this is actually so for the methods which are of fundamental importance for Fourier series, namely those of Cesàro and Abel.

### 3. Summability of $S[f]$ and $\tilde{S}[f]$ by the method of the first arithmetic mean

The kernel corresponding to the (C, 1) method for (2.4) is

$$K_n(t) = \frac{1}{n+1} \sum_{\nu=0}^n D_\nu(t) = \frac{1}{n+1} \sum_{\nu=0}^n \frac{\sin(\nu + \frac{1}{2})t}{2 \sin \frac{1}{2}t}. \quad (3.1)$$

Multiplying the numerator and denominator of the right-hand side by  $2 \sin \frac{1}{2}t$  and replacing the products of sines in the numerator by differences of cosines we easily find

$$K_n(t) = \frac{1}{n+1} \frac{1 - \cos(n+1)t}{(2 \sin \frac{1}{2}t)^2} = \frac{2}{n+1} \left\{ \frac{\sin \frac{1}{2}(n+1)t}{2 \sin \frac{1}{2}t} \right\}^2. \quad (3.2)$$

Thus the (C, 1) kernel is non positive.

We shall from now on always use the symbol  $K_n(t)$  for the (C, 1) kernel (and later  $\tilde{K}_n$  for the corresponding conjugate kernel).  $K_n$  is also called the *Fejér kernel*.  $K_n$  has the properties:

- (A)  $\frac{1}{\pi} \int_{-\pi}^{\pi} K_n(t) dt = 1$ ;    (B)  $K_n(t) \geq 0$ ;  
 (C)  $\mu_n(\delta) \rightarrow 0$  for each  $0 < \delta \leq \pi$ ,

where

$$\mu_n(\delta) = \max_{\delta < t < \pi} K_n(t).$$

Condition (A) follows from the corresponding property of the  $D$ , (see (3.1)), and (c) from the inequality

$$\mu_n(\delta) \leq 1/2(n+1) \sin^2 \frac{1}{2}\delta.$$

Hence, with the terminology of the previous section, the kernel  $K_n(t)$  is positive and satisfies condition (c). In the following theorem, which is a consequence of (2.16) and (2.21),

$$\sigma_n(x) = \sigma_n(x; f) = \frac{2}{\pi(n+1)} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin \frac{1}{2}(n+1)t}{2 \sin \frac{1}{2}t} \right)^2 dt \quad (3.3)$$

denotes the (C, 1) mean of  $S[f]$ , a notation we adhere to from now on.

(3.4) **THEOREM OF FEJÉR.** *At every point  $x_0$  at which the limits  $f(x_0 \pm 0)$  exist (and, if both are infinite, are of the same sign) we have*

$$\sigma_n(x_0) \rightarrow \frac{1}{2}\{f(x_0+0) + f(x_0-0)\}.$$

*In particular,  $\sigma_n(x_0) \rightarrow f(x_0)$  at every point of continuity of  $f$ . The convergence of the  $\sigma_n$  is uniform over every closed interval of points of continuity. In particular,  $\sigma_n(x)$  converges uniformly to  $f(x)$  if  $f$  is everywhere continuous.*

*If  $m \leq f(x) \leq M$  for all  $x$ , then*

$$m \leq \sigma_n(x) \leq M \quad (n=0, 1, \dots). \quad (3.5)$$

Since  $K_n(t)$  is zero only at a finite number of points, we easily see that, if  $f \neq \text{constant}$ , (3.5) can be replaced by the stronger inequality

$$m < \sigma_n(x) < M.$$

(If  $f \equiv C$ , then  $\sigma_n(x) = C$  for all  $x$  and  $n$ .)

The theorem of Fejér has a number of important applications, some of which we now give.

*If  $S[f]$  converges at a point  $x_0$  of continuity of  $f$ , then its sum must necessarily be  $f(x_0)$ . More generally, if  $S[f]$  converges at a point  $x_0$  of simple discontinuity of  $f$  then its sum is  $s = \frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$ .*

For at  $x_0$  the series is certainly summable (C, 1) to  $s$  and so, if it converges, its sum must be  $s$ .

A similar argument shows that *if at  $x_0$  the function  $f$  is continuous or has a simple discontinuity, the number  $\frac{1}{2}\{f(x_0+0) + f(x_0-0)\}$  is contained between the limits of indetermination of  $\{S_n(x_0; f)\}$ .*

*If  $\tilde{S}[f]$  is a Fourier series,  $\tilde{S}[f] = S[g]$ , then  $f + ig$  cannot have a simple discontinuity at any point. For if, for example,  $f(x_0 \pm 0)$  exist, are finite and different, and if, say,  $f(x_0+0) - f(x_0-0) > 0$ , then, by Chapter II, (8.13),  $\tilde{S}_n(x_0; f) \rightarrow -\infty$ , and so also  $\tilde{\sigma}_n(x_0; f) = \sigma_n(x_0; g) \rightarrow -\infty$ , which is impossible since  $g$  is bounded near  $x_0$ . In particular, if both  $f$  and  $g$  are of bounded variation, they are continuous.*

*The trigonometric system is complete* (Chapter I, § 6). For if all coefficients of a continuous function  $f$  are zero, the  $\sigma_n(x; f)$  vanish identically, and so does  $f(x) = \lim \sigma_n(x; f)$ . For  $f$  discontinuous, we apply the same argument as in Chapter I, § 6.

(3-6) **THEOREM OF WEIERSTRASS.** *If  $f$  is periodic and continuous, then for every  $\epsilon > 0$  there is a trigonometric polynomial  $T(x)$  such that  $|f(x) - T(x)| < \epsilon$  for all  $x$ .*

We may take  $T(x) = \sigma_n(x; f)$ , with  $n$  sufficiently large.

(3-7) **THEOREM.** *If  $f(x)$  is bounded and has Fourier coefficients  $O(1/n)$  (in particular, if  $f$  is of bounded variation) the partial sums of  $S[f]$  are uniformly bounded.*

For the  $\sigma_n$  are uniformly bounded and the assumption about the coefficients implies (see (1-25)) that the  $s_n - \sigma_n$  are uniformly bounded.

If we use Theorem (1-26) and the fact that the coefficients of a function of bounded variation are  $O(1/n)$  (cf. also the Remark at the end of § 1), the Dirichlet-Jordan theorem (8-1) of Chapter II becomes a corollary of Fejér's.

Theorem (8-6) of Chapter II may be generalized as follows:

(3-8) **THEOREM.** *Suppose that the Fourier coefficients of  $f$  are  $O(1/n)$  and that  $x_0$  is a point of continuity of  $f$ . Then  $S[f]$  converges uniformly at  $x_0$ .*

This is a consequence of the general fact that, if a series  $\Sigma u_n(x)$  is uniformly summable (C, 1) at  $x_0$  to sum  $f(x_0)$  (that  $S[f]$  is uniformly summable (C, 1) at every point of continuity of  $f$  follows from Theorem (3-30)), and if the  $u_n(x)$  are uniformly  $O(1/n)$ , then the series converges uniformly at  $x_0$ . For with the notation of the proof of Theorem (1-26) we have  $|\sigma_{n,k}(x) - s_n(x)| < A\epsilon$  for all  $x$ . Since  $\sigma_n(x)$  converges uniformly at  $x_0$  to limit  $f(x_0)$ , the same holds for  $\sigma_{n,k}(x)$ , after (1-30), and we have  $|\sigma_{n,k}(x) - f(x_0)| < \epsilon$  for  $|x - x_0| < \delta$  and  $n > n_0$ . For such  $n$  and  $x$  we have  $|s_n(x) - f(x_0)| < (A + 1)\epsilon$  and (3-8) is established.

(3-9) **THEOREM OF LEBESGUE.**  *$S[f]$  is summable (C, 1) to  $f(x)$  at every point  $x$  where  $\Phi_x(h) = o(h)$  (in particular, almost everywhere).*

We note that

$$K_n(t) < n + 1, \quad K_n(t) \leq \frac{A}{(n+1)t^2} \quad (0 < t \leq \pi; A \text{ an absolute const.}), \quad (3-10)$$

the first inequality following from (3-1) and the estimate  $|D_n| \leq \nu + \frac{1}{2} < n + 1$ , and the second from (3-2). The first inequality (3-10) is suitable for  $t$  not too large in comparison with  $1/n$ , the second for the remaining  $t$ . Applying this to the formula

$$\sigma_n(x) - f(x) = \frac{2}{\pi} \int_0^\pi \phi_x(t) K_n(t) dt \quad (3-11)$$

(cf. (2-15)), we see that  $|\sigma_n(x) - f(x)|$  is majorized by

$$\frac{2}{\pi} \int_0^\pi |\phi_x(t)| K_n(t) dt \leq \frac{2(n+1)}{\pi} \int_0^{1/n} |\phi_x(t)| dt + \frac{2A}{\pi} \int_{1/n}^\pi \frac{|\phi_x(t)|}{(n+1)t^2} dt = P + Q. \quad (3-12)$$

$$\text{Clearly,} \quad P \leq (n+1) \Phi(1/n) \leq 2n \Phi(1/n) \rightarrow 0, \quad (3-13)$$

and integrating by parts we find that  $Q$  does not exceed

$$\frac{2A}{\pi(n+1)} [\Phi(t)t^{-2}]_{1/n}^\pi + \frac{4A}{\pi(n+1)} \int_{1/n}^\pi \frac{\Phi(t)}{t^3} dt \leq \frac{2A}{\pi^2(n+1)} \Phi(\pi) + \frac{1}{n} \int_{1/n}^\pi o\left(\frac{1}{t^2}\right) dt = o(1). \quad (3-14)$$

Thus  $P + Q = o(1)$  and the theorem is proved.

As an application we obtain a new proof of Parseval's formula for the trigonometric system (Chapter II, § 1). For let  $f \in L^2$ . Then

$$\frac{1}{2\pi} \int_0^{2\pi} |\sigma_n(x; f)|^2 dx = \sum_{\nu=-n}^{+n} \left(1 - \frac{|\nu|}{n+1}\right)^2 |c_\nu|^2 \leq \sum_{\nu=-\infty}^{+\infty} |c_\nu|^2,$$

and since  $\sigma_n \rightarrow f$  almost everywhere, Fatou's lemma (Chapter I, (11.2)) gives

$$\frac{1}{2\pi} \int_0^{2\pi} |f|^2 dx \leq \sum_{\nu=-\infty}^{+\infty} |c_\nu|^2.$$

This combined with the opposite inequality of Bessel gives Parseval's formula.

The following theorem completes (3.4):

(3.15) THEOREM. *If  $f \in \Lambda_\alpha$ ,  $0 < \alpha < 1$ , then  $\sigma_n(x) - f(x) = O(n^{-\alpha})$  uniformly in  $x$ . If  $f \in \Lambda_*$  (in particular, if  $f \in \Lambda_1$ ), then  $\sigma_n(x) - f(x) = O(n^{-1} \log n)$ .*

For in the majorant (3.12) for  $|\sigma_n - f|$  we have  $\phi(t) = O(t^\alpha)$ , which immediately gives  $P = O(n^{-\alpha})$  or  $P = O(n^{-1})$ , and  $Q = O(n^{-\alpha})$  or  $Q = O(n^{-1} \log n)$ , according as  $f \in \Lambda_\alpha$  or  $f \in \Lambda_*$ .

A slight generalization of the first part of (3.15) will be needed later.

(3.16) THEOREM. *Let  $\omega^*(t)$  be a non-negative and increasing function defined in a right-hand neighbourhood of  $t = 0$ . Suppose that  $\omega^*(t)t^{-\alpha}$  is decreasing for some  $\alpha$ ,  $0 < \alpha < 1$ . Let  $\omega(t)$  be the modulus of continuity for a periodic  $f$ . Then, if  $\omega(t) = O(\omega^*(t))$  as  $t \rightarrow +0$ , we have  $\sigma_n - f = O(\omega^*(1/n))$ . Similarly,  $\omega(t) = o(\omega^*(t))$  implies*

$$\sigma_n - f = o(\omega^*(1/n)).$$

Consider, for example, the 'o' case. Without loss of generality we may suppose that  $\omega^*(t)$  is defined and satisfies the required conditions in  $(0, \pi)$ . For if the initial interval of definition is  $(0, \epsilon)$ , with  $0 < \epsilon < \pi$ , it is enough to set  $\omega^*(t) = \omega^*(\epsilon)$  for  $\epsilon \leq t \leq \pi$ . As in (3.12), we consider the terms  $P$  and  $Q$ . Clearly,

$$\begin{aligned} \frac{1}{2}\pi P &\leq (n+1) \int_0^{1/n} \omega\left(\frac{t}{n}\right) dt = o\left(\omega^*\left(\frac{1}{n}\right)\right), \\ \frac{1}{2}\pi Q &\leq \frac{A}{n+1} \int_{1/n}^\pi \frac{\omega(t)}{t^2} dt = \frac{A}{n+1} \int_{1/n}^\pi o\left(\frac{\omega^*(t)}{t^{\alpha+2}}\right) dt \\ &\leq \frac{A}{n+1} \frac{\omega^*(1/n)}{(1/n)^\alpha} \int_{1/n}^\pi o(t^{\alpha-2}) dt = \frac{A}{n+1} \frac{\omega^*(1/n)}{(1/n)^\alpha} o(n^{1-\alpha}) = o\left(\omega^*\left(\frac{1}{n}\right)\right), \end{aligned}$$

which completes the proof.

We turn to the  $(C, 1)$  summability of  $\tilde{S}[f]$ . To avoid repetition we state once for all that in taking arithmetic (or any linear) means of a trigonometric series we shall always take into account the constant term with which the series begins, even if that term (as in  $\tilde{S}[f]$  or  $S'[f]$ ) happens to be zero (cf. (2.5)).

The conjugate Fejér kernel is

$$\begin{aligned} \tilde{K}_n(t) &= \frac{1}{n+1} \sum_{\nu=0}^n D_\nu(t) = \frac{1}{2} \cot \frac{1}{2}t - \frac{1}{n+1} \sum_{\nu=0}^n \frac{\cos(\nu + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \\ &= \frac{1}{2} \cot \frac{1}{2}t - \frac{1}{n+1} \frac{\sin(n+1)t}{(2 \sin \frac{1}{2}t)^2}, \end{aligned} \tag{3.17}$$

by an argument similar to the one used to prove (3.2). The inequality

$$\sin(n+1)t \leq (n+1) \sin t$$

applied to 
$$K_n(t) = \frac{(n+1)\sin t - \sin(n+1)t}{(n+1)(2\sin \frac{1}{2}t)^2}$$

gives 
$$\tilde{K}_n(t) > 0 \text{ for } 0 < t < \pi, n = 1, 2, \dots, \tag{3-18}$$

so that  $\tilde{K}_n(t) \operatorname{sign} t \geq 0$  in  $(-\pi, \pi)$ .

The (C, 1) means of  $\tilde{S}[f]$  are given by the formula

$$\tilde{\sigma}_n(x) = \tilde{\sigma}_n(x; f) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \tilde{K}_n(t) dt = -\frac{2}{\pi} \int_0^{\pi} \psi_x(t) \tilde{K}_n(t) dt. \tag{3-19}$$

It is clear that if  $f$  is integrable and  $\tilde{S}[f] = S[f]$ , then  $\tilde{\sigma}_n(x; f) = \sigma_n(x; f)$ .

(3-20) THEOREM. *At every point at which  $\Psi_x(h) = o(h)$  (in particular, almost everywhere) we have*

$$\tilde{\sigma}_n(x) - f(x; 1/n) \rightarrow 0. \tag{3-21}$$

For let  $\tilde{K}_n(t) = \frac{1}{2} \cot \frac{1}{2}t - H_n(t)$ . The inequalities

$$|\tilde{K}_n(t)| \leq \frac{1}{2}n, \quad |H_n(t)| \leq \frac{A}{(n+1)t^2} \quad (0 < t \leq \pi) \tag{3-22}$$

are immediate (see (3-17)). By (3-19) and (3-17),  $|\tilde{\sigma}_n(x) - f(x; 1/n)|$  does not exceed

$$\begin{aligned} & \frac{2}{\pi} \left| \int_0^{1/n} \psi(t) \tilde{K}_n(t) dt - \int_{1/n}^{\pi} \psi(t) H_n(t) dt \right| \\ & \leq \frac{n}{\pi} \int_0^{1/n} |\psi(t)| dt + \frac{2A}{\pi(n+1)} \int_{1/n}^{\pi} \frac{|\psi(t)|}{t^2} dt = P^* + Q^*, \end{aligned}$$

and an argument similar to (3-13) and (3-14) gives  $P^* + Q^* = o(1)$ . This proves the theorem.

Suppose that  $1/(n+1) \leq h \leq 1/n$ . Since  $f(x; h) - f(x; 1/n) \rightarrow 0$  at every point at which  $\Psi_x(h) = o(h)$  (see Chapter II, (11-8)), we conclude that *at such points the summability (C, 1) of  $\tilde{S}[f]$  is equivalent to the existence of  $f(x)$* . It will be shown later (see Chapter IV, §3, and Chapter VII, §1) that  $f(x)$  exists almost everywhere for any integrable  $f$ . Assuming this we can state the following:

(3-23) THEOREM.  $\tilde{S}[f]$  is summable (C, 1) to sum  $f(x)$  almost everywhere.

It is of interest to consider the integral (C, 1) means of  $S[f]$  and  $\tilde{S}[f]$  (see p. 83). Returning to the formulae (7-2) and (7-3) of Chapter II, we find that, except when  $\omega$  is an integer, the right-hand sides there are the sum functions for  $S[f]$  and  $\tilde{S}[f]$ . The left-hand sides are uniformly bounded for  $\omega$  confined to an arbitrary finite interval, and converge uniformly except in the neighbourhood of integral values of  $\omega$ . Hence if we integrate the equations with respect to  $\omega$ , we can interchange the order of integrations on the left, obtaining the formulae

$$\sum_{\nu < \omega} \left(1 - \frac{\nu}{\omega}\right) A_{\nu}(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \frac{2\sin^2 \frac{1}{2}\omega t}{\omega t^2} dt, \tag{3-24}$$

$$\sum_{\nu < \omega} \left(1 - \frac{\nu}{\omega}\right) B_{\nu}(x) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} f(x+t) \left\{ \frac{1}{t} - \frac{\sin \omega t}{\omega t^2} \right\} dt, \tag{3-25}$$

analogous to (3-3) and (3-19). The left-hand sides here are continuous functions of  $\omega$  and the first integral converges absolutely.

4. Convergence factors

In Chapter VIII we shall see that a Fourier series may diverge almost everywhere. We may therefore ask about *convergence factors* of the Fourier series, that is to say sequences  $\{\lambda_n\}$  such that, for each Fourier series  $\Sigma(a_n \cos nx + b_n \sin nx)$ , the series

$$\frac{1}{2}a_0\lambda_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \lambda_n$$

converges almost everywhere.

A sequence of numbers  $\lambda_0, \lambda_1, \dots$  is said to be *convex* if  $\Delta^2\lambda_n \geq 0$  for all  $n$ , where

$$\Delta\lambda_n = \lambda_n - \lambda_{n+1}, \quad \Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}.$$

Geometrically, this amounts to saying that the polygonal line with vertices at the points  $(n, \lambda_n)$  is convex. We shall show that if  $\{\lambda_n\}$  is convex and bounded, then it is non-increasing. By hypothesis,  $\Delta\lambda_n$  is non-increasing. It cannot be negative for any value of  $n$ , for then it would be less than a negative constant for all subsequent values of  $n$ , which would imply that  $\lambda_n \rightarrow +\infty$ , contrary to hypothesis. Thus  $\Delta\lambda_n = \lambda_n - \lambda_{n+1} \geq 0$  for all  $n$ , so that

$$\lambda_n \geq \lambda_{n+1} \rightarrow \lambda > -\infty.$$

In the equation  $\lambda_0 - \lambda = \Delta\lambda_0 + \Delta\lambda_1 + \dots$

the terms on the right are non-increasing, and so, by the classical theorem of Abel,  $n\Delta\lambda_n \rightarrow 0$ . Taking this into account and summing the series  $1 \cdot \Delta\lambda_0 + 1 \cdot \Delta\lambda_1 + \dots$  by parts we get:

(4.1) THEOREM. *If  $\{\lambda_n\}$  is convex and bounded, then  $\{\lambda_n\}$  is non-increasing,  $n\Delta\lambda_n \rightarrow 0$ , and the series*

$$\sum_{n=0}^{\infty} (n+1) \Delta^2\lambda_n \tag{4.2}$$

converges to sum  $\lambda_0 - \lim \lambda_n$ .

It is geometrically obvious that if a function  $\lambda(x)$  is convex, the sequence  $\{\lambda_n\} = \{\lambda(n)\}$  is convex.

In particular, if we take  $\lambda_n = 1/\log n$  for  $n = 2, 3, \dots$  and choose  $\lambda_0, \lambda_1$  suitably,  $\{\lambda_n\}$  will be convex.

(4.3) THEOREM. *Let  $s_n$  and  $\sigma_n$  be the partial sums and the (C, 1) means of a series  $u_n + u_{n+1} + \dots$ . If  $\sigma_n$  converges, and if  $s_n = o(\mu_n)$ , where  $\{1/\mu_n\}$  is convex and tends to 0, then  $u_0\mu_0^{-1} + u_1\mu_1^{-1} + \dots$  converges.*

For applying summation by parts twice to the  $n$ th partial sum of the last series we find that it is equal to

$$\sum_{k=0}^{n-2} (k+1) \sigma_k \Delta^2 \frac{1}{\mu_k} + n\sigma_{n-1} \Delta \frac{1}{\mu_{n-1}} + s_n \frac{1}{\mu_n} = \sum_{k=0}^n (k+1) \sigma_k \Delta^2 \frac{1}{\mu_k}.$$

Take  $\mu_n = \log n$  for  $n \geq 2$ . From (4.3), (3.9), (3.23) and Theorem (11.9) of Chapter II, we deduce:

(4.4) THEOREM. *If  $a_k, b_k$  are the Fourier coefficients of a function, both series*

$$\sum_{k=2}^{\infty} \frac{a_k \cos kx + b_k \sin kx}{\log k}, \quad \sum_{k=2}^{\infty} \frac{a_k \sin kx - b_k \cos kx}{\log k} \tag{4.5}$$

converge almost everywhere.

The result obviously holds if  $\log k$  here is replaced by  $k^\alpha$ ,  $\alpha > 0$ .

The first series (4.5) converges at every point where  $\Phi_x(h) = o(h)$ , in particular at every point of continuity of  $f$ . If  $f$  is continuous in  $(a, b)$ , the series is uniformly convergent in every  $(a + \epsilon, b - \epsilon)$ ,  $\epsilon > 0$ .

5. Summability (C,  $\alpha$ )

(5.1) THEOREM OF M. RIESZ. The theorems (3.4) of Fejér (except for the last sentence) and (3.9) of Lebesgue hold if summability (C, 1) is replaced by (C,  $\alpha$ ),  $\alpha > 0$ .

Let  $K_n^\alpha(t)$  denote the (C,  $\alpha$ ) kernel and  $\sigma_n^\alpha(x) = \sigma_n^\alpha(x; f)$  the (C,  $\alpha$ ) means of  $S[f]$ . Then

$$K_n^\alpha(t) = \sum_{\nu=0}^n A_n^{\alpha-1} D_\nu(t) / A_n^\alpha \tag{5.2}$$

$$\sigma_n^\alpha(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) K_n^\alpha(t) dt, \tag{5.3}$$

$$\sigma_n^\alpha(x) - f(x) = \frac{2}{\pi} \int_0^\pi \phi_x(t) K_n^\alpha(t) dt, \tag{5.4}$$

the last equation being a consequence of the validity of condition (A) (p. 85) for  $K_n^\alpha(t)$ .

It is enough to consider the case  $0 < \alpha < 1$ . We shall show that then

$$|K_n^\alpha(t)| < n + 1 \leq 2n, \quad |K_n^\alpha(t)| \leq A_\alpha n^{-\alpha} t^{-(\alpha+1)} \tag{5.5}$$

for  $n = 1, 2, \dots$ ,  $0 < t \leq \pi$ , with  $A_\alpha$  depending on  $\alpha$  only.

These inequalities are analogous to (3.10) and reduce to the latter for  $\alpha = 1$ . Once (5.5) is established, the proof of the extended (3.9) goes as before. Similarly, to extend (3.4) it is enough to show that the kernel  $K_n^\alpha(t)$  is quasi-positive and satisfies condition (c) (p. 86). Both these facts are corollaries of (5.5): for

$$\int_0^\pi |K_n^\alpha(t)| dt < 2n \int_0^{1/n} dt + A_\alpha n^{-\alpha} \int_{1/n}^\pi t^{-\alpha-1} dt < 2 + A_\alpha/\alpha,$$

and

$$\max_{\delta < t < \pi} |K_n^\alpha(t)| \leq A_\alpha n^{-\alpha} \delta^{-(\alpha+1)} \rightarrow 0.$$

It remains to prove (5.5). The first part follows from (5.2) and the estimate

$$|D_\nu| \leq \nu + \frac{1}{2} < n + 1.$$

For the second we have, from (5.2),

$$\begin{aligned} K_n^\alpha(t) &= \frac{1}{2A_n^\alpha \sin \frac{1}{2}t} \sum_{\nu=0}^n A_n^{\alpha-1} e^{i(\nu+1/2)t} = \frac{e^{i(n+1/2)t}}{2A_n^\alpha \sin \frac{1}{2}t} \sum_{\nu=0}^n A_\nu^{\alpha-1} e^{-i\nu t} \\ &= \mathcal{J} \left\{ \frac{e^{i(n+1/2)t}}{2A_n^\alpha \sin \frac{1}{2}t} \left[ (1 - e^{-it})^{-\alpha} - \sum_{\nu=-n+1}^{\infty} A_\nu^{\alpha-1} e^{-i\nu t} \right] \right\}. \end{aligned} \tag{5.6}$$

Since  $A_\nu^{\alpha-1}$  decreases monotonically to 0, the last series converges for  $0 < t \leq \pi$  and the modulus of its sum is  $\leq 2A_{n+1}^{\alpha-1} |1 - e^{-it}|^{-1}$  (see Chapter I, (2.2)). So, since  $|\mathcal{J}z| \leq |z|$ ,  $|K_n^\alpha(t)|$  is majorized by

$$\left\{ (2 \sin \frac{1}{2}t)^{-\alpha-1} \frac{1}{A_n^\alpha} + \frac{2A_{n+1}^{\alpha-1}}{A_n^\alpha} (2 \sin \frac{1}{2}t)^{-1} \right\} \leq A_\alpha \{ n^{-\alpha} t^{-\alpha-1} + n^{-1} t^{-\alpha} \} \tag{5.7}$$

for  $0 < t \leq \pi$ . If  $nt \geq 1$ , then  $nt^2 = (nt)^{1-\alpha} n^{\alpha t^{\alpha+1}} \geq n^{\alpha t^{\alpha+1}}$ ,

the right-hand side of (5.7) does not exceed  $2A_\alpha n^{-\alpha t^{\alpha+1}}$ , and the second part of (5.5) follows. For  $0 < t \leq 1/n$  the second part of (5.5) is a consequence of the first. This completes the proof of (5.1).

Let  $\bar{\sigma}_n^\alpha(x) = \bar{\sigma}_n^\alpha(x; f)$  be the (C,  $\alpha$ ) means of  $\bar{S}[f]$ . The theorem which follows is an extension of (3.20) and (3.23) to summability (C,  $\alpha$ ).

(5.8) THEOREM. Let  $0 < \alpha < 1$ . At every point  $x$  at which  $\Psi_\alpha(h) = o(h)$  we have

$$\bar{\sigma}_n^\alpha(x) - f(x; 1/n) \rightarrow 0.$$

In particular,  $\bar{S}[f]$  is almost everywhere summable (C,  $\alpha$ ) to sum  $f(x)$ .

The proof is analogous to that of (3.20). Let  $\bar{K}_n^\alpha(t)$  be the conjugate (C,  $\alpha$ ) kernel. Then

$$\bar{\sigma}_n^\alpha(x) = -\frac{2}{\pi} \int_0^\pi \psi_x(t) \bar{K}_n^\alpha(t) dt, \tag{5.9}$$

$$\bar{K}_n^\alpha(t) = \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} \bar{D}_\nu(t) / A_n^\alpha, \tag{5.10}$$

$$\bar{K}_n^\alpha(t) = \frac{1}{2} \cot \frac{1}{2} t - \frac{1}{A_n^\alpha} \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} \frac{\cos(\nu + \frac{1}{2})t}{2 \sin \frac{1}{2} t} = \frac{1}{2} \cot \frac{1}{2} t - H_n^\alpha(t), \tag{5.11}$$

say. We show that (for  $0 < \alpha < 1$ )

$$|\bar{K}_n^\alpha(t)| \leq n, \quad |H_n^\alpha(t)| \leq A_\alpha n^{-\alpha t^{-(\alpha+1)}} \quad (n^{-1} \leq t \leq \pi). \tag{5.12}$$

The first inequality here follows from (5.10) and the estimate  $|\bar{D}_\nu| \leq \nu \leq n$ . For the second inequality we note that for  $H_n^\alpha(t)$  we have a formula analogous to (5.6) with  $\mathcal{S}$  replaced by  $\mathcal{S}$ , and the previous argument is still applicable.

To prove (5.8) we write (5.9) in the form

$$\bar{\sigma}_n^\alpha(x) - f(x; 1/n) = -\frac{2}{\pi} \int_0^{1/n} \psi_x(t) \bar{K}_n^\alpha(t) dt + \frac{2}{\pi} \int_{1/n}^\pi \psi_x(t) H_n^\alpha(t) dt, \tag{5.13}$$

so that, as in the proof of (3.20),

$$|\bar{\sigma}_n^\alpha(x) - f(x; 1/n)| \leq \frac{2}{\pi} n \int_0^{1/n} |\psi_x(t)| dt + \frac{2A_\alpha}{\pi n^\alpha} \int_{1/n}^\pi \frac{|\psi_x(t)|}{t^{\alpha+1}} dt \rightarrow 0,$$

and the theorem follows.

For later applications we shall need a refinement of (5.6), namely, if  $-1 < \alpha < 1$  we have

$$K_n^\alpha(t) = \frac{1}{A_n^\alpha} \frac{\sin[(n + \frac{1}{2} + \frac{1}{2}\alpha)t - \frac{1}{2}\pi\alpha]}{(2 \sin \frac{1}{2} t)^{\alpha+1}} + \frac{2\theta\alpha}{n(2 \sin \frac{1}{2} t)^\alpha} \quad (|\theta| \leq 1). \tag{5.14}$$

Applying repeated summation by parts to the last series in (5.6) we obtain more and more accurate approximations for  $K_n^\alpha(t)$ . For example,

$$\begin{aligned} K_n^\alpha(t) &= \mathcal{S} \left\{ \frac{e^{i(n+\frac{1}{2})t}}{A_n^\alpha (2 \sin \frac{1}{2} t)^\alpha} \left[ \frac{1}{(1-e^{-u})^\alpha} - A_{n+1}^{\alpha-1} \frac{e^{-i(n+1)t}}{1-e^{-u}} - \sum_{\nu=n+1}^\infty A_{\nu+1}^{\alpha-2} \frac{e^{-i(\nu+1)t}}{1-e^{-u}} \right] \right\} \\ &= \frac{1}{A_n^\alpha} \frac{\sin[(n + \frac{1}{2} + \frac{1}{2}\alpha)t - \frac{1}{2}\pi\alpha]}{(2 \sin \frac{1}{2} t)^{\alpha+1}} + \frac{\alpha}{n+1} \frac{1}{(2 \sin \frac{1}{2} t)^2} + \frac{2\theta\alpha(1-\alpha)}{(n+1)(n+2)(2 \sin \frac{1}{2} t)^\alpha}. \end{aligned} \tag{5.15}$$

Similar formulae hold for  $\bar{K}_n^\alpha(t)$ . We may add that the estimates (5.5) and (5.12) hold also for  $-1 < \alpha \leq 0$ .

### 6. Abel summability

Let  $a_\nu, b_\nu$  be the Fourier coefficients of  $f$ . The Abel (or, simply, the A) means of  $S[f]$  and  $\tilde{S}[f]$  are the functions

$$\left. \begin{aligned} f(r, x) &= \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) r^\nu \\ \tilde{f}(r, x) &= \sum_{\nu=1}^{\infty} (a_\nu \sin \nu x - b_\nu \cos \nu x) r^\nu \end{aligned} \right\} \quad (0 \leq r < 1), \quad (6.1)$$

and we wish to investigate their limits as  $r \rightarrow 1$ . Since  $a_\nu, b_\nu \rightarrow 0$ , the series converge absolutely and uniformly for  $0 \leq r \leq 1 - \delta, \delta > 0$ . Thus  $f(r, x)$  and  $\tilde{f}(r, x)$  (a notation which we shall use systematically) are continuous functions of the point  $re^{ix}$  for  $r < 1$ .

The Abel means of the even series  $\frac{1}{2} + \sum \cos \nu t$  and the odd series  $\sum \sin \nu t$  are

$$P(r, t) = \frac{1}{2} + \sum_{\nu=1}^{\infty} r^\nu \cos \nu t = \frac{1}{2} \frac{1-r^2}{1-2r \cos t + r^2}, \quad (6.2)$$

$$Q(r, t) = \sum_{\nu=1}^{\infty} r^\nu \sin \nu t = \frac{r \sin t}{1-2r \cos t + r^2} \quad (6.3)$$

(see Chapter I, § 1). They are called the *Poisson kernel* and the *Poisson conjugate kernel* respectively. The standard formulæ (2.12) and (2.13) (where now the continuous variable  $r$  plays the role of the former  $n$ ) will now be

$$f(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) P(r, t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) P(r, t-x) dt, \quad (6.4)$$

$$\tilde{f}(r, x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) Q(r, t) dt = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) Q(r, t-x) dt. \quad (6.5)$$

The right-hand sides here are usually called the *Poisson integral* and the *conjugate Poisson integral* of  $f$ . Thus the expressions 'Abel mean of  $S[f]$ ' and 'Poisson integral of  $f$ ' are synonymous.

The denominator  $\Delta(r, t) = 1 - 2r \cos t + r^2$  ( $0 \leq r < 1$ ) of  $P$  and  $Q$  is positive for all  $t$ . It follows that

$$\left. \begin{aligned} P(r, t) &> 0 \quad \text{for all } t, \\ Q(r, t) &> 0 \quad \text{for } 0 < t < \pi. \end{aligned} \right\} \quad (6.6)$$

Hence  $P$  is a positive kernel. For fixed  $r$ , its maximum and minimum are attained at  $t=0$  and  $t=\pi$  respectively. Thus

$$\frac{1}{2} \frac{1-r}{1+r} \leq P(r, t) \leq \frac{1}{2} \frac{1+r}{1-r}. \quad (6.7)$$

It is sometimes convenient to use the inequality

$$P(r, t) \leq A \frac{\delta}{\delta^2 + t^2} \quad (\delta = 1-r, (t) \leq \pi), \quad (6.8)$$

in which  $A$  is an absolute constant. For  $0 \leq r \leq \frac{1}{2}$  it is immediate, since both  $P(r, t)$  (see (6.7)) and  $\delta/(\delta^2 + t^2)$  are then contained between two positive constants. For  $\frac{1}{2} \leq r < 1$ ,

$$P(r, t) = \frac{1}{2} \frac{(1+r)(1-r)}{(1-r)^2 + 4r \sin^2 \frac{1}{2}t} < \frac{\delta}{\delta^2 + 4 \cdot \frac{1}{4}(\pi-t)^2} < \frac{1}{4} \pi^2 \frac{\delta}{\delta^2 + t^2},$$

and (6.8) holds again. In particular, (6.7) and (6.8) imply

$$P(r, t) \leq \frac{1}{\delta}, \quad P(r, t) \leq \frac{A\delta}{t^2} \quad (0 < t \leq \pi, 0 \leq r < 1), \tag{6.9}$$

inequalities similar to those satisfied by Fejér's kernel (see (3.10)) if we replace  $\delta$  by  $1/(n+1)$ .

The kernel  $P(r, t)$  is positive, satisfies condition (A), (p. 85), i.e.

$$\frac{1}{\pi} \int_{-\pi}^{\pi} P(r, t) dt = 1 \tag{6.10}$$

(as is seen by termwise integration of the series in (6.2)), and also condition (C) (as is seen from the second inequality (6.9)). Thus all the assumptions which led us to Theorem (3.4) from (2.21) are valid, and we have the following result:

**(6.11) THEOREM.** *Theorem (3.4) of Fejér holds if we replace the (C, 1) means of  $S[f]$  by the Abel means.*

Of course, this result, like some of the results established below, is also a consequence of the theorem of Fejér and of the fact that summability (C, 1) implies summability A for any series. But a direct study of summability A of Fourier series is of interest for two reasons. First, summability A of  $S[f]$  may hold under weaker conditions for  $f$  than summability (C, 1); secondly, summability A of  $S[f]$  has special features which are absent in (C, 1). For example, we may consider not only the radial but also the non-tangential, and even unrestricted, limit of  $f(r, x)$  as  $(r, x)$  tends to a point on the unit circle.

The functions  $f(r, x)$  and  $\hat{f}(r, x)$  of (6.1) are the real and imaginary parts of the function

$$\Phi(z) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_{\nu} - ib_{\nu})z^{\nu}, \quad z = re^{ix},$$

regular for  $|z| < 1$ . Thus  $f(r, x)$  and  $\hat{f}(r, x)$  are harmonic, that is as functions of Cartesian co-ordinates  $\xi, \eta$  they satisfy Laplace's equation

$$u_{\xi\xi} + u_{\eta\eta} = 0.$$

Each real-valued function harmonic in the interior of a circle is the real part of a regular function†. Hence, if  $u(r, x)$  is harmonic in the circle  $0 \leq r < R$ , we have

$$u(r, x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x) r^{\nu} \quad (0 \leq r < R).$$

Let  $f(x)$  be a continuous and periodic function and let  $(r, x)$  be the polar co-ordinates of a point. Theorem (6.11) asserts that the Poisson integral  $f(r, x)$  of  $f(x)$  tends uniformly to  $f(x)$  as  $r \rightarrow 1$ . In other words, Poisson's integral gives, for the case of a circle, a solution (or indeed, as is shown in the theory of harmonic functions, the unique solution) of the following problem of Dirichlet. Given

- (i) a plane region  $D$  limited by a simple closed curve  $C$ ,
- (ii) a function  $f(p)$  defined and continuous for  $p \in C$ ,

find a function  $F(p)$  harmonic in  $D$ , continuous in  $D + C$ , and coinciding with  $f(p)$  on  $C$ . As we shall see below, in the case of the unit circle the Poisson integral solves a more general Dirichlet problem in which  $f(p)$  is an arbitrary integrable function.

† See, e.g., Littlewood, *Lectures on the theory of functions*, p. 84.

(6·12) THEOREM. If  $m \leq f(x) \leq M$  for all  $x$ , then

$$m \leq f(r, x) \leq M \quad (0 \leq r < 1, 0 \leq x \leq 2\pi).$$

In particular,  $f(r, x) \geq 0$  if  $f \geq 0$ .

If  $m \leq f(x) \leq M$  for  $x \in (a, b)$ , then for every  $\epsilon, \eta > 0$  there is a number  $r_0$  such that

$$m - \epsilon < f(r, x) < M + \epsilon \quad \text{for } r_0 \leq r < 1, x \in (a + \eta, b - \eta).$$

This is a special case of (2·16) and (2·28).

Since  $P(r, x)$  is strictly positive for  $r < 1$ , it follows that if  $m \leq f(x) \leq M$  in  $(0, 2\pi)$  and  $f \neq \text{const.}$ , then we have the sharper estimate

$$m_1 < f(r, x) < M \quad (0 \leq r < 1).$$

Let  $m(x_0)$  and  $M(x_0)$  be the minimum and maximum of  $f$  at  $x_0$  (see p. 88).

(6·13) THEOREM. If  $L$  is any path leading from the interior of the unit circle to the point  $(1, x_0)$ , then the limits of indetermination of  $f(r, x)$  as the point  $(r, x)$  approaches  $(1, x_0)$  along  $L$  are contained between  $m(x_0)$  and  $M(x_0)$ .

For if  $(r_n, x_n)$  is any sequence of points, with  $r_n < 1$ , approaching  $(1, x_0)$ , then (see (2·30))

$$m(x_0) \leq \liminf f(r_n, x_n) \leq \limsup f(r_n, x_n) \leq M(x_0).$$

A special case of (6·13) asserts that, if  $f$  is continuous at  $x_0$ , then  $f(r, x)$  tends to  $f(x_0)$  along  $L$ .

Suppose that  $f$  has at  $x_0$  a discontinuity of the first kind and that

$$f(x_0) = \frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}.$$

Let

$$d = f(x_0 + 0) - f(x_0 - 0)$$

be the jump of  $f$  at  $x_0$ . Without loss of generality we may suppose that  $x_0 = 0$ . The function

$$\phi(x) \sim \sum_1^{\infty} \frac{\sin \nu x}{\nu}$$

has jump  $\pi$  at  $x = 0$  (see p. 9). It follows that

$$g(x) = f(x) - \frac{d}{\pi} \phi(x)$$

is continuous at  $x = 0$ , and  $g(0) = f(0)$ . Moreover,

$$f(r, x) = g(r, x) + \frac{d}{\pi} \phi(r, x) = g(r, x) + \frac{d}{\pi} \arctan \frac{r \sin x}{1 - r \cos x} \quad (6·14)$$

(Chapter I, § 1). Along any path  $L$  leading to  $(1, 0)$  from the interior of the unit circle we have  $\lim g(r, x) = g(0) = f(0)$  so that the behaviour of  $f(r, x)$  along  $L$  depends on that of the last term in (6·14).

The arctan in (6·14) is the angle, numerically  $< \frac{1}{2}\pi$  and reckoned clockwise, which the segment joining  $(r, x)$  to  $(1, 0)$  makes with the negatively directed real axis. Hence:

(6·15) THEOREM. Suppose that  $f(x)$  has at  $x_0$  a discontinuity of the first kind and that  $f(x_0) = \frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}$ . Let  $L$  be any path approaching the point  $A(1, x_0)$  from inside

the unit circle and making at  $A$  an angle  $\theta$ ,  $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$ , reckoned clockwise, with the radius at  $A$  directed inwards. Then  $\lim f(r, x)$  along  $L$  exists and equals

$$f(x_0) + \frac{d}{\pi}\theta, \quad \text{where } d = f(x_0 + 0) - f(x_0 - 0). \tag{6.16}$$

If  $L$  does not have a tangent at  $A$ ,  $f(r, x)$  oscillates finitely as  $(r, x) \rightarrow (1, x_0)$  along  $L$ .

If  $L$  is tangent to the unit circle at  $A$ , then  $\theta = \frac{1}{2}\pi$  or  $\theta = -\frac{1}{2}\pi$ , and correspondingly (6.16) reduces to  $f(x_0 + 0)$  or  $f(x_0 - 0)$ .

Let  $\chi(x)$  be the characteristic function of an interval  $(\alpha, \beta)$ . The Poisson integral

$$\chi(r, x) = \frac{1}{\pi} \int_{\alpha}^{\beta} P(r, t - x) dt \tag{6.17}$$

of  $\chi$  has a simple and useful interpretation.

Let  $C$  be the unit circumference with centre 0,  $z$  and  $\zeta$  points inside and on  $C$  respectively. The point  $\zeta'$  at which the ray  $\zeta z$  intersects  $C$  will be called *opposite* to  $\zeta$ , with respect to  $z$ . If  $s$  is an arc of  $C$ , the arc  $s'$  consisting of the points opposite to those of  $s$  will be called *opposite* to  $s$  (with respect to  $z$ ).

(6.18) **THEOREM.** *With the notation of (6.17),  $2\pi\chi(r, x)$  is the length of the arc  $s' = (e^{i\alpha'}, e^{i\beta'})$  opposite to  $s = (e^{i\alpha}, e^{i\beta})$  with respect to  $z = r e^{ix}$ .*

Let  $s, s'$  also denote the lengths of  $s, s'$ . The angle  $\gamma$  which  $s$  subtends at  $z$  is  $\frac{1}{2}(s + s')$ . Clearly it is also

$$\int_{\alpha}^{\beta} d \log (\zeta - z) = \int_{\alpha}^{\beta} \frac{d\zeta}{\zeta - z} = \int_{\alpha}^{\beta} \frac{dt}{1 - r e^{i(t-x)}} = \int_{\alpha}^{\beta} \left\{ \frac{1}{2} + P(r, t - x) \right\} dt.$$

Thus

$$\frac{1}{2}(s + s') = \frac{1}{2}s + \pi\chi(r, x),$$

from which the theorem follows.

The level curves of  $\chi(r, x)$ , being the curves on which  $\gamma$  is constant, are those arcs of the circles through  $e^{i\alpha}, e^{i\beta}$  which are inside  $C$ .

### 7. Abel summability (cont.)

From Theorem (3.9) we deduce that  $S[f]$  is summable  $A$  at every point  $x$  for which  $\Phi_x(h) = o(h)$ , in particular almost everywhere. This result will be superseded by a somewhat stronger one (see (7.9) below). We shall first prove results about summability  $A$  of formally differentiated Fourier series.

As in Chapter I, p. 22, if

$$\lim_{h \rightarrow +0} \frac{F(x_0 + h) - F(x_0 - h)}{2h} = D_1 F(x_0) \tag{7.1}$$

exists it is called the first symmetric derivative of  $F$  at  $x_0$ . In the general case the upper and lower limits, as  $h \rightarrow 0$ , of the ratio in (7.1), are called the *upper* and *lower* first symmetric derivatives. We shall denote them by  $\bar{D}_1 F(x_0)$  and  $\underline{D}_1 F(x_0)$  respectively. If  $F'(x_0)$  exists so does  $D_1 F(x_0)$ , and their values are equal.

(7.2) **FATOU'S THEOREM.** *Suppose that*

$$F(x) \sim \frac{1}{2}A_0 + \Sigma(A_n \cos nx + B_n \sin nx)$$

and that  $D_1 F(x_0)$  exists, finite or infinite. Then  $S[F]$  is summable A at  $x_0$  to sum  $D_1 F(x_0)$ , that is,

$$\sum_{\nu=1}^{\infty} \nu (B_{\nu} \cos \nu x_0 - A_{\nu} \sin \nu x_0) r^{\nu} \quad (7.3)$$

tends to  $D_1 F(x_0)$  as  $r \rightarrow 1$ .

More generally, the limits of indetermination of (7.3) as  $r \rightarrow 1$  are contained between  $\underline{D}_1 F(x_0)$  and  $\bar{D}_1 F(x_0)$ .

It is enough to prove the second part. If  $F(r, x)$  is the Poisson integral of  $F$ , then (7.3) is  $\{\partial F(r, x)/\partial x\}_{x=x_0}$ . From

$$F(r, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) P(r, t-x) dt$$

$$\text{we get } \frac{1}{r} \left( \frac{\partial F(r, x)}{\partial x} \right)_{x=x_0} = \frac{1}{\pi r} \int_{-\pi}^{\pi} F'(t) P'(r, t-x_0) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) K(r, t) dt, \quad (7.4)$$

where the dash denotes differentiation with respect to  $t$ ,

$$g(t) = \{F'(x_0 + t) - F'(x_0 - t)\} / 2 \sin t,$$

$$K(r, t) = -r^{-1} P'(r, t) \sin t = (1-r^2) \sin^2 t / \Delta^2(r, t).$$

We note that  $K(r, t)$  has the properties (A), (B), (C) of kernels. Property (B) is immediate; so is (C), even for  $P'(r, t)$ . In order to show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} K(r, t) dt = 1, \quad (7.5)$$

we take  $F(x) = \sin x$ ,  $x_0 = 0$ . Then  $F(r, x) = r \sin x$ ,  $g(t) = 1$  and a comparison of the extreme terms in (7.4) gives (7.5).

Since the maximum and minimum of  $g(t)$  at  $t=0$  are  $\bar{D}_1 F(x_0)$  and  $\underline{D}_1 F(x_0)$ , (7.2) would follow from (2.30) (with  $x_n = x_0$  for all  $n$ ) if we knew that  $g(t)$  was integrable. The latter is not necessarily true (except when, for example,  $\underline{D}_1 F(x_0)$  and  $\bar{D}_1 F(x_0)$  are both finite), but this does not affect the proof, as we see from the following argument. Let  $0 < \delta < \pi$ . Since  $P'(r, t)$  satisfies condition (c), the right-hand side in (7.4) is

$$\frac{1}{\pi} \int_{-\delta}^{\delta} g(t) K(r, t) dt + o(1).$$

The integral here is contained between the upper and lower bounds of  $g(t)$  in  $(-\delta, \delta)$  times  $\frac{1}{\pi} \int_{-\delta}^{\delta} K dt$ . But the latter integral tends to 1 as  $r \rightarrow 1$ . Hence the limits of indetermination of (7.3), as  $r \rightarrow 1$ , are contained between the upper and lower bounds of  $g(t)$  in  $(-\delta, \delta)$ . Taking  $\delta$  arbitrarily small we may replace these bounds by  $\bar{D}_1 F(x_0)$  and  $\underline{D}_1 F(x_0)$ , and (7.2) is established.

(7.6) THEOREM. *If  $F'(x_0)$  exists and is finite, then*

$$\partial F(r, x) / \partial x \rightarrow F'(x_0)$$

as  $(r, x)$  approaches  $(1, x_0)$  non-tangentially.

We may suppose that  $x_0 = 0$ ,  $F(0) = 0$ . Let us also temporarily suppose that  $F'(0) = 0$ .

Given any  $\epsilon > 0$ , let  $\sigma$  be such that  $|F(u)| \leq \epsilon |u|$  for  $|u| \leq 2\sigma$ . From (7.4) and using the property (c) of  $P'(r, t)$  we have

$$\frac{\partial F(r, x)}{\partial x} = -\frac{1}{\pi} \int_{-\sigma}^{\pi} F(x+t) P'(r, t) dt = -\frac{1}{\pi} \int_{-\sigma}^{\sigma} F(x+t) P'(r, t) dt + o(1) = A + o(1). \quad (7.7)$$

Suppose that  $|x| \leq \sigma$ . Then  $|F(x+t)| \leq \epsilon |x+t|$  in  $A$  and

$$\begin{aligned} |A| &\leq \frac{\epsilon}{\pi} \int_{-\sigma}^{\sigma} (|x| + |t|) |P'| dt \\ &< -\frac{2\epsilon}{\pi} \int_0^{\pi} (|x| + t) P' dt < \frac{2\epsilon}{\pi} \left\{ |x| P(r, 0) + \int_0^{\pi} P dt \right\} < \frac{2\epsilon}{\pi} \left\{ \frac{|x|}{1-r} + \frac{\pi}{2} \right\}. \end{aligned} \quad (7.8)$$

The expression in curly brackets remains bounded in a non-tangential approach. Hence, taking  $\epsilon$  arbitrarily small, we see that (7.7) tends to 0, under the hypothesis  $F'(0) = 0$ .

In the general case, we write

$$F(x) = \{F(x) - F'(0) \sin x\} + F'(0) \sin x.$$

The derivative of the expression in curly brackets at  $x = 0$  is zero, and for the function  $F'(0) \sin x$ , whose Poisson integral is  $F'(0) r \sin x$ , the theorem is obvious.

(7.9) THEOREM. Let  $F(x)$  be the indefinite integral of an integrable and periodic  $f$ . Then

(i)  $S[f]$  is summable  $A$  to sum  $D_1 F(x_0)$  at every point  $x_0$  at which  $D_1 F(x_0)$  exists, finite or infinite;

(ii) at every point at which  $F'(x_0) = f(x_0)$  exists and is finite (in particular almost everywhere) the Poisson integral  $f(r, x)$  of  $f$  tends to  $f(x_0)$  as  $(r, x) \rightarrow (1, x_0)$  non-tangentially.

For supposing, as we may, that the constant term of  $S[f]$  is zero, we have  $S[f] = S'[F]$ ,  $f(r, x) = \partial F(r, x) / \partial x$ , and the theorem follows from (7.2) and (7.6). Part (i) here is not a consequence of (3.9), since the condition  $\Phi_x(h) = o(h)$  is more stringent than the existence of the symmetric derivative.

It is sometimes important to know the behaviour of the Poisson integral for a tangential approach. The following result, in which for simplicity we take  $x_0 = 0$ , will be useful later and indicates the type of estimate one can expect.

(7.10) THEOREM. Suppose that

$$\left| \frac{1}{h} \int_0^h f(t) dt \right| \leq M \quad \text{for} \quad |h| \leq \pi. \quad (7.11)$$

Then, with  $\delta = 1 - r$  and denoting by  $K$  an absolute constant,

$$|f(r, x)| \leq KM \left( 1 + \frac{|x|}{\delta} \right), \quad (7.12)$$

$$\left| \int_0^x f(r, u) du \right| \leq KM |x|. \quad (7.13)$$

Suppose that the constant term of  $S[f]$  is zero. Then  $F(x) = \int_0^x f dt$  is periodic. For  $f(r, x) = \partial F(r, x) / \partial x$  we have (7.7) with  $\sigma = \pi$  and no term  $o(1)$ . (7.8) with  $\epsilon = M$  then shows that we have (7.12) with  $K = 1$ .

In the general case we put  $f = f_1 + f_2$ , where

$$f_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f dt$$

is a constant. Clearly  $|f_2| \leq M$ , and  $f_1 = f - f_2$  satisfies (7.11) with  $2M$  instead of  $M$ ; also the constant term of  $S[f_1]$  is zero. It follows that  $f(r, x) = f_1(r, x) + f_2(r, x)$  satisfies (7.12) with  $K = 3$ .

For (7.13) we first suppose as before that  $F(x) = \int_0^x f dt$  is periodic. From the equation preceding (7.4) we obtain

$$\int_0^x f(r, u) du = F(r, x) - F(r, 0) = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \{P(r, t-x) - P(r, t)\} dt,$$

so that the left-hand side of (7.13) does not exceed

$$M \frac{1}{\pi} \int_{-\pi}^{\pi} |t| |P(r, t-x) - P(r, t)| dt = MI(r, x),$$

say, and it is enough to show that

$$I(r, x) \leq K|x| \quad (|x| \leq \pi). \tag{7.14}$$

The cases  $x > 0$  and  $x < 0$  in (7.13) are equivalent and we may suppose that  $0 < x \leq \pi$ , or even  $0 < x \leq \frac{1}{2}\pi$ , since otherwise (7.14) is obvious.

Now

$$\pi I(r, x) = \int_{-x}^x + \int_x^{\pi} + \int_{-\pi+x}^{-x} + \int_{-\pi}^{-\pi+x} = I_1 + I_2 + I_3 + I_4.$$

Clearly,

$$\begin{aligned} I_1 &\leq x \int_{-\pi}^{\pi} \{P(r, t) + P(r, t-x)\} dt = 2\pi x, \\ I_2 &= \int_0^{\pi-x} (t+x)P(r, t) dt - \int_x^{\pi} tP(r, t) dt \\ &< x \int_x^{\pi-x} P(r, t) dt + \int_0^x (t+x)P(r, t) dt \\ &< x \int_0^{\pi} P(r, t) dt + 2x \int_0^x P(r, t) dt = \frac{3}{2}\pi x. \end{aligned}$$

The same argument gives  $I_3 < \frac{3}{2}\pi x$ . Since the integrand in  $I_4$  is uniformly bounded we see that  $I_4(x) = O(x)$ . Collecting results we obtain (7.14). For general  $f$ , we apply the same decomposition  $f = f_1 + f_2$  as above.

The series

$$\sum_{\nu=1}^{\infty} \nu(A, \cos \nu x + B, \sin \nu x)$$

is both the conjugate of  $S'[F]$  and the formal derivative of  $\tilde{S}[F]$ . We shall denote it by  $\tilde{S}'[F]$ . Obviously,

$$\sum_{\nu=1}^{\infty} \nu(A, \cos \nu x + B, \sin \nu x) r^{\nu} = \partial \tilde{F}(r, x) / \partial x.$$

(7.15) THEOREM. *If  $F$  is periodic and integrable the difference*

$$\frac{\partial \tilde{F}(r, x)}{\partial x} - \left( -\frac{1}{\pi} \int_{1-r}^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{4 \sin^2 \frac{1}{2}t} dt \right) \tag{7.16}$$

tends to 0 with  $1-r$  at every  $x$  at which  $F$  is smooth, i.e. at which

$$F(x+t) + F(x-t) - 2F(x) = o(t). \quad (7.17)$$

The formula (6.5), with  $F$  for  $f$ , gives

$$\frac{\partial \tilde{F}(r, x)}{\partial x} = -\frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \frac{\partial}{\partial x} Q(r, t-x) dt = \frac{1}{\pi} \int_0^{\pi} \{F(x+t) + F(x-t) - 2F(x)\} Q'(r, t) dt,$$

since  $Q'$  is even and the integral of  $Q'$  over  $(-\pi, \pi)$  is 0. We note that

$$\left. \begin{aligned} Q'(r, t) &= \frac{r[(1+r^2)\cos t - 2r]}{(1-2r\cos t + r^2)^2}, & Q'(1, t) &= -\frac{1}{2(1-\cos t)}, \\ |Q'(r, t)| &\leq r + 2r^2 + 3r^3 + \dots = r/(1-r)^2. \end{aligned} \right\} \quad (7.18)$$

Let  $\xi(t) = F(x+t) + F(x-t) - 2F(x)$ ,  $\delta = 1-r$ .

By (7.17),  $\xi(t) = o(t)$ . We split the integral last written into two, denoted by  $A, B$  with ranges respectively  $0 \leq t \leq \delta$ ,  $\delta \leq t \leq \pi$ . Then (see (7.18))

$$|A| \leq \frac{r}{\pi \delta^2} \int_0^{\delta} |\xi(t)| dt = \delta^{-2} \int_0^{\delta} o(t) dt = o(1),$$

$$B = \frac{1}{\pi} \int_{\delta}^{\pi} \xi(t) Q'(1, t) dt + \frac{1}{\pi} \int_{\delta}^{\pi} \xi(t) [Q'(r, t) - Q'(1, t)] dt = B_1 + B_2,$$

say. Here  $B_1$  equals the expression in parentheses in (7.16), and (7.15) will be proved if we show that  $B_2 \rightarrow 0$ . Collecting separately the terms with  $\cos t$  and  $\cos^2 t$  in the numerator we find

$$Q'(r, t) - Q'(1, t) = \frac{(1-r)^2 [(1+r)^2 - 2r\cos t - 2r\cos^2 t]}{2(1-\cos t) \Delta^2(r, t)} = \frac{\delta^2 [\Delta(r, t) + 2r\sin^2 t]}{2(1-\cos t) \Delta^2(r, t)}.$$

Since  $\Delta > 4r\sin^2 \frac{1}{2}t$ , the last expression is  $O(\delta^2 t^{-4})$ . Hence

$$|B_2| \leq \frac{1}{\pi} \int_{\delta}^{\pi} o(t) O(\delta^2 t^{-4}) dt = \delta^2 \int_{\delta}^{\pi} o(t^{-3}) dt = o(1),$$

which completes the proof of Theorem (7.15).

Thus, under the hypothesis (7.17) (in particular, if  $F'(x)$  exists and is finite) the summability  $A$  of  $\tilde{S}[F]$  at  $x$  is equivalent to the existence of the integral

$$F''(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{4\sin^2 \frac{1}{2}t} dt = \lim_{\delta \rightarrow +0} \left\{ -\frac{1}{\pi} \int_{\delta}^{\pi} \right\}. \quad (7.19)$$

We show below (Chapter IV, § 3) that if  $F'(x)$  exists at every point of a set  $E$  then the integral (7.19) exists almost everywhere in  $E$ ; we infer that the series  $\tilde{S}[F]$  is then summable  $A$  almost everywhere in  $E$ .

(7.20) THEOREM. *If  $f$  is integrable and  $F$  the indefinite integral of  $f$ , then*

$$f(r, x) - \left( -\frac{1}{\pi} \int_{1-r}^{\pi} [f(x+t) - f(x-t)] \frac{1}{2} \cot \frac{1}{2}t dt \right) \rightarrow 0 \quad (r \rightarrow 1) \quad (7.21)$$

at every point where  $F$  is smooth, in particular where  $f$  is continuous. If  $f$  is everywhere continuous, the convergence is uniform.

This is a strengthening of (3·20), since the condition of smoothness, which can be written  $\int_0^h \psi_x(t) dt = o(h)$ , is less stringent than  $\Psi_x(h) = o(h)$ .

Suppose that the constant term of  $S[f]$  is 0. Then  $F$  is periodic and  $f(r, x) = \partial \tilde{F}(r, x) / \partial x$ . Integration by parts gives

$$\int_{\delta}^{\pi} \frac{\xi(t)}{4 \sin^2 \frac{1}{2}t} dt = \xi(\delta) \frac{1}{2} \cot \frac{1}{2}\delta + \int_{\delta}^{\pi} \frac{\psi(t)}{\tan \frac{1}{2}t} dt. \quad (7\cdot22)$$

where  $\xi(t) = F(x+t) + F(x-t) - 2F(x)$ . The integrated term is  $o(1)$  by hypothesis and (7·20) follows from (7·15).

Theorem (7·20) shows that under the smoothness condition (7·17)  $\tilde{S}[f]$  is summable A at  $x$  if and only if  $f(x)$  exists. We show now that *the mere existence of  $f(x)$  implies (7·17) and consequently (by (7·20)) the summability A of  $\tilde{S}[f]$  at  $x$ .*

Since the ratio  $t/\tan \frac{1}{2}t$  and its reciprocal are both bounded and monotone near  $t=0$ , an application of the second mean-value theorem shows that the existence of  $f(x)$  is equivalent to the existence of the integral  $\int_0^{\infty} t^{-1}\psi(t) dt$ . The relation  $\xi(t) = o(t)$  will follow if we apply to the latter integral the following lemma, with  $\alpha=1$  and  $h(u) = u^{-1}\psi(u)$ :

(7·23) LEMMA. Suppose that  $h(u)$ ,  $0 < u \leq a$ , is integrable over each interval  $(\epsilon, a)$ ,  $\epsilon > 0$ , and that the (improper) integral  $\int_0^a h du = \lim_{\epsilon \rightarrow +0} \int_{\epsilon}^a h du$  exists. Then, if  $\alpha > 0$ ,

$$\int_0^t h(u) u^{\alpha} du = o(t^{\alpha}) \quad (t \rightarrow 0). \quad (7\cdot24)$$

Let  $H(u) = \int_0^u h(v) dv$ . For  $t > \epsilon$  integration by parts gives

$$\int_{\epsilon}^t u^{\alpha} h(u) du = [u^{\alpha} H(u)]_{\epsilon}^t - \alpha \int_{\epsilon}^t u^{\alpha-1} H(u) du.$$

If we make  $\epsilon \rightarrow 0$  and observe that  $H(u) = o(1)$ , (7·24) follows.

Let  $\Xi(t) = \int_0^t \xi(u) du$ . A minor modification in the proof of (7·15) (integration by parts so as to have  $\Xi(t)$  instead of  $\xi(t)$ ) shows that (7·16) tends to 0 if (7·17) is replaced by  $\Xi(t) = o(t^2)$ . If (7·19) exists, so does  $\int_0^{\infty} t^{-2}\xi(t) dt$ . By (7·23), with  $\alpha=2$ , we then have  $\Xi(t) = o(t^2)$ . Hence *the existence of the integral (7·19) implies summability A of  $\tilde{S}[f]$  at  $x$ .*

Let  $\Sigma_c z^{\alpha} = \phi(z)$  be the power series whose real and imaginary parts on  $|z|=1$  are  $S[f]$  and  $\tilde{S}[f]$  respectively. If this power series is summable (C, 1) at a point  $e^{i\alpha_0}$ , then  $\phi(z)$  tends to a limit as  $z$  approaches  $e^{i\alpha_0}$  non-tangentially (cf. (1·34)). Thus, considering the imaginary part of  $\phi(z)$  and applying (3·9) and (3·23), we get:

(7·25) THEOREM. If  $f \in L$ , then for almost all  $x$  the harmonic function  $f(r, x)$  tends to a limit as  $(r, x)$  approaches  $(1, x_0)$  non-tangentially.

The fact that  $S[f]$  is almost everywhere summable A to  $f$  can be complemented as follows:

(7·26) THEOREM. Given any set  $E$  in  $(0, 2\pi)$  of measure zero, there is a periodic and integrable  $f(x) \geq 0$  such that for every  $x_0 \in E$  we have  $f(r, x) \rightarrow +\infty$  as  $re^{ix}$  approaches  $e^{ix_0}$  from the interior of the unit circle.

For let  $G_n$  be an open set containing  $E$  and such that  $|G_n| < 1/n^4$ . Let  $f_n(x) = n^2$  in  $G_n$ ,  $f_n(x) = 0$  elsewhere. Let  $f(x) = \sum f_n(x)$ . Obviously,  $f \geq 0$ ,  $f_n \leq f$  for every  $n$  and

$$\int_0^{2\pi} f dz = \sum \int_0^{2\pi} f_n dx < \sum n^{-4} \cdot n^2 < \infty,$$

so that  $f \in L$ . If  $x_0 \in E$ , then  $x_0 \in G_n$  and so, for  $re^{ix} \rightarrow e^{ix_0}$ ,

$$\liminf f(r, x) \geq \liminf f_n(r, x) = n^2,$$

so that  $\lim f(r, x) = +\infty$ .

A similar argument shows that  $S[f]$  is summable  $(C, 1)$  to  $+\infty$  at every point of  $E$ .

### 8. Summability of $S[dF]$ and $\tilde{S}[dF]$

Let  $F(x)$ ,  $0 \leq x \leq 2\pi$ , be a function of bounded variation. From (7·6) we see that at every point where  $F'(x)$  exists and is finite,  $S[dF]$  is summable A to  $\text{sum } F'(x)$ . Similarly, Theorem (7·15) implies that at every such point summability A of  $\tilde{S}[dF]$  is equivalent to the existence of the integral (7·19).

(8·1) THEOREM. Let  $\sigma_n^\alpha(x)$  and  $\tilde{\sigma}_n^\alpha(x)$  be the  $\alpha$ -th Cesàro means of  $S[dF]$  and  $\tilde{S}[dF]$ . If  $0 < \alpha \leq 1$ , then

$$\sigma_n^\alpha(x) \rightarrow F'(x), \tag{8·2}$$

$$\tilde{\sigma}_n^\alpha(x) - \left\{ -\frac{1}{\pi} \int_{1/n}^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{(2 \sin \frac{1}{2}t)^2} dt \right\} \rightarrow 0 \tag{8·3}$$

for almost all  $x$ .

We shall only sketch the proof, which is similar to those of (5·1) and (5·8). First we prove the following analogue of Theorem (11·1) of Chapter II:

(8·4) THEOREM. Let  $F(x)$  be of bounded variation,

$$F_x(t) = F(x+t) - F(x-t) - 2tF'(x),$$

$$G_x(t) = F(x+t) + F(x-t) - 2F(x),$$

and let  $\Phi_x(h)$ ,  $\Psi_x(h)$  be the total variations of the functions  $F_x(t)$ ,  $G_x(t)$  over the interval  $0 \leq t \leq h$ . Then

$$\Phi_x(h) = o(h), \quad \Psi_x(h) = o(h)$$

for almost all  $x$ .

Let  $\gamma$  be any number, and let  $V_\gamma(t)$  be the total variation of the function  $F(t) - \gamma t$ . For almost all  $x$ , we have  $V_\gamma'(x) = |F'(x) - \gamma|$ , that is,

$$h^{-1} \int_0^h |d_t \{F(x \pm t) - \gamma(\pm t)\}| \rightarrow |F'(x) - \gamma| \quad \text{as } h \rightarrow +0,$$

where the suffix  $t$  indicates that the variation is taken with respect to the variable  $t$ .

Considering rational values of  $\gamma$  and arguing as in the proof of Theorem (11.1) in Chapter II, we prove that

$$\int_0^h |d_t\{F(x \pm t) - (\pm t)F'(x)\}| = o(h),$$

and hence

$$\int_0^h |d_t F_x(t)| = o(h). \quad \int_0^h |d_t G_x(t)| = o(h),$$

for almost all  $x$ .

It is now easy to prove (8.2) for all  $x$  with  $\Phi_x(h) = o(h)$ . For

$$\sigma_n^\alpha(x) = \frac{1}{\pi} \int_{-\pi}^\pi K_n^\alpha(x-t) dF(t) = \frac{1}{\pi} \int_0^\pi K_n^\alpha(t) d_t\{F(x+t) - F(x-t)\},$$

$$\sigma_n^\alpha(x) - F'(x) = \frac{1}{\pi} \int_0^\pi K_n^\alpha(t) d_t F_x(t).$$

$$|\sigma_n^\alpha(x) - F'(x)| \leq \frac{1}{\pi} \int_0^\pi |K_n^\alpha(t)| |d_t F_x(t)| = \frac{1}{\pi} \int_0^{1/n} + \frac{1}{\pi} \int_{1/n}^\pi = A + B,$$

say. Here  $A \leq 2n\Phi_x(1/n) = o(1)$ . Integration by parts, in view of (5.5), gives

$$B \leq C\pi^{-1}n^{-\alpha}[\Phi_x(t)t^{-\alpha-1}]_{1/n}^\pi + C(1+\alpha)\pi^{-1}n^{-\alpha} \int_{1/n}^\pi \Phi_x(t)t^{-\alpha-2} dt = o(1),$$

and this yields (8.2). To obtain (8.3) we note that

$$\tilde{\sigma}_n^\alpha(x) = -\frac{1}{\pi} \int_0^\pi \tilde{K}_n^\alpha(t) d_t[F(x+t) + F(x-t)] = -\frac{1}{\pi} \int_0^\pi \tilde{K}_n^\alpha(t) d_t G_x(t),$$

$$\tilde{\sigma}_n^\alpha(x) - \left(-\frac{1}{\pi} \int_{1/n}^\pi \frac{d_t G_x(t)}{2 \tan \frac{1}{2}t}\right) = -\frac{1}{\pi} \int_0^{1/n} \tilde{K}_n^\alpha(t) d_t G_x(t) + \frac{1}{\pi} \int_{1/n}^\pi H_n^\alpha(t) d_t G_x(t)$$

(cf. (5.13)). The two terms on the right are  $o(1)$ , since  $\Psi_x(h) = o(h)$ . Integration by parts gives

$$\int_{1/n}^\pi \frac{d_t G_x(t)}{2 \tan \frac{1}{2}t} dt - \int_{1/n}^\pi \frac{G_x(t)}{(2 \sin \frac{1}{2}t)^2} dt = \left[ \frac{G_x(t)}{2 \tan \frac{1}{2}t} \right]_{1/n}^\pi = o(1) \tag{8.5}$$

for all  $x$  at which  $F$  is smooth, and this proves (8.3).

Arguing as in the proofs of Theorem (11.9), Chapter II, we find that the *partial sums of  $S[dF]$  and  $\tilde{S}[dF]$  are  $o(\log n)$  almost everywhere.*

Taking for granted (again) that the integral (7.19) exists almost everywhere (see Chapter VII, (1.6)) we see that  $\tilde{S}[dF]$  is summable  $(C, \alpha)$ ,  $\alpha > 0$ , at almost all points. This implies, in turn, that Theorems (4.4) and (7.25) are valid for Fourier-Stieltjes series.

### 9. Fourier series at simple discontinuities

Given a numerical sequence  $s_0, s_1, s_2, \dots$ , consider the numbers

$$\tau_n = \left( \sum_{\nu=1}^n \frac{s_\nu}{\nu} \right) / \log(n+1) \quad (n = 1, 2, \dots). \tag{9.1}$$

We can verify that the matrix transforming  $\{s_n\}$  into  $\{\tau_n\}$  satisfies the conditions of regularity (§1). The method of summability defined by (9.1) is called the *logarithmic mean*.

Let  $\sigma_n = (s_0 + s_1 + \dots + s_n) / (v + 1)$ . Substituting  $s_n = (v + 1)\sigma_n - v\sigma_{n-1}$  into (9.1), we get

$$\tau_n = \left( -\sigma_0 + \sum_{v=1}^{n-1} \frac{\sigma_v}{v} + \frac{n+1}{n} \sigma_n \right) / \log(n+1). \tag{9.2}$$

The matrix transforming  $\{\sigma_n\}$  into  $\{\tau_n\}$  again satisfies the conditions of regularity, so that the logarithmic mean is at least as strong as (C, 1). If  $\sigma_n = (-1)^n$ , then  $\tau_n \rightarrow 0$  though  $\lim \sigma_n$  does not exist; thus the method of the logarithmic mean is actually stronger than (C, 1).

Theorem (8.13) of Chapter II can be restated as follows: at every point  $x$  where  $f$  has a jump  $d$ , the terms

$$v(b_v \cos vx - a_v \sin vx)$$

of  $S'[f]$  are summable by the logarithmic mean to  $d/\pi$ . Thus the terms of the differentiated Fourier series determine the jumps of the function.

It can be shown by examples (see p. 314, Example 3) that in general we cannot here replace the logarithmic mean by (C, 1). This, however, can be done if the function is of bounded variation. In this case we assume, slightly more generally, that  $F(x)$  is not necessarily periodic but satisfies the condition

$$F(x + 2\pi) - F(x) = \text{const.} \quad (-\infty < x < +\infty),$$

and is of bounded variation in  $(0, 2\pi)$ .

(9.3) THEOREM. Let  $dF(x) \sim \sum c_n e^{inx}$ . Then

$$\frac{1}{n} \sum_{v=-n}^n c_v e^{ivx_0} \rightarrow \pi^{-1} [F(x_0 + 0) - F(x_0 - 0)]. \tag{9.4}$$

If  $F \sim \frac{1}{2}A_0 + \sum (A_v \cos vx + B_v \sin vx),$

then  $S[dF] = S'[F]$  and (9.4) can be written

$$\frac{1}{n} \sum_{v=-n}^n v(B_v \cos vx_0 - A_v \sin vx_0) \rightarrow \pi^{-1} [F(x_0 + 0) - F(x_0 - 0)]. \tag{9.5}$$

In the proof of (9.4) we may suppose that  $x_0 = 0$ . We verify (9.4) (or (9.5)) for the function

$$\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2} \sin 2x + \dots \quad (0 < x < 2\pi).$$

Subtracting a multiple of it from  $F$  we may suppose that  $F$  is continuous at  $x = 0$ , and we have to show that

$$\sum_{v=-n}^n c_v = \frac{1}{\pi} \int_{-\eta}^{\eta} D_n(t) dF(t) = \frac{1}{\pi} \int_{-\eta}^{\eta} + \frac{1}{\pi} \int_{\eta < |t| \leq \pi} = A + B$$

is  $o(n)$ . We choose  $\eta$  so small that the total variation of  $F$  over  $(-\eta, \eta)$  is  $\leq \epsilon$ . Then

$$|A| \leq \pi^{-1} (n + \frac{1}{2}) \int_{-\eta}^{\eta} |dF(t)| < 2n\pi^{-1}\epsilon < \epsilon n,$$

$$|B| \leq \pi^{-1} \max_{\eta < |t| \leq \pi} |D_n(t)| \int_{\eta < |t| \leq \pi} |dF(t)| = O(1) = o(n).$$

Hence  $A + B = o(n)$  and (9.3) follows.

We apply Theorem (9.3) to the function  $F^*$  associated with  $F$  by means of the formula (1.28) of Chapter II. Using the results (1.29) and (1.30) of Chapter II, we obtain:

(9·6) THEOREM. Let  $dF(x) \sim \sum c_\nu e^{i\nu x}$ , and let  $d_1, d_2, \dots$  be all the jumps of  $F(x)$  in  $0 \leq x < 2\pi$ . Then

$$\lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{\nu=-n}^n |c_\nu|^2 = (2\pi)^{-2} \sum |d_j|^2. \quad (9\cdot7)$$

and  $F$  is everywhere continuous if and only if

$$R_n = \frac{1}{2n+1} \sum_{\nu=-n}^n |c_\nu|^2 \rightarrow 0. \quad (9\cdot8)$$

In particular,  $F$  is continuous if  $c_\nu \rightarrow 0$ .

The condition  $R_n \rightarrow 0$  is equivalent to

$$R'_n = \frac{1}{2n+1} \sum_{\nu=-n}^n |c_\nu| \rightarrow 0. \quad (9\cdot9)$$

For  $R_n \leq R'_n \max |c_k|$ , and so (9·9) implies (9·8). The converse follows from Schwarz's inequality:

$$R'_n = (2n+1)^{-1} \sum_{\nu=-n}^n |c_\nu| \cdot 1 \leq (2n+1)^{-1} \left( \sum_{\nu=-n}^n |c_\nu|^2 \right)^{\frac{1}{2}} (2n+1)^{\frac{1}{2}} = R_n^{\frac{1}{2}}.$$

Return to the hypothesis  $f \in L$ . For applications it is of interest to investigate the Abel summability of the sequence  $\{\nu(b_\nu \cos \nu x - a_\nu \sin \nu x)\}$ , that is, the existence of the limit of

$$(1-r) \sum_{\nu=1}^{\infty} \nu(b_\nu \cos \nu x - a_\nu \sin \nu x) r^\nu = (1-r) \frac{\partial}{\partial x} f(r, x) \quad (9\cdot10)$$

as  $r \rightarrow 1$ . Instead of  $d(x_0) = \lim_{t \rightarrow +0} [f(x_0+t) - f(x_0-t)]$  we may consider the *generalized jump*

$$\delta(x_0) = \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h [f(x_0+t) - f(x_0-t)] dt = \lim_{h \rightarrow 0} \frac{F(x_0+h) + F(x_0-h) - 2F(x_0)}{h},$$

where  $F$  is the integral of  $f$ . Thus  $\delta(x_0) = 0$  is the same thing as the smoothness of  $F$  at  $x_0$ . The existence of  $d(x_0)$  implies that of  $\delta(x_0)$  and both numbers are equal.

(9·11) THEOREM. If  $\delta(x_0)$  exists and is finite then  $(1-r)f_x(r, x)$ , with  $x = x_0$ , tends to  $\delta(x_0)/\pi$  as  $r \rightarrow 1$ .

We may assume that  $a_0 = 0$ , so that  $F$  is periodic and  $f_x(r, x) = F_{xx}(r, x)$ . We may also take  $x_0 = 0$ . If  $f \sim \sum \nu^{-1} \sin \nu x$ , then (9·11) is obviously valid and so we may confine ourselves to the case  $\delta(x_0) = 0$ . Since

$$P''(r, t) = (1-r^2) r \frac{[2r(1+\sin^2 t) - (1+r^2) \cos t]}{(1-2r \cos t + r^2)^2} \quad (9\cdot12)$$

is even in  $t$ , and since  $\int_0^\pi P'' dt = P'(r, \pi) - P'(r, 0) = 0$ , we have

$$\frac{\partial^2 F(r, x)}{\partial x^2} = \frac{1}{\pi} \int_{-\pi}^\pi F(t) \frac{\partial^2}{\partial x^2} P(r, t-x) dt = \frac{1}{\pi} \int_{-\pi}^\pi \frac{1}{2} \{F(x+t) + F(x-t) - 2F(x)\} P''(r, t) dt. \quad (9\cdot13)$$

It is enough to show that the expression

$$K(r, t) = (1-r)tP''(r, t)$$

has the properties (B') and (C) of kernels (see § 2). (Property (A) is not needed because  $\delta(x_0) = 0$ .)

Property (c) follows from (9.12). The latter formula also shows that  $P''(r, t)$  changes sign in  $(0, \pi)$  once only, namely, for  $t = \tau = \tau(r)$  satisfying

$$\frac{\cos \tau}{1 + \sin^2 \tau} = \frac{2r}{1 + r^2},$$

so that  $\tau \rightarrow 0$  as  $r \rightarrow 1$ . Furthermore,

$$\frac{(1-r)^2}{1+r^2} = \frac{1 - \cos \tau + \sin^2 \tau}{1 + \sin^2 \tau} \approx \frac{3}{2} \tau^2,$$

so that

$$\tau \approx \sqrt{\frac{2}{3}}(1-r). \tag{9.14}$$

For (B') we have to show that

$$\int_0^\pi t |P''(r, t)| dt \leq \frac{C}{1-r}.$$

The left-hand side here is

$$\left(-\int_0^\tau + \int_\tau^\pi\right) t P'' dt = -2\tau P'(r, \tau) + 2P(r, \tau) - P(r, 0) - P(r, \pi) < -2\tau P'(r, \tau) + 2P(r, \tau).$$

Since  $P(r, \tau) \leq 1/(1-r)$ , it is enough to obtain a similar estimate for  $-\tau P'(r, \tau)$ , which is easy by means of (9.14). This completes the proof of (9.11).

It is to be observed that, in the argument beginning with (9.13), we did not make use of the fact that  $F$  was an integral. Thus the reasoning shows that for any integrable  $F$  which is smooth at  $x$  we have

$$F_{xx}(r, x) = o\{(1-r)^{-1}\}. \tag{9.15}$$

With the obvious extension to uniformity we can state the following result:

(9.16) THEOREM. *If  $F \in \lambda_*$ , we have (9.15), uniformly in  $x$ . If  $F \in \Lambda_*$ , the conclusion holds with 'O' instead of 'o'.*

### 10. Fourier sine series

If  $f(x)$  is odd, its Poisson integral may be written

$$f(r, x) = \frac{1}{\pi} \int_0^\pi f(t) [P(r, x-t) - P(r, x+t)] dt. \tag{10.1}$$

Since  $P(r, u)$  is even and decreases in  $0 < u < \pi$ , the difference in square brackets is positive for  $0 < x < \pi$ . Thus, if  $f(t)$  is non-negative and  $f(t) \neq 0$  in  $(0, \pi)$ ,  $f(r, x)$  is strictly positive for  $0 < x < \pi$ . (Of course,  $f(r, 0) = f(r, \pi) = 0$ .) If  $m \leq f(t) \leq M$ , and  $f(t) \neq \text{const.}$  for  $0 \leq t \leq \pi$ , then (10.1) implies

$$\frac{m}{\pi} \int_0^\pi [P(r, x-t) - P(r, x+t)] dt < f(r, x) < \frac{M}{\pi} \int_0^\pi [P(r, x-t) - P(r, x+t)] dt$$

for  $0 < x < \pi$ . These inequalities may be rewritten

$$m\mu(r, x) < f(r, x) < M\mu(r, x),$$

where  $\mu(r, x)$  (positive for  $0 < x < \pi$ ) is the Poisson integral of the function

$$\mu(t) = \text{sign } t \quad (|t| < \pi).$$

This results holds if summability A is replaced by summability (C, 3):

(10.2) THEOREM. If  $f(x) \neq 0$  is odd and non-negative in  $(0, \pi)$ , then the third arithmetic means of  $S[f]$  are strictly positive for  $0 < x < \pi$ . More generally, if  $f(x) \neq \text{const.}$  and  $m \leq f(x) \leq M$  in  $(0, \pi)$ , then

$$m\sigma_n^3(x; \mu) < \sigma_n^3(x; f) < M\sigma_n^3(x; \mu) \quad (0 < x < \pi; n = 1, 2, \dots).$$

For the proof it is enough (arguing as in the case of Poisson's kernel) to show that the kernel  $K_n^3(t)$  is strictly decreasing in  $(0, \pi)$ , or,  $K_n^3(t)$  being a polynomial, that  $\{K_n^3(t)\}' \leq 0$  there. The expression  $\{K_n^3(t)\}'$  is the Cesàro mean  $S_n^3(t)/A_n^3$  of the series  $\frac{1}{2} + \cos t + \cos 2t + \dots$  differentiated term by term. Then (1.9) gives the identity

$$\sum_{n=0}^{\infty} S_n^3(t) r^n = (1-r)^{-4} P'(r, t) = - \left[ \frac{1}{2} \frac{1-r^2}{(1-r)^2 \Delta(r, t)} \right]^3 \frac{4r \sin t}{1-r^2} \quad (10.3)$$

where

$$\Delta(r, t) = 1 - 2r \cos t + r^2.$$

Using (1.9) again we see that the expression in square brackets is the power series

$$K_0(t) + 2K_1(t)r + 3K_2(t)r^2 + \dots,$$

where  $K_n(t)$  is Fejér's kernel and is non-negative. Since  $r/(1-r^2) = r + r^3 + \dots$  also has non-negative coefficients, we see that  $S_n^3(t) \leq 0$  in  $(0, \pi)$ , and (10.2) follows.

## 11. Gibbs's phenomenon for the method $(C, \alpha)$

This phenomenon was defined in §9 of Chapter II. Let  $M(x_0)$  and  $m(x_0)$  be the maximum and minimum of  $f$  at  $x_0$  (see §2). Since for every  $\{x_n\} \rightarrow x_0$  the limits of indetermination of  $\sigma_n(x_n)$  are contained between  $m(x_0)$  and  $M(x_0)$  (see (2.30)), the  $(C, 1)$  mean of  $S[f]$  does not show the phenomenon. It is easy to see that if the phenomenon for  $(C, \alpha)$  is not shown for  $\alpha = \alpha_1$  then it is not shown for any  $\alpha > \alpha_1$ . For if

$$m(x_0) - \epsilon \leq \sigma_n^\alpha(x) \leq M(x_0) + \epsilon$$

for  $|x - x_0| \leq \eta$ ,  $n > n_0$ , then

$$m(x_0) - 2\epsilon \leq \sigma_n^\alpha(x) \leq M(x_0) + 2\epsilon$$

for  $|x - x_0| \leq \eta$ ,  $n > n_1$  (see (1.5)). It is therefore enough to consider the range  $0 < \alpha < 1$ .

(11.1) THEOREM. There is an absolute constant  $\alpha_0$ ,  $0 < \alpha_0 < 1$ , with the following property: if  $f(x)$  has a simple discontinuity at a point  $\xi$ , the means  $\sigma_n^\alpha(x; f)$  show Gibbs's phenomenon at  $\xi$  for  $\alpha < \alpha_0$  but not for  $\alpha \geq \alpha_0$ .

Since  $S[f]$  is uniformly summable  $(C, \alpha)$  at every point of continuity, it is enough (as in Chapter II, §9) to prove (11.1) for the function

$$f(x) \sim \sin x + \frac{1}{2} \sin 2x + \dots$$

at  $\xi = 0$ . Observing that  $S[f] = \cos x + \cos 2x + \dots$ , we find

$$\left. \begin{aligned} \sigma_n^\alpha(x) &= -\frac{1}{2}x + \int_0^x K_n^\alpha(t) dt, \\ \sigma_n^\alpha(x) &= \frac{1}{2}(\pi - x) - \int_x^\pi K_n^\alpha(t) dt. \end{aligned} \right\} \quad (11.2)$$

Let us first take  $\alpha = 1$ . The first formula then gives

$$\begin{aligned} \sigma_n(x) &= -\frac{1}{2}x + \frac{2}{n+1} \int_0^x \frac{\sin^2 \frac{1}{2}(n+1)t}{t^2} dt + \frac{2}{n+1} \int_0^x \sin^2 \frac{1}{2}(n+1)t \left[ \frac{1}{(2 \sin \frac{1}{2}t)^2} - \frac{1}{t^2} \right] dt \\ &= -\frac{1}{2}x + \int_0^{(n+1)x} \left( \frac{\sin u}{u} \right)^2 du + R_n(x), \end{aligned} \tag{11.3}$$

where  $R_n(x) = O(1/n)$  uniformly in  $x$ . Observing that  $\sigma_n(x) \rightarrow \frac{1}{2}(\pi - x)$  for  $0 < x < 2\pi$ , we deduce the formula

$$\int_0^\infty \left( \frac{\sin u}{u} \right)^2 du = \frac{1}{2}\pi. \tag{11.4}$$

(This is an analogue of (8.4) in Chapter II and can be deduced from it by integration by parts.) From (11.3) and (11.4) we get

$$\sigma_n(x) = \frac{1}{2}(\pi - x) - \int_{\frac{1}{2}(n+1)x}^\infty \left( \frac{\sin u}{u} \right)^2 du + R_n(x) < \frac{1}{2}\pi - \int_{\frac{1}{2}(n+1)x}^\infty \left( \frac{\sin u}{u} \right)^2 du + R_n(x),$$

for  $n \geq 1$ . Hence

(i) given any  $l > 0$ , there is a  $\delta = \delta(l) > 0$  and an  $n_0 = n_0(l)$  such that

$$\sigma_n(x) < \frac{1}{2}\pi - \delta \quad \text{for } 0 \leq x \leq l/n, \quad n > n_0.$$

We shall now use the approximate formulae (5.15) for  $K_n^\alpha(t)$ . Integrating the right-hand side there over  $(x, \pi)$ , applying the second mean-value theorem to the first integral and using the second formula (11.2), we get for  $\sigma_n^\alpha(x)$  the value

$$\frac{1}{2}(\pi - x) - \frac{\alpha}{n+1} \frac{1}{2} \cot \frac{1}{2}x + \frac{2\theta_1}{nA_n^\alpha(2 \sin \frac{1}{2}x)^{\alpha+1}} + \frac{B}{n^2x^2}, \tag{11.5}$$

where  $|\theta_1| \leq 1$  and  $|B|$  is less than an absolute constant. Since  $A_n^\alpha \geq Cn^\alpha$  for  $n \geq 1$  and  $0 \leq \alpha \leq 1$ , we see that, for  $nx$  large enough, of the three last terms in (11.5) the first is the largest in absolute value. Therefore

(ii) if  $\frac{1}{2} \leq \alpha \leq 1$ , there is an  $l_1$  such that  $|\sigma_n^\alpha(x)| \leq \frac{1}{2}\pi$  for  $l_1/n \leq x \leq \pi$ ,  $n \geq n_1$ .

We shall now show that

(iii) if  $1 - \alpha$  is small enough then  $|\sigma_n^\alpha(x)| \leq \frac{1}{2}\pi$  for  $0 \leq x \leq l_1/n$ .

This, in conjunction with (ii), will prove that if  $\alpha$  is close enough to 1 the  $\sigma_n^\alpha(x)$  do not show Gibbs's phenomenon. First of all we verify the inequality

$$A_k^\alpha/A_n^\alpha \geq A_k^\beta/A_n^\beta \quad \text{for } -1 < \alpha < \beta, \quad 0 \leq k \leq n.$$

From it we deduce that  $|\sigma_n^\alpha(x) - \sigma_n^\beta(x)|$  is less than

$$\sum_{\nu=1}^n \left( \frac{A_{n-\nu}^\alpha}{A_n^\alpha} - \frac{A_{n-\nu}^\beta}{A_n^\beta} \right) \frac{|\sin \nu x|}{\nu} \leq x \sum_{\nu=1}^n \left( \frac{A_{n-\nu}^\alpha}{A_n^\alpha} - \frac{A_{n-\nu}^\beta}{A_n^\beta} \right) = x \left( \frac{A_n^{\alpha+1}}{A_n^\alpha} - \frac{A_n^{\beta+1}}{A_n^\beta} \right) = \frac{nx(\beta - \alpha)}{(x+1)(\beta+1)}. \tag{11.6}$$

For  $\beta = 1$  the last term is less than  $\frac{1}{2}nx(1 - \alpha)$ , and so, using (i), it is enough to take  $\alpha$  such that  $\frac{1}{2}(1 - \alpha)l_1 \leq \delta(l_1)$ .

In order to show that for  $\alpha$  positive and small enough the phenomenon does occur, we consider the difference  $\sigma_n^\alpha - \sigma_n^0 = \sigma_n^\alpha - s_n$ , which, by (11.6), is numerically less than  $nx\alpha/(\alpha + 1) < nx\alpha$ . Since  $s_n(\pi/n)$  tends to a limit greater than  $\frac{1}{2}\pi$  (Chapter II, § 9), it follows that  $\liminf \sigma_n^\alpha(\pi/n) > \frac{1}{2}\pi$ , and so the phenomenon does occur, for  $\alpha$  small enough.

We have therefore shown the existence of  $\alpha_0$ .  $0 < \alpha_0 < 1$ , such that for  $\alpha < \alpha_0$  we have the phenomenon, while for  $\alpha > \alpha_0$  we do not. If we show that the set  $G$  of  $\alpha$  for which the phenomenon occurs is open, it cannot occur for  $\alpha = \alpha_0$ , and the proof of (11.1) will be complete.

Let  $0 < \alpha' < \alpha_0$ . As in (ii) we see that there is an  $l'$  such that

$$|\sigma_n^\alpha(x)| \leq \frac{1}{2}\pi \quad \text{for } \alpha' \leq \alpha \leq 1, \quad l'/n \leq x \leq \pi, \quad n \geq n'.$$

From the majorant (11.6) for  $|\sigma_n^\alpha - \sigma_n^\beta|$  we deduce that  $\sigma_n^\alpha(x)$  is a uniformly continuous function of  $\alpha$  in the range  $0 \leq \alpha \leq 1, 0 \leq x \leq l'/n, n = 1, 2, \dots$ . If the Gibbs's phenomenon occurs for a value  $\alpha > \alpha'$ , that is, if there is an  $x_n \rightarrow 0$  such that  $\sigma_n^\alpha(x_n) > \frac{1}{2}\pi + \epsilon$  for all  $n$  large enough, then, first,  $0 < x_n \leq l'/n$ , and secondly, if  $\beta - \alpha$  is small enough,  $\sigma_n^\beta(x_n) \geq \frac{1}{2}\pi + \frac{1}{2}\epsilon$ . This shows that  $G$  is open and completes the proof of (11.1).

### 12. Theorems of Rogosinski

Let  $\lambda(t)$  be a function with  $\lambda(0) = 1$ . Any series  $u_0 + u_1 + \dots$  may be considered as the series

$$\sum_{\nu=0}^{\infty} u_\nu \lambda(\nu t) \tag{12.1}$$

at the point  $t = 0$ . Let  $s_n(t)$  be the  $n$ th partial sum of (12.1), and  $s_n$  the  $n$ th partial sum of  $\Sigma u_\nu$ . We shall investigate the behaviour of  $s_n(t)$  as  $n \rightarrow \infty$  and simultaneously  $t \rightarrow 0$ .

(12.2) THEOREM. Suppose that  $\alpha_n = O(1/n)$ . (i) If  $\lambda(t)$  is continuous at  $t = 0$  and is of bounded variation over every finite interval, then  $s_n \rightarrow s$  implies  $s_n(\alpha_n) \rightarrow s$ . (ii) If  $\lambda''(t)$  exists and is bounded over every finite interval, then the summability (C, 1) of  $u_0 + u_1 + \dots$  to sum  $s$  implies  $s_n(\alpha_n) - (s_n - s) \lambda(n\alpha_n) \rightarrow s$ . (12.3)

(i) Summation by parts gives

$$s_n(\alpha_n) = \sum_{\nu=0}^{n-1} s_\nu [\lambda(\nu\alpha_n) - \lambda((\nu+1)\alpha_n)] + s_n \lambda(n\alpha_n). \tag{12.4}$$

This is a linear transformation of  $\{s_\nu\}$  which satisfies the conditions of regularity, and (i) follows.

(ii) Let  $\sigma_n$  be the (C, 1) means of  $u_0 + u_1 + \dots$ . Summation by parts gives

$$\begin{aligned} & s_n(\alpha_n) - s_n \lambda(n\alpha_n) + \sigma_n \lambda(n\alpha_n) \\ &= \sum_{\nu=0}^{n-2} (\nu+1) \sigma_\nu [\lambda(\nu\alpha_n) - 2\lambda((\nu+1)\alpha_n) + \lambda((\nu+2)\alpha_n)] + n\sigma_{n-1} [\lambda((n-1)\alpha_n) - \lambda(n\alpha_n)] \\ & \qquad \qquad \qquad + \sigma_n \lambda(n\alpha_n), \end{aligned} \tag{12.5}$$

and the right-hand side is a linear transformation of  $\{\sigma_\nu\}$ . The example  $u_0 = 1, u_1 = u_2 = \dots = 0$  gives  $s_0 = s_1 = \dots = 1, \sigma_0 = \sigma_1 = \dots = 1$  and shows that the sum of the coefficients of the  $\sigma_\nu$  on the right is 1. This proves condition (iii) of regularity.

We now observe that for any fixed  $x$ ,

$$\begin{aligned} \lambda(x+u) - \lambda(x) &= u\lambda'(x+\theta u), \\ \lambda(x+u) + \lambda(x-u) - 2\lambda(x) &= \frac{1}{2}u^2\lambda''(x; \theta_1 u), \end{aligned}$$

where  $\theta$  and  $|\theta_1|$  are between 0 and 1. Let  $|n\alpha_n| \leq h$  for all  $n$ , and denote by  $M$  the common bound of  $|\lambda(t)|$ ,  $|\lambda'(t)|$  and  $|\lambda''(t)|$  in the interval  $(-h, h)$ ; then we find that the sum of the moduli of the coefficients of the  $\sigma$ , on the right in (12.5) is at most

$$M \cdot \frac{1}{2} \alpha_n^2 \sum_0^{n-2} (\nu+1) + nM |\alpha_n| + M \leq M(\frac{1}{2}h^2 + h + 1).$$

This proves condition (ii) of regularity. Condition (i) follows from the continuity of  $\lambda$  at  $t=0$ . Hence the left-hand side of (12.5) tends to  $s$  and (12.3) follows.

We note that if  $\alpha_n = \alpha/n$ , where  $\alpha$  is a zero of  $\lambda(t)$ , then (12.3) simplifies.

We also observe that if the terms of  $u_0 + u_1 + \dots$  depend on a parameter, and if the hypotheses concerning this series are satisfied uniformly, the conclusions also hold uniformly.

Suppose now that  $\lambda(t)$  satisfies the same conditions as before, except for the condition  $\lambda(0) = 1$ . The case  $\lambda(0) \neq 0$  reduces to  $\lambda(0) = 1$  by considering  $\lambda(t)/\lambda(0)$ . If, however,  $\lambda(0) = 0$ , condition (iii) of regularity is no longer satisfied and the matrices generating the transformations (12.4) and (12.5) will have sums 0 in each row.

The result for this case is:

(12.6) THEOREM. *If  $\lambda(0) = 0$ , and if the other conditions of (12.2) are satisfied, we have*

$$s_n(\alpha_n) \rightarrow 0, \quad s_n(\alpha_n) - (s_n - s) \lambda(n\alpha_n) \rightarrow 0$$

*respectively, according as  $u_0 + u_1 + \dots$  is convergent or summable (C, 1) to sum  $s$ .*

The most important special cases are  $\lambda(t) = \cos t$  and  $\lambda(t) = \sin t$ . The reason for this is that, if  $S_n(x)$  denotes the partial sum of any series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x), \tag{12.7}$$

$$\text{then } \left. \begin{aligned} \frac{1}{2}[S_n(x + \alpha_n) + S_n(x - \alpha_n)] &= \frac{1}{2}a_0 + \sum_{\nu=1}^n (a_\nu \cos \nu x + b_\nu \sin \nu x) \cos \nu \alpha_n \\ \frac{1}{2}[S_n(x + \alpha_n) - S_n(x - \alpha_n)] &= - \sum_{\nu=1}^n (a_\nu \sin \nu x - b_\nu \cos \nu x) \sin \nu \alpha_n \end{aligned} \right\} \tag{12.8}$$

are  $s_n(\alpha_n)$  with  $\lambda(t) = \cos t, \sin t$ . Thus from the first formula we get:

(12.9) THEOREM. *Let  $\alpha_n = O(1/n)$ , and let  $S_n(x)$  be the partial sums of (12.7). Then*

$$\frac{1}{2}[S_n(x + \alpha_n) + S_n(x - \alpha_n)] \rightarrow s$$

*at every point where  $S_n(x) \rightarrow s$ ; and*

$$\frac{1}{2}[S_n(x + \alpha_n) + S_n(x - \alpha_n)] - (S_n(x) - s) \cos n\alpha_n \tag{12.10}$$

*tends to  $s$  at every point where (12.7) is summable (C, 1) to  $s$ . In particular, if (12.7) is  $S[f]$ , (12.10) tends to  $f(x)$  at every point of continuity of  $f$ , and the convergence is uniform over any closed interval of continuity.*

If  $\alpha_n = \frac{1}{2}p\pi/n$ , where  $p$  is any fixed odd integer, (12.10) becomes

$$\frac{1}{2}[S_n(x + \frac{1}{2}p\pi/n) + S_n(x - \frac{1}{2}p\pi/n)], \tag{12.11}$$

*and this expression gives a method of summability of trigonometric series not weaker than the method (C, 1).*

If  $\alpha_n = \frac{1}{2}p\pi/n + O(n^{-2})$ , the last term in (12.10) is  $o(1)$ , since  $S_n(x) = o(n)$  wherever (12.7) is summable (C, 1). In particular, what was said about (12.11) also applies to

$$\frac{1}{2} \left[ S_n \left( x + \frac{p\pi}{2n+1} \right) + S_n \left( x - \frac{p\pi}{2n+1} \right) \right] \quad (p = 1, 3, 5, \dots). \quad (12.12)$$

(12.13) **THEOREM.** Let (12.7) be  $S[f]$  and let  $\xi$  be a point of continuity of  $f$ . Then, for any sequence  $h_n \rightarrow 0$ , the expression

$$\tau_n(\xi) = \frac{1}{2} [S_n(\xi + h_n + \frac{1}{2}\pi/n) + S_n(\xi + h_n - \frac{1}{2}\pi/n)] \quad (12.14)$$

tends to  $f(\xi)$ . For (12.5), with  $\lambda(t) = \cos t$ ,  $\alpha_n = \pi/2n$  shows that  $\limsup |\sigma_n(\alpha_n)| \leq A \limsup |\sigma_\nu|$ , where  $A$  is an absolute constant. Similarly

$$\limsup |\tau_n(\xi)| \leq A \limsup |\sigma_\nu(\xi + h_n; f)|.$$

We may suppose that  $f(\xi) = 0$ . Taking into account that  $\sigma_n(\xi + h_n; f) \rightarrow 0$  (see (2.30)), we have  $\tau_n(\xi) \rightarrow 0$ .

In Chapter VIII we shall see that  $S_n(x; f)$  may diverge at a point  $\xi$  of continuity of  $f$ . Theorem (12.13) indicates the existence of a certain symmetry in the behaviour of the curves  $y = S_n(x)$  near  $\xi$ : *The mean of the values of  $S_n(x)$  at the end-points of any interval of length  $\pi/n$  differs little from  $f(\xi)$ , if the distance of its midpoint from  $\xi$  is small.*

These are applications of  $\lambda(t) = \cos t$ . Now let  $\lambda(t) = \sin t$ . From the second formula (12.8), and from (12.6), we get

(12.15) **THEOREM.** Let  $\alpha_n = O(1/n)$ , and let  $S_n(x)$  and  $\bar{S}_n(x)$  be the partial sums of (12.7) and of the conjugate series. Then

$$S_n(x + \alpha_n) - S_n(x - \alpha_n) \rightarrow 0$$

at every point  $x$  where the conjugate series converges. At every point where it is summable (C, 1) to  $\bar{s}$ , we have  $\frac{1}{2} [S_n(x + \alpha_n) - S_n(x - \alpha_n)] + (\bar{S}_n(x) - \bar{s}) \sin n\alpha_n \rightarrow 0$ .

Hence, if  $q$  is any fixed integer,

$$S_n(x + q\pi/n) - S_n(x - q\pi/n) \rightarrow 0$$

at every point where  $\{\bar{S}_n(x)\}$  is summable (C, 1), and in particular almost everywhere if (12.7) is an  $S[f]$ .

(12.16) **THEOREM.** Let  $S_n(x)$  be the partial sums of  $\sum_0^\infty c_\nu e^{i\nu x}$  and let  $\alpha_n = O(1/n)$ . Then  $S_n(x + \alpha_n) \rightarrow s$  if  $S_n(x) \rightarrow s$ ; and

$$S_n(x + \alpha_n) - (S_n(x) - s) e^{in\alpha_n} \rightarrow s \quad (12.17)$$

if  $\{S_n(x)\}$  is summable (C, 1) to  $s$ .

Apply (12.2), with  $\lambda(t) = e^{it}$ , to  $S_n(x + \alpha_n) = \sum_0^n c_\nu e^{i\nu x} \cdot e^{i\nu\alpha_n}$ .

### 13. Approximation to functions by trigonometric polynomials

Given a periodic and continuous function  $f(x)$ , the deviation  $\delta(f, T)$  of a trigonometric polynomial  $T(x)$  from  $f$  is defined by the formula

$$\delta(f, T) = \max |f(x) - T(x)|.$$

The lower bound of the numbers  $\delta(f, T)$  for all polynomials

$$T(x) = \frac{1}{2}a_0 + \sum_1^n (a_n \cos nx + b_n \sin nx)$$

of given order  $n$  will be denoted by  $E_n[f]$  and called the *best approximation* of  $f$  of order  $n$ . By (3.6),  $E_n[f]$  tends (monotonically) to 0 as  $n \rightarrow \infty$ .

(13.1) THEOREM. Given  $f$  and  $n$ ,  $E_n[f]$  is 'attained'; that is to say there is a polynomial  $T^*(x) = T^*(x; f, n)$  of order  $n$  such that  $\delta(f, T^*) = E_n[f]$ .

For let  $T^1, T^2, \dots$ , be a sequence of polynomials of order  $n$  such that

$$\delta(f, T^k) \leq E_n[f] + 1/k \quad (k = 1, 2, \dots). \tag{13.2}$$

In particular, the  $T^k$  are uniformly bounded. Thus if  $a_0^k, \dots, b_n^k$  are the coefficients of  $T^k$ , these numbers are all bounded. By the theorem of Bolzano-Weierstrass, there is a subsequence of the points  $(a_0^k, \dots, b_n^k)$  of  $(2n + 1)$ -dimensional space which tends to a limit  $(a_0^*, \dots, b_n^*)$ . The corresponding  $T^k(x)$  then tend uniformly to a polynomial  $T^*(x)$  of order  $n$ . From (13.2) we get  $\delta(f, T^*) \leq E_n[f]$ , and since the opposite inequality is obvious (13.1) follows.

Let us write

$$f(x) = T^*(x) + R(x),$$

so that  $|R(x)| \leq E_n = E_n[f]$ . Let  $s_k(x)$  and  $r_k(x)$  denote the partial sums, and  $\sigma_k(x)$  and  $\rho_k(x)$  the (C, 1) means, of  $S[f]$  and  $S[R]$  respectively. For  $k \geq n$  we have  $s_k = T^* + r_k$ , so that

$$h^{-1} \sum_n^{n+h-1} s_k = T^* + h^{-1} \sum_n^{n+h-1} r_k, \tag{13.3}$$

$$\left(1 + \frac{n}{h}\right) \sigma_{n+h-1} - \frac{n}{h} \sigma_{n-1} = T^* + \left(1 + \frac{n}{h}\right) \rho_{n+h-1} - \frac{n}{h} \rho_{n-1}. \tag{13.4}$$

Since  $|\rho_k| \leq E_n$  for all  $k$ , the right-hand side of (13.4) differs from  $T^*$  by not more than  $(1 + 2n/h)E_n$ , and so from  $f$  by not more than  $2(1 + n/h)E_n$ . The left-hand side of (13.4) is a delayed arithmetic mean of  $S[f]$  (see p. 80). For  $h = n$  we get:

(13.5) THEOREM. Let  $\sigma_n(x) = \sigma_n(x; f)$ . Then the difference between  $f$  and

$$\tau_n(x) = 2\sigma_{2n-1}(x) - \sigma_{n-1}(x)$$

never exceeds  $4E_n[f]$ .

We know that  $\tau_n(x)$  is obtained by adding to  $S_n(x; f)$  a simple linear combination of the next  $n - 1$  terms of  $S[f]$ . In this way we obtain a polynomial whose approximation to  $f$  is almost as good as the best approximation  $E_n$ . (One must not forget, however, that  $\tau_n$  is of order  $2n - 1$ .)

(13.6) THEOREM. Let  $f(x)$  be periodic and  $k$  times differentiable. If  $|f^{(k)}(x)| \leq M$ , then

$$E_n[f] \leq A_k M n^{-k} \quad (n = 1, 2, \dots). \tag{13.7}$$

If  $f^{(k)}$  is continuous and has modulus of continuity  $\omega(\delta)$ , then

$$E_n[f] \leq B_k \omega\left(\frac{2\pi}{n}\right) n^{-k} \quad (n = 1, 2, \dots). \tag{13.8}$$

The constants  $A_k$  and  $B_k$  here depend on  $k$  only.

Let  $\tau_m = 2\sigma_{2m-1} - \sigma_{m-1}$ . Using the formula (3.24) with  $\omega = 2m$  and with  $\omega = m$ , we get

$$\tau_m(x) = \frac{2}{\pi m} \int_{-\infty}^{+\infty} f(x+t) \frac{h(mt)}{t^2} dt, \tag{13.9}$$

where

$$h(t) = \sin^2 t - \sin^2 \frac{1}{2}t = \frac{1}{2}(\cos t - \cos 2t),$$

and, by (11.4), 
$$\tau_m(x) - f(x) = \frac{2}{\pi} \int_0^{\infty} \left\{ f\left(x + \frac{t}{m}\right) + f\left(x - \frac{t}{m}\right) - 2f(x) \right\} \frac{h(t)}{t^2} dt. \tag{13.10}$$

We introduce the functions

$$H_0(t) = h(t)/t^2, \quad H_i(t) = \int_t^{\infty} H_{i-1}(u) du \quad (i = 1, 2, \dots),$$

and temporarily take for granted that

- (i) the integral defining  $H_i(t)$  is absolutely convergent for  $i = 1, 2, \dots$ ;
- (ii)  $H_1(0) = H_2(0) = H_3(0) = \dots = 0$ .

Then, integrating by parts as many times as the existence of derivatives of  $f$  permits, we find for  $\tau_m - f$  the values

$$\begin{aligned} \frac{2}{\pi} \int_0^{\infty} \left\{ f\left(x + \frac{t}{m}\right) + f\left(x - \frac{t}{m}\right) - 2f(x) \right\} H_0(t) dt &= \frac{2}{\pi m} \int_0^{\infty} \left\{ f'\left(x + \frac{t}{m}\right) - f'\left(x - \frac{t}{m}\right) \right\} H_1(t) dt \\ &= \frac{2}{\pi m^2} \int_0^{\infty} \left\{ f''\left(x + \frac{t}{m}\right) + f''\left(x - \frac{t}{m}\right) \right\} H_2(t) dt \\ &= \frac{2}{\pi m^3} \int_0^{\infty} \left\{ f'''\left(x + \frac{t}{m}\right) - f'''\left(x - \frac{t}{m}\right) \right\} H_3(t) dt \\ &= \dots \end{aligned}$$

Hence 
$$|\tau_m(x) - f(x)| \leq C_k m^{-k} M \quad \text{with} \quad C_k = \frac{4}{\pi} \int_0^{\infty} |H_k(t)| dt,$$

so that  $E_{2m-1} \leq C_k m^{-k} M$ . The same inequality holds for  $E_{2m} (\leq E_{2m-1})$ , and it follows for any  $n$  (whether even or odd) that

$$E_n \leq C_k \left(\frac{1}{2}n\right)^{-k} M,$$

which is (13.7) with  $A_k = 2^k C_k$ .

We now prove (i). It is enough to show that each  $H_i(t)$  is  $O(t^{-2})$  near  $t = \infty$ . (Since  $H_0(t)$  is bounded, this will also imply that each  $H_i(t)$  is bounded for  $0 \leq t < \infty$ .) The fact is obvious for  $H_0(t)$ . Let now  $h_i(t)$  denote that  $i$ th integral of  $h(t)$  which is periodic and has no constant term;  $h_i(t)$  is either

$$\pm \frac{1}{2}(\cos t - 2^{-i} \cos 2t) \quad \text{or} \quad \pm \frac{1}{2}(\sin t - 2^{-i} \sin 2t).$$

Integration by parts gives

$$H_1(t) = -\frac{h_1(t)}{t^2} - 2! \frac{h_2(t)}{t^3} - \dots - p! \frac{h_p(t)}{t^{p+1}} + (p+1)! \int_t^{\infty} \frac{h_p(u)}{u^{p+2}} du. \tag{13.11}$$

Here  $p$  is any positive integer. Since  $|h_p(u)| \leq 1$ , the last term is numerically not greater than  $p! t^{-(p+1)}$ . Hence  $H_1(t) = O(t^{-2})$ . If we integrate (13.11) over  $(t, +\infty)$ , integrating the terms on the right by parts, we find that  $H_2(t)$  is a sum of a linear combination of  $h_2(t)t^{-2}, h_3(t)t^{-3}, \dots$  and a remainder  $O(t^{-p})$ . Similarly,  $H_3(t)$  is a sum of a linear combination of  $h_3(t)t^{-2}, h_4(t)t^{-3}, \dots$  and a remainder  $O(t^{-(p-1)})$ , and so on. This proves (i).

To prove (ii), we apply (13.9) to the functions  $f(t) = 1$  and  $f(t) = \cos t$ . Correspondingly,  $\tau_m(x) = 1$  and  $\tau_m(x) = \cos x$ ,  $m \geq 1$ . This gives, at  $x = 0$ ,

$$\frac{4}{\pi} \int_0^\infty H_0(t) dt = 1, \quad \frac{4}{\pi} \int_0^\infty \cos \frac{t}{m} H_0(t) dt = 1.$$

Integrating the second integral by parts twice and using the first identity, we get

$$\int_0^\infty \cos \frac{t}{m} H_2(t) dt = 0.$$

In this,  $\cos(t/m) \rightarrow 1$  uniformly over any finite interval as  $m \rightarrow \infty$ , so that  $\int_0^\infty H_2 dt = 0$ .

Similarly we prove that  $\int_0^\infty H_4 dt = 0$ , and so on. This gives (ii).

The second part of (13.6) is obtainable from the first by a simple device. Given any periodic and integrable  $f(x)$ , whose integral is  $F(x)$ , and a number  $\delta > 0$ , let

$$f_\delta(x) = \frac{1}{2\delta} \int_{-x}^x f(x+t) dt = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(t) dt = \frac{F(x+\delta) - F(x-\delta)}{2\delta}. \quad (13.12)$$

The function  $f_\delta(x)$ , which is also periodic, is called the *moving average* of  $f(x)$ . If  $f$  has  $k$  continuous derivatives,  $f_\delta$  has  $k+1$  such derivatives. For  $f$  absolutely continuous, we have  $(f_\delta)' = (f')_\delta$ . Though we shall not need the fact here, we observe that the modulus of continuity of  $f_\delta$  never exceeds that of  $f$ . Clearly,

$$|f_\delta(x) - f(x)| \leq \omega(\delta; f). \quad (13.13)$$

Returning to the second part of (13.6) we write

$$f(x) = f_\delta(x) + g(x).$$

Then  $f_\delta(x)$  has  $k+1$  derivatives, and, by (13.12) and (13.13),

$$|f_\delta^{(k+1)}(x)| = \left| \frac{f^{(k)}(x+\delta) - f^{(k)}(x-\delta)}{2\delta} \right| \leq (2\delta)^{-1} \omega(2\delta; f^{(k)}) \leq \delta^{-1} \omega(\delta; f^{(k)}),$$

$$|g^{(k)}(x)| = |f^{(k)}(x) - f_\delta^{(k)}(x)| \leq \omega(\delta; f^{(k)}).$$

Hence, by (13.7),

$$E_n[f] \leq E_n[f_\delta] + E_n[g] = A_{k+1} n^{-k-1} \delta^{-1} \omega(\delta; f^{(k)}) + A_k n^{-k} \omega(\delta; f^{(k)}),$$

and setting here  $\delta = 2\pi/n$ , we get (13.8) with  $B_k = A_k + A_{k+1}/2\pi$ .

(13.14) THEOREM. If  $f$  has a continuous  $k$ -th derivative ( $k = 0, 1, \dots$ ), and if  $f^{(k)} \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , then

$$E_n[f] = O(n^{-k-\alpha}). \quad (13.15)$$

This inequality, with  $\alpha = 1$ , holds if  $f^{(k)}$  merely belongs to  $\Lambda_*$ .

It is only the last statement that requires a proof. Suppose that

$$|f^{(k)}(x+t) + f^{(k)}(x-t) - 2f^{(k)}(x)| \leq Mt,$$

where  $M$  is independent of  $x$  and  $t$ . Let  $f_{2\delta}(x)$  be the moving average of  $f_\delta$ , and let  $f(x) = f_{2\delta}(x) + g(x)$ . Thus  $f_{2\delta}$  has  $k+2$  derivatives and

$$|f_{2\delta}^{(k+2)}(x)| = \left| \frac{f_\delta^{(k+1)}(x+\delta) - f_\delta^{(k+1)}(x-\delta)}{2\delta} \right| = \left| \frac{f^{(k)}(x+2\delta) + f^{(k)}(x-2\delta) - 2f^{(k)}(x)}{4\delta^2} \right| \leq \frac{M}{2\delta}.$$

It follows from (13.12) that

$$f_{\delta}(x) = \frac{1}{4\delta^2} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} f(x+u+v) du dv = \frac{1}{4\delta^2} \int_{-2\delta}^{2\delta} f(x+t) (2\delta - |t|) dt$$

$$= \frac{1}{4\delta^2} \int_0^{2\delta} \{f(x+t) + f(x-t)\} (2\delta - t) dt.$$

and since the operation  $\delta$  commutes with differentiation,

$$|g^{(k)}(x)| = |f_{\delta}^{(k)}(x) - f^{(k)}(x)| = \frac{1}{4\delta^2} \left| \int_0^{2\delta} \{f^{(k)}(x+t) + f^{(k)}(x-t) - 2f^{(k)}(x)\} (2\delta - t) dt \right|$$

The last integrand is numerically not greater than  $Mt(2\delta - t) \leq M\delta^2$ , so that  $|g^{(k)}(x)| \leq \frac{1}{4} M\delta$ . Hence, by (13.7),

$$E_n[f] \leq E_n[f_{\delta}] + E_n[g] \leq A_{k-2} n^{-k-2} (2\delta)^{-1} M + \frac{1}{2} A_k n^{-k} M\delta.$$

On setting  $\delta = 2\pi/n$  here, we get

$$E_n[f] \leq BMn^{-k-1}. \quad B = A_{k-2}/4\pi + \pi A_k.$$

Our next aim is the converse of (13.14).

(13.16) LEMMA. Let  $T(x)$  be a polynomial of order  $n$ , and  $M = \max |T(x)|$ . Then

$$|T'(x)| \leq 2nM. \quad |T''(x)| \leq 2nM. \tag{13.17}$$

This lemma is purely utilitarian; in Chapter X, § 3, we shall show that the factors 2 on the right are superfluous. We write

$$T(x) = \frac{1}{\pi} \int_0^{2\pi} T(t) D_n(x-t) dt, \quad T'(x) = -\frac{1}{\pi} \int_0^{2\pi} T(\tau+t) D'_n(t) dt.$$

Since  $T$  is a polynomial of order  $n$ , in the last integral we can add to

$$D'_n(t) = -\sum_1^n k \sin kt$$

any polynomial  $Q$  all of whose terms have rank greater than  $n$ . If we choose

$$Q = -\sum_1^{n-1} k \sin (2n-k)t$$

and take together the terms  $k \sin kt$  and  $k \sin (2n-k)t$  we get

$$\left. \begin{aligned} T'(x) &= \frac{2n}{\pi} \int_0^{2\pi} T(x-t) \sin nt K_{n-1}(t) dt, \\ |T'| &\leq 2n \frac{1}{\pi} \int_0^{2\pi} M K_{n-1}(t) dt = 2nM, \end{aligned} \right\} \tag{13.18}$$

as desired. For the second part we take the formula

$$T'(x) = \frac{1}{\pi} \int_0^{2\pi} T(x-t) D'_n(t) dt = \frac{1}{\pi} \int_0^{2\pi} T(x+t) \left\{ \sum_1^n k \cos kt \right\} dt,$$

and add to the expression in curly brackets the polynomial  $\sum_1^{n-1} k \cos (2n-k)t$ , obtaining

$$T'(x) = \frac{2n}{\pi} \int_0^{2\pi} T(x+t) \cos nt K_{n-1}(t) dt, \quad |T'| \leq 2nM. \tag{13.19}$$

(13·20) THEOREM. Suppose that  $f$  satisfies (13·15) for some  $k=0, 1, 2, \dots$ , and  $0 < \alpha \leq 1$ . Then  $f$  has  $k$  continuous derivatives. If  $\alpha < 1$ , then  $f^{(k)} \in \Lambda_\alpha$ . If  $\alpha = 1$  then  $f^{(k)}$  belongs to  $\Lambda_*$  (though not necessarily to  $\Lambda_1$ ) and  $\omega(\delta; f^{(k)}) = O(\delta \log \delta)$ .

Suppose that  $E_n[f] \leq M n^{-(k+\alpha)}$ ,  $n = 1, 2, \dots$ . Let  $T_n$  be a polynomial of best approximation of order  $n$  for  $f$ . Then  $f(x) = \lim_n T_{2^n}(x)$  or

$$f(x) = T_2 + (T_4 - T_2) + \dots + (T_{2^n} - T_{2^{n-1}}) + \dots, \tag{13·21}$$

the series converging uniformly. Since  $u_n = T_{2^n} - T_{2^{n-1}}$  is a polynomial of order  $2^n$  with absolute value not greater than

$$|T_{2^n} - f| + |f - T_{2^{n-1}}| \leq 2E_{2^{n-1}}[f] = O(2^{-n(k+\alpha)}),$$

the first inequality (13·17) applied  $j$  times shows that  $u_n^{(j)} = O(2^{-n(k-j+\alpha)})$ . Hence the series (13·21) differentiated termwise  $j$  times,  $j \leq k$ , converges absolutely and uniformly. In particular,  $f^{(k)}$  exists and is continuous.

We set  $f - T_2 = g$ . It is enough to show that the conclusions of (13·20) are satisfied by  $g$ . Let  $0 < h \leq \frac{1}{2}$ , and let  $N$  be the positive integer satisfying  $2^{-N} < h \leq 2^{-(N-1)}$ . Since  $g = u_2 + u_3 + \dots$ , where  $u_n = T_{2^n} - T_{2^{n-1}}$ , we have

$$g^{(k)}(x+h) - g^{(k)}(x) = \sum \{u_n^{(k)}(x+h) - u_n^{(k)}(x)\} = \sum_2^N + \sum_{N+1}^\infty = P + Q. \tag{13·22}$$

The polynomial  $u_n$  is of order  $2^n$ , and  $u_n = O(2^{-n(k+\alpha)})$ . Hence, by the mean-value theorem and the first inequality (13·17),

$$\begin{aligned} |P| &\leq h \sum_2^N \max |u_n^{(k+1)}(x)| \leq h \sum_2^N (2 \cdot 2^n)^{k+1} \cdot O(2^{-n(k+\alpha)}) \\ &= h \sum_2^N O(2^{n(1-\alpha)}) = hO(2^{N(1-\alpha)}) = hO(h^{\alpha-1}) = O(h^\alpha), \end{aligned}$$

provided  $0 < \alpha < 1$ . Next, and this is true for  $0 < \alpha \leq 1$ , we have

$$\begin{aligned} |Q| &\leq \sum_{N+1}^\infty 2 \max |u_n^{(k)}(x)| \leq \sum_{N+1}^\infty 2(2 \cdot 2^n)^k O(2^{-n(k+\alpha)}) \\ &= \sum_{N+1}^\infty O(2^{-n\alpha}) = O(2^{-N\alpha}) = O(h^\alpha). \end{aligned}$$

Hence  $g^{(k)} \in \Lambda_\alpha$  for  $0 < \alpha < 1$ .

If  $\alpha = 1$ , we still have  $Q = O(h)$ , and the estimate above for  $P$  becomes

$$P = O(hN) = O(h \log h).$$

Hence  $P + Q = O(h \log h)$  and  $\omega(\delta; g^{(k)}) = O(\delta \log \delta)$ .

It remains to prove that  $g^{(k)} \in \Lambda_*$  for  $\alpha = 1$ . With the same relation between  $h$  and  $N$  we write

$$\begin{aligned} g^{(k)}(x+h) + g^{(k)}(x-h) - 2g^{(k)}(x) &= \sum \{u_n^{(k)}(x+h) + u_n^{(k)}(x-h) - 2u_n^{(k)}(x)\} \\ &= \sum_2^N + \sum_{N+1}^\infty = P_1 + Q_1. \end{aligned} \tag{13·23}$$

The terms of  $Q_1$  are numerically not greater than  $4 \max |u_n^{(k)}(x)|$ , so that automatically

we get the same estimate as for  $Q$ , namely  $Q_1 = O(h)$ . By the mean-value theorem, and arguing as for  $P$ ,

$$|P_1| \leq h^2 \sum_{\frac{1}{2}}^N \max |u_n^{(k+2)}| = h^2 \sum_{\frac{1}{2}}^N O(2^n) = O(h^2 2^N) = O(h).$$

Hence  $P_1 + Q_1 = O(h)$  and  $g^{(k)} \in \Lambda_*$ .

*Remarks.* (a) From (13.6) and (13.20) we see that a necessary and sufficient condition that  $E_n[f] = O(n^{-(k+\alpha)})$ , where  $k$  is a non-negative integer and  $0 < \alpha \leq 1$ , is that  $f$  should have a continuous  $k$ -th derivative belonging to  $\Lambda_\alpha$  if  $\alpha < 1$ , and to  $\Lambda_*$  if  $\alpha = 1$ .

In particular, a necessary and sufficient condition for  $f \in \Lambda_\alpha$ ,  $0 < \alpha < 1$ , is  $E_n[f] = O(n^{-\alpha})$ , and for  $f \in \Lambda_*$  is  $E_n = O(n^{-1})$ .

(b) Since the class  $\Lambda_1$  is a proper subset of  $\Lambda_*$  (Chapter II, (4.9)), we cannot replace  $\Lambda_*$  by  $\Lambda_1$  in (13.20). Taking for instance  $k = 0$ , we can also verify this by a simple example. Let  $f(x) = \sum_1^\infty 2^{-m} \cos 2^m x$ ,  $2^N \leq n < 2^{N+1}$ . Then

$$|f(x) - S_n(x; f)| = \left| \sum_{N+1}^\infty 2^{-m} \cos 2^m x \right| \leq \sum_{N+1}^\infty 2^{-m} = 2^{-N} < 2/n.$$

In particular,  $E_n[f] < 2/n$ , so that, by (13.20),  $f \in \Lambda_*$  (a fact which we have verified directly in Chapter II, p. 47). However,  $f$  is not in  $\Lambda_1$ , since  $S[f]$  is not a Fourier series. Thus there is no simple characterization of the class  $\Lambda_1$  in terms of the order of best approximation.

(c) If  $f \in \Lambda_*$ , then  $E_n[f] = O(1/n)$  and so  $\omega(\delta; f) = O(\delta \log \delta)$  by (13.20). It follows that every  $f \in \Lambda_*$  has modulus of continuity  $O(\delta \log \delta)$ , and, in particular, belongs to  $\Lambda_\alpha$ ,  $0 < \alpha < 1$ . This result is not new (see Chapter II, (3.4)).

(d) The proof of (3.15) shows that, if

$$f(x+t) + f(x-t) - 2f(x) = O(t^\alpha) \quad (t > 0) \quad (13.24)$$

for  $0 < \alpha < 1$ , then  $\sigma_n[f] - f = O(n^{-\alpha})$ . In particular  $E_n[f] = O(n^{-\alpha})$ , so that  $f \in \Lambda_\alpha$ , that is,  $f(x+t) - f(x) = O(t^\alpha)$ . Since the latter condition implies (13.24), we see that  $\Lambda_\alpha$ ,  $0 < \alpha < 1$ , can be defined as the class of continuous functions satisfying (13.24).† It is only for  $\alpha = 1$  that condition (13.24) yields a new class,  $\Lambda_*$ , larger than  $\Lambda_1$ .

(e) For every continuous  $f$ ,

$$|f(x) - S_n(x; f)| \leq (L_n + 1) E_n[f], \quad (13.25)$$

where  $L_n$  is the Lebesgue constant (Chapter II, § 12). For let  $T_n$  be a polynomial of best approximation of order  $n$  for  $f$ , and let  $f = T_n + g$ . Then

$$\begin{aligned} |f - S_n[f]| &= |T_n - S_n[T_n] + g - S_n[g]| = |g - S_n[g]| \leq |g| + |S_n[g]| \\ &\leq \max |g| + L_n \max |g| = (L_n + 1) E_n[f]. \end{aligned}$$

In particular, since  $L_n = O(\log n)$ ,

$$f(x) - S_n(x; f) = O(n^{-k-\alpha} \log n) \quad (k = 0, 1, \dots; 0 < \alpha \leq 1), \quad (13.26)$$

provided  $f$  has  $k$  derivatives, and  $f^{(k)} \in \Lambda_\alpha$  for  $\alpha < 1$ ,  $f^{(k)} \in \Lambda_*$  for  $\alpha = 1$ .

The inequality (13.25) shows that the approximation of  $f$  by  $S_n[f]$  is at most  $L_n + 1 = O(\log n)$  times worse than the best approximation. For  $k = 0$ , (13.26) gives

† This can also be shown directly by the method which gave Theorem (3.6) of Chapter II.

Theorem (10·8) of Chapter II. If  $\omega(\delta; f) = o(|\log \delta|^{-1})$ , we have  $E_n = o(1/\log n)$  by (13·6), and (13·25) shows that  $S[f]$  converges uniformly to  $f$ . This is Theorem (10·3) of Chapter II.

(13·27) THEOREM. Under the hypothesis of (13·20),  $f$  satisfies the same conclusions as  $f$ .

Denote the series (13·21) again by  $u_1 + u_2 + \dots$ . We shall show that

$$\tilde{f} = \tilde{u}_1 + \tilde{u}_2 + \dots \quad (\tilde{u}_n = \tilde{T}_{2^n} - \tilde{T}_{2^{n-1}} \text{ for } n > 1). \quad (13\cdot28)$$

Let  $\tilde{L}_n$  be the constant introduced in Chapter II, (12·3). Since  $\tilde{L}_n = O(\log n)$ ,

$$\max_x |\tilde{u}_n(x)| \leq \tilde{L}_n \max |u_n(x)| = O(n) O(2^{-n(k+\alpha)}) = O(n2^{-n\alpha}),$$

which shows that the series  $\Sigma \tilde{u}_n$  converges uniformly to a function  $f^*(x)$ . If  $f_n$  are the partial sums of the series  $u_1 + u_2 + \dots$ , the partial sums in (13·28) are  $\tilde{f}_n$ . Since  $f_n \rightarrow f$  and  $\tilde{f}_n \rightarrow f^*$ , both uniformly, each Fourier coefficient of  $f_n$  tends to the corresponding coefficient of  $f$ , and the coefficients of  $\tilde{f}_n$  tend to those of  $f^*$ . Hence  $f^* = \tilde{f}$ , and (13·28) follows.

The series (13·28) differentiated termwise  $j$  times,  $j \leq k$ , also converges uniformly. For

$$\max |\tilde{u}_n^{(j)} x| \leq (2 \cdot 2^n)^j \cdot \max |\tilde{u}_n(x)| = O(n2^{-n(k-j+\alpha)}) = O(n2^{-n\alpha}).$$

It remains to show that  $\tilde{f}^{(k)}$  satisfies the same conclusions as  $f^{(k)}$  in (13·20). Suppose first that  $k \geq 1$ . Then, as in (13·22),

$$\tilde{g}^{(k)}(x+h) - \tilde{g}^{(k)}(x) = \Sigma \{\tilde{u}_n^{(k)}(x+h) - \tilde{u}_n^{(k)}(x)\};$$

and from this point on the proof remains exactly the same as before, since in terms of  $\max |u_n|$  we have the same estimates for the derivatives of  $\tilde{u}_n$  as we had for the derivatives of  $u_n$ . (We use the second inequality (13·17) instead of the first.) Similarly for the analogue of (13·23).

Suppose now that  $k = 0$  and that  $\alpha < 1$ . Since subtracting a constant from  $f$  does not affect  $E_n[f]$  we may suppose that the constant term of  $S[f]$  is 0, so that the integral  $F$  of  $f$  is periodic. By (13·6),  $E_n[F] = O(n^{-1-\alpha})$ , and, by the case just disposed of,  $\tilde{F}$  has a derivative, obviously  $\tilde{f}$ , belonging to  $\Lambda_\alpha$ . The argument holds when  $k = 0$ ,  $\alpha = 1$ .

On taking  $k = 0$  in (13·14) and (13·27), we obtain the following result:

(13·29) THEOREM. If  $f$  belongs to  $\Lambda_\alpha$ ,  $0 < \alpha < 1$ , so does  $\tilde{f}$ ; if  $f$  belongs to  $\Lambda_1$  so does  $\tilde{f}$ .

It must be added that if  $f \in \Lambda_1$ , the function  $\tilde{f}$  need not belong to  $\Lambda_1$ , since the function conjugate to a bounded function need not be bounded.

Theorem (13·29) can also be proved directly; we shall confine our attention to the first part, slightly generalizing it.

(13·30) THEOREM. If the modulus of continuity of  $f$  is  $\omega(h)$ , that of  $\tilde{f}$  does not exceed

$$A \left[ \int_0^h t^{-1} \omega(t) dt + h \int_h^\pi t^{-2} \omega(t) dt \right] = A \int_0^h dt \int_t^\pi u^{-2} \omega(u) du \quad (h \leq \frac{1}{2}\pi), \quad (13\cdot31)$$

where  $A$  is an absolute constant.

We may suppose that  $\omega(t)/t$  is integrable near 0, so that the integral defining  $f$  is absolutely convergent, since otherwise there is nothing to prove. Now the identity (13.31) follows by integration by parts. (We note that  $\int_1^{\infty} u^{-2}\omega(u) du = o(t^{-1})$  as  $t \rightarrow 0$ .)

For the proof of (13.30) we consider the formulae

$$f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t) - f(x)}{2 \tan \frac{1}{2}t} dt, \quad f(x+h) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t) - f(x+h)}{2 \tan \frac{1}{2}(t-h)} dt.$$

The first is obvious since  $\cot \frac{1}{2}t$  is odd, and the second follows from the first on replacing  $x$  by  $x+h$  and  $t$  by  $t-h$ . The integrands are respectively majorized by  $\omega(|t|)|t|^{-1}$  and  $\omega(|t-h|)|t-h|^{-1}$ . Thus if we cut out the interval  $(-2h, 2h)$  from the interval of integration  $(-\pi, \pi)$  and write  $I(h) = \frac{1}{\pi} \int_0^h t^{-1}\omega(t) dt$ , we commit errors at most  $2I(2h) \leq 4I(h)$  in the first integral and at most  $I(h) + I(3h) \leq 4I(h)$  in the second. It follows that with an error not greater than  $8I(h)$  we have

$$\begin{aligned} f(x+h) - f(x) &= -\frac{1}{\pi} \left( \int_{-\pi}^{-2h} + \int_{2h}^{\pi} \right) [f(x+t) - f(x)] \left[ \frac{1}{2} \cot \frac{1}{2}(t-h) - \frac{1}{2} \cot \frac{1}{2}t \right] dt \\ &\quad + \frac{1}{\pi} [f(x+h) - f(x)] \left( \int_{-\pi}^{-2h} + \int_{2h}^{\pi} \right) \frac{1}{2} \cot \frac{1}{2}(t-h) dt. \end{aligned}$$

The first term on the right is absolutely less than

$$\frac{1}{2\pi} \left( \int_{-\pi}^{-2h} + \int_{2h}^{\pi} \right) \frac{\omega(|t|) \sin \frac{1}{2}h}{\left| \sin \frac{1}{2}(t-h) \sin \frac{1}{2}t \right|} dt = \int_{2h}^{\pi} O(h t^{-2}) \omega(t) dt = O\left(h \int_h^{\pi} t^{-2}\omega(t) dt\right).$$

A simple integration shows that the coefficient of  $f(x+h) - f(x)$  in the remaining term is  $O(1)$ , so that the total contribution from it is  $O(\omega(h))$ . Since

$$\omega(h) \leq \omega(h) 2h \int_h^{\pi} t^{-2} dt \leq 2h \int_h^{\pi} t^{-2}\omega(t) dt,$$

we see, collecting results, that  $|f(x+h) - f(x)|$  does not exceed the first expression (13.31), and since the latter increases with  $h$ , the inequality for  $\omega(h; f)$  follows.

We know that the approximation of  $f$  by  $S_n[f]$  is only  $O(\log n)$  times worse than the best approximation  $E_n[f]$  of  $f$ . The approximation of  $f$  by  $\tau_n = 2\sigma_{2n-1} - \sigma_{n-1}$  is of the same order as  $E_n[f]$ . It is curious that the  $\sigma_n$  themselves give only mediocre approximations, though they converge uniformly to every continuous  $f$ .

(13.32) THEOREM. If  $\sigma_n(x; f) - f(x) = o(1/n)$  uniformly in  $x$ , then  $f \equiv \text{const}$ .

For if  $c_k$  are the complex coefficients of  $f$ , then

$$(2\pi)^{-1} \int_0^{2\pi} \{f(x) - \sigma_n(x)\} e^{-ikx} dx = |k| c_k / (n+1) \quad (|k| \leq n),$$

and the relation  $f - \sigma_n = o(1/n)$  implies that the left-hand side here is  $o(1/n)$ , which means that  $c_k = 0$  for  $k \neq 0$ , that is,  $f \equiv c_0$ .

Thus if  $f \neq \text{const.}$ , the (C, 1) means of  $S[f]$  never give approximation with error  $o(1/n)$ . A similar argument shows that the Abel means  $f(r, x)$  of  $S[f]$  never give approximation with error  $o(1-r)$ , unless  $f \equiv \text{const.}$  Generally, let us consider a matrix (with finite or infinite rows)

$$\begin{pmatrix} \gamma_{00} & \gamma_{01} & \dots & \gamma_{0m_0} & \dots & \dots \\ \gamma_{10} & \gamma_{11} & \dots & \dots & \gamma_{1m_1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n0} & \gamma_{n1} & \dots & \dots & \dots & \gamma_{nm_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{13.33}$$

If  $f \sim \sum A_k(x)$ , and if  $f_n = \sum_0^{m_n} \gamma_{nk} A_k(x)$  is to approach  $f$  with an error  $o(\rho_n)$ , or  $O(\rho_n)$ , where  $\rho_n \rightarrow 0$ , then in each column of (13.33)  $1 - \gamma_{nk}$  must be  $o(\rho_n)$  or  $O(\rho_n)$ , as the case may be. If  $f_n$  is either  $S_n(x; f)$  or  $r_n(x; f)$ , then  $\gamma_{nk} = 1$  for  $k$  fixed and  $n$  large enough, so that the condition is satisfied. This condition is only necessary, not sufficient, but it explains the fact that for trigonometric series with coefficients rapidly tending to 0 ordinary partial sums may give a better approximation than stronger methods of summability.

Let  $0 < \alpha < 1$ . By (3.15) and (13.20), we have  $\sigma_n[f] - f = O(n^{-\alpha})$  if and only if  $f \in \Lambda_\alpha$ . We shall now prove the following theorem.

(13.34) THEOREM. A necessary and sufficient condition for  $\sigma_n[f] - f = O(n^{-1})$  is that  $f \in \Lambda_1$ .

*Necessity.* The hypothesis  $\sigma_n[f] - f = O(1/n)$  indicates that  $f$  exists and is continuous. It will be convenient to write  $T_n(x)$  for  $\sigma_n(x; f)$ . Let  $f = T_n + g_n$ . Then

$$\sigma_n[f] - f = \{\sigma_n[T_n] - T_n\} + \{\sigma_n[g_n] - g_n\} = O(1/n).$$

The hypothesis  $g_n = O(1/n)$  leads to  $\sigma_n[g_n] = O(1/n)$ , and so the same estimate holds for  $T_n - \sigma_n[T_n]$ . From the general formula (1.25) we deduce that

$$T_n - \sigma_n[T_n] = \hat{T}'_n/(n+1),$$

and since the left-hand side is  $O(1/n)$ , it follows that  $\hat{T}'_n(x) = O(1)$ . Hence the  $\hat{T}_n(x)$  satisfy condition  $\Lambda_1$  uniformly in  $n$ , and so  $f = \lim \hat{T}_n$  is in  $\Lambda_1$ .

*Sufficiency.* Interchanging the roles of  $f$  and  $\hat{f}$  it is enough to show that if  $f \in \Lambda_1$ , then  $\hat{\sigma}_n[f] - \hat{f} = O(1/n)$ . From (3.17) we see that  $\hat{\sigma}_n(x) - \hat{f}(x)$  equals

$$\frac{1}{\pi(n+1)} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{\sin(\frac{n+1}{2}t)}{(2 \sin \frac{1}{2}t)^2} dt = \int_0^{1/n} + \int_{1/n}^\pi = P + Q,$$

say. Assuming that  $|f(x+t) - f(x)| \leq M|t|$ , we have

$$|P| \leq \pi^{-1}(n+1)^{-1} \int_0^{1/n} 2Mt(n+1)t \left(\frac{2}{\pi}t\right)^{-2} dt = O(1/n).$$

By the second mean-value theorem,

$$\left| \int_t^\pi \frac{\sin(n+1)u}{(2 \sin \frac{1}{2}u)^2} du \right| \leq \frac{2}{n+1} \frac{1}{(2 \sin \frac{1}{2}t)^2} \leq \frac{A}{\pi t^2}.$$

Denote the integral on the left by  $R_n(t)$ . Since  $f'$  exists almost everywhere and  $|f'| \leq M$ , integration by parts gives

$$Q = \pi^{-1}(n+1)^{-1} \left\{ [-R_n(t)(f(x+t) - f(x-t))]_{1/n}^{\pi} + \int_{1/n}^{\pi} [f'(x+t) + f'(x-t)] R_n(t) dt \right\},$$

$$|Q| \leq \pi^{-1}(n+1)^{-1} \left\{ |R_n(1/n)| \cdot 2Mn^{-1} + \int_{1/n}^{\pi} 2M \cdot An^{-1}t^{-2} dt \right\} = O(n^{-1}).$$

Hence  $P + Q = O(1/n)$ ,  $\bar{\sigma}_n - \bar{f} = O(1/n)$ .

(13.35) THEOREM. If  $S[f]$  is of power series type, then  $f - \sigma_n[f] = O(1/n)$  if and only if  $f \in \Lambda_1$ .

For in this case  $\bar{f} = -i(f - c_0)$ .

#### MISCELLANEOUS THEOREMS AND EXAMPLES

1. Let a simple, closed, convex curve  $L$  be given by the equations  $x = \phi(t)$ ,  $y = \psi(t)$ ,  $0 \leq t \leq 2\pi$ . Show that if  $\phi_n(t)$  and  $\psi_n(t)$  are the  $(C, 1)$  means of  $S[\phi]$  and  $S[\psi]$ , then the curves  $x = \phi_n(t)$ ,  $y = \psi_n(t)$  are in the interior of  $L$ . (Fejér [5].)

[If  $A, B, C$  are constants and  $A\phi(t) + B\psi(t) + C$  is non-negative, but not identically zero, then  $A\phi_n(t) + B\psi_n(t) + C > 0$ . Here, as in Example 2 below, the result is immediately extensible to any non-negative kernel, in particular to Poisson's.]

2. Let  $\phi(t)$  and  $\psi(t)$  be periodic functions,  $\phi_n(t)$  and  $\psi_n(t)$  the  $(C, 1)$  means of  $S[\phi]$  and  $S[\psi]$ , and  $L, L_n$  the lengths of the curves  $x = \phi(t)$ ,  $y = \psi(t)$  and  $x = \phi_n(t)$ ,  $y = \psi_n(t)$ ,  $0 \leq t \leq 2\pi$ , respectively. Show that  $L_n \leq L$  for all  $n$ . (Compare Pólya and Szegő, *Aufgaben und Lehrsätze*, I, p. 56. Problem 89.)

[Let  $0 = t_0 < t_1 < \dots < t_k = 2\pi$ . Let  $\Delta_j \phi = \phi(t_j) - \phi(t_{j-1})$ ,  $\Delta_j \psi(t) = \phi(t_j + t) - \phi(t_{j-1} + t)$ , and similarly for  $\psi$ ,  $\phi_n$ ,  $\psi_n$ . Then

$$|\Delta_j \phi_n \cos \alpha + \Delta_j \psi_n \sin \alpha| \leq \pi^{-1} \int_0^{2\pi} |\Delta_j \phi(t) \cos \alpha + \Delta_j \psi(t) \sin \alpha| K_n(t) dt$$

for all  $\alpha$  and  $j$ . We integrate this with respect to  $\alpha$  over  $0 \leq \alpha \leq 2\pi$ , interchange the order of integration on the right, and use the equation  $\int_0^{2\pi} |a \cos \alpha + b \sin \alpha| d\alpha = 4(a^2 + b^2)^{1/2}$ . Summation with respect to  $j$  gives

$$\sum_j \{(\Delta_j \phi_n)^2 + (\Delta_j \psi_n)^2\}^{1/2} \leq \pi^{-1} \int_0^{2\pi} \sum_j \{(\Delta_j \phi(t))^2 + (\Delta_j \psi(t))^2\}^{1/2} K_n(t) dt \leq \pi^{-1} \int_0^{2\pi} L K_n(t) dt = L,$$

so that  $L_n \leq L$ .]

3. Let  $f(x)$  be periodic, integrable and equal to 0 for  $x_0 < x < x_0 + h$ . Let  $\Gamma$  be any circle tangent internally to the unit circle  $\Gamma_1$  at the point  $e^{i\alpha}$ . Show that  $f(re^{ix})$  tends to 0 as  $re^{ix}$  approaches  $e^{i\alpha}$  through that part  $\Delta$  of the cuspidal region between  $\Gamma$  and  $\Gamma_1$  for which  $x > x_0$ . What localization theorem does this give? (Hardy and Rogosinski, *Fourier series*, p. 65.)

[Let  $x_0 = 0$ . For  $re^{ix}$  tending to  $e^{i\alpha}$  through  $\Delta$  we have

$$f(r, x) = \pi^{-1} \int_{-h}^0 P(r, x-t) f(t) dt + o(1).$$

If  $re^{ix}$  belongs to  $\Delta$ , so does  $re^{i(x-t)}$  for  $t < 0$ . It is now enough to observe that

$$P(r, u) = \frac{1}{2} \zeta \{ (1 + re^{iu}) / (1 - re^{iu}) \}$$

is bounded for  $z = re^{iu}$  situated between  $\Gamma$  and  $\Gamma_1$  (the function  $\zeta = \frac{1}{2}(1+z)/(1-z)$  maps this domain into a vertical strip of the  $\zeta$  plane), and that  $\int_{-h}^0 |f| dt$  is small with  $h$ .]

4. The  $n$ th partial sum of  $\frac{1}{2} + r \cos x + r^2 \cos 2x + \dots$  is non-negative for  $0 \leq r \leq \frac{1}{2}$ , though not necessarily for  $r > \frac{1}{2}$ . (Fejér [6].)

[The partial sum is

$$P_n(r, x) = \frac{1 - r^{2n} - 2r^{n+1}(\cos(n+1)x - r \cos nx)}{2(1 - 2r \cos x + r^2)}.$$

The sum  $\frac{1}{2} + r \cos x$  is negative for  $x = \pi$  if  $r > \frac{1}{2}$ .]

5. Let  $f_n(r, x)$  be the  $n$ th partial sum of the series  $f(r, x)$  in (6.1). Show that if  $m \leq f(x) \leq M$  for all  $x$ , then  $m \leq f_n(r, x) \leq M$  for  $r \leq \frac{1}{2}$ , but not necessarily for  $r > \frac{1}{2}$ . (Fejér [6].)

[A corollary of Example 4.]

6. For any  $N = 1, 2, \dots$  there is a number  $r_N$  with the following property. Under the hypotheses of Example 5, we have  $m \leq f_n(r, x) \leq M$  for  $r_N \leq r < 1, n \geq N$ . Moreover,  $r_N \leq r_{N+1}, r_N \rightarrow 1$  as  $N \rightarrow \infty$ . (Schur and Szegő [1]. See Example 4.)

7. The (C, 1) means  $\sigma_n(x)$  of the series  $\sum n^{-1} \sin nx$  are positive and less than  $\frac{1}{2}(\pi - x)$  in the interval  $0 < x < \pi$ .

[See Chapter II, (9.4). Also

$$\sigma_n(x) = -\frac{1}{2}x + \int_0^x K_n(t) dt < -\frac{1}{2}x + \int_0^\pi K_n(t) dt = \frac{1}{2}(\pi - x).]$$

8. If  $u_0 + u_1 + u_2 + \dots$  is summable (C,  $\alpha$ ), or A, to sum  $s$ , so is  $0 + u_0 + u_1 + \dots$ , while  $u_1 + u_2 + \dots$  is summable to  $s - u_0$ .

9. For any series  $u_0 + u_1 + \dots$  with partial sums  $s_n$ , let

$$e_n^* = \frac{1}{2}(s_n + s_{n-1}) = u_0 + u_1 + \dots + \frac{1}{2}u_n$$

be the modified partial sums. Let  $\sigma_n^{\alpha} = e_n^*/A_n^{\alpha}$  be the (C,  $\alpha$ ) means of the sequence  $(e_n^*)$ . Show that

$$\sum_0^{\infty} \sigma_n^{\alpha} r^n = \frac{1}{2} \frac{1+r}{(1-r)^{\alpha+1}} \sum_0^{\infty} u_n r^n.$$

10. Let  $\sigma_n^{\alpha}(x; f)$  be the (C,  $\alpha$ ) means of  $S_n^{\alpha}(x; f)$ . Show that under the hypotheses of (10.2),

$$m \sigma_n^{\alpha}(x; \mu) \leq \sigma_n^{\alpha}(x; f) \leq M \sigma_n^{\alpha}(x; \mu) \quad (0 \leq x \leq \pi).$$

[For the termwise differentiated series  $\frac{1}{2} + \cos t + \cos 2t + \dots$  we have

$$-\sum_0^{\infty} S_n^{\alpha}(t) r^n = \frac{1}{2} \frac{1+r}{(1-r)^{\alpha}} \frac{1-r^2}{\Delta^{\alpha}(r, t)} r \sin t = \left[ \frac{1}{2} \frac{1-r^2}{(1-r)^{\alpha}} \Delta^{\alpha} \right]^{-1} 2r \sin t,$$

so that  $S_n^{\alpha}(t) < 0$  for  $0 < t < \pi$ .]

11. The (C, 3) means in (10.2) cannot be replaced by (C, 2) means. (Fejér [4].)

[( $K_n^{\alpha}(t)$ )' is positive if  $\sin(n + \frac{1}{2})t = 0, \cos(n + \frac{1}{2})t = -1, \cos \frac{1}{2}t < \frac{1}{2}$ .]

12. If  $F'(x_0) = \lim_{h \rightarrow 0} [F(x_0 + h) - F(x_0 - h)]/2h$  exists and is finite, then at the point  $x_0$   $S[F]$  is summable by the logarithmic mean to sum  $F'(x_0)$ .

13. Let  $f(z) = c_0 + c_1 z + c_2 z^2 + \dots$  be regular for  $|z| < 1$ , continuous for  $|z| \leq 1$ . Let  $a, b, \alpha, \beta$  be numbers, real or complex, satisfying  $\alpha + \beta = 1, \alpha e^{\alpha} + \beta e^{\beta} = 0$ . Show that then  $\alpha s_n(z e^{\alpha/n}) + \beta s_n(z e^{\beta/n})$  converges uniformly to  $f(z)$  for  $|z| \leq 1$ . Here  $s_n(z) = c_0 + c_1 z + \dots + c_n z^n$ . (Rogosinski and Szegő [1].)

[The argument closely resembles that of § 12.]

14. Let  $S_n^{\alpha}(x)$  be the modified partial sums of  $S[f]$ . At every point  $x$  at which  $\Phi_n(t) = o(t)$ , a necessary and sufficient condition for the convergence of the series  $\sum (S_n^{\alpha} - f)/k$  is the convergence of the integral  $\int_0^{\pi} \frac{\phi_n(t)}{2 \sin \frac{1}{2}t} dt$ . (See Hardy and Littlewood [7].)

[Let  $u_n(x) = \sin x + 2^{-1} \sin 2x + \dots + n^{-1} \sin nx, r_n(x) = \frac{1}{2}(\pi - x) - u_n(x)$ .

Plainly  $|u_n(x)| \leq nx$ , and applying summation by parts we get  $r_n(x) = O(1/nx)$ . Let  $T_n(x)$  be the  $n$ th partial sum of the given series. Then

$$T_n(x) = \frac{2}{\pi} \int_0^\pi \frac{\phi_n(t)}{2 \tan \frac{1}{2}t} u_n(t) dt = \frac{2}{\pi} \int_0^{1/n} + \frac{2}{\pi} \int_{1/n}^\pi = A + B.$$

Here  $A \rightarrow 0$ , and on account of the inequality for  $\tau_n$ ,

$$T_n(x) - \frac{2}{\pi} \int_{1/n}^\pi \frac{\phi_x(t)}{2 \tan \frac{1}{2}t} \frac{\pi - t}{2} dt \rightarrow 0.]$$

15. Suppose that  $F_n(h)$  is the integral of  $\phi_n(t)$  over  $0 \leq t \leq h$  and that  $\Phi_n(h)$  has its usual meaning. Neither of the conditions

$$(i) F_n(h) = o(h), \quad (ii) \Phi_n(h) = O(h)$$

taken separately implies the summability (C, 1) of  $S[f]$  at  $x$ . Show that if both of them are satisfied, then  $S[f]$  is summable (C, 1) at  $x$  to sum  $f(x)$ . (Hardy and Littlewood [8].)

This generalization of Theorem (3.9) is typical and many other results can be generalized similarly.

[The proof is similar to that of (3.9) except that now we split the integral (3.11) into integrals extended over intervals  $(0, k/n)$ ,  $(k/n, \pi)$ , where  $k$  is large but fixed. By (ii) and the second estimate (3.10), the second integral is small with  $1/k$ . The Fejér kernel has a bounded number of maxima and minima in  $(0, k/n)$  and so, by the second mean-value theorem and (i), the first integral tends to 0.]

16. Let  $f$  be periodic and  $k$  times continuously differentiable, and let  $T_n(x)$  be a polynomial of best approximation of order  $n$  to  $f$ . Then  $T_n^{(k)}(x)$  tends uniformly to  $f^{(k)}(x)$ . (E. Stein [1].)

[If  $\tau_n(x) = \tau_n(x; f)$  is the delayed (C, 1) mean of  $S[f]$ , then

$$T_n^{(k)}(x) - f^{(k)}(x) = [T_n(x) - \tau_n(x; f)]^{(k)} + [\tau_n(x; f)^{(k)} - f^{(k)}(x)].$$

The second term on the right tends uniformly to 0. The preceding term is, by (13.16),

$$O(n^k) \max_x |T_n(x) - \tau_n(x; f)|,$$

and it is enough to observe that  $T_n - f$  and  $\tau_n - f$  are both  $o(n^{-k})$ .]

## CHAPTER IV

## CLASSES OF FUNCTIONS AND FOURIER SERIES

1. The class  $L^2$ 

Let  $\phi_1(x), \phi_2(x), \dots$  be a system orthonormal in  $(a, b)$ . If  $c_1, c_2, \dots$  are the Fourier coefficients of an  $f \in L^2$ , with respect to  $\{\phi_n\}$ , the series  $\sum |c_n|^2$  converges. The converse is one of the most important results of the Lebesgue theory of integration.

(1·I) THEOREM OF RIESZ AND FISCHER. *Let  $\phi_1, \phi_2, \dots$  be an orthonormal set of functions in  $(a, b)$  and let  $c_1, c_2, \dots$  be any sequence of numbers such that  $\sum |c_n|^2$  converges. Then there is a function  $f \in L^2(a, b)$  such that the Fourier coefficient of  $f$  with respect to  $\phi_n$  is  $c_n$  for all  $n$ , and moreover*

$$\int_a^b |f|^2 dx = \sum_{n=1}^{\infty} |c_n|^2, \quad (1\cdot2)$$

$$\int_a^b |f - s_n|^2 dx \rightarrow 0, \quad (1\cdot3)$$

where  $s_n$  is the  $n$ -th partial sum of the series  $c_1\phi_1 + c_2\phi_2 + \dots$

The equation

$$\int_a^b |s_{n+k} - s_n|^2 dx = \sum_{n+1}^{n+k} |c_n|^2$$

implies that  $\mathfrak{M}_2[s_{n+k} - s_n] \rightarrow 0$  as  $m, n \rightarrow \infty$ . By Theorem (11·1) of Chapter I, there is a function  $f \in L^2$  such that  $\mathfrak{M}_2[f - s_n] \rightarrow 0$ . If  $n \geq j$ ,

$$c_j = \int_a^b s_n \bar{\phi}_j dx = \int_a^b f \bar{\phi}_j dx + \int_a^b (s_n - f) \bar{\phi}_j dx.$$

By Schwarz's inequality, the last integral does not exceed  $\mathfrak{M}_2[s_n - f] = o(1)$  in absolute value, and making  $n \rightarrow \infty$  we see that  $c_j$  is the Fourier coefficient of  $f$  with respect to  $\phi_j$ . Since  $s_n$  is now the  $n$ th partial sum of the Fourier series of  $f$ , the left-hand side of (1·3) is

$$\int_a^b |f|^2 dx - (|c_1|^2 + \dots + |c_n|^2)$$

(Chapter I, (7·4)), and (1·2) follows on making  $n \rightarrow \infty$ .

In Chapter I, §3, we defined complete orthonormal systems. A system  $\{\phi_n\}$  orthonormal in  $(a, b)$  is said to be *closed* if for each  $f \in L^2(a, b)$  we have the Parseval formula

$$\int_a^b |f|^2 dx = \sum_{n=1}^{\infty} |c_n|^2 \quad \left( c_n = \int_a^b f \bar{\phi}_n dx \right). \quad (1\cdot4)$$

In the domain of functions of the class  $L^2$  the notions of 'closed' and 'complete' systems are equivalent. Every closed system is obviously complete. To prove the converse, let  $c_1, c_2, \dots$  be the Fourier coefficients of an  $f \in L^2(a, b)$  with respect to  $\{\phi_n\}$ . Since  $\sum |c_n|^2$  converges, there is, by (1·1), a  $g \in L^2$  with Fourier coefficients  $c_n$  and such that  $\mathfrak{M}_2^2[g] = |c_1|^2 + |c_2|^2 + \dots$ . Since  $f$  and  $g$  have the same Fourier coefficients and  $\{\phi_n\}$  is complete, we have  $f \equiv g$  and (1·4) follows.

If the system  $\{\phi_n\}$  in (1.1) is complete, the function  $f$  there is uniquely determined. Suppose now that  $\{\phi_n\}$  is not complete and let  $\{\psi_n\}$  be one of its completions, so that the system  $\phi_1, \phi_2, \dots, \psi_1, \psi_2, \dots$  is orthonormal and complete in  $(a, b)$ . Let  $d_n = \int_a^b f \bar{\psi}_n dx$ .

From the Parseval formula,

$$\int_a^b |f|^2 dx = \sum |c_n|^2 + \sum |d_n|^2,$$

and from (1.2) we get  $d_1 = d_2 = \dots = 0$ . Thus, if  $\{\phi_n\}$  is not complete, the function  $f$  of the Riesz-Fischer theorem is uniquely determined by the condition that its Fourier coefficients with respect to the  $\phi_n$  are  $c_n$ , and with respect to any system completing  $\{\phi_n\}$  are zero.

(1.5) THEOREM. A system  $\{\phi_n\}$  orthonormal in  $(a, b)$  is complete if and only if for any  $f \in L^2(a, b)$  and any  $\epsilon > 0$  there is a linear combination  $S = \gamma_1 \phi_1 + \dots + \gamma_n \phi_n$  with constant coefficients such that  $\mathfrak{R}_2[f - S] < \epsilon$ .

For the completeness is equivalent to (1.4), and this in turn is equivalent to  $\mathfrak{R}_2[f - s_n] \rightarrow 0$ , where  $s_n$  is the partial sum of the Fourier series  $c_1 \phi_1 + c_2 \phi_2 + \dots$  of  $f$ . Hence if  $\{\phi_n\}$  is complete we can find an  $S = s_n$  such that  $\mathfrak{R}_2[f - S] < \epsilon$ . Conversely, if  $\mathfrak{R}_2[f - S] < \epsilon$  for some  $S = \gamma_1 \phi_1 + \dots + \gamma_n \phi_n$  then  $\mathfrak{R}_2[f - s_n] \leq \mathfrak{R}_2[f - S] < \epsilon$  (Chapter I, (7.3)), so that  $\mathfrak{R}_2[f - s_n] \rightarrow 0$ .

The trigonometric system is complete (Chapter I, § 6). It is thus closed, and we get one more proof of Parseval's formula for this system (see also § 1 of Chapter II and § 3 of Chapter III).

Let  $a_n, b_n$  be the trigonometric Fourier coefficients of an  $f \in L^2$ . By (1.1),

$$\tilde{S}[f] = \sum (a_n \sin nx - b_n \cos nx) \quad (1.6)$$

is the Fourier series of a function of class  $L^2$ . Thus  $\tilde{S}[f]$  is summable  $(C, 1)$  almost everywhere and, by Theorem (3.20) of Chapter III, the conjugate function  $\tilde{f}(x)$  exists and is equal to the  $(C, 1)$  sum of (1.6) almost everywhere. Hence  $\tilde{S}[f] = S[\tilde{f}]$  and, by Parseval's formula,

$$\frac{1}{\pi} \int_0^{2\pi} f^2 dx = \frac{1}{2} a_0^2 + \frac{1}{\pi} \int_0^{2\pi} \tilde{f}^2 dx. \quad (1.7)$$

Though the problem of the convergence almost everywhere of Fourier series will be discussed only in a later chapter (see Chapter XIII), a special result may be mentioned here.

(1.8) THEOREM. The series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_0^{\infty} A_n(x) \quad (1.9)$$

converges almost everywhere if  $\sum (a_n^2 + b_n^2) \log^2 n$  is finite.

For the condition implies that  $\sum A_n(x) \log n$  is a Fourier series, and it is enough to apply Theorem (4.4) of Chapter III.

We shall prove in Chapter XIII, § 1, that the finiteness of  $\sum (a_n^2 + b_n^2) \log n$  is sufficient for the convergence almost everywhere of (1.9); but the proof of that is much less simple. Whether the hypothesis  $\sum (a_n^2 + b_n^2) < \infty$  is sufficient remains an open question.

## 2. A theorem of Marcinkiewicz

The fact that every integrable function is almost everywhere the derivative of its indefinite integral is fundamental in questions about the representation of functions by their Fourier series. But certain problems require more than even the strengthened forms (11.1) and (11.3) of Chapter II, and we need to go more deeply into the structure of functions and point sets. The following result is particularly useful:

(2.1) **THEOREM OF MARCINKIEWICZ.** *Let  $P$  be a closed set in a finite interval  $(a, b)$  and let  $\chi(t) = \chi_P(t)$  be the distance of the point  $t$  from  $P$ . Then*

(i) *for every  $\lambda > 0$  the integral*

$$J_\lambda(x) = I_\lambda(x, P) = \int_a^b \frac{\chi^\lambda(t)}{|t-x|^{\lambda+1}} dt \quad (2.2)$$

*is finite at almost all points of  $P$ ; more generally, if  $f$  is an integrable function in  $(a, b)$ , the integral*

$$J_\lambda(x) = J_\lambda(x, f, P) = \int_a^b \frac{f(t) \chi^\lambda(t)}{|t-x|^{\lambda+1}} dt \quad (2.3)$$

*converges absolutely at almost all points of  $P$  and*

$$\int_P |J_\lambda(x)| dx \leq 2\lambda^{-1} \int_a^b |f(x)| dx. \quad (2.4)$$

(ii) *If all intervals contiguous to  $P$  are of length less than 1, the integrals*

$$I_0(x) = \int_a^b \frac{\{\log 1/\chi(t)\}^{-1}}{|t-x|} dt, \quad J_0(x) = \int_a^b \frac{f(t) \{\log 1/\chi(t)\}^{-1}}{|t-x|} dt \quad (2.5)$$

*converge absolutely at almost all points of  $P$  and*

$$\int_P |J_0(x)| dx \leq A \int_a^b |f(x)| dx, \quad (2.6)$$

*where  $A$  is a positive constant independent of  $f$ .*

(i) It is enough to consider  $J_\lambda$ . We may suppose that  $f \geq 0$ . Then  $0 \leq J_\lambda(x) \leq \infty$ , and if we prove (2.4) the finiteness of  $J_\lambda(x)$  at almost all points of  $P$  will follow. Observe that the function  $\chi(t)$  vanishes on  $P$  and its graph over any interval  $d$  contiguous to  $P$  is an isosceles triangle of height  $\frac{1}{2}|d|$ ; also  $\chi(t)$  is linear to the right and left of  $P$ .

Integration in (2.3) may be confined to the set  $Q = (a, b) - P$ . We have

$$\int_P J_\lambda(x) dx = \int_Q f(t) \chi^\lambda(t) \left\{ \int_P \frac{dx}{|t-x|^{\lambda+1}} \right\} dt, \quad (2.7)$$

the interchange of the order of integration being justified by the positiveness of the integrand. To estimate the inner integral, fix a point  $t$  interior to an interval  $(\alpha, \beta)$  contiguous to  $P$  and suppose, say, that  $t$  is closer to  $\alpha$  than to  $\beta$ . Then

$$\int_P \frac{dx}{|t-x|^{\lambda+1}} \leq 2 \int_{t-\alpha}^{\infty} u^{-\lambda-1} du = 2\lambda^{-1}(t-\alpha)^{-\lambda} = 2\lambda^{-1}\chi^{-\lambda}(t), \quad (2.8)$$

an estimate which still holds if  $t$  is to the right or left of  $P$ . Substituting this in (2.7) we obtain (2.4).

(ii) We consider  $J_0$  and suppose that  $f \geq 0$ . If  $\lambda = 0, l = b - a$ , the left-hand side of (2.8) does not exceed

$$2 \int_{t-a}^t u^{-1} du = 2 \log l - 2 \log (t - a) = 2 \log l + 2 \log 1/\chi(t) < A \log 1/\chi(t),$$

and this, combined with

$$\int_P J_0(x) dx = \int_Q f(t) \{\log 1/\chi(t)\}^{-1} \left\{ \int_P \frac{dx}{|t-x|} \right\} dt$$

immediately gives (2.6).

A modification of the function  $\chi(t)$  is sometimes useful. Denote by  $\chi^*(t)$  ( $= \chi_P^*(t)$ ) the function equal to 0 in  $P$ , equal to  $|d|$  if  $t$  is in an interval  $d$  contiguous to  $P$ , and equal, say, to 0 to the left and right of  $P$ .

(2.9) THEOREM. *With the hypotheses of Theorem (2.1) the integrals*

$$J_\lambda^*(x) = \int_a^b \frac{f(t) \chi^{*\lambda}(t)}{|t-x|^{\lambda+1}} dt, \quad J_0^*(x) = \int_a^b \frac{f(t) \{\log 1/\chi^*(t)\}^{-1}}{|t-x|} dt$$

converge at almost all points of  $P$ .

It is enough to consider  $J_\lambda^*$ , the proof for  $J_0^*$  being similar. We may suppose that  $f \geq 0$ . Since  $\chi(t) \leq \frac{1}{2} \chi^*(t)$  for  $t$  between the extreme points of  $P$ , the convergence of  $J_\lambda^*$  is a stronger result than the convergence of  $J_\lambda$ . We can, however, deduce the former from the latter. Given any  $\epsilon > 0$ , let  $Q_\epsilon$  denote the union of the intervals making up  $Q$ , each expanded concentrically in the ratio  $1 + \epsilon$  (in this process some of the expanded intervals may become overlapping). Let  $P_\epsilon$  be the closed set complementary to  $Q_\epsilon$  with respect to  $(a, b)$ . Since  $Q_\epsilon \supset Q, |Q_\epsilon - Q| \leq (b-a)\epsilon \rightarrow 0$  with  $\epsilon$ , we have

$$P_\epsilon \subset P, |P - P_\epsilon| \rightarrow 0.$$

We easily see that

$$\chi_P^*(t) \leq 2\epsilon^{-1} \chi_{P_\epsilon}(t).$$

Since  $J_\lambda(x, f, P_\epsilon)$  is finite almost everywhere in  $P_\epsilon$ , the same holds for  $J_\lambda^*(x, f, P)$ . Making  $\epsilon$  approach 0 we see that  $J_\lambda^*(x, f, P)$  is finite almost everywhere in  $P$ .

*Remarks.* (a) In the case of sets  $P$  having period  $2\pi$  it is sometimes more convenient to use the integral

$$J'_\lambda(x) = \int_0^{2\pi} \frac{f(t) \chi^\lambda(t)}{|2 \sin \frac{1}{2}(x-t)|^{\lambda+1}} dt = \int_{-\pi}^{\pi} \frac{f(x+t) \chi^\lambda(x+t)}{|2 \sin \frac{1}{2}t|^{\lambda+1}} dt \tag{2.10}$$

instead of  $J_\lambda(x)$ , and to make a corresponding modification of  $J_0$ . Theorem (2.1) changes little; the factor  $2/\lambda$  in (2.4) must be replaced by another factor depending on  $\lambda$ :

$$\int_P |J'_\lambda(x)| dx \leq A_\lambda \int_0^{2\pi} |f(x)| dx. \tag{2.11}$$

The proof remains the same.

(b) Though we shall not use the fact here, it is of interest to observe that an analogue of (2.1) holds in Euclidean space of any number of dimensions. Suppose, for instance, that  $P$  is a closed set contained in a finite circle  $K$  and that  $\chi(t)$  is the distance of the point  $t$  in the plane from  $P$ . If  $f$  is integrable over  $K$ , then the two integrals

$$\int_K \frac{f(t) \chi^\lambda(t)}{|t-x|^{\lambda+1}} d\sigma \quad (\lambda > 0), \quad \int_K \frac{f(t) \{\log 1/\chi(t)\}^{-1}}{|t-x|^2} d\sigma,$$

where  $|t-x|$  denotes the distance between  $t$  and  $x$  and  $d\sigma$  is an element of area, converge absolutely almost everywhere in  $P$ . When the dimension of the space increases by 1, so do the exponents in the denominators of the integrals considered.

(c) The convergence of the integral  $I_\lambda^*(x)$  has a simple geometric interpretation. Let  $d_1, d_2, \dots$  be the intervals contiguous to a bounded closed and non-dense set  $P$ . For any  $x \in P$ , denote by  $\delta_i(x)$  the distance of  $x$  from  $d_i = (a_i, b_i)$ ; thus  $\delta_i(x) = \min(|x - a_i|, |x - b_i|)$ . Almost every point  $x \in P$  is a point of density of  $P$ , and at a point of density  $|d_i| = o(\delta_i(x))$  as  $d_i$  approaches  $x$ . The finiteness of  $I_\lambda^*(x)$  almost everywhere in  $P$  may be interpreted as follows: the series

$$\sum (|d_i|/\delta_i(x))^{\lambda+1} \tag{2.12}$$

converges for every  $\lambda > 0$  and almost all  $x \in P$ .

For let  $x$  be a point of density of  $P$ , and  $\eta > 0$  so small that  $|d_i| < \delta_i(x)$  for all the  $d_i$  situated entirely in  $(x - \eta, x + \eta)$ . Let  $d'_1, d'_2, \dots$  be all the  $d$ 's situated, say, in  $(x, x + \eta)$ . We may suppose that  $x + \eta$  is not interior to any  $d'$ . Let  $\delta'_i(x)$  be the distance of  $x$  from  $d'_i$ . Then

$$\sum |d'_i|^\lambda \cdot |d'_i| \cdot [2\delta'_i(x)]^{-\lambda-1} < \int_x^{x+\eta} \frac{\chi^{\lambda+1}(t)}{|t-x|^{\lambda+1}} dt < \sum |d'_i|^\lambda \cdot |d'_i| \cdot [\delta'_i(x)]^{-\lambda-1}$$

and the convergence of  $\sum (|d'_i|/\delta'_i(x))^{\lambda+1}$  is equivalent to that of the integral. Similarly for  $(x - \eta, x)$ . Since the series (2.12) and the integral  $I_\lambda^*$  extended over the  $d$ 's outside  $(x - \eta, x + \eta)$  are finite in any case, the assertion follows. Similar interpretations can be given for  $I_0^*(x)$  and  $J_\lambda^*(x)$ .

(d) If in (2.3) and (2.6) we replace  $f(t) dt$  by  $dF(t)$ , where  $F$  is of bounded variation, the resulting integrals converge almost everywhere in  $P$ . More interesting for applications, however, is the following result in which, for simplicity, we consider integrals of the type (2.10).

Suppose that  $\mu(t)$  is a positive measure on the circumference of the unit circle such that

$$\left| \int_0^t d\mu \right| < A |t| \tag{2.13}$$

for all  $t$ . Then, if  $\lambda > 0$ , the integral

$$\int_{-\pi}^{\pi} f(x+t) \chi^\lambda(x+t) \frac{d\mu(t)}{(2 \sin \frac{1}{2}t)^{\lambda+1}} = \int_{-\pi}^{\pi} f(t) \chi^\lambda(t) \frac{d_\mu \mu(t-x)}{|2 \sin \frac{1}{2}(t-x)|^{\lambda+1}}$$

converges almost everywhere in  $P$ . The proof remains unchanged if we note (using integration by

parts) that (2.13) implies  $\int_{\delta < |t| < \pi} |t|^{-\lambda-1} d\mu(t) = O(\delta^{-\lambda})$ .

### 3. Existence of the conjugate function

(3.1) THEOREM. If  $f \in L$ , then

$$f(x) = -\frac{1}{\pi} \int_0^\pi [f(x+t) - f(x-t)] \frac{1}{2} \cot \frac{1}{2}t dt = -\frac{1}{\pi} \lim_{\epsilon \rightarrow +0} \int_\epsilon^\pi \tag{3.2}$$

exists for almost all  $x$ .

This result was already stated and used in Chapter III, § 3, and we shall give two proofs of it, one now and the other in Chapter VII, § 1. The latter proof is much the shorter of the two, but it uses the theory of analytic functions. On the other hand, what we prove here is more general, and is not so easily accessible by complex methods.

(3.3) THEOREM. Suppose that  $F \in L$  is periodic and has a finite derivative at every point of a set  $E$  of positive measure. Then the integral

$$F^*(x) = \frac{1}{\pi} \int_0^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{(2 \sin \frac{1}{2}t)^2} dt = \frac{1}{\pi} \lim_{\epsilon \rightarrow +0} \int_\epsilon^\pi \tag{3.4}$$

exists at almost all points of  $E$ .

To see that (3.1) follows from (3.3) we may suppose that  $a_0 = 0$ , which does not affect  $f$ . The indefinite integral  $F$  of  $f$  is then periodic, and

$$\int_{\epsilon}^{\pi} \frac{F(x+t) + F(x-t) - 2F(x)}{(2 \sin \frac{1}{2}t)^{2\alpha}} dt = \frac{F(x+\epsilon) + F(x-\epsilon) - 2F(x)}{2 \tan \frac{1}{2}\epsilon} + \int_{\epsilon}^{\pi} \frac{f(x+t) - f(x-t)}{2 \tan \frac{1}{2}t} dt. \quad (3.5)$$

At every point  $x$  where  $F$  is differentiable the integrated term tends to zero with  $\epsilon$ , and the existence of  $F^*(x)$  is equivalent to that of  $f(x)$ . Moreover,  $F^*(x) = f(x)$ .

We turn now to (3.3). The special case  $F = \int f dx$ ,  $f \in L^2$ , was given in § 1. It follows that  $F^*(x)$  exists almost everywhere if  $F$  is the integral of a function in  $L^2$ , and in particular if  $F \in \Lambda_1$ .

We write 
$$\rho(x, h) = [F(x+h) - F(x)]/h$$

and denote by  $E_k$  the set of  $x \in E$  such that  $|\rho(x, h)| \leq k$  for  $|h| < 1/k$ . Hence

$$E_1 \subset E_2 \subset \dots \subset E_k \subset \dots \subset E, \quad |E_k| \rightarrow |E|.$$

Fix  $k$ , say  $k = M$ , and consider any closed subset  $P$  of  $E_M$ . We shall prove that  $F^*(x)$  exists almost everywhere in  $P$ . Since  $|E - P|$  may be arbitrarily small, (3.3) will follow.

By hypothesis,

$$|F(x+h) - F(x)| \leq M|h| \quad \text{for } x \in P, |h| \leq 1/M. \quad (3.6)$$

Let  $G(x)$  be the function coinciding with  $F$  on  $P$  and linear in the closed intervals  $d_1, d_2, \dots$  contiguous to  $P$ . We prove that  $G(x) \in \Lambda_1$ , and for this it is enough to prove

$$|G(x+h) - G(x)| \leq A|h| \quad \text{for } |h| \leq 1/M. \quad (3.7)$$

with  $A$  independent of  $x, h$ . Suppose, for example, that  $h > 0$ . We consider first two special cases: (i) both  $x$  and  $x+h$  belong to  $P$ ; (ii) the interior of  $(x, x+h)$  contains no points of  $P$ . In case (i), (3.7), with  $A = M$ , follows from (3.6). In case (ii),  $(x, x+h)$  is contained in an interval  $d$  contiguous to  $P$ , and  $G$  is linear there. Thus, if  $|d| \leq 1/M$ , (3.7), with  $A = M$ , again follows from (3.6). Since there are only a finite number (if any) of  $d$ 's with  $|d| > 1/M$ , (3.7) is always true in case (ii), provided  $A$  is large enough.

If neither (i) nor (ii) holds,  $(x, x+h)$  contains points of  $P$  in its interior. Let  $x+h_1$  and  $x+h_2$ ,  $h_1 \leq h_2$ , be the extreme points of  $P$  in  $(x, x+h)$ . The absolute increment of  $G$  over  $(x, x+h)$  does not exceed the sum of the increments over the intervals  $(x, x+h_1)$ ,  $(x+h_1, x+h_2)$ ,  $(x+h_2, x+h)$ , and each of the latter is at most  $A$  times the length of the corresponding interval, in virtue of cases (i) and (ii). This leads again to (3.7). Hence  $G \in \Lambda_1$ .

We set  $H(x) = F(x) - G(x)$ , so that

$$F(x) = G(x) + H(x). \quad (3.8)$$

From (3.6) and (3.7) we see that  $H(x)$  satisfies an inequality analogous to (3.6), with  $M' = M + A$  for  $M$ . Since, however,  $H(x)$  vanishes in  $P$ , this implies that, except in a finite number of intervals exterior to  $P$ ,

$$|H(x)| \leq M'\chi(x), \quad (3.9)$$

where  $\chi(x)$  is the distance of  $x$  from  $P$ .

The functions  $G'$  and  $H$  being periodic, (3.3) will follow if we show that the integrals  $G^*(x)$  and  $H^*(x)$  exist almost everywhere in  $P$ . This has already been proved for  $G^*$ , since  $G \in \Lambda_1$ . Consider  $H^*(x)$ . If  $x \in P$ , then  $H(x) = 0$  and (3.9) gives

$$\int_0^{1/M} \left| \frac{H(x+t) + H(x-t) - 2H(x)}{(2 \sin \frac{1}{2}t)^2} \right| dt \leq M' \int_{-1/M}^{1/M} \frac{\lambda(x+t)}{(2 \sin \frac{1}{2}t)^2} dt.$$

The integral on the right is finite almost everywhere in  $P$ . (See (2.10) and (2.11) with  $\lambda = 1, f = 1$ ; we could also use the finiteness of the  $I_\lambda$  in (2.2).) The same therefore holds for the integral on the left, and the finiteness is not affected if on the left we replace the interval of integration  $(0, 1/M)$  by  $(0, \pi)$ . Hence the integral  $H^*(x)$  converges, even absolutely, almost everywhere in  $P$ , which completes the proof of (3.3).

In this proof we tacitly assumed that the sets  $E_\alpha$  were measurable (to ensure the existence of closed subsets  $P$ ). To prove the measurability it is enough to show that for any fixed  $\alpha$  the function

$$\rho(x) = \rho(x, \alpha) = \sup_{0 < |t| < \alpha} \frac{|F(x+t) - F(x)|}{t}$$

is lower semi-continuous, that is that  $\liminf_{x \rightarrow x_0} \rho(x) \geq \rho(x_0)$ . The inequality is immediate if we interpret  $[F(x+t) - F(x)]/t$  as the slope of a chord. For if, e.g.,  $F$  is continuous at  $x_0$  and if  $x_1, |x_1 - x_0| < \alpha$ , is such that the absolute value of the slope of the chord joining the point  $P_0(x_0, F(x_0))$  to  $P_1(x_1, F(x_1))$  exceeds  $\rho(x_0) - \epsilon$ , then the absolute value of the slope of the chord  $PP_1$  exceeds  $\rho(x_0) - 2\epsilon$ , provided the abscissa  $x$  of  $P$  is close enough to  $x_0$ , and the inequality follows. If  $F$  is discontinuous at  $x_0$  both sides of the inequality are  $+\infty$ .

Immediate consequences of (3.1) are Theorems (5.8) and (3.23) of Chapter III, initially stated without proof. From (3.3) and from Theorem (7.15) of Chapter III we also deduce

(3.10) THEOREM. *If  $F(x)$ , periodic and integrable, is differentiable in a set  $E$  of positive measure, the series  $\tilde{S}'[F]$  is summable  $A$  to sum (3.4) almost everywhere in  $E$ .*

The existence of  $f(x)$  is not trivial even if  $f(x)$  is continuous. The existence of the  $f$  is due not to the smallness of  $f(x+t) - f(x-t)$  for small  $|t|$  but to the interference of positive and negative values; in fact, as we will show, there exist continuous functions  $f$  such that the integral

$$\int_0^\pi \left| \frac{f(x+t) - f(x-t)}{t} \right| dt \tag{3.11}$$

diverges at every  $x$ . It will slightly simplify the notation if we consider functions of period 1 and replace the upper limit of integration  $\pi$  in (3.11) by 1. We first need the following lemma:

(3.12) LEMMA. *Let  $g(x)$  be a function of period 1 such that  $|g(x)| \leq 1, |g'(x)| \leq 1$ , and that for no value of  $x$  does the difference  $g(x+u) - g(x-u)$  vanish identically in  $u$ .† Then*

$$\int_{1/n}^1 \left| \frac{g(nx+nt) - g(nx-nt)}{t} \right| dt \geq C \log n, \quad \int_0^1 \left| \frac{g(nx+nt) - g(nx-nt)}{t} \right| dt \leq C_1 \log n$$

for  $n = 2, 3, \dots$ ,  $C$  and  $C_1$  being positive constants independent of  $n$ .

Let  $nx = y, nt = u$ . Since  $g$  is periodic, the first integral is

$$\int_0^1 |g(y+u) - g(y-u)| \sum_{\nu=1}^{n-1} \frac{1}{u+\nu} du \geq \left( \sum_{\nu=2}^n \frac{1}{\nu} \right) \int_0^1 |g(y+u) - g(y-u)| du.$$

The first factor on the right exceeds a multiple of  $\log n$ , and the second, as a periodic, continuous and nowhere vanishing function of  $y$ , is bounded below by a positive number. This gives the first part of the lemma. Similarly we obtain the second part, observing that

$$\int_0^1 |g(y+u) - g(y-u)| u^{-1} du < \infty.$$

† For  $g(x), 0 \leq x \leq 1$ , we may take, for example, the polygonal line with vertices  $(0, 0), (\frac{1}{2}, \frac{1}{2}), (1, 0)$ .

We now set

$$f(x) = \sum_{n=1}^{\infty} a_n g(\lambda_n x). \quad (3.13)$$

where the numbers  $a_n > 0$  and the integers  $0 < \lambda_1 < \lambda_2 < \dots$  will be determined in a moment. Then

$$\begin{aligned} \int_{1/\lambda_\nu}^1 \frac{|f(x+t) - f(x-t)|}{t} dt &\geq a_\nu \int_{1/\lambda_\nu}^1 \frac{|g(\lambda_\nu x + \lambda_\nu t) - g(\lambda_\nu x - \lambda_\nu t)|}{t} dt \\ &\quad - \left( \sum_{n=1}^{\nu-1} + \sum_{n=\nu+1}^{\infty} \right) a_n \int_{1/\lambda_\nu}^1 \frac{|g(\lambda_n x + \lambda_n t) - g(\lambda_n x - \lambda_n t)|}{t} dt \\ &\geq C a_\nu \log \lambda_\nu - C \sum_{n=1}^{\nu-1} a_n \log \lambda_n - 2 \log \lambda_\nu \sum_{n=\nu+1}^{\infty} a_n. \end{aligned} \quad (3.14)$$

since  $|g(\lambda_n x + \lambda_n t) - g(\lambda_n x - \lambda_n t)| \leq 2$ . If we take  $a_n = 1/n!$ ,  $\lambda_n = 2^{n\nu}$ , the right-hand side of (3.14) divided by  $\nu!$  tends to  $C \log 2 > 0$ , and this shows that (3.11) diverges everywhere.

It is interesting to observe that the integrals

$$\int_0^\nu \frac{f(x+t) - f(x)}{t} dt \quad \text{and} \quad \int_0^\nu \frac{f(x+t) + f(x-t) - 2f(x)}{t} dt, \quad (3.15)$$

though apparently similar to (3.2), can diverge everywhere for a continuous  $f$ . The proof is analogous to that given above, but slightly less simple.

The theorem which follows will find an application in Chapter XII. Its proof is similar to that of (3.1) but the details are somewhat more elaborate.

(3.16) THEOREM. If  $f \in L$ ,  $y > 0$ , then the measure of the set  $E_y = E_y(f)$  where  $|f(x)| > y$  satisfies

$$|E_y| \leq \frac{A}{y} \int_0^{2\pi} |f| dx, \quad (3.17)$$

where  $A$  is an absolute constant.

We may suppose that  $f \geq 0$ . For if  $f = f_1 + f_2$ , then  $E_{2y}(f) \subset E_y(f_1) + E_y(f_2)$ ,

$$|E_{2y}(f)| \leq |E_y(f_1)| + |E_y(f_2)|. \quad (3.18)$$

Hence if  $f_1$  and  $f_2$  are the positive and negative parts of  $f$ , and if the theorem holds for  $f_1$  and  $f_2$ , it holds for  $f$ .

We may also suppose that

$$\int_0^{2\pi} f dx = 1. \quad (3.19)$$

The function  $F(x) = \int_0^x f dt$  is non-decreasing in  $(-\infty, +\infty)$ , and

$$f(x) = -\frac{1}{\pi} \int_0^\pi \frac{F(x+t) + F(x-t) - 2F(x)}{(2 \sin \frac{1}{2}t)^2} dt \quad (3.20)$$

almost everywhere.

Fix  $y$  and denote by  $Q$  the set of  $x$  for which  $\frac{F(\xi) - F(x)}{\xi - x} > y$  (3.21)

for some  $\xi$  in the interior of  $(x, x + 2\pi)$ .  $Q$  is open (possibly empty) and periodic. The complement  $P$  of  $Q$  is closed. If  $P$  and  $Q$  are not empty,  $Q$  is the union of a family  $\{(a_i, b_i)\}$  of disjoint open intervals such that

$$\frac{F(b_i) - F(a_i)}{b_i - a_i} = y. \quad (3.22)$$

This fact is proved in the same way as Lemma (13.8) of Chapter I and the Remark to it

Write now  $F = G + H$ , where  $G$  coincides with  $F$  on  $P$  and is linear in each interval  $(a_i, b_i)$ ; hence  $H = 0$  on  $P$ . If  $Q$  is empty, we write  $F = G$ ,  $H = 0$ , and disregard  $H$  in the argument which follows.

The function  $G(x)$  is in  $\Lambda_1$ . More precisely,

$$0 \leq \frac{G(x+h) - G(x)}{h} \leq y \quad \text{for } 0 < h < 2\pi.$$

The first inequality is obvious since  $G$ , like  $F$ , is non-decreasing. The second inequality is immediate if both  $x$  and  $x + h$  are either in  $P$  or in the same interval contiguous to  $P$ ; the general case follows from these two by the argument used on p. 132.

Hence  $G$  is the indefinite integral of a (periodic) function  $g = G'$ . We have  $0 \leq g(x) \leq y$  for almost all  $x$ , and  $g(x) = f(x)$  almost everywhere in  $P$  (since  $G = F$  in  $P$ ). Clearly  $H$  is the indefinite integral of  $h = H'$ , and

$$f = g + h, \quad \tilde{f} = \tilde{g} + \tilde{h}.$$

Since  $H = 0$  in  $P$ , the integral of  $h$  over each interval  $(a_i, b_i)$  is  $H(b_i) - H(a_i) = 0$ , and

$$\int_{a_i}^{b_i} f dx = \int_{a_i}^{b_i} g dx \quad (i = 1, 2, \dots), \quad \int_0^{2\pi} f dx = \int_0^{2\pi} g dx. \quad (3.23)$$

Since  $|E_y(f)| \leq |E_y(g)| + |E_y(h)|$  it is enough to show that each of the two terms on the right is majorized by

$$\frac{A}{y} \int_0^{2\pi} f dx = \frac{A}{y}.$$

For  $g$  we have

$$\begin{aligned} |E_y(g)| &\leq y^{-2} \int_0^{2\pi} \tilde{g}^2 dx \leq y^{-2} \int_0^{2\pi} g^2 dx \\ &\leq y^{-1} \int_0^{2\pi} g dx = y^{-1} \int_0^{2\pi} f dx, \end{aligned}$$

by (3.23).

It remains to estimate  $|E_y(h)|$ . First, summing from (3.22) over all  $(a_i, b_i)$  in a period, we get

$$|Q| = \sum(b_i - a_i) \leq \frac{F(2\pi) - F(0)}{y} = \frac{1}{y} \int_0^{2\pi} f(x) dx = \frac{1}{y}. \quad (3.24)$$

Next, let  $\chi^*(x)$  be the function equal to  $b_i - a_i$  in each  $(a_i, b_i)$ , and to 0 in  $P$ . We show that

$$H(x) \leq y\chi^*(x). \quad (3.25)$$

This is obvious for  $x$  in  $P$ , since both sides are 0. If  $a_i < x < b_i$ , then

$$0 \leq F(x) - F(a_i) \leq y(x - a_i), \quad 0 \leq G(x) - G(a_i) \leq y(x - a_i);$$

and the equations  $H(a_i) = 0$ ,  $H = F - G$  imply

$$|H(x)| = |H(x) - H(a_i)| \leq y(x - a_i) < y(b_i - a_i),$$

which gives (3.25).

Write

$$I(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\chi^*(t)}{\{2 \sin \frac{1}{2}(x-t)\}^2} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\chi^*(x+t)}{(2 \sin \frac{1}{2}t)^2} dt.$$

In view of (3.25), if we apply (3.20) to  $h$  we obtain

$$|\tilde{h}(x)| \leq yI(x) \quad \text{for } x \in P. \quad (3.26)$$

Let  $Q^*$  be the set obtained by expanding each  $(a_i, b_i)$  concentrically three times, and let  $P^*$  be the complement of  $Q^*$ . We have

$$\int_{P^*} I(x) dx \leq B |Q|, \quad (3.27)$$

where  $B$  is an absolute constant; this is a result analogous to (2.11) and the proof is essentially the same.

It is now easy to estimate  $|E_\nu(h)|$ . The intersection of  $E_\nu(h)$  with  $Q^*$  has measure not greater than  $|Q^*| \leq 3|Q|$ . In  $P$ , and *a fortiori* in  $P^*$ , we have (3.26), and so, if  $|\hat{h}(x)| > y$ , then  $I(x) > 1$ . But, by (3.27), the subset of  $P^*$  where  $I(x) > 1$  has measure not greater than  $B|Q|$ . Hence, collecting results and using (3.24), we find

$$|E_\nu(h)| \leq 3|Q| + B|Q| \leq (B+3)y^{-1}.$$

This completes the proof of (3.16).

#### 4. Classes of functions and (C, 1) means of Fourier series

We know that the necessary and sufficient condition for the numbers  $c_\nu$  ( $\nu = 0, \pm 1, \pm 2, \dots$ ) to be the Fourier coefficients of a function  $L^2$  is that the sum  $\sum |c_\nu|^2$  be finite. It is natural to ask whether anything so simple can be proved for the classes  $L^r$  with  $r \neq 2$ . The answer is no, and it is this fact which makes the Parseval formula and the Riesz-Fischer theorem such exceptionally powerful tools of investigation. We shall now consider criteria of a different kind involving the Cesàro or Abel means of the series considered.

One point must be made clear. What matters in the proofs that follow is that the (C, 1) kernel, and Abel's kernel, satisfy conditions (A), (B), (C) stated in § 2 of Chapter III (and in particular are positive), and also that  $S[f]$  is summable by these methods. The arguments are therefore applicable without change to any other kernel with these properties. Logically the (C, 1) method is simpler than Abel's, but the latter is often more significant, especially in applications to harmonic and analytic functions.

We pursue the following course: in §§ 4 and 5 the results will be proved for the (C, 1) means, and in § 6 the analogues for Abel means will be stated without proof.

Besides the classes  $L_p, L^r$  introduced in § 9 of Chapter I, we shall consider other classes of functions. We shall denote by B, C, A and V the classes of periodic functions which are respectively bounded, continuous, absolutely continuous, and of bounded variation. If

$$\sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu x} \quad (4.1)$$

is the Fourier series of a function of a definite class, we say that the series itself belongs to that class. By S we denote the class of Fourier-Stieltjes series. The (C, 1) means of (4.1) will be denoted by  $\sigma_n(x)$ .

**(4.2) THEOREM.** (i) A necessary and sufficient condition for  $\sum c_\nu e^{i\nu x}$  to belong to class C is the uniform convergence of  $\{\sigma_n(x)\}$ .

(ii) A necessary and sufficient condition for  $\sum c_\nu e^{i\nu x}$  to belong to class B is the uniform boundedness of the  $\sigma_n(x)$ .

The necessity of the condition in (i) is Fejér's theorem (Chapter III. (3.4)). To prove the sufficiency, we note that, for  $n \geq |k|$ ,

$$\left(1 - \frac{|k|}{n+1}\right) c_k = \frac{1}{2\pi} \int_0^{2\pi} \sigma_n(x) e^{-ikx} dx.$$

As  $n \rightarrow \infty$  the left-hand side tends to  $c_k$  and the right-hand side to the  $k$ th Fourier coefficient of the continuous function  $f(x) = \lim \sigma_n(x)$ .

The necessity of the condition in (ii) is contained in Theorem (2.30) of Chapter III: if  $K$  is the essential upper bound of  $|f|$  then  $|\sigma_n(x)| \leq K$ . Conversely, if  $|\sigma_n| \leq K$ , then, for large  $n$ ,

$$K^2 \geq \frac{1}{2\pi} \int_0^{2\pi} |\sigma_n|^2 dx = \sum_{k=-n}^n |c_k|^2 \left(1 - \frac{|k|}{n+1}\right)^2 \geq \sum_{k=-\nu}^{\nu} |c_k|^2 \left(1 - \frac{|k|}{n+1}\right)^2,$$

where  $\nu$  is any fixed positive integer not exceeding  $n$ . Making  $n \rightarrow \infty$  we get

$$|c_{-\nu}|^2 + \dots + |c_0|^2 + \dots + |c_{\nu}|^2 \leq K^2,$$

and this is true for every  $\nu$ . Hence  $\sum |c_k|^2$  converges, and, by the Riesz-Fischer theorem, (4.1) is an  $S[f]$  with  $f \in L^2$ . Therefore  $\sigma_n(x) \rightarrow f(x)$  almost everywhere, and the inequalities  $|\sigma_n(x)| \leq K$  imply that  $|f(x)| \leq K$  almost everywhere.

**(4.3) THEOREM.** *The series  $\sum c_\nu e^{i\nu x}$  belongs to class S if and only if  $\mathfrak{M}[\sigma_n] = O(1)$ .*

We first suppose that  $\sum c_\nu e^{i\nu x}$  is an  $S[dF]$ . Then

$$\begin{aligned} \sigma_n(x) &= \frac{1}{\pi} \int_0^{2\pi} K_n(t-x) dF(t), \\ |\sigma_n(x)| &\leq \frac{1}{\pi} \int_0^{2\pi} K_n(t-x) |dF(t)|, \end{aligned} \tag{4.4}$$

where  $|dF(t)|$  stands for  $dV(t)$ ,  $V(t)$  denoting the total variation of  $F$  over  $(0, t)$ . Let  $V = V(2\pi)$ . The right-hand side of the last inequality is a trigonometric polynomial in  $x$  whose constant term is  $V/2\pi$ . Integration over  $0 \leq x \leq 2\pi$  therefore gives

$$\mathfrak{M}[\sigma_n] \leq \int_0^{2\pi} |dF(t)| = V, \tag{4.5}$$

and one part of (4.3) is established. For the other we need the following classical result, which we take for granted here:

**(4.6) THEOREM OF HELLY.** *Let  $\{F_n(x)\}$  be a sequence of functions uniformly bounded and of uniformly bounded variation in an interval  $(a, b)$ . Then there is a subsequence  $\{F_{n_k}(x)\}$  converging at every point of  $(a, b)$  to a function  $F(x)$  of bounded variation.*

The hypothesis of uniform boundedness may be replaced by the boundedness of  $\{F_n\}$  at a single point  $x$ , since the former is implied by the latter together with the uniformly bounded variation.

Returning to Theorem (4.3), suppose that  $\mathfrak{M}[\sigma_n] \leq V$  for all  $n$  and let

$$F_n(x) = \int_0^x \sigma_n(t) dt.$$

The functions  $F_n(x)$  are of uniformly bounded variation in  $(0, 2\pi)$  and vanish at  $x=0$ . Hence there is a subsequence  $\{F_{n_j}(x)\}$ , uniformly bounded and everywhere convergent to a function  $F(x)$  of bounded variation (total variation  $\leq V$ ) in  $(0, 2\pi)$ . If  $|k| \leq n_j$ , integrating by parts and making  $j \rightarrow \infty$  we get

$$\left(1 - \frac{|k|}{n_j+1}\right) c_k = \frac{1}{2\pi} \int_0^{2\pi} \sigma_{n_j} e^{-ikx} dx = \frac{1}{2\pi} F_{n_j}(2\pi) + \frac{ik}{2\pi} \int_0^{2\pi} F_{n_j} e^{-ikx} dx,$$

$$c_k = \frac{1}{2\pi} F(2\pi) + \frac{ik}{2\pi} \int_0^{2\pi} F e^{-ikx} dx = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} dF(x),$$

since  $F(0)=0$ . Hence (4.1) is  $S[dF]$  and the proof of (4.3) is complete. The result can be stated in the following equivalent form.

(4.7) THEOREM. A necessary and sufficient condition for  $\sum c_n e^{inx}$  to belong to class  $V$  is that  $\Re[\sigma'_n] = O(1)$ , that is, that the  $\sigma_n$  be of uniformly bounded variation.

The following result completes (4.3):

(4.8) THEOREM. A necessary and sufficient condition for  $\sum c_n e^{inx}$  to be an  $S[dF]$  with  $F$  non-decreasing is that  $\sigma_n \geq 0$  for all  $n$ .

The necessity follows from (4.4) since  $K_n(u) \geq 0$ . Conversely, if  $\sigma_n(x) \geq 0$ , the  $F_n(x)$  in the proof of (4.3) are non-decreasing and so is  $F(x) = \lim F_{n_j}(x)$ .

(4.9) THEOREM. A necessary and sufficient condition for  $\sum c_n e^{inx}$  to be an  $S[dF]$  with  $F$  non-decreasing is that

$$\sum_{\mu, \nu=0}^n c_{\mu-\nu} \xi_\mu \bar{\xi}_\nu \geq 0 \quad (4.10)$$

for all  $n \geq 0$  and for all (complex)  $\xi_0, \xi_1, \dots, \xi_n$ .

If  $c_\nu = (2\pi)^{-1} \int_0^{2\pi} e^{-i\nu x} dF$ , with  $F$  non-decreasing, then

$$2\pi \sum_{\mu, \nu=0}^n c_{\mu-\nu} \xi_\mu \bar{\xi}_\nu = \int_0^{2\pi} \left( \sum_{\mu, \nu=0}^n e^{-i(\mu-\nu)x} \xi_\mu \bar{\xi}_\nu \right) dF = \int_0^{2\pi} \left| \sum_{\mu=0}^n e^{-i\mu x} \xi_\mu \right|^2 dF \geq 0.$$

Conversely, if we take  $\xi_\nu = e^{i\nu x}$  for all  $\nu$ , and denote by  $\sigma_n$  the  $(C, 1)$  means of the series  $\sum c_n e^{inx}$ , the left-hand side of (4.10) becomes

$$(n+1)c_0 + n(c_1 e^{ix} + c_{-1} e^{-ix}) + \dots + (c_n e^{inx} + c_{-n} e^{-inx}) = (n+1)\sigma_n(x),$$

and it is enough to apply (4.8).

Let  $u^+$  and  $u^-$  denote respectively  $\max(u, 0)$  and  $\max(-u, 0)$ , so that

$$u^+ = \frac{1}{2}(|u| + u), \quad u^- = \frac{1}{2}(|u| - u). \quad (4.11)$$

Since the integral of  $\sigma_n$  over  $(0, 2\pi)$  is constant (being  $2\pi c_0$ ), the first equation shows that the conditions

$$\Re[\sigma_n] = O(1), \quad \Re[\sigma_n^+] = O(1)$$

are equivalent.

(4.12) THEOREM. Suppose that  $\sum c_n e^{inx}$  is an  $S[dF]$  and that

$$F(x) = \frac{1}{2}[F(x+0) + F(x-0)] \quad (4.13)$$

for all  $x$ . Let  $V, P, N$  denote respectively the total, positive and negative variations of  $F$  over an interval  $\alpha \leq x \leq \beta$ . Then

$$\int_{\alpha}^{\beta} |\sigma_n| dx \rightarrow V, \quad \int_{\alpha}^{\beta} \sigma_n^+ dx \rightarrow P, \quad \int_{\alpha}^{\beta} \sigma_n^- dx \rightarrow N. \quad (4.14)$$

Since  $S[dF] = S'[F]$  (Chapter II, (2.4)), (4.13) implies that

$$\int_{\alpha}^{\beta} \sigma_n dx \rightarrow F(\beta) - F(\alpha) \quad (4.15)$$

(Chapter III, (3.4)). It is enough to prove the first formula (4.14), since the other two relations in (4.14) follow from this, combined with (4.15), (4.11) and

$$2P = V + (F(\beta) - F(\alpha)), \quad 2N = V - (F(\beta) - F(\alpha)).$$

That  $\liminf \mathfrak{M}[\sigma_n; \alpha, \beta] \geq V$  is clear. For by (4.15) the functions  $F_n(x) = \int_{\alpha}^x \sigma_n dt$  converge to  $F(x) - F(\alpha)$  on  $(\alpha, \beta)$  and  $\mathfrak{M}[\sigma_n; \alpha, \beta]$  is the total variation of  $F_n$  over  $(\alpha, \beta)$ . The total variation  $V$  of the limit cannot exceed the limit inferior of the total variations of the  $F_n$ . It remains therefore to show that  $\limsup \mathfrak{M}[\sigma_n; \alpha, \beta] \leq V$ . If  $(\alpha, \beta)$  coincides with  $(0, 2\pi)$ , this follows from (4.5), and (4.12) is established in this particular case. If  $\beta - \alpha < 2\pi$ , suppose that the inequality we want to prove is false. If  $V'$  is the total variation of  $F$  over the closed interval  $(\beta, \alpha + 2\pi)$  we therefore have

$$\int_{\alpha}^{\alpha+2\pi} |\sigma_n| dx \rightarrow V + V', \quad \limsup \int_{\alpha}^{\beta} |\sigma_n| dx > V.$$

This implies that  $\liminf \mathfrak{M}[\sigma_n; \beta, \alpha + 2\pi] < V'$ , contrary to the opposite inequality which we have already proved (with  $\alpha, \beta$  for  $\beta, \alpha + 2\pi$ ). This proves (4.12).

We know that  $\sigma_n(x; dF) \rightarrow F'(x)$  for almost all  $x$  (Chapter III, (8.1)), and we shall sometimes write  $\sigma(x)$  ( $= \lim \sigma_n(x)$ ) instead of  $F'(x)$ .

Let  $P(x)$  be the positive variation of  $F(x)$  over  $(\alpha, x)$  and let

$$P(x) = P_a(x) + P_s(x)$$

be the decomposition of  $P$  into its absolutely continuous and singular parts. Both  $P_a(x)$  and  $P_s(x)$  are non-negative and non-decreasing for  $x \geq \alpha$ . Moreover, as is well known, we have almost everywhere

$$P'_s(x) = 0, \quad P'_a(x) = P'(x) = (F'(x))^+ = \sigma^+(x).$$

By (4.12),

$$\int_{\alpha}^{\beta} \sigma_n^+ dx \rightarrow \{P_a(\beta) - P_a(\alpha)\} + \{P_s(\beta) - P_s(\alpha)\} = \int_{\alpha}^{\beta} \sigma^+ dx + P_s(\beta). \quad (4.16)$$

A necessary and sufficient condition for  $P(x)$  to be absolutely continuous over  $(\alpha, \beta)$  is that  $P_s(\beta) = 0$ , or

$$\int_{\alpha}^{\beta} \sigma_n^+ dx \rightarrow \int_{\alpha}^{\beta} \sigma^+ dx. \quad (4.17)$$

Similarly, a necessary and sufficient condition for the negative variation  $N(x)$  of  $F$  to be absolutely continuous in  $(\alpha, \beta)$  is

$$\int_{\alpha}^{\beta} \sigma_n^- dx \rightarrow \int_{\alpha}^{\beta} \sigma^- dx. \quad (4.18)$$

If both  $P(x)$  and  $N(x)$  are absolutely continuous over  $(\alpha, \beta)$ , adding (4.17) and (4.18) we get

$$\int_{\alpha}^{\beta} |\sigma_n| dx \rightarrow \int_{\alpha}^{\beta} |\sigma| dx. \quad (4.19)$$

and conversely this relation implies both (4.17) and (4.18). (For then, from (4.16) and from a similar formula for  $\sigma_n^-$ , we see that  $P_3(\beta) = N_3(\beta) = 0$ .) Thus (4.19) is both necessary and sufficient for  $F(x)$  to be absolutely continuous over  $(\alpha, \beta)$ . Hence:

**(4.20) THEOREM.** *Let  $\Sigma c_n e^{inx}$  be an  $S[dF]$ , where  $F$  satisfies (4.13). The conditions (4.19), (4.17) and (4.18) are necessary and sufficient for the function  $F$ , its positive variation, and its negative variation, respectively, to be absolutely continuous over  $(\alpha, \beta)$ .*

Let  $F_k(x)$ ,  $0 < x < 2\pi$ , be a sequence of uniformly bounded functions. If  $F_k(x)$  tends almost everywhere to a limit  $F(x)$ , then  $C_n^k \rightarrow C_n$  as  $k \rightarrow \infty$ , where  $C_n^k$  and  $C_n$  denote the  $n$ th coefficients of  $F_k$  and  $F$  respectively. The converse is obviously false. If, for instance,  $I_1, I_2, \dots$  is any sequence of intervals whose length tends to zero, such that every point in  $(0, 2\pi)$  belongs to infinitely many  $I_k$ , then the sequence of the characteristic functions  $F_k(x)$  of the  $I_k$  diverges at every  $x$ , though  $|C_n^k| \leq |I_k|/2\pi \rightarrow 0$ , as  $k \rightarrow \infty$  (uniformly in  $n$ ). The converse is, however, true if the functions  $F_k$  are monotone.

**(4.21) THEOREM OF CARATHÉODORY.** *Let  $\{F_k(x)\}$ ,  $0 < x < 2\pi$ , be a sequence of uniformly bounded and non-decreasing functions, and let  $C_n^k$  be the (complex) Fourier coefficients of  $F_k$ . If  $\lim_{k \rightarrow \infty} C_n^k = C_n$  exists for every  $n$ , then the numbers  $C_n$  are the Fourier coefficients of a bounded non-decreasing function  $F(x)$ ,  $0 < x < 2\pi$ , and  $F_k(x) \rightarrow F(x)$  at every point  $x$  at which  $F$  is continuous.*

By (4.6) there is a subsequence of  $\{F_k\}$  converging to a non-decreasing  $F(x)$ ,  $0 < x < 2\pi$ . Obviously the Fourier coefficients of  $F$  are the  $C_n$ , and we have only to show that  $F_k(\xi) \rightarrow F(\xi)$  for any point  $\xi$  of continuity of  $F$  interior to  $(0, 2\pi)$ . Suppose that  $F_k(\xi)$  does not tend to  $F(\xi)$ . We can then find a subsequence  $\{F_{k_j}\}$  such that  $\lim F_{k_j}(\xi)$  exists and differs from  $F(\xi)$ , e.g. is greater than  $F(\xi)$ . We can select a subsequence  $\{F_{k_j}(x)\}$  of  $\{F_{k_j}(x)\}$  such that  $\lim F_{k_j}(x) = G(x)$  exists everywhere. The Fourier coefficients of  $G(x)$  are again  $C_n$ , so that  $F(x) \equiv G(x)$ . On the other hand,

$$G(\xi) = \lim F_{k_j}(\xi) = \lim F_{k_j}(\xi) > F(\xi),$$

and since  $G(x)$  is non-decreasing and  $F(x)$  is continuous at  $x = \xi$ , we have  $G(x) > F(x)$  in some interval to the right of  $\xi$ , so that  $G(x) \neq F(x)$ . This contradiction shows that  $F_k(\xi) \rightarrow F(\xi)$ .

We shall now extend (4.21) to Fourier-Stieltjes series. Except when otherwise stated, every non-decreasing function  $\Phi$  considered below will be defined for all  $x$  and will satisfy the condition

$$\Phi(x + 2\pi) - \Phi(x) = \Phi(2\pi) - \Phi(0).$$

**(4.22) THEOREM.** *Let  $F_1(x), F_2(x), \dots$  be a sequence of non-decreasing functions and let  $c_n^k$  be the Fourier coefficients of  $dF_k$ . Then*

- (i) *If  $\lim_{k \rightarrow \infty} c_n^k = c_n$  exists for every  $n$ , there is a non-decreasing function  $F(x)$  such that*

the Fourier coefficients of  $dF$  are  $c_n$ . Moreover, there are constants  $B_k$  such that  $\{F_k(x) - B_k\}$  converges to  $F(x)$  at every point of continuity of  $F$ .

(ii) Conversely, if for a sequence of constants  $B_k$  the sequence  $\{F_k(x) - B_k\}$  converges to a (non-decreasing) function  $F(x)$  at every point of continuity of  $F$ , then, denoting by  $c_n$  the Fourier coefficients of  $dF$ , we have  $c_n^k \rightarrow c_n$  for all  $n$ .

(i) Let  $B_k$  be the constant term of  $S[F_k]$ . The  $F_k - B_k$  being uniformly bounded, there is a subsequence  $\{F_{k_j}(x) - B_{k_j}\}$  converging to a limit  $F(x)$  everywhere in  $0 \leq x \leq 2\pi$ . Let  $C_n^k$  be the Fourier coefficients of  $F_k - B_k$ . Then  $C_0^k = 0$  for all  $k$ , and for  $n \neq 0$  integration by parts gives

$$C_n^k = \frac{1}{2\pi} \int_0^{2\pi} (F_k - B_k) e^{-inx} dx = (c_n^k - c_0^k) / in. \quad (4.23)$$

Hence  $\lim C_n^k = C_n$  exists for every  $n$ . By Theorem (4.21),  $F_k(x) - B_k$  converges to  $F(x)$  at every point of continuity of  $F$  interior to  $(0, 2\pi)$ . Hence if in (4.23) we make  $k \rightarrow \infty$ , we get  $C_n = (c_n - c_0) / in$  for  $n \neq 0$ . On the other hand, if  $\gamma_n$  are the Fourier coefficients of  $dF$ , integration by parts gives

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} F e^{-inx} dx = (\gamma_n - \gamma_0) / in.$$

so that  $c_n - c_0 = \gamma_n - \gamma_0$  for  $n \neq 0$ . Moreover,

$$2\pi\gamma_0 = F(2\pi) - F(0) = \lim \{F_{k_j}(2\pi) - F_{k_j}(0)\} = \lim 2\pi c_0^k = 2\pi c_0,$$

so that  $\gamma_0 = c_0$ . Hence  $c_n = \gamma_n$  for all  $n$ .

Let us now continue  $F(x)$  outside  $(0, 2\pi)$  by the condition

$$F(x + 2\pi) - F(x) = F(2\pi) - F(0).$$

This, together with

$$F_k(x + 2\pi) - F_k(x) = 2\pi c_0^k, \quad F(x + 2\pi) - F(x) = 2\pi c_0, \quad c_0^k \rightarrow c_0,$$

implies that  $F_k(x) - B_k$  converges to  $F(x)$  at every point of continuity of  $F$  distinct from  $0 \pmod{2\pi}$ . This in turn implies convergence also at the points congruent to  $0 \pmod{2\pi}$ , if  $F$  is continuous there.

(ii) Let us write  $F_k$  instead of  $F_k - B_k$ , which does not change the Fourier-Stieltjes coefficients. Let  $C_n^k, C_n$  be the Fourier coefficients of  $F_k, F$  considered in  $(0, 2\pi)$ . Obviously,  $C_n^k \rightarrow C_n$ . If  $F$  is continuous at  $x$ , then

$$2\pi c_0^k = F_k(x + 2\pi) - F_k(x) = F(x + 2\pi) - F(x) = 2\pi c_0,$$

and so  $c_0^k \rightarrow c_0$ . For  $n \neq 0$ ,

$$(c_n^k - c_0^k) / in = C_n^k \rightarrow C_n = (c_n - c_0) / in.$$

which gives  $c_n^k \rightarrow c_n$ .

We can apply Theorem (4.22) to the problem of the distribution mod 1 of sequences

$$x_1, x_2, \dots, x_k, \dots \quad (4.24)$$

of real numbers. We wind the real axis around the circle  $\Gamma$  of length 1, and consider (4.24) as points on  $\Gamma$ , not distinguishing points congruent mod 1. Given any semi-open arc  $\alpha < x \leq \beta$  on  $\Gamma$ ,  $0 \leq \beta - \alpha \leq 1$ , we denote by  $r_k(x, \beta)$  the number of points among  $x_1, x_2, \dots, x_k$  which fall in that arc. We shall say that a function  $F(x)$  is a *distribution*

function of (4.24), if  $F(x)$  is non-decreasing over  $(-\infty, +\infty)$  and satisfies the condition  $F(x+1) - F(x) = 1$ , and if  $\nu_k(\alpha, \beta)/k \rightarrow F(\beta) - F(\alpha)$

for any arc  $(\alpha, \beta)$  whose end-points are points of continuity of  $F$ . If (4.24) has a distribution function, the latter is determined except for an arbitrary additive constant. If  $F(x) = x + C$ , we say that (4.24) is equidistributed mod 1, or simply *equidistributed*.

**(4.25) THEOREM.** *A necessary and sufficient condition for (4.24) to have a distribution function is that the limits*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \{e^{-2\pi i n x_1} + e^{-2\pi i n x_2} + \dots + e^{-2\pi i n x_k}\} = c_n \tag{4.26}$$

exist for  $n = 0, \pm 1, \pm 2, \dots$ . If these limits do exist, they are the Fourier-Stieltjes coefficients with respect to the interval  $(0, 1)$  of the distribution function of (4.24).

*Sufficiency.* Let  $F_k(x)$  be the non-decreasing function defined by the conditions  $F_k(x) = \nu_k(0, x)/k$  for  $0 < x \leq 1$  and  $F_k(x+1) - F_k(x) = 1$  for all  $x$ . In particular,  $F_k(0) = 0, F_k(1) = 1$ .  $F_k(x)$  is a step function having jumps at the points  $x_1, \dots, x_k$  and the point congruent to them mod 1; the expression under the limit sign in (4.26) is  $c_n^k = \int_0^1 e^{-2\pi i n x} dF_k$ . If  $c_n^k \rightarrow c_n$  for all  $n$ , (4.22) implies that the  $c_n$  are the Fourier-Stieltjes coefficients of a non-decreasing  $F$  satisfying  $F(x+1) - F(x) = 1$ . (Since  $c_0^k = 1$  for all  $k$ , we also have  $c_0 = 1$ .) Moreover, there are constants  $B_k$  such that  $F_k(x) - B_k \rightarrow F(x)$  at the points of continuity of  $F$ . It follows that for any arc  $(\alpha, \beta)$  at whose end-points  $F$  is continuous,

$$\nu_k(\alpha, \beta)/k = F_k(\beta) - F_k(\alpha) \rightarrow F(\beta) - F(\alpha).$$

*Necessity.* Suppose (4.24) has a distribution function  $F$ . Let  $x$  be any point of continuity of  $F$  and let  $F_k$  be the functions defined above. If  $x$  is any point of continuity of  $F$  situated in  $(\alpha, \alpha + 1)$ , the expression  $F_k(x) - F_k(\alpha) = \nu_k(\alpha, x)/k$  tends to a limit. Since  $F_k(x+1) - F_k(x) = 1$ , it must tend to a limit for every point of continuity of  $F$ . By (4.22), the Fourier-Stieltjes coefficients of  $F_k(x) - F_k(\alpha)$  must tend to limits as  $k \rightarrow \infty$ . This proves (4.26), since the ratio there is  $\int_0^1 e^{-2\pi i n x} dF_k(x)$ .

**(4.27) THEOREM.** *A necessary and sufficient condition for (4.24) to be equidistributed is that the limits (4.26) exist for  $n = \pm 1, \pm 2, \dots$  and are all equal to 0.*

This can be seen at once if we note that the Fourier-Stieltjes series of  $x + C$  consists of the constant term 1 only and that the limit  $c_0$  in (4.26) always exists and equals 1.

A corollary of (4.27) is that for any irrational  $x$  the sequence  $x, 2x, 3x, \dots, kx, \dots$  is equidistributed. For if  $x_s = sx$ , and if  $s \neq 0$ , the absolute value of the expression under the limit sign in (4.26) is

$$k^{-1} \left| \sum_{s=1}^k e^{-2\pi i s n x} \right| \leq 2k^{-1} |1 - e^{-2\pi i n x}|^{-1} = o(1).$$

Similarly we prove the fact (which will be used in Chapter VIII, § 4) that if  $x$  is irrational, the sequence  $x, 3x, 5x, 7x, \dots$  is equidistributed.

**(4.28) THEOREM.** *Let  $m_1, m_2, \dots$  be any sequence of distinct positive integers, and let  $\alpha_1, \alpha_2, \dots$  be any sequence of real numbers. Then for almost all  $x$  the sequence  $m_s(x - \alpha_s)$  is equidistributed.*

It is enough to prove that for almost all  $x$  we have

$$\frac{1}{k} \sum_{n=1}^k e^{2\pi i m_n(x-\alpha)} = o(1) \quad (k \rightarrow \infty; n = \pm 1, \pm 2, \dots).$$

and this will follow (cf. footnote on p. 78) if we show that all the series

$$\sum_{s=1}^{\infty} \frac{e^{2\pi i m_s(x-\alpha)}}{s} \quad (n = \pm 1, \pm 2, \dots)$$

converge almost everywhere. The latter, in turn, is a corollary of the following lemma

**(4.29) LEMMA.** *Let  $\phi_1(x), \phi_2(x), \dots$  be a system orthonormal and uniformly bounded in  $(a, b)$ . Then the series*

$$\sum_{s=1}^{\infty} \frac{\phi_s(x)}{s} \tag{4.30}$$

converges almost everywhere in  $(a, b)$ .

Let  $s_N$  be the partial sums of (4.30), and let  $f(x)$  be the function such that  $\mathfrak{M}_1\{f - s_N\} \rightarrow 0$  (§ 1). For  $N = k^2$  we have

$$\int_a^b |f - s_N|^2 = \sum_{N+1}^{\infty} \frac{1}{s^2} < \frac{1}{N} = \frac{1}{k^2}.$$

Thus the series  $\sum \int_a^b |f - s_N|^2 dx$  converges, which implies that  $s_N \rightarrow f$  almost everywhere (Chapter I, (11.5)). For general  $N$  we find a  $k$  such that  $k^2 \leq N < (k+1)^2$ . Then  $s_N$  is obtained by augmenting  $s_k$  by less than  $(k+1)^2 - k^2 = O(k)$  terms, each of which is  $O(1/k^2)$ . Thus the contribution of the additional terms is  $O(k)O(1/k^2) = o(1)$ , and  $s_N \rightarrow f$  almost everywhere.

**5. Classes of functions and (C, 1) means of Fourier series (cont.)**

Let  $\mathfrak{F}$  be a family of functions  $F(x), \alpha \leq x \leq \beta$ , having the following property: for every  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$|\sum [F(b_k) - F(a_k)]| < \epsilon \tag{5.1}$$

for every  $F \in \mathfrak{F}$  and every finite system  $S$  of non-overlapping subintervals  $(a_k, b_k)$  of  $(\alpha, \beta)$  satisfying  $\sum(b_k - a_k) < \delta$ . We shall then say that the functions  $F \in \mathfrak{F}$  are *uniformly absolutely continuous* in  $(\alpha, \beta)$ . Clearly, the limit of an everywhere convergent sequence of functions from  $\mathfrak{F}$  is absolutely continuous.

Let  $\phi(u)$  be non-negative and non-decreasing for  $u \geq 0$ , and such that  $\phi(u)/u \rightarrow \infty$  with  $u$ . Let  $\mathfrak{f}$  be a family of functions  $f(x)$  defined on  $(0, 2\pi)$  and such that

$$\int_0^{2\pi} \phi(|f(x)|) dx \leq C,$$

where  $C$  is independent of  $f$ . The integrals  $F$  of the  $f \in \mathfrak{f}$  are then uniformly absolutely continuous.

We have to show that the sums in (5.1), which are  $\int_S f dx$ , are uniformly small with  $|S|$ . Given any  $M > 0$ , let  $u_0$  be such that  $\phi(u)/u \geq M$  for  $u \geq u_0$ . We set  $|f| = f_1 + f_2$ , where  $f_1 = |f|$  if  $|f| \leq u_0$  and  $f_1 = 0$  otherwise. Thus the values of  $f_2$  are either 0 or else at least  $u_0$ . Now

$$\left| \int_S f dx \right| \leq \int_S f_1 dx + \int_S f_2 dx \leq u_0 |S| + M^{-1} \int_S \phi(f_2) dx \leq u_0 |S| + CM^{-1}.$$

The last sum is small if we first take  $M$  large, but fixed, and then  $|S|$  small.

(5.2) THEOREM. A necessary and sufficient condition for

$$\frac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_0^{\infty} A_n(x) \quad (5.3)$$

to belong to  $L$  is that the functions

$$F_n(x) = \int_0^x \sigma_n(t) dt \quad (n = 1, 2, \dots) \quad (5.4)$$

should be uniformly absolutely continuous in  $(0, 2\pi)$ .

If the  $F_n$  are uniformly absolutely continuous, then *a fortiori* they are of uniformly bounded variation,  $\mathfrak{M}[\sigma_n] = O(1)$ , and  $\sum A_n(x)$  is an  $S[dF]$ . Since  $F$  is the limit of an everywhere convergent subsequence of  $F_n$ ,  $F$  is absolutely continuous, and  $S[dF] = S[f]$ , where  $f = F'$ .

Conversely, suppose that  $\sum A_n(x)$  is an  $S[f]$ . Suppose for simplicity that  $a_0 = 0$ . The functions  $F_n$  in (5.4) are then, except for an additive constant depending on  $n$ , the (C, 1) means of the Fourier series of the integral  $F$  of  $f$ . Thus

$$\begin{aligned} & |\Sigma[F_n(b_k) - F_n(a_k)]| \\ &= \left| \frac{1}{\pi} \int_{-\pi}^{\pi} \Sigma[F(b_k + t) - F(a_k + t)] K_n(t) dt \right| \leq \max_t |\Sigma[F(b_k + t) - F(a_k + t)]|, \end{aligned}$$

which is small with  $S$ . This proves (5.2).

Obviously,  $\sum A_n(x)$  belongs to class A if and only if the  $\sigma_n$  are uniformly absolutely continuous.

(5.5) THEOREM. (i) A necessary and sufficient condition for  $\sum A_n(x)$  to belong to  $L$  is that  $\mathfrak{M}[\sigma_m - \sigma_n] \rightarrow 0$  as  $m, n \rightarrow \infty$ .

(ii) If  $\sum A_n(x)$  is  $S[f]$ , then  $\mathfrak{M}[\sigma_n - f] \rightarrow 0$ .

Suppose that  $\sum A_n(x)$  is  $S[f]$ . Integrating the inequality

$$|\sigma_n(x) - f(x)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)| K_n(t) dt \quad (5.6)$$

over  $0 \leq x \leq 2\pi$ , we find

$$\mathfrak{M}[\sigma_n - f] \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \eta(t) K_n(t) dt, \quad \text{where } \eta(t) = \int_0^{2\pi} |f(x+t) - f(x)| dx.$$

Since  $\eta(t)$  is continuous and vanishes at  $t = 0$  (Chapter I, (11.8)), and since the right-hand side of the last inequality is the (C, 1) mean of  $S[\eta]$  at  $t = 0$ , we find that  $\mathfrak{M}[\sigma_n - f] \rightarrow 0$ . This proves (ii) and also the necessity of the condition in (i), since

$$\mathfrak{M}[\sigma_m - \sigma_n] \leq \mathfrak{M}[\sigma_m - f] + \mathfrak{M}[\sigma_n - f] \rightarrow 0 \quad \text{as } m, n \rightarrow \infty.$$

Conversely, if  $\mathfrak{M}[\sigma_m - \sigma_n] \rightarrow 0$  there is an  $f \in L$  such that  $\mathfrak{M}[\sigma_n - f] \rightarrow 0$  (Chapter I, (11.1)). For  $n > |k|$ ,

$$2\pi \left(1 - \frac{|k|}{n+1}\right) c_k = \int_{-\pi}^{\pi} \sigma_n e^{-ikx} dx = \int_{-\pi}^{\pi} f e^{-ikx} dx + \int_{-\pi}^{\pi} (\sigma_n - f) e^{-ikx} dx.$$

Making  $n \rightarrow \infty$  and observing that the absolute value of the last term does not exceed  $\mathfrak{M}[\sigma_n - f]$ , we see that  $c_k$  is the  $k$ th coefficient of  $f$ . This proves (5.5).

(5.7) THEOREM. Let  $\phi(u), u \geq 0$ , be convex, non-negative, non-decreasing, and such that  $\phi(u)/u \rightarrow \infty$  with  $u$ . A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to  $L_\phi$  is that

$$\int_0^{2\pi} \phi(|\sigma_n(x)|) dx \leq C, \quad (5.8)$$

where  $C$  is finite and independent of  $n$ .

To prove the necessity of the condition, we consider

$$|\sigma_n(x)| \leq \frac{1}{\pi} \int_0^{2\pi} K_n(x-t) |f(t)| dt. \quad (5.9)$$

By Jensen's inequality, taking into account that the integral of the function  $p(t) = K_n(x-t)$  over  $(0, 2\pi)$  is  $\pi$ , we find

$$\phi(|\sigma_n(x)|) \leq \frac{1}{\pi} \int_0^{2\pi} K_n(x-t) \phi(|f(t)|) dt. \quad (5.10)$$

If we integrate this with respect to  $x$  and invert the order of integration on the right we get

$$\int_0^{2\pi} \phi(|\sigma_n|) dx \leq \int_0^{2\pi} \phi(|f|) dt. \quad (5.11)$$

which proves the necessity of the condition.

As regards the sufficiency, Jensen's inequality

$$\phi\left(\frac{1}{2\pi} \int_0^{2\pi} |\sigma_n| dx\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(|\sigma_n|) dx \leq \frac{C}{2\pi}$$

implies that  $\mathfrak{M}[\sigma_n] = O(1)$ , so that  $\Sigma A_n(x)$  is an  $S[dF]$ . Moreover, the functions (5.4) are uniformly absolutely continuous. Hence  $F$  is absolutely continuous, and  $\Sigma A_n(x)$  is  $S[f]$ ,  $f = F'$ . Since  $\sigma_n(x) \rightarrow f(x)$  almost everywhere, (5.8) implies that  $\mathfrak{M}[\phi(|f|)] \leq C$ , that is,  $f \in L_\phi$ .

In particular, a necessary and sufficient condition for (5.3) to belong to  $L^r$ ,  $r > 1$ , is  $\mathfrak{M}_r[\sigma_n] = O(1)$ . As (4.3) shows, the result fails for  $r = 1$ .

(5.12) THEOREM. Suppose that  $\phi(u), u \geq 0$ , is convex, non-negative and non-decreasing. If  $f \in L_\phi$ , then

$$\int_0^{2\pi} \phi(|\sigma_n|) dx \rightarrow \int_0^{2\pi} \phi(|f|) dx.$$

In particular, if  $f \in L^r$ ,  $r \geq 1$ , then  $\mathfrak{M}_r[\sigma_n] \rightarrow \mathfrak{M}_r[f]$ .

After (5.11), it is enough to show that  $\liminf \int_0^{2\pi} \phi(|\sigma_n|) dx \geq \int_0^{2\pi} \phi(|f|) dx$ . Let  $E$  be any set of points at which the  $\sigma_n$  are uniformly bounded. Since  $\sigma_n \rightarrow f$  almost everywhere, we have  $\int_E \phi(|\sigma_n|) dx \rightarrow \int_E \phi(|f|) dx$  and hence

$$\liminf \int_0^{2\pi} \phi(|\sigma_n|) dx \geq \int_E \phi(|f|) dx.$$

The right-hand side here can be made arbitrarily close to  $\int_0^{2\pi} \phi(|f|) dx$ , since  $|E|$  can be made arbitrarily close to  $2\pi$ . This completes the proof of (5.12).

Suppose that a convex and non-negative function  $\phi(u)$ ,  $u \geq 0$ , has  $\phi(0) = 0$  and is non-decreasing. Supposing that  $\Sigma A_n(x)$  belongs to  $L_\phi$ , we may ask whether

$$\int_0^{2\pi} \phi(|\sigma_n - f|) dx \rightarrow 0. \quad (5.13)$$

Applying Jensen's inequality to (5.6) we see that (5.13) holds provided the function  $\eta(t) = \int_{-\pi}^{\pi} \phi(|f(x+t) - f(x)|) d\tau$  is integrable and tends to 0 with  $t$ . This is not always true if  $\phi(u)$  increases too rapidly with  $u$ , but we can save the situation by adding a harmless factor to the argument of  $\phi$  in (5.13): If  $f \in L_\phi$ , then

$$\eta^*(t) = \int_{-\pi}^{\pi} \phi\left\{\frac{1}{2} |f(x+t) - f(x)|\right\} dx$$

is integrable and tends to 0 with  $t$ .

In fact, let  $f = g + h$ , where  $g$  is bounded and  $\int_{-\pi}^{\pi} \phi(|h|) dx < \epsilon$ . By Jensen's inequality,  $\int_0^{2\pi} \phi\left\{\frac{1}{2} |f(x+t) - f(x)|\right\} dx \leq \frac{1}{2} \int_0^{2\pi} \phi\left\{\frac{1}{2} |g(x+t) - g(x)|\right\} dx + \frac{1}{2} \int_0^{2\pi} \phi\left\{\frac{1}{2} |h(x+t) - h(x)|\right\} dx$ , where the last term does not exceed  $\frac{1}{4} \int_0^{2\pi} \phi(|h(x+t)|) dx + \frac{1}{4} \int_0^{2\pi} \phi(|h(x)|) dx < \frac{1}{2}\epsilon$ , and the preceding term is bounded and tends to 0 with  $t$ . (Our hypothesis about  $\phi$  implies that in every interval  $0 \leq u \leq a$  we have  $\phi(u) \leq Mu$ , where  $M = \phi(a)/a$ .) Hence the total is less than  $\epsilon$  for  $|t|$  small, which proves the assertion. We thus obtain the following:

(5.14) THEOREM. Suppose that  $\phi(u)$ ,  $u \geq 0$ , is convex, non-negative, non-decreasing and that  $\phi(0) = 0$ . If  $\Sigma A_n(x)$  is an  $S[f]$  with  $f \in L_\phi$ , then

$$\int_0^{2\pi} \phi\left\{\frac{1}{2} |f - \sigma_n|\right\} dx \rightarrow 0.$$

In particular, if  $f \in \mathcal{I}$ ,  $r \geq 1$ , then  $\mathfrak{M}_r[f - \sigma_n] \rightarrow 0$ .

(5.15) THEOREM. Suppose that  $\Sigma A_n(x)$  is an  $S[dF]$  with  $F(x) = \frac{1}{2}[F(x+0) + F(x-0)]$  for all  $x$ .

(i) Either of the following two conditions is both necessary and sufficient for  $F$  to be absolutely continuous over a closed interval  $(\alpha, \beta)$ :

(a) the functions  $F_n(x) = \int_\alpha^x \sigma_n dt$  are uniformly absolutely continuous over  $(\alpha, \beta)$ ;

(b)  $\mathfrak{M}[\sigma_n - \sigma_n; \alpha, \beta] \rightarrow 0$ .

(ii) If the functions  $\sigma_n(t)$  in (a) and (b) are replaced by  $\sigma_n^+(t)$ , we obtain necessary and sufficient conditions ((a'), (b')), say, for the positive variation of  $F$  to be absolutely continuous in  $(\alpha, \beta)$ .

It is easy to see that (b) implies (a), so that for (i) it is enough to prove the sufficiency of (a) and the necessity of (b). The former is immediate, since then the function  $F$  in the proof of (4.3) is absolutely continuous in  $(\alpha, \beta)$ .

Suppose then, that  $\Sigma A_n(x)$  is an  $S[dF]$  and that  $F$  is absolutely continuous in  $\alpha \leq x \leq \beta$ . If we show that  $\mathfrak{M}[\sigma_n - \sigma_n; \alpha', \beta'] \rightarrow 0$  for any interval  $(\alpha', \beta')$  interior to  $(\alpha, \beta)$ , the necessity of (b) will follow. For, by (4.12),  $\mathfrak{M}[\sigma_n; \alpha, \alpha']$  and  $\mathfrak{M}[\sigma_n; \beta', \beta]$  tend to the

total variations of  $F$  over  $(\alpha, \alpha')$  and  $(\beta', \beta)$ , and so are small with  $\alpha' - \alpha, \beta - \beta'$ . The same follows for the integrals of  $|\sigma_m - \sigma_n|$  over  $(\alpha, \alpha')$  and  $(\beta', \beta)$ .

Let  $f(x) = F'(x)$ . To show that  $\mathfrak{M}[\sigma_m - \sigma_n; \alpha', \beta'] \rightarrow 0$ , it is enough to prove that  $\mathfrak{M}[\sigma_m - f; \alpha', \beta'] \rightarrow 0$ . We observe that

$$\sigma_m(x) = \frac{1}{\pi} \int_0^{2\pi} K_m(x-t) dF(t) = \frac{1}{\pi} \int_{\alpha}^{\beta} + \frac{1}{\pi} \int_{\beta}^{\alpha+2\pi} = v_m + w_m, \quad (5.16)$$

say. For  $x \in (\alpha', \beta')$  and  $t \in (\beta, x + 2\pi)$  the integrand of  $w_m$  tends uniformly to 0, so that  $\mathfrak{M}[w_m; \alpha', \beta'] \rightarrow 0$ . Since  $v_m = \sigma_m(x; f^*)$ , where  $f^* = f$  in  $(\alpha, \beta)$ ,  $f^* = 0$  elsewhere, we have  $\mathfrak{M}[v_m - f^*; 0, 2\pi] \rightarrow 0$ , and so also  $\mathfrak{M}[v_m - f; \alpha', \beta'] \rightarrow 0$ . Hence

$$\mathfrak{M}[\sigma_m - f; \alpha', \beta'] \leq \mathfrak{M}[v_m - f; \alpha', \beta'] + \mathfrak{M}[w_m; \alpha', \beta'] \rightarrow 0,$$

and (i) is proved.

Analogously, for (ii) we must show the sufficiency of (a') and necessity of (b'). If the functions  $F_n^*(x) = \int_{\alpha}^x \sigma_n^+ dt$  are uniformly absolutely continuous in  $(x, \beta)$ , their limit, which represents the positive variation of  $F$  over  $(\alpha, x)$  (cf. (4.12)), is absolutely continuous there.

To prove the necessity of condition (b'), we begin with the case  $(\alpha, \beta) = (0, 2\pi)$ . Let  $V(x)$ ,  $P(x)$  and  $N(x)$  be the total, positive and negative variations of  $F$  over  $(0, x)$ . Then

$$\sigma_n = \sigma_n' - \sigma_n'', \quad \text{where } \sigma_n' = \sigma_n[dP] \geq 0, \quad \sigma_n'' = \sigma_n[dN] \geq 0. \quad (5.17)$$

The relations  $P' + N' = V' = |F'|$ ,  $P' - N' = F'$  (known to be true almost everywhere) show that  $P' = F'^+$ ,  $N' = F'^-$  almost everywhere.

The inequalities  $\sigma_n \leq \sigma_n'$ ,  $\sigma_n'' \geq 0$  show that  $0 \leq \sigma_n^+ \leq \sigma_n'$ . If we define  $\theta_n(x)$  by

$$\sigma_n^+(x) = \theta_n(x) \sigma_n'(x)$$

at the points where  $\sigma_n' \neq 0$ , and  $\theta_n(x) = 1$  elsewhere, then  $0 \leq \theta_n(x) \leq 1$  for all  $x$  and  $n$ . We observe that, almost everywhere,  $\sigma_n \rightarrow F'$  (Chapter III, (8.1)), and so also  $\sigma_n^+ \rightarrow F'^+$ . The same fact applied to  $\sigma_n[dP]$  gives  $\sigma_n' \rightarrow P' = F'^+$ . Hence  $\theta_n(x)$  tends to 1 at almost all points where  $p(x) = P'(x) \neq 0$ .

Using now (for the first time) the hypothesis that  $P(x)$  is absolutely continuous, we show that  $\mathfrak{M}[\sigma_n^+ - p; 0, 2\pi] \rightarrow 0$ . In fact,

$$\int_0^{2\pi} |\sigma_n^+ - p| dx = \int_0^{2\pi} |\sigma_n' \theta_n - p| dx \leq \int_0^{2\pi} |\sigma_n' - p| \theta_n dx + \int_0^{2\pi} |\theta_n - 1| p dx.$$

The first integral on the right is majorized by  $\mathfrak{M}[\sigma_n' - p] \rightarrow 0$ . The last integral on the right also tends to 0, since the integrand  $|\theta_n(x) - 1| p(x)$  is majorized by  $p(x) \in L$ , and tends to 0 almost everywhere. Thus

$$\mathfrak{M}[\sigma_n^+ - p] \rightarrow 0, \quad \mathfrak{M}[\sigma_m^+ - \sigma_n^+] \leq \mathfrak{M}[\sigma_m^+ - p] + \mathfrak{M}[\sigma_n^+ - p] \rightarrow 0,$$

and the necessity of condition (b') is proved when  $(\alpha, \beta) = (0, 2\pi)$ .

To remove this restriction, we proceed as in case (b). It is enough to show that  $\mathfrak{M}[\sigma_m^+ - p; \alpha', \beta'] \rightarrow 0$  for any  $(\alpha', \beta')$  interior to  $(\alpha, \beta)$ . Assume for simplicity that  $(\alpha, \beta)$  is included in  $(0, 2\pi)$ , and return to (5.16). The  $v_m$  there is  $\sigma_m(x; dF^*)$ , where  $F^*$  equals  $F(x)$  in  $(\alpha, \beta)$ ,  $F(\alpha)$  in  $(0, \alpha)$  and  $F(\beta)$  in  $(\beta, 2\pi)$ . The positive variation  $P^*$  of  $F^*$

is absolutely continuous, so that, if  $p^* = P^*$ ,  $\mathfrak{M}[v_m^+ - p; \alpha', \beta'] \leq \mathfrak{M}[v_m^+ - p^*; 0, 2\pi] \rightarrow 0$ . Since  $v_m$  tends uniformly to 0 over  $(\alpha', \beta')$  we get  $\mathfrak{M}[\sigma_m^+ - p; \alpha', \beta'] \rightarrow 0$ .

Condition (b) is satisfied if there is a non-negative non-decreasing convex function  $\phi(u)$ ,  $u \geq 0$ , such that  $\phi(u)/u \rightarrow \infty$  with  $u$ , and if  $\mathfrak{M}[\phi(|\sigma_n|); \alpha, \beta] = O(1)$ . Similarly condition (b') is satisfied if  $\mathfrak{M}[\phi(\sigma_n^+); \alpha, \beta] = O(1)$ .

Many results of this and the preceding section hold, though some inequalities become less precise, for the kernel  $(C, \alpha)$ ,  $0 < \alpha < 1$ . Let

$$\lambda_n = \lambda_n^\alpha = \frac{1}{\pi} \int_0^{2\pi} |K_n^\alpha(t)| dt, \quad \lambda = \sup_n \lambda_n \quad (\alpha > 0).$$

The proofs of the following results for  $0 < \alpha < 1$  are essentially the same as for  $\alpha = 1$ .

(5·18) THEOREM. Let  $\phi$  be the same as in (5·7). If  $\mathfrak{M}[\phi(|\sigma_n^\alpha|)] = O(1)$ , then (5·3) belongs to  $L_\phi$ . If (5·3) is an  $S[f]$ ,  $f \in L_\phi$ , then  $\mathfrak{M}[\phi(|\sigma_n^\alpha|/\lambda)] = O(1)$ . If, in addition,  $\phi(0) = 0$ , then  $\mathfrak{M}[\phi(|f - \sigma_n^\alpha|/\lambda)] \rightarrow 0$  as  $n \rightarrow \infty$ .

(5·19) THEOREM. A necessary and sufficient condition for (5·3) to belong to  $S$  is  $\mathfrak{M}[\sigma_n^\alpha] = O(1)$ . A necessary and sufficient condition for (5·3) to belong to  $L$  is  $\mathfrak{M}[\sigma_m^\alpha - \sigma_n^\alpha] \rightarrow 0$  as  $m, n \rightarrow \infty$ .

If we replace the  $\sigma_n$  by the partial sums  $s_n$  in the theorems of this and the preceding section, the conditions we obtain remain sufficient, though no longer necessary. The proofs of sufficiency remain the same, except at one point; we cannot use the fact that  $s_n(x; f) \rightarrow f(x)$  almost everywhere, for this is false (see Chapter VIII, § 3). But for this purpose it is enough to know that there is a subsequence  $\{s_{n_k}(x; f)\}$  converging to  $f(x)$  almost everywhere, and we shall see in Chapter VII, § 6, that this is true.

Another observation on the sufficiency conditions in the theorems of this and the preceding section is also useful. In showing that a certain behaviour of the  $\sigma_n$  (or  $s_n$ ) implies that the series belongs to a definite class, it is not really necessary to consider all positive integers  $n$ ; it is enough to suppose that the condition is satisfied for some sequence  $\{n_k\}$  tending to  $+\infty$ . Thus, if  $\{\sigma_{n_k}(x)\}$  or  $\{s_{n_k}(x)\}$  converges uniformly, the series belongs to class  $C$  (see (4·2)); if  $\mathfrak{M}[s_{n_k}] = O(1)$ , it is an  $S[dF]$  (see (4·3)); if the  $s_{n_k}(x)$  are non-negative, the series is an  $S[dF]$  with  $F$  non-decreasing (see (4·8)), etc.

This makes it possible to state in a slightly different form some of the theorems proved above. For example, a necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to class  $C$  is that the  $\sigma_n(x)$  are uniformly continuous. The necessity follows from the inequality (5·9), which, applied to  $f(x+h) - f(x)$ , yields

$$\omega(\delta; \sigma_n) \leq \omega(\delta; f).$$

Conversely, if the  $\sigma_n(x)$  are uniformly continuous there is, by Arzelà's well-known theorem, a subsequence  $\{\sigma_{n_k}(x)\}$  converging uniformly to a continuous function  $f(x)$ , and so  $\Sigma A_n(x)$  is an  $S[f]$ ,  $f \in C$ .

(5·20) THEOREM. If  $\mathfrak{M}[s_{n_k}] = O(1)$  for a sequence of partial sums of  $\Sigma A_n(x)$  (in particular, if the  $s_{n_k}$  are non-negative), the series is an  $S[dF]$  with  $F$  continuous.

We know already that  $\Sigma A_n(x)$  is an  $S[dF]$ , and so need only prove the continuity of  $F$ . Suppose that  $F(x_0 + 0) - F(x_0 - 0) = d \neq 0$  for some  $x_0$ , and suppose for simplicity

that  $x_0 = 0$  and that  $2F(0) = F(+0) + F(-0)$ . Let  $\phi(x) \sim \Sigma \nu^{-1} \sin \nu x$  (see Chapter I, (4.12)). We may write

$$F(x) = \{F(x) - (d/\pi) \phi(x)\} + (d/\pi) \phi(x) = F_1(x) + F_2(x),$$

say, where  $F_1$  is continuous at  $x = 0$ . (Correspondingly

$$S[dF] = S[dF_1] + S[dF_2], \quad s_n = s_n^1 + s_n^2.$$

Since  $\phi(0) = \phi(2\pi)$ , we have  $S[dF_2] = S'[F_2]$  and the  $n$ th partial sum of  $S[dF_2]$  is  $(d/\pi)[D_n(x) - \frac{1}{2}]$ . Thus whatever the value of  $\epsilon > 0$ ,  $\mathfrak{R}[s_n^2; -\epsilon, \epsilon] \simeq C \log n$ , where  $C$  is a positive constant (Chapter II, (12.2)). If we can show that for  $\epsilon$  small enough and  $n > n_0$  we have  $\mathfrak{R}[s_n^1; -\epsilon, \epsilon] < \frac{1}{2} C \log n$ , it will follow that  $\mathfrak{R}[s_n; -\epsilon, \epsilon]$ , and so also  $\mathfrak{R}[s_n]$ , tends to  $\infty$ , contrary to hypothesis.

Let  $I = (-\epsilon, \epsilon)$ ,  $I' = (-2\epsilon, 2\epsilon)$ . If  $x \in I$ , then

$$|s_n^1(x)| = \left| \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(x-t) dF_1(t) \right| \leq \frac{1}{\pi} \int_{I'} |D_n(x-t)| |dF_1(t)| + O(1),$$

$D_n(u)$  being uniformly bounded for  $\epsilon \leq |u| \leq \pi$ . Integrating this over  $I$  and writing  $L_n$  for Lebesgue's constant, we have

$$\int_I |s_n^1(x)| dx \leq \frac{1}{\pi} \int_{I'} |dF(t)| \int_I |D_n(x-t)| dx \leq L_n \int_{I'} |dF_1(t)|.$$

Since  $L_n = O(\log n)$ , and the variation of  $F_1$  over  $I$  is small with  $\epsilon$ , owing to the continuity of  $F_1$  at 0, we have  $\mathfrak{R}[s_n^1; -\epsilon, \epsilon] < \frac{1}{2} C \log n$  for  $\epsilon$  small enough and  $n > n_0$ . This proves Theorem (5.20).

**6. Classes of functions and Abel means of Fourier series**

Let 
$$f(\rho, x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \rho^n \quad (0 \leq \rho < 1) \tag{6.1}$$

be the harmonic function associated with the series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x). \tag{6.2}$$

The analogues for Abel means of the results obtained in the preceding two sections may be stated as follows (As was explained in § 4, we omit the proofs.)

**(6.3) THEOREM.** *A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to class C or, what is the same thing, for*

$$f(\rho, x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(x-t) + \rho^2} f(t) dt \quad (0 \leq \rho < 1) \tag{6.4}$$

with  $f(t)$  continuous, is that  $f(\rho, x)$  should converge uniformly as  $\rho \rightarrow 1$ . A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to class B is that  $f(\rho, x)$  should be bounded for  $0 \leq \rho < 1$ .

**(6.5) THEOREM.** *A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to class S, or, what is the same thing, for*

$$f(\rho, x) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(x-t) + \rho^2} dF(t) \quad (0 \leq \rho < 1). \tag{6.6}$$

where  $F(t)$  is of bounded variation, is that the integral

$$\int_0^{2\pi} |f(\rho, x)| dx = \mathfrak{M}[f(\rho, x)] \quad (6.7)$$

should be bounded as  $\rho \rightarrow 1$ . The latter condition is equivalent to  $\mathfrak{M}[f^+(\rho, x)] = O(1)$ . If we have (6.6), then

$$\int_0^{2\pi} |f(\rho, x)| dx \leq \int_0^{2\pi} |dF(x)|.$$

(6.8) THEOREM. A necessary and sufficient condition for  $f(\rho, x)$  to be representable by (6.6), with  $F(t)$  non-decreasing, is that  $f(\rho, x) \geq 0$  for  $0 \leq \rho < 1$ .

(6.9) THEOREM. If  $f(\rho, x)$  is given by (6.6), and if

$$F(x) = \frac{1}{2}\{F(x+0) + F(x-0)\}, \quad (6.10)$$

then 
$$\int_{\alpha}^{\beta} |f(\rho, x)| dx \rightarrow V, \quad \int_{\alpha}^{\beta} f^+(\rho, x) dx \rightarrow P, \quad \int_{\alpha}^{\beta} f^-(\rho, x) dx \rightarrow N,$$

where  $V, P, N$  are the total, positive and negative variations of  $F$  over  $(\alpha, \beta)$ .

This result leads to the following:

(6.11) THEOREM. Let  $F(\rho, x)$  be the Poisson integral of a periodic  $F$  of bounded variation satisfying (6.10). Then the total (positive, negative) variation of  $F(\rho, x)$  over an arc  $\alpha \leq x \leq \beta$  tends to the total (positive, negative) variation of  $F(x)$  over  $\alpha \leq x \leq \beta$  as  $\rho \rightarrow 1$ .

(6.12) THEOREM. Each of the following conditions is both necessary and sufficient for  $\Sigma A_n(x)$  to belong to  $L$  (that is, for (6.4) to hold with an  $f \in L$ ):

(i)  $\int_0^x f(\rho, u) du$  is a uniformly absolutely continuous function of  $x$  for  $0 \leq \rho < 1$ ;

(ii)  $\int_0^{2\pi} |f(\rho, x) - f(\rho', x)| dx \rightarrow 0$  as  $\rho, \rho' \rightarrow 1$ .

(6.13) THEOREM. Let  $\phi(u)$  be non-negative, convex, and non-decreasing for  $u \geq 0$ , and let  $\phi(u)/u \rightarrow \infty$  with  $u$ . A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to  $L_{\phi}$  is

$$\int_0^{2\pi} \phi(|f(\rho, x)|) dx = O(1) \quad (0 \leq \rho < 1). \quad (6.14)$$

(6.15) THEOREM. If  $\Sigma A_n(x)$  is an  $S[f]$ ,  $f \in L_{\phi}$ , where  $\phi(u)$  is convex non-negative and non-decreasing for  $u \geq 0$ , then

$$\int_0^{2\pi} \phi(|f(\rho, x)|) dx \rightarrow \int_0^{2\pi} \phi(|f|) dx \quad (\rho \rightarrow 1). \quad (6.16)$$

If in addition  $\phi(0) = 0$ , then

$$\int_0^{2\pi} \phi\left(\frac{1}{2}|f(\rho, x) - f(x)|\right) dx \rightarrow 0 \quad (\rho \rightarrow 1).$$

(6.17) THEOREM. A necessary and sufficient condition for  $\Sigma A_n(x)$  to belong to  $L^r$ ,  $r > 1$ , is

$$\int_0^{2\pi} |f(\rho, x)|^r dx = O(1) \quad (\rho \rightarrow 1).$$

If  $\Sigma A_n(x)$  is an  $S[f]$  with  $f \in L^r$ ,  $r \geq 1$ , then

$$\int_0^{2\pi} |f(\rho, x) - f(x)|^r dx \rightarrow 0 \quad (\rho \rightarrow 1).$$

(6.18) THEOREM. If  $\Sigma A_n(x)$  is an  $S[dF]$  with  $F$  satisfying (6.10), each of the conditions (i), (ii) below is both necessary and sufficient for  $F$  to be absolutely continuous in  $(\alpha, \beta)$ :

(i) The functions  $\int_\alpha^x f(\rho, u) du$  are uniformly absolutely continuous in  $(\alpha, \beta)$ :

(ii)  $\mathfrak{M}[f(\rho, x) - f(\rho', x); \alpha, \beta] \rightarrow 0$  as  $\rho, \rho' \rightarrow 1$ .

If  $f(\rho, x)$  is replaced by  $f^+(\rho, x)$ , we obtain necessary and sufficient conditions for the absolute continuity in  $(\alpha, \beta)$  of the positive variation of  $F$ .

(6.19) THEOREM. Let  $\Sigma A_n(x)$  be an  $S[dF]$ , let  $F$  satisfy (6.10), and let

$$f(x) = \lim f(\rho, x) = F'(x).$$

Of the two conditions

$$\int_\alpha^\beta |f(\rho, x)| dx \rightarrow \int_\alpha^\beta |f(x)| dx, \quad \int_\alpha^\beta f^+(\rho, x) dx \rightarrow \int_\alpha^\beta f^+(x) dx,$$

the first is necessary and sufficient for  $F$  to be absolutely continuous in  $(\alpha, \beta)$ , the second for the positive variation of  $F$  to be absolutely continuous there.

The analogue of (5.11) for Abel means is

$$\int_0^{2\pi} \phi(|f(\rho, x)|) dx \leq \int_0^{2\pi} \phi(|f(x)|) dx. \quad (6.20)$$

Let  $0 \leq \rho < \rho' < 1$ , so that  $\rho = \rho'R$ , with  $0 < R < 1$ . From (6.1) we see that  $f(\rho, x)$  is the Poisson integral of  $f(\rho', x)$ , and (6.20) implies that

$$\int_0^{2\pi} \phi(|f(\rho, x)|) dx \leq \int_0^{2\pi} \phi(|f(\rho', x)|) dx \quad (0 \leq \rho < \rho' < 1). \quad (6.21)$$

Thus

(6.22) THEOREM. If  $\phi(u)$  is non-negative, non-decreasing, and convex for  $u \geq 0$ , and  $f(\rho, x)$  is harmonic for  $\rho < 1$ , the integral  $\int_0^{2\pi} \phi(|f(\rho, x)|) dx$  is a non-decreasing function of  $\rho$ .

The case  $\phi(u) = u^r$ ,  $r \geq 1$ , is particularly important.

If  $f(\rho, x)$  is given by (6.6), then writing  $F = F_1 - F_2$ , where  $F_1, F_2$  are non-decreasing, we represent  $f(\rho, x)$  as a difference of two non-negative harmonic functions. If  $f(\rho, x)$  is non-negative, the integral (6.7) is bounded (being in fact  $\pi a_0$ ). The same holds if  $f(\rho, x)$  is a difference of two non-negative harmonic functions. Thus

(6.23) THEOREM. A necessary and sufficient condition for a harmonic function  $f(\rho, x)$ ,  $0 \leq \rho < 1$ , to be representable in the form (6.6), with  $F$  of bounded variation, is that  $f(\rho, x)$  should be a difference of two non-negative harmonic functions.

Let  $z = \rho e^{ix}$ . The Poisson kernel  $P(\rho, x)$  is the real part of

$$\frac{1}{2} + z + z^2 + \dots = \frac{1}{2}(1+z)/(1-z).$$

Thus, the harmonic function (6.6) is the real part of the function

$$\Phi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{iu} + z}{e^{it} - z} dF(t) \quad (z = \rho e^{ix}), \quad (6.24)$$

regular in  $|z| < 1$ . The imaginary part of  $\Phi(z)$  is

$$f(\rho, x) = \sum_{\nu=1}^{\infty} (a_{\nu} \cos \nu x - b_{\nu} \sin \nu x) \rho^{\nu} = \frac{1}{\pi} \int_0^{2\pi} \frac{\rho \sin(x-t)}{1 - 2\rho \cos(x-t) + \rho^2} dF(t), \quad (6.25)$$

the harmonic function conjugate to  $f(\rho, x)$  and vanishing at the origin. Hence

(6.26) **THEOREM.** A function  $\Phi(z)$ , with  $\Re\Phi(0) = 0$ , regular for  $|z| < 1$ , has a non-negative real part there if and only if  $\Phi(z)$  is given by the formula (6.24) with  $F(t)$  non-decreasing and bounded.

The boundedness of the integral (6.7) does not imply the boundedness of the integral with  $f(\rho, x)$ , as we see by the example

$$f(\rho, x) = P(\rho, x), \quad \bar{f}(\rho, x) = Q(\rho, x).$$

(That  $\Re\{Q(\rho, x)\} \in O(1)$  may be verified either directly, or by observing that  $\sin x + \sin 2x + \dots$  is not an  $S[dF]$ .) However:

(6.27) **THEOREM.** Suppose that the integral (6.7) does not exceed  $C$  for  $0 \leq \rho < 1$ . Then the integral of  $\rho^{-1} |f(\rho, x)|$  (and a fortiori that of  $|f(\rho, x)|$ ) along any diameter of the unit circle does not exceed  $\frac{1}{2}C$ .

The result is quite elementary, and in order not to use the representation (6.6), whose proof is rather deep, let us suppose first that  $f(\rho, x)$  is continuous for  $\rho \leq 1$ . Then  $f(\rho, x)$  is the Poisson integral of  $f(x) = f(1, x)$ , and

$$\begin{aligned} f(\rho, x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t+x) Q(\rho, t) dt. \\ \int_0^1 \rho^{-1} (|f(\rho, x)| + |f(\rho, x+\pi)|) d\rho &\leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f(t+x)| \\ &\quad \times \left\{ \int_0^1 \rho^{-1} (|Q(\rho, t)| + |Q(\rho, t+\pi)|) d\rho \right\} dt. \end{aligned}$$

In estimating the term in curly brackets we may suppose that  $0 < t < \pi$ . Then  $Q(\rho, t) > 0$ ,  $Q(\rho, t+\pi) < 0$ , the term in question is

$$\begin{aligned} \lim_{R \rightarrow 1} \int_0^R \rho^{-1} [Q(\rho, t) - Q(\rho, t+\pi)] d\rho &= \lim_{R \rightarrow 1} \int_0^R \left[ \sum_1^{\infty} \rho^{\nu-1} \sin \nu t - \sum_1^{\infty} (-1)^{\nu} \rho^{\nu-1} \sin \nu t \right] d\rho \\ &= \lim_{R \rightarrow 1} 2 \sum_{\nu=1}^{\infty} R^{2\nu-1} \frac{\sin(2\nu-1)t}{2\nu-1} = \frac{1}{2}\pi \end{aligned}$$

(see Chapter I, (4.13)), and the whole expression on the right is

$$\frac{1}{\pi} \frac{\pi}{2} \int_{-\pi}^{\pi} |f(t+x)| dt = \frac{1}{2} \int_{-\pi}^{\pi} |f(t)| dt \leq \frac{1}{2}C.$$

In the general case we fix  $R$ ,  $0 < R < 1$ , and apply the result obtained to the function

$$f_1(\rho, x) = f(\rho R, x)$$

harmonic and continuous for  $\rho \leq 1$ . Since  $\mathfrak{M}[f_1(\rho, x); 0, 2\pi] \leq C$ , the integral

$$\int_0^1 \rho^{-1} (|f_1(\rho, x)| + |f_1(\rho, x + \pi)|) d\rho = \int_0^R \rho^{-1} (|f(\rho, x)| + |f(\rho, x + \pi)|) d\rho$$

does not exceed  $\frac{1}{2}C$ . The proof is completed by letting  $R$  tend to 1.

Let  $U(\rho, x)$  be any function harmonic for  $\rho < 1$  and let  $V(\rho, x)$  be the conjugate function. The harmonic function  $v(\rho, x) = V_x(\rho, x)$  vanishes at the origin (observe that  $V$  is of the form  $\sum A_n(x)\rho^n$ ) and is the conjugate of  $u(\rho, x) = U_x(\rho, x)$ . Suppose  $U$  satisfies

$$\mathfrak{M}[U_x(\rho, x); 0, 2\pi] \leq C \quad \text{for } 0 \leq \rho < 1.$$

Then, by (6.27) and the Cauchy-Riemann equations,

$$\int_D |\rho^{-1}v| d\rho = \int_D |\rho^{-1}V_x| d\rho = \int_D |U_\rho| d\rho \leq \frac{1}{2}C,$$

the integration being along any diameter  $D$  of the unit circle. The last integral is the total variation of  $U$  over  $D$ . Thus (6.27) may be re-stated as follows:

**(6.28) THEOREM.** Let  $U(\rho, x)$  be harmonic for  $\rho < 1$ . If the total variation of  $U$  over any circle  $\rho = \rho_0 < 1$  does not exceed  $C$ , the total variation of  $U$  over any diameter of the unit circle does not exceed  $\frac{1}{2}C$ .

Consider the Poisson integral  $f(\rho, x)$  of an  $f$  in  $L^p$ ,  $p \geq 1$  (Cf. Chapter III, (6.4)), and suppose that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(\rho, x)|^p dx \leq M^p \quad (0 \leq \rho < 1). \quad (6.29)$$

We shall deduce from this an estimate for  $\mathfrak{A}_r[f(\rho, x)]$  for  $r > p$ . Let us apply to (6.4) Theorem (1.15) of Chapter II. If  $q$  is defined by  $1/r = 1/p + 1/q - 1$  (so that  $q > 1$ ), then

$$\mathfrak{A}_r[f(\rho, x)] \leq \mathfrak{A}_p[f] \mathfrak{A}_q[2P(\rho, t)]. \quad (6.30)$$

In order to estimate  $\mathfrak{A}_q[P(\rho, t)]$  we use the inequalities (6.9) of Chapter III, where we may suppose that  $A > 1$ , and find

$$\begin{aligned} \mathfrak{A}_q^q[P(\rho, t)] &\leq \frac{2}{2\pi} \int_0^\delta \delta^{-q} dt + \frac{2}{2\pi} A^q \delta^q \int_\delta^\infty t^{-2q} dt \leq A^q \delta^{1-q}. \\ \mathfrak{A}_q[P(\rho, t)] &\leq A \delta^{-1/q}. \end{aligned} \quad (6.31)$$

Hence, observing that  $1/q' = 1/p - 1/r$ , we get

**(6.32) THEOREM.** If (6.29) holds for some  $p \geq 1$ , then

$$\mathfrak{A}_r[f(\rho, x)] \leq BM(1 - \rho)^{1/r - 1/p}$$

for  $r > p$ ,  $B$  denoting an absolute constant.

Subtracting from  $f$  a suitable polynomial, we may make  $M$  as small as we please. Thus,

**(6.33) THEOREM.** If  $\sum A_n(x)$  is in  $L^p$ ,  $p \geq 1$ , then

$$\mathfrak{A}_r[f(\rho, x)] = o\{(1 - \rho)^{1/r - 1/p}\}, \text{ as } \rho \rightarrow 1.$$

The following result generalizes (6.32):

(6.34) THEOREM. If  $\mathfrak{A}_p[f(\rho, x)] \leq M(1-\rho)^{-\beta}$  for some  $p \geq 1$ ,  $\beta > 0$ , then

$$\mathfrak{A}_r[f(\rho, x)] \leq MB_\beta(1-\rho)^{-\beta+1/r-1/p} \quad \text{for } r > p,$$

with  $B_\beta$  depending on  $\beta$  only.

Let  $0 < \rho < 1$ ,  $\rho_1 = \rho^{\frac{1}{2}}$ ,  $g(x) = f(\rho_1, x)$ .

Since  $\rho_1 > \rho$ ,  $f(\rho, x)$  is the Poisson integral of  $g(x)$ :  $f(\rho, x) = g(\rho_1, x)$ . By hypothesis,

$$\mathfrak{A}_p[g] = \mathfrak{A}_p[f(\rho_1, x)] \leq M(1-\rho_1)^{-\beta},$$

and by (6.32) applied to  $g$ ,

$$\begin{aligned} \mathfrak{A}_r[f(\rho, x)] &= \mathfrak{A}_r[g(\rho_1, x)] \leq BM(1-\rho_1)^{-\beta}(1-\rho_1)^{1/r-1/p} \\ &= BM(1-\rho_1)^{-\beta+1/r-1/p}. \end{aligned}$$

Since  $(1-\rho_1)/(1-\rho)$  is contained between  $\frac{1}{2}$  and 1, (6.34) follows with  $B_\beta = 2^{\beta+1}B$ . The 'O' in the conclusion is not replaceable by 'o' here, as it is in (6.32); see Example 6 at the end of the chapter.

The theorem which follows is an analogue of (6.32) for trigonometric polynomials. It suggests that to estimates of harmonic functions  $f(\rho, x)$  there should correspond estimates for polynomials of order  $n \sim 1/(1-\rho)$ .

(6.35) THEOREM. If  $T$  is a polynomial of order  $n$ , then

$$\mathfrak{A}_r[T] \leq Bn^{1/p-1/r} \mathfrak{A}_p[T] \quad (6.36)$$

for  $r > p \geq 1$ , with  $B$  an absolute constant.

The Fejér kernel  $K_n(t)$  satisfies an inequality

$$\mathfrak{A}_q[K_n] \leq An^{1/q} \quad (6.37)$$

analogous to (6.31), since, as we have already observed (p. 97), the estimates for  $K_n(t)$  and  $P(\rho, t)$  are similar if we identify  $n$  and  $1/(1-\rho)$ . If the  $\sigma_k$  are the (C, 1) means of  $T$ , we have the inequalities (compare (6.30))

$$\mathfrak{A}_r[\sigma_k] \leq \mathfrak{A}_p[T] \mathfrak{A}_q[2K_k] \leq A \mathfrak{A}_p[T] k^{1/q}. \quad (6.38)$$

For the delayed means  $\tau_n = 2\sigma_{2n-1} - \sigma_{n-1}$  (p. 80) we have therefore

$$\mathfrak{A}_r[\tau_n] \leq A \mathfrak{A}_p[T] \{2(2n)^{1/q} + n^{1/q}\} \leq 5A \mathfrak{A}_p[T] n^{1/q},$$

and it is enough to observe that  $\tau_n = T$ .

## 7. Majorants for the Abel and Cesàro means of $S[f]$

These means have simple estimates in terms of the non-negative function

$$M(x) = M_r(x) = \sup_{|t| \leq \pi} \frac{1}{t} \int_0^t |f(x+u)| du$$

introduced in Chapter I, § 13. The proofs will be based on the following lemma:

(7.1) LEMMA. Let  $\chi(t, p)$ ,  $-\pi \leq t \leq \pi$ , be a non-negative function depending on a parameter  $p$  and satisfying the conditions

$$(i) \int_{-\pi}^{\pi} \chi(t, p) dt \leq K, \quad (ii) \int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} \chi(t, p) \right| dt \leq K_1. \quad (7.2)$$

where  $K$  and  $K_1$  are independent of  $p$ . If we set

$$h(x, p) = \int_{-\pi}^{\pi} f(x+t) \chi(t, p) dt, \tag{7.3}$$

then

$$\sup_p |h(x, p)| \leq AM(x), \tag{7.4}$$

where  $A$  depends only on  $K$  and  $K_1$ .

For, fixing  $x$ , let  $F(t) = \int_0^t f(x+u) du$ . Then integrating in (7.3) by parts and using the inequality  $|F(t)| \leq |t| M(x)$ , we get

$$|h(x, p)| \leq M(x) \left\{ \int_{-\pi}^{\pi} \left| t \frac{\partial}{\partial t} \chi(t, p) \right| dt + [\pi \chi(\pi, p) + \pi \chi(-\pi, p)] \right\}.$$

The expression in square brackets does not exceed  $K + K_1$ , as we see by writing (7.2) (i) in the form

$$- \int_{-\pi}^{\pi} t \frac{\partial \chi}{\partial t} dt + \pi[\chi(\pi, p) + \chi(-\pi, p)] \leq K$$

and applying (7.2) (ii). Summing up,

$$|h(x, p)| \leq (2K_1 + K) M(x),$$

and (7.4) is established.

It is useful to observe that if  $t \partial \chi / \partial t$  is of constant sign and if  $\chi(\pm \pi, p)$  are bounded functions of  $p$ , then (7.2) (ii) is a consequence of (7.2) (i). This follows at once if we drop the absolute value sign in (7.2) (ii) and integrate by parts.

Combining (7.4) with the inequalities (13.17) of Chapter I, we get the following:

(7.5) THEOREM. Under the hypotheses of (7.1), the function

$$N(x) = \sup_p |h(x, p)|$$

satisfies the inequalities

$$\left. \begin{aligned} \int_{-\pi}^{\pi} N^r(x) dx &\leq A_r \int_{-\pi}^{\pi} |f|^r dx \quad (r > 1), \\ \int_{-\pi}^{\pi} N^\alpha(x) dx &\leq A_\alpha \left( \int_{-\pi}^{\pi} |f| dx \right)^\alpha \quad (0 < \alpha < 1), \\ \int_{-\pi}^{\pi} N(x) dx &\leq A \int_{-\pi}^{\pi} |f| \log^+ |f| dx + A, \end{aligned} \right\} \tag{7.6}$$

where the constants depend only on the indices shown explicitly, and on  $K$  and  $K_1$ .

It is useful to note that  $A_r$  remains bounded as  $r \rightarrow +\infty$ .

We note some special functions  $\chi$ . The Poisson kernel  $P(\rho, t)$  is one; the first inequality (7.2) is familiar, and the second follows from it since  $t dP/dt \leq 0$  and

$$P(\rho, \pm \pi) = O(1).$$

The Fejér kernel  $K_n(t)$  satisfies the first inequality but not the second. The same holds for the kernel  $K_n^\delta(t)$ ,  $0 < \delta \leq 1$ , which, in addition, is not of constant sign if  $\delta < 1$ . The kernel  $K_n^\delta(t)$  can, however, be majorized by a function satisfying (7.2), namely,

$$|K_n^\delta(t)| \leq \frac{c(\delta)n}{(1+n|t|)^{\delta+1}} \quad \text{for } n \geq 1, |t| \leq \pi, \tag{7.7}$$

where  $c(\delta)$  depends on  $\delta$  only ( $0 < \delta \leq 1$ ). For let  $H_n(t)$  be the expression on the right. It exceeds at least one of  $2^{-\delta-1} c(\delta) n$  and  $c(\delta)/2^{\delta+1} n^{\delta} |t|^{1+\delta}$ . Hence, by Chapter III, (5.5), it exceeds  $|K_n^{\delta}(t)|$ , provided that  $c(\delta)$  is large enough. It is easy to see that  $H_n(t)$  satisfies the first inequality (7.2), from which the second follows since  $|tH_n'(t)| \leq (1+\delta)H_n(t)$ .

Thus:

(7.8) THEOREM. *The inequalities (7.6) hold if  $N(x)$  is one of the functions*

$$\sup_{\rho < 1} |f(\rho, x)|, \quad \sup_{n > 1} |\sigma_n^{\delta}(x)|. \dagger$$

The constants here depend again only on the indices shown explicitly and, in the second case, also on  $\delta$ . ‡

Let  $\zeta = \rho e^{i\theta}$ . For any  $0 \leq \sigma < 1$ , let  $\Omega_{\sigma}$  denote the open domain bounded by the two tangents from  $\zeta = 1$  to the circle  $|\zeta| = \sigma$ , and by the more distant arc of the circle between the points of contact. By  $\Omega_{\sigma}(x)$  we mean the domain  $\Omega_{\sigma}$  rotated around the origin by an angle  $x$ . If  $f(\rho, \theta)$  is the Poisson integral of  $f$ , we set

$$N(x) = N_{\sigma, \rho}(x) = \sup_{\zeta \in \Omega_{\sigma}(x)} |f(\rho, \theta)|. \quad (7.9)$$

Clearly,  $N$  is an increasing function of  $\sigma$ .

(7.10) THEOREM. *The function  $N(x)$  in (7.9) satisfies the inequalities (7.6), where the constants will also depend on  $\sigma$ .*

Fix  $x$ , and let  $\zeta = \rho e^{i\theta}$ ,  $p = \rho e^{i(\theta-x)}$ . For  $\zeta \in \Omega_{\sigma}(x)$  we have

$$f(\rho, \theta) = \int_{-\pi}^{\pi} f(x+t) \chi(t, p) dt, \quad \text{where } \chi(t, p) = \frac{1}{\pi} P(\rho, t+x-\theta).$$

The expression  $\chi(t, p)$  here depends on the variable  $t$  and on the parameter  $p$  which is a point of  $\Omega_{\sigma}$ . That (7.2) (i) holds is obvious. The left-hand side of (7.2) (ii), with  $\xi = x - \theta$ ,  $P' = dP/dt$ , is

$$\begin{aligned} & \frac{1}{\pi} \int_{-\pi}^{\pi} |tP'(\rho, t+\xi)| dt \leq \frac{2}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \pi |\sin \frac{1}{2} t P'(\rho, t+\xi)| dt \\ & = \int_{-\pi}^{\pi} |\sin \frac{1}{2} (t-\xi) P'(\rho, t)| dt \leq \frac{1}{2} \int_{-\pi}^{\pi} |tP'(\rho, t)| dt + \frac{1}{2} |\xi| \int_{-\pi}^{\pi} |P'(\rho, t)| dt. \end{aligned}$$

The penultimate integral is, as we know, bounded. The last term is

$$-|\xi| \int_0^{\xi} \frac{d}{dt} P(\rho, t) dt \leq \frac{|\xi|}{1-\rho} = \frac{|x-\theta|}{1-\rho}.$$

Considering separately the cases  $\rho \geq \sigma$  and  $\rho < \sigma$  we see that the last expression does not exceed a constant depending on  $\sigma$  only. This proves (7.2) (ii) and so also the theorem.

The most important special case of (7.10) is  $\sigma = 0$ , when  $\Omega_{\sigma}$  degenerates into a radius of the unit circle and (7.10) reduces to (7.8).

Results of this section can be extended to Fourier-Stieltjes series, and the generalizations do not require new ideas. For simplicity we confine our attention to Theorem (7.8).

† The conclusion holds actually for  $n \geq 0$ . It is enough, for example, to replace  $n$  by  $n+1$  on the right of (7.7) and the inequality will hold for  $n \geq 0$ . The point is clearly without importance.

‡ The result holds for  $\delta > 1$ . This follows from the fact, easy to verify (see Chapter III, (1.10)(ii)) that  $N(x) = N^{\delta}(x)$  is a non-increasing function of  $\delta$  for  $\delta > 1$ .

(7·11) THEOREM. Let  $\sigma_n$  and  $f(r, x)$  be the (C, 1) and Abel means of an  $S[dF]$ , and let  $N(x)$  be one of the functions of Theorem (7·8). Then

$$\mathfrak{M}_\alpha[N] \leq C_\alpha \int_0^{2\pi} |dF(x)| \quad (0 < \alpha < 1), \quad (7\cdot12)$$

$$\text{and} \quad \int_0^{2\pi} |\sigma_n(x) - F'(x)|^\alpha dx \rightarrow 0, \quad \int_0^{2\pi} |f(r, x) - F'(x)|^\alpha dx \rightarrow 0 \quad (0 < \alpha < 1) \quad (7\cdot13)$$

Let  $0 < R < 1$ ,  $N_R(x) = \max_{r < R} |f(r, x)|$ . By (7·8) and the last inequality in theorem (6·5),

$$\mathfrak{M}_\alpha[N_R(x)] \leq C_\alpha \int_0^{2\pi} |f(R, x)| dx \leq C_\alpha \int_0^{2\pi} |dF(x)|,$$

and making  $R \rightarrow 1$  we obtain (7·12) for Abel means. By considering the (C, 1) means of  $f(R, x)$  and making  $R \rightarrow 1$  we prove (7·12) in the remaining case.

The relations (7·13) follow from the fact that  $|\sigma_n(x) - F'(x)|^\alpha$  and  $|f(r, x) - F'(x)|^\alpha$  tend to 0 almost everywhere (see Chapter III, (7·2) and § 8) and are majorized by integrable functions.

### 8. Parseval's formula

Let  $f(x)$  and  $g(x)$  be periodic and of class  $L^2$ . If their coefficients are respectively  $c_\nu$  and  $c'_\nu$ , we have the Parseval formula (Chapter II, (1·13))

$$\frac{1}{2\pi} \int_0^{2\pi} fg dx = \sum_{\nu=-\infty}^{+\infty} c_\nu c'_\nu, \quad (8\cdot1)$$

or, what is the same thing,

$$\frac{1}{2\pi} \int_0^{2\pi} f\bar{g} dx = \sum_{\nu=-\infty}^{+\infty} c_\nu \bar{c}'_\nu. \quad (8\cdot2)$$

Both series on the right here converge absolutely. If  $f$  and  $g$  are real-valued, and if  $f \sim \frac{1}{2}a_0 + \sum(a_\nu \cos \nu x + b_\nu \sin \nu x)$ ,  $g \sim \frac{1}{2}a'_0 + \sum(a'_\nu \cos \nu x + b'_\nu \sin \nu x)$ , we have

$$\frac{1}{\pi} \int_0^{2\pi} fg dx = \frac{1}{2}a_0 a'_0 + \sum_{\nu=1}^{\infty} (a_\nu a'_\nu + b_\nu b'_\nu). \quad (8\cdot3)$$

The above formulæ hold in other cases besides the one in which  $f \in L^2$ ,  $g \in L^2$ . Two classes  $K$  and  $K_1$  of functions will be called *complementary*, if (8·1) holds for every  $f \in K$ ,  $g \in K_1$  in the sense that the series on the right is summable by some method of summation. It will appear that the Fourier series of functions belonging to complementary classes have in many cases the same or analogous properties; and the Parseval formula (8·1), in which  $f$  and  $g$  enter symmetrically, is the means for discovering these related properties. The formula is obvious (by termwise integration) if  $f$  is a trigonometric polynomial and  $g$  any integrable function.

Let  $\dots, \mu_{-1}, \mu_0, \mu_1, \dots$  be a two-way infinite sequence of numbers. Suppose that along with  $c_\nu, c'_\nu$  the numbers  $c_\nu, \mu_\nu, c'_\nu, \mu_{-\nu}$  are also Fourier coefficients, say of functions  $f^*, g_*$ , and that Parseval's formula for  $f^*$  and  $g_*$  is valid. Then (8·1) gives

$$\int_0^{2\pi} fg dx = \int_0^{2\pi} f^* g_* dx. \quad (8\cdot4)$$

The number  $\mu_{-\nu}$  is necessarily distinct from 0 if  $c'_\nu \neq 0$ , but if  $c'_\nu = 0$  the value ascribed to

$c'_0/\mu_{-v}$  has no influence upon the result, and for the sake of simplicity we may take it equal to 0 even if  $\mu_{-v} = 0$ .

Suppose, for example, that  $c'_0 = 0$ , and that  $\mu_v = iv$  for each  $v$ . Then  $f^* = f'$  and  $g_* = -G$ , where  $G$  is the indefinite integral of  $g$ , and is in our case a periodic function. Thus

$$\int_0^{2\pi} fg dx = - \int_0^{2\pi} f'G dx, \quad (8.5)$$

a formula which, of course, may also be obtained by integration by parts. (Less trivial is the analogue of (8.5) for fractional derivatives and integrals: see Chapter XII §8.) If  $\mu_v = -i \operatorname{sign} v$ , then  $f^* = f$ ,  $J_* = \hat{g}$ , and, formally,

$$\int_0^{2\pi} fg dx = c_0 c'_0 + \int_0^{2\pi} f \hat{g} dx. \quad (8.6)$$

(8.7) THEOREM. *The following are pairs of complementary classes: (i)  $L^r$  and  $L^r$  ( $r > 1$ ); (ii)  $B$  and  $L$ . (iii)  $L_\Phi$  and  $L_\Psi$ , if  $\Phi$  and  $\Psi$  are complementary functions in the sense of Young; (iv)  $C$  and  $S$ . In all these cases the series in (8.1) are summable  $(C, 1)$ .*

Part (iv) here is to be understood in the sense that if  $c_n$  are the coefficients of an  $S[f]$ , and  $c'_n$  the coefficients of an  $S[dG]$ , we have (8.2) with  $f\hat{g}$  replaced by  $f\hat{g}'$ . Part (ii) is a limiting case ( $r = \infty$ ) of (i).

Let  $\sigma_n(x)$  be the  $(C, 1)$  means of  $S[f]$ ,  $\tau_n$  the (symmetric)  $(C, 1)$  means of the series in (8.1), and  $\Delta_n$  the difference between the integral in (8.1) and  $\tau_n$ . Then

$$\Delta_n = \frac{1}{2\pi} \int_0^{2\pi} (f - \sigma_n) g dx, \quad (8.8)$$

and, by Hölder's inequality,

$$2\pi |\Delta_n| \leq \mathfrak{M}_r[f - \sigma_n] \mathfrak{M}_r[g].$$

Hence  $\Delta_n \rightarrow 0$  as  $n \rightarrow \infty$  (cf. (5.14)), and (i) follows. The argument holds for  $r = 1$  (using (5.5)), which proves (ii). To prove (iii), which generalizes (i), we apply Young's inequality (Chapter I, (9.1)) to  $\Delta_n/16$ :

$$2\pi |\Delta_n|/16 \leq \mathfrak{M}[\Phi(\frac{1}{4}|f - \sigma_n|)] + \mathfrak{M}[\Psi(\frac{1}{4}|g|)].$$

By virtue of (5.14), we get  $\limsup |\Delta_n| \leq 8\pi^{-1} \mathfrak{M}[\Psi(\frac{1}{4}|g|)]$ . Let  $g = g' + g''$ , where  $g'$  is a trigonometric polynomial and  $\mathfrak{M}[\Psi(\frac{1}{4}|g''|)] < \epsilon$ . (By (5.14) we may take  $g' = \sigma_m(x; g)$  with  $m$  sufficiently large.) Substituting  $g'$  and  $g''$  for  $g$  in (8.3) we find expressions  $\Delta'_n$  and  $\Delta''_n$  such that  $\Delta_n = \Delta'_n + \Delta''_n$ . Since  $g'$  is only a polynomial,  $\Delta'_n \rightarrow 0$ . On the other hand,

$$\limsup |\Delta''_n| \leq 8\pi^{-1} \mathfrak{M}[\Psi(\frac{1}{4}|g''|)] < 8\pi^{-1}\epsilon.$$

Thus  $\limsup |\Delta_n| \leq 8\pi^{-1}\epsilon$ , so that  $\Delta_n \rightarrow 0$ .

In (iv),  $g(x)$  is replaced by  $dG(x)$  in (8.1) and  $f$  is continuous. Then  $2\pi |\Delta_n|$  does not exceed  $\max |f(x) - \sigma_n(x)|$  multiplied by the total variation of  $G$  over  $(0, 2\pi)$ . Thus  $\Delta_n \rightarrow 0$ .

Let  $g(x)$  be the characteristic function of a set  $E$  and  $f(x)$  an integrable function. Parseval's formulae (8.1) and (8.3) can then be written

$$\int_E f dx = \sum_{v=-\infty}^{\infty} c_v \int_E e^{ivx} dx = \frac{1}{2} a_0 |E| + \sum_{v=1}^{\infty} \int_E (a_v \cos vx + b_v \sin vx) dx.$$

Hence

(8·9) THEOREM. *If  $S[f]$  is integrated termwise over any measurable set  $E$ , the resulting series is summable (C, 1) to sum  $\int_E f dx$ .*

Applying (8·1) to the functions  $f(x-t)$  and  $g(t)$  of the variable  $t$ , we find that in each of the cases listed in (8·7) we have

$$\frac{1}{2\pi} \int_0^{2\pi} f(x-t)g(t) dt = \sum_{\nu=-\infty}^{+\infty} c_\nu c'_\nu e^{i\nu x}, \tag{8·10}$$

where the series on the right is uniformly summable (C, 1). Moreover:

(8·11) THEOREM. *Given any pair of integrable functions  $f, g$ , the formula (8·10) holds, in the (C, 1) sense, almost everywhere in  $x$ .*

The proof follows from the fact that the left-hand side  $h(x)$  of (8·10) is an integrable function, and that the series on the right is  $S[h]$  (Chapter II, (1·5)).

Let us substitute  $g(x)e^{-inx}$  for  $g(x)$  in (8·1), and let  $c'_\nu$  be the coefficients of  $g(x)e^{-inx}$ . Since  $c'_{-\nu} = c'_\nu$ , we find

$$\frac{1}{2\pi} \int_0^{2\pi} fg e^{-inx} dx = \sum_{\nu=-\infty}^{+\infty} c_\nu c'_{-\nu}. \tag{8·12}$$

Thus:

(8·13) THEOREM. *The Fourier series of the product of the functions  $f \in L_\Phi, g \in L_\Psi$  ( $\Phi$  and  $\Psi$  being complementary functions in the sense of Young) is obtained by the formal multiplication of  $S[f]$  and  $S[g]$  by Laurent's rule. The series (8·12) defining the coefficients of  $fg$  are summable (C, 1). The result holds if  $f \in B, g \in L$ .*

It is obvious that each of the inequalities  $\sum |c_\nu| < \infty, \sum |c'_\nu| < \infty$  implies the absolute convergence of the series in (8·12). If both inequalities hold,  $S[fg]$  converges absolutely.

Let  $f(x)$  be continuous and  $G(x)$  of bounded variation. If  $c_\nu, c'_\nu$  are the coefficients of  $S[f], S[dG]$ , we have the following analogue of (8·12):

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f dG = \sum_{\nu=-\infty}^{+\infty} c_\nu c'_{-\nu}. \tag{8·14}$$

In the results above we may replace summability (C, 1) by (C,  $\alpha$ ),  $\alpha > 0$ . The problem of replacing summability (C,  $\infty$ ) by ordinary convergence is more delicate. Going over the proofs of parts (i) and (ii) of (8·7), we see that we may replace summability (C, 1) there by convergence provided  $\mathfrak{M}_r[f - s_n] \rightarrow 0$ , where  $s_n = S_n(x; f)$ . In Chapter VII, § 6, we shall see that this in fact happens if  $f \in L^r, r > 1$  (though not for  $r = 1$ ; see Chapter V, (1·12)). Thus, at least in (8·7) (i), the Parseval series converges. In particular, if  $f \in L^r, g \in L^{r'}, r > 1$ , we have convergence in (8·12). The proof of the following theorem is much easier:

(8·15) THEOREM. *If  $f$  is integrable and  $g$  of bounded variation, the series in (8·1) converges.*

Let  $\delta_n$  be the difference between the integral and the  $n$ th partial sum of the series in (8·1). Then

$$|\delta_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} [g(x) - S_n(x; g)] f(x) dx \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |g - S_n[g]| |f| dx.$$

Since the  $S_n[g]$  are uniformly bounded and tend to  $g$  outside a denumerable set, the last integrand is majorized by an integrable function and tends to 0 almost everywhere. Thus  $\delta_n \rightarrow 0$ .

From (8.15) we obtain:

(8.16) THEOREM. *If  $f$  is integrable and periodic,  $(\alpha, \beta)$  is a finite interval, and  $g(x)$  any function of bounded variation in  $(\alpha, \beta)$ , not necessarily periodic, then*

$$\int_{\alpha}^{\beta} fg dx = \sum_{\nu=-\infty}^{+\infty} c_{\nu} \int_{\alpha}^{\beta} g e^{i\nu x} dx. \quad (8.17)$$

Thus Fourier series can be integrated term by term after multiplication by any function of bounded variation. If  $\beta - \alpha = 2\pi$  this is nothing but (8.1), and the case  $\beta - \alpha < 2\pi$  may be included by setting  $g(x) = 0$  in  $\beta < x < \alpha + 2\pi$ ; in the general case we break up  $(\alpha, \beta)$  into a finite number of intervals of length not exceeding  $2\pi$ .

The last result can be extended to the case of an infinite interval. Without loss of generality we may take  $(\alpha, \beta) = (-\infty, +\infty)$ . We have in fact

(8.18) THEOREM. *The formula*

$$\int_{-\infty}^{+\infty} fg dx = \sum_{\nu=-\infty}^{+\infty} c_{\nu} \int_{-\infty}^{+\infty} g(x) e^{i\nu x} dx \quad (8.19)$$

holds, and the series on the right converges for any integrable and periodic  $f$ , provided that  $g(x)$  is (i) integrable, and (ii) of bounded variation, over  $(-\infty, +\infty)$ .

Let  $G(x) = \sum_{k=-\infty}^{+\infty} g(x + 2k\pi)$ . If the series converges at some point, it converges uniformly and its sum is of bounded variation over  $(0, 2\pi)$  (Chapter II, § 13). On the other hand, since

$$\sum_{k=-\infty}^{+\infty} \int_0^{2\pi} |g(x + 2k\pi)| dx = \int_{-\infty}^{+\infty} |g(x)| dx < \infty,$$

the series defining  $G(x)$  certainly has points of convergence.

Let  $c'_{\nu}$  be the Fourier coefficients of  $G(x)$ . We may replace  $g$  by  $G$  in (8.1). Since a uniformly convergent series can be integrated term by term over  $(0, 2\pi)$  after multiplication by any integrable function, and since  $f$  is periodic, it follows from the definition of  $G$  that

$$\int_0^{2\pi} fG dx = \int_{-\infty}^{+\infty} fg dx, \quad \int_0^{2\pi} G(x) e^{-inx} dx = \int_{-\infty}^{+\infty} g(x) e^{-inx} dx,$$

and Parseval's formula for  $f$  and  $G$  takes the form (8.19).

The hypothesis that  $g$  is integrable over  $(-\infty, +\infty)$  is of course essential for the validity of (8.19). However, if  $c_0 = 0$ , condition (i) in (8.18) may be replaced by the condition (i')  $g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

For let  $g^*(x) = g(2k\pi)$  for  $2k\pi \leq x < 2(k+1)\pi$ ,  $k = 0, \pm 1, \dots$ , and let  $v_k$  be the total variation of  $g(x)$  over  $2k\pi \leq x < 2(k+1)\pi$ . The function  $g^*(x)$  is of bounded variation over  $(-\infty, +\infty)$ . Since  $\gamma(x) = g(x) - g^*(x)$  does not exceed  $v_k$  in absolute value for  $2k\pi \leq x < 2(k+1)\pi$ ,  $\gamma(x)$  is both integrable and of bounded variation over  $(-\infty, +\infty)$ .

Apply (8-19) to  $f$  and  $\gamma$ . Since the integral of  $f$  over a period is zero and  $g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , it is easy to verify that

$$\int_{-\sigma}^{+\infty} f\gamma dx = \int_{-\infty}^{+\infty} f\gamma dx, \quad \int_{-\sigma}^{+\infty} \gamma e^{-ivx} dx = \int_{-\infty}^{+\infty} g e^{-ivx} dx$$

for  $\nu = \pm 1, \pm 2, \dots$ , and the formula (8-19) for  $f$  and  $\gamma$  reduces to that for  $f$  and  $g$ .

Equations (8-1) and (8-10) can be extended to the case of several factors. Consider a finite set of functions  $f, f_1, f_2, \dots$  with Fourier coefficients  $c_\nu, c'_\nu, c''_\nu, \dots$ , respectively. Multiplying formally  $S[f], S[f_1], S[f_2], \dots$ , and integrating the result over  $(0, 2\pi)$ , we get

$$\frac{1}{2\pi} \int_0^{2\pi} f f_1 f_2 \dots dt = \sum_{\lambda+\mu+\nu+\dots=0} c_\lambda c'_\mu c''_\nu \dots, \tag{8-20}$$

a formula which in the case of two functions reduces to (8-1). The argument is valid if the series  $\sum |c'_\mu|, \sum |c''_\nu|, \dots$  all converge. (Nothing is assumed about  $\sum |c_\lambda|$ .) For let  $F = f f_1 f_2 \dots$ . Then  $S[F] = S[f_1] S[f_2] \dots = \sum \gamma_n e^{int}$ , where  $\gamma_n = \sum c'_\mu c''_\nu \dots$  for  $\mu + \nu + \dots = n$ . The series for  $\gamma_n$  converges absolutely, and  $\sum |\gamma_n| < \infty$ . The left-hand side of (8-20) is thus

$$\frac{1}{2\pi} \int_0^{2\pi} f F dt = \sum_{\lambda+n=0} c_\lambda \gamma_n = \sum_{\lambda+\mu+\nu+\dots=0} c_\lambda c'_\mu c''_\nu \dots,$$

and the series here are absolutely convergent. In particular, (8-20) holds if all the functions  $f, f_1, f_2, \dots$ , except possibly one, are trigonometric polynomials.

Restricting ourselves to three functions we also have the following result:

(8-21) THEOREM. Let  $x(t) \sim \sum x_\nu e^{i\nu t}, y(t) \sim \sum y_\nu e^{i\nu t}, h(t) \sim \sum h_\nu e^{i\nu t}$ , and let  $x \in L^2, y \in L^2, h \in B$ . Then

$$\frac{1}{2\pi} \int_0^{2\pi} x(t) y(t) h(t) dt = \sum_{\lambda+\mu+\nu=0} x_\lambda y_\mu h_\nu, \tag{8-22}$$

provided the sum on the right is treated as  $\lim_{L, M \rightarrow \infty} \sum_{\lambda=-L}^L \sum_{\mu=-M}^M x_\lambda y_\mu h_{-\lambda-\mu}$ .

Denote the last sum by  $S_{L, M}$ , and let  $X_L(t)$  and  $Y_M(t)$  denote the partial sums of  $S[x]$  and  $S[y]$ . The integral in (8-22) with  $x, y$  replaced by  $X_L, Y_M$  becomes  $S_{L, M}$ . Let  $H = \sup |h(t)|$ . Then

$$\left| \int_0^{2\pi} xyh dt - \int_0^{2\pi} X_L Y_M h dt \right| \leq H \left\{ \int_0^{2\pi} |x - X_L| |y| dt + \int_0^{2\pi} |X_L| |y - Y_M| dt \right\} \\ \leq H \{ \mathfrak{M}_2[x - X_L] \mathfrak{M}_2[y] + \mathfrak{M}_2[X_L] \mathfrak{M}_2[y - Y_M] \} \rightarrow 0$$

as  $L, M \rightarrow \infty$ . This proves (8-21).

If  $y(t) = \bar{x}(t) \sim \sum \bar{x}_{-n} e^{in t}$ , (8-22) gives

$$\frac{1}{2\pi} \int_0^{2\pi} |x(t)|^2 h(t) dt = \sum_{\lambda, \mu=-\infty}^{+\infty} x_\lambda \bar{x}_\mu h_{\mu-\lambda}, \tag{8-23}$$

a formula with applications in the theory of quadratic forms.

(8-24) THEOREM. With the notation of (8-21), suppose that  $x(t) \in L^2, y(t) \in L^2, h(t) \in \mathcal{L}$ ; then

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} x(u) y(v) h(-u-v) du dv = \sum_{n=-\infty}^{+\infty} x_n y_n h_n,$$

where the series on the right converges absolutely.

For the proof, we apply (8.1) to the product of  $x(u)$  and  $y_1(u) = \frac{1}{2\pi} \int_0^{2\pi} y(v) h(-u-v) dv$ . The latter function belongs to  $L^2$  and has coefficients  $y_{-n} h_{-n}$  (Chapter II, (1.15), (1.5)).

## 9. Linear operations

We are now going to prove a number of results on linear operations which will later find application to trigonometric series.

We consider a set  $E$  of arbitrary elements  $x, y, z, \dots$ . It is often convenient to call  $E$  a *space*, and its elements  $x, y, z, \dots$  *points*.  $E$  will be called a *metric space* if to every pair of points  $x, y$  of  $E$  corresponds a non-negative number  $d(x, y)$ , called the *distance* between the points  $x$  and  $y$ , satisfying the following conditions:

- (i)  $d(x, y) = d(y, x)$ ;
- (ii)  $d(x, z) \leq d(x, y) + d(y, z)$  (*triangle inequality*);
- (iii)  $d(x, y) = 0$  if and only if  $x = y$ .

We say that a sequence  $\{x_n\}$  of points of  $E$  tends to limit  $x, x \in E$ , and write  $\lim x_n = x$ , or  $x_n \rightarrow x$ , if  $d(x, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Once distance has been introduced, there are various associated notions familiar from the theory of Euclidean spaces. First, by the *sphere* with centre  $x_0$  and radius  $\rho$  we mean the set of points  $x \in E$  such that  $d(x, x_0) \leq \rho$ ; this sphere will be denoted by  $S(x_0, \rho)$ . This notion enables us, in turn, to introduce various kinds of point sets, such as *open, closed, non-dense, dense, everywhere dense*, the definitions being the same as in Euclidean spaces. Furthermore, we may consider sets of the *first category*, i.e. denumerable sums of non-dense sets, and sets of the *second category*, i.e. sets which are not of the first category (cf. Chapter I, § 12).

A metric space  $E$  is said to be *complete*, if for any sequence of points  $x_n$  such that  $d(x_m, x_n) \rightarrow 0$  as  $m, n \rightarrow \infty$  there is a point  $x$  such that  $d(x_m, x) \rightarrow 0$ . The inequality  $d(x, x') \leq d(x, x_m) + d(x_m, x')$  shows that such a point  $x$  must be unique. It is a very important fact that a *complete metric space*  $E$  is of the *second category*, i.e. is not a sum of a sequence of sets non-dense in  $E$ . The proof of this in the general case is essentially the same as in the case (discussed in Chapter I, § 12) when  $E$  is a one-dimensional Euclidean space.

$E$  is called *separable* if there is a denumerable set dense in  $E$ .

A space  $E$ , not necessarily metric, will be called *linear* if the following conditions are satisfied:

- (i) there is a commutative and associative operation called *addition*, denoted by  $+$  and applicable to any two points  $x, y$  of  $E$ ; whenever  $x$  and  $y$  belong to  $E$ , so does  $x + y$ ;
- (ii) there is a unique element  $o$  (the *null element*) such that  $x + o = x$  for every  $x \in E$ ;
- (iii) there is an operation called *multiplication*, applicable to every  $x \in E$  and every scalar†  $\alpha$ , and denoted by  $\cdot$ . Instead of  $\alpha \cdot x$  we often write  $\alpha x$ . Multiplication is assumed to have the properties

$$1 \cdot x = x, \quad 0 \cdot x = o, \quad \alpha \cdot x \in E \quad \text{if } x \in E,$$

and further to be distributive in  $\alpha$  and in  $x$ , and associative in  $\alpha$ . The latter means that  $\beta \cdot (\alpha \cdot x) = \beta \alpha \cdot x$ .

The formula  $x - y = x + (-1)y$  defines *subtraction* of elements of  $E$ .

† The only fields of scalars we use are the complex numbers or the real numbers.

Suppose that to every element  $x$  of a linear space  $E$  corresponds a unique non-negative number  $\|x\|$  called the *norm* of  $x$  satisfying the conditions

$$\|x+y\| \leq \|x\| + \|y\|, \quad \|\alpha x\| = |\alpha| \|x\|,$$

$$\|x\| = 0 \quad \text{if and only if } x = o.$$

If the distance between any two points  $x, y$  of our linear space  $E$  is defined by the formula

$$d(x, y) = \|x - y\|,$$

this distance satisfies conditions (i), (ii), (iii) imposed above, and  $E$  becomes a *normed linear space*. A complete normed linear space is usually called a *Banach space*.

We shall now give a few examples of spaces. In each case the points of  $E$  will be either numbers or functions, and addition and multiplication have their usual interpretation. No confusion will arise if the null point  $o$  is denoted by 0.

(a) Let  $E$  be the set of all complex (or only all real) numbers. If  $\|x\| = |x|$ , we have a Banach space.

(b) Let  $E$  be the set  $C$  of all continuous functions  $x(t)$  defined in a fixed interval  $(a, b)$ , and let  $\|x\| = \sup |x(t)|$  for  $t \in (a, b)$ . Then  $E$  is a Banach space. The relation  $x_n \rightarrow x$  means that  $x_n(t)$  converges uniformly to  $x(t)$ .

(c) Let  $E$  be the set of all complex-valued functions  $x(t)$  defined and essentially bounded in  $(a, b)$ , and let  $\|x\|$  be the essential upper bound of  $|x(t)|$  in  $(a, b)$  (cf. Chapter I, §9).  $E$  is again a Banach space, and  $x_n \rightarrow x$  means that  $x_n(t)$  converges to  $x(t)$  uniformly outside a set of measure 0.

(d) Let  $p \geq 1$ , and let  $E$  be the set of all complex-valued functions  $x(t) \in L^p(a, b)$ . Let

$$\|x\| = \|x\|_p = \mathfrak{M}_p[x; a, b].$$

The space is linear, normed and complete (cf. Chapter I (9-11), (11-1)). For  $p = \infty$ , we obtain case (c).

(e) Let  $1 \leq p < \infty$ , and let  $E$  be the set of all sequences  $x = \{x_k\}$  of complex numbers such that  $\sum |x_k|^p < \infty$ . Let  $\|x\| = \|x\|_p = (\sum |x_k|^p)^{1/p}$ .

The space is linear, normed, and (as is easily seen) complete. It is often denoted by  $l^p$ .

(f) Let  $E$  be the set of all bounded sequences  $x = \{x_k\}$  of complex numbers. If we set  $\|x\| = \sup_k |x_k|$ , we get a Banach space. It is the limiting case  $p = \infty$  of  $l^p$ .

(g) Let  $E$  be the set of all convergent sequences  $x = \{x_k\}$  of complex numbers, and once again let  $\|x\| = \sup |x_k|$ . The set  $E$  (a subset of the preceding  $E$ ) is a normed linear space. It is also complete. For suppose that  $x^m = \{x_1^m, x_2^m, \dots\} \in E$  for  $m = 1, 2, \dots$ , and that  $\|x^m - x^n\| \rightarrow 0$  as  $m, n \rightarrow \infty$ . This is equivalent to saying that  $|x_k^m - x_k^n| \rightarrow 0$  as  $m, n \rightarrow \infty$ , uniformly in  $k$ . This implies the existence of an  $x = \{x_k\}$  such that  $|x_k^m - x_k| \rightarrow 0$  as  $m \rightarrow \infty$ , uniformly in  $k$ . We shall show that (i)  $\{x_k\}$  is convergent, (ii)  $\|x^m - x\| \rightarrow 0$ .

To prove (i) observe that

$$|x_k - x_l| \leq |x_k - x_k^m| + |x_k^m - x_l^m| + |x_l^m - x_l|.$$

The first and third terms on the right are less than  $\epsilon$  for  $m$  large enough, uniformly in  $k$  and  $l$ . Having fixed such a large  $m$ , we make the second term less than  $\epsilon$  by taking  $k$  and  $l$  large. Hence  $|x_k - x_l| < 3\epsilon$  for  $k, l$  large, and so  $\{x_k\}$  is convergent. Assertion (ii) follows from the fact, established above, that  $|x_k^m - x_k| \rightarrow 0$  as  $m \rightarrow \infty$ , uniformly in  $k$ .

(h) Let  $p \geq 1$ , and let  $H_p$  be the class of all functions  $x(t)$  of period  $2\pi$ , of class  $L^p$ , with  $S[x]$  of power series type. If  $\|x\| = \mathfrak{M}_p[x; 0, 2\pi]$ , the space becomes a Banach space.

(i) Let  $X$  be the set of all characteristic functions in  $(a, b)$ , that is, of functions  $x(t)$  taking almost everywhere in  $(a, b)$  the values 0 and 1 only. The set  $X$  is not a linear space, since  $2x$  need not be a characteristic function if  $x$  is one. However, if for  $x \in X$ ,  $y \in X$  we set  $d(x, y) = \mathfrak{M}[x - y; a, b]$ ,  $X$  becomes a complete metric space.

Let us consider in addition to the space  $E$  another space  $U$ . If to every  $x \in E$  corresponds a uniquely determined point  $u = u(x)$  in  $U$ , we say that  $u(x)$  is a functional operation (or transformation) defined in  $E$ . If the spaces  $E$  and  $U$  are linear, and if for any numbers  $\lambda_1$  and  $\lambda_2$  we have

$$u(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 u(x_1) + \lambda_2 u(x_2),$$

the operation  $u(x)$  is called *linear*. If both  $E$  and  $U$  are metric, and if whenever  $x_n \rightarrow x$  we have  $u(x_n) \rightarrow u(x)$ , we say that  $u$  is *continuous* at the point  $x$ . If a linear operation is continuous at some point, it is continuous at any other point, i.e. is continuous everywhere.

(9.1) THEOREM. A necessary and sufficient condition for a linear operation  $u(x)$  to be continuous in  $E$  is the existence of a finite number  $M$  such that

$$\|u(x)\| \leq M \|x\| \quad \text{for every } x \in E. \quad (9.2)$$

The sufficiency of the condition is obvious. To prove its necessity, suppose that the ratio  $\|u(x)\|/\|x\|$  is unbounded. Then there is a sequence of points  $x_n$ ,  $x_n \neq 0$ , such that  $\|u(x_n)\| \geq n \|x_n\|$ . Multiplying  $x_n$  by a suitable constant we may assume that  $\|x_n\| = 1/n$ . Thus  $x_n \rightarrow 0$ , while the preceding inequality gives  $\|u(x_n)\| \geq 1$ , contradicting the continuity of  $u$  at  $x = 0$ .

The norms on the two sides of (9.2) may have different meanings, since the spaces  $E$  and  $U$  may be different.

A linear operation which is continuous is usually called *bounded*.

If  $U$  is the space of all complex, or all real, numbers and  $\|u(x)\| = |u(x)|$ , the linear operation  $u$  is called a *functional*.

The smallest number  $M$  satisfying (9.2) for all  $x \in E$  will be called the *norm of the operation* and often denoted by  $M_u$ .†

(9.3) THEOREM. Let  $E$  be a normed linear space and  $L$  a linear subspace of  $E$  dense in  $E$ . Let  $u = u(x)$  be a linear operation defined for  $x \in L$ , taking values from a Banach space  $U$  and satisfying an inequality

$$\|u(x)\| \leq M \|x\| \quad (x \in L). \quad (9.4)$$

Then  $u(x)$  can be uniquely extended as a linear operation to the whole of  $E$  without increasing the  $M$  in (9.4).

For let  $x_*$  be a point of  $E$ ,  $\{x_n\}$  a sequence of points from  $L$  such that  $\|x_* - x_n\| \rightarrow 0$ ,

† One often denotes the norm of the operation  $u(x)$  by  $\|u\|$ . This terminology and notation—very natural in a systematic study of the subject—arise from the fact that the set of all linear operations  $u(x)$  defined on  $E$  may be considered as a new space, to whose points  $u$  are assigned norms  $\|u\|$ . Of course,  $\|u\|$  and  $\|u(x)\|$  mean different things.

and  $u_n = u(x_n)$ ,  $n = 1, 2, \dots$ . We have  $\|x_m - x_n\| \rightarrow 0$  and so, by (9.4),  $\|u_m - u_n\| \rightarrow 0$ . The completeness of  $U$  implies the existence of a  $u_* = \lim u_n$ , and we set  $u_* = u(x_*)$ .

The number  $u_*$  is independent of the choice of  $\{x_n\} \rightarrow x_*$ . For if we take another sequence  $\{x'_n\} \rightarrow x_*$  and set  $u'_* = \lim u(x'_n)$ , the joint sequence  $\{x''_n\} = x_1, x'_1, x_2, x'_2, \dots$  will also converge to  $x_*$  and the number  $u''_* = \lim u(x''_n)$  will be equal to both  $u_*$  and  $u'_*$ . Hence  $u_* = u'_*$ .

The validity of (9.4) at the point  $x_n$ , together with the relations  $\|x_n\| \rightarrow \|x_*\|$ ,  $\|u_n\| \rightarrow \|u_*\|$  (consequences of  $x_n \rightarrow x_*$ ,  $u_n \rightarrow u_*$ ), implies its validity at  $x_*$ . Similarly, the validity of

$$u(\alpha x + \beta y) = \alpha u(x) + \beta u(y)$$

at the points  $x, y$  of  $L$  is preserved in  $E$ . This shows that the extended operation satisfies (9.4) and is additive. The inequality (9.4) implies the continuity of  $u$  and there is at most one continuous extension of  $u(x)$  from a set dense in  $E$ .

The following theorem is basic for the theory of linear operations:

(9.5) THEOREM OF BANACH-STEINHAUS. *Let  $\{u_n(x)\}$  be a sequence of bounded linear operations defined in a Banach space  $E$ , and let  $M_{u_n}$  be the norm of the operation  $u_n$ . If  $\sup \|u_n(x)\|$  is finite for every point  $x$  belonging to a set  $F$  of the second category in  $E$  (in particular if it is finite for every  $x \in E$ ), then the sequence  $M_{u_n}$  is bounded. In other words, there is a constant  $M$  such that*

$$\|u_n(x)\| \leq M \|x\| \quad \text{for } x \in E \text{ and } n = 1, 2, \dots$$

The proof is based on two lemmas.

(9.6) LEMMA. *Let  $\{u_n(x)\}$  be a sequence of bounded linear operations defined in a normed linear space  $E$ . If  $F$  is the set of points  $x$  at which  $\sup \|u_n(x)\| < \infty$ , then  $F = F_1 + F_2 + \dots$ , where the sets  $F_i$  are closed and the sequence  $\{\|u_n(x)\|\}$  is uniformly bounded on each of them.*

For let  $F_{mn}$  be the set of points  $x$  such that  $\|u_m(x)\| \leq n$ . The operations  $u_m$  being continuous, the sets  $F_{mn}$  are closed. So are the products  $F_n = F_{1n} F_{2n} F_{3n} \dots$ . We note that  $\|u_m(x)\| \leq n$  for  $x \in F_n$ ,  $m = 1, 2, \dots$ , and that  $F = F_1 + F_2 + \dots$ .

(9.7) LEMMA. *If the set  $F$  in Lemma (9.6) is of the second category, then there is a sphere  $S(x_0, \rho)$ ,  $\rho > 0$ , and a number  $K$  such that  $\|u_m(x)\| \leq K$  for  $x \in S(x_0, \rho)$  and  $m = 1, 2, \dots$ .*

For since  $F = F_1 + F_2 + \dots$  and  $F$  is of the second category, at least one of the sets  $F_1, F_2, \dots$ , say  $F_K$ , is not non-dense. Thus there is a sphere  $S(x_0, \rho)$  in which  $F_K$  is dense. Since  $F_K$  is closed,  $F_K$  contains  $S(x_0, \rho)$  and consequently  $\|u_m(x)\| \leq K$  for  $x \in S(x_0, \rho)$  and  $m = 1, 2, \dots$ .

Returning to the proof of (9.5), let  $S(x_0, \rho)$  be the sphere of (9.7). Every  $x \in S(0, \rho)$  can be written in the form  $(x_0 + x) - x_0 = x_1 - x_0$ , say, where  $x_1 \in S(x_0, \rho)$ . Hence  $\|u_n(x)\| \leq \|u_n(x_1)\| + \|u_n(x_0)\| \leq 2K$  for  $n = 1, 2, \dots$ . It follows that

$$\|u_n(x)\| \leq 2K/\rho = M$$

on the sphere  $\|x\| = 1$ , so that  $\|u_n(x)\| \leq M \|x\|$  for all  $x$  and  $n$ .

The theorem may also be stated as follows: *If the sequence  $\{\|u_n(x)\|\}$  is unbounded at some point, the set of points  $x$  at which the sequence is bounded is of the first category in  $E$ .*

We shall apply (9.5) to the functionals

$$u(x) = \int_a^b x(t)y(t) dt, \quad (9.8)$$

where the function  $x = x(t)$  is the variable point of a Banach space  $E$ , and  $y = y(t)$  is a fixed function such that the product  $x(t)y(t)$  is integrable over  $(a, b)$  for any  $x \in E$ . For  $x(t) \in L^r$  we define  $\|x\|$  as  $\mathfrak{M}_r[x; a, b]$ .

(9.9) LEMMA. (i) *If the integral (9.8) exists for every bounded, or even only for every bounded and continuous function  $x(t)$ , then  $y \in L(a, b)$ .*

(ii) *Conversely, if (9.8) exists for every  $x \in L(a, b)$ , then the function  $y$  is essentially bounded.*

(iii) *If (9.8) exists for every  $x \in L^r(a, b)$ ,  $r > 1$ , then  $y \in L^r(a, b)$ , with  $r' = r/(r-1)$ .*

Part (i) is trivial (take  $x(t) \equiv 1$ ). For (ii), let  $y^n(t)$  be the function  $y(t)$  truncated by  $n$  (that is, equal to  $y(t)$  wherever the latter function is absolutely not greater than  $n$ , and equal to 0 elsewhere). The existence of (9.8) implies that of

$$u_n(x) = \int_a^b x(t)y^n(t) dt \quad (n = 1, 2, \dots) \quad (9.10)$$

for every  $x \in L(a, b)$ . These formulae define a sequence of functionals in  $L(a, b)$ , and we easily find that  $M_{u_n}$  is the essential upper bound of  $|y^n(t)|$ . The existence of (9.8) implies that  $u_n(x)$  converges for each  $x \in L(a, b)$ . Thus, by (9.5),  $M_{u_n} = O(1)$ , that is,  $y(t)$  is essentially bounded. For (iii), let  $y^n$  and  $u_n(x)$  have the same meaning as before. Then  $u_n(x)$  is a functional in  $L^r(a, b)$  with norm  $M_{u_n} = \mathfrak{M}_{r'}[y^n; a, b]$  (cf. Chapter I, (9.14)). Thus, by (9.5),  $\mathfrak{M}_{r'}[y^n] = O(1)$ , that is,  $\mathfrak{M}_{r'}[y] < \infty$ .

(9.11) THEOREM. (i) *If the sequence*

$$u_n(x) = \int_a^b x(t)y_n(t) dt \quad (9.12)$$

*is bounded for every bounded, or even only merely for every bounded and continuous function  $x(t)$ ,  $t \in (a, b)$ , then  $\mathfrak{M}[y_n; a, b] = O(1)$ .*

(ii) *If  $u_n(x)$  is bounded for every  $x \in L(a, b)$ , then the essential upper bounds of the  $|y_n|$  are bounded.*

(iii) *If  $u_n(x)$  is bounded for every  $x \in L^r(a, b)$ ,  $1 < r < \infty$ , then  $\mathfrak{M}_r[y_n; a, b] = O(1)$ .*

For (i), we observe that by (9.9) (i) each of the functions  $y_n$  is integrable, so that  $u_n(x)$  is a functional in the space  $B$  or the space  $C$ . Taking  $x = \overline{\text{sign}} y_n$ , we see that the norm  $M_{u_n}$  in  $B$  equals  $\mathfrak{M}[y_n]$ , and an application of (9.5) gives  $\mathfrak{M}[y_n] = O(1)$ . The same proof holds in the case of  $x \in C$ , provided we can show that  $M_{u_n} = \mathfrak{M}[y_n]$  also in this case. The latter is, however, obvious, since the function  $\text{sign } y_n(t)$  is almost everywhere the limit of a convergent sequence of functions continuous and not exceeding 1 in absolute value.

In case (ii) we proceed similarly: each of the functions  $y_n$  is essentially bounded (cf. (9.9) (ii)), and  $M_{u_n}$  is the essential upper bound of  $|y_n|$ .

Finally, in case (iii) each  $y_n$  belongs to  $L^r(a, b)$ ,  $u_n$  is a functional in  $L^r$ , and  $M_{u_n} = \mathfrak{M}_r[y_n]$  (Chapter I, (9.14)).

(9·13) THEOREM. (i) Suppose that  $y_n(t) \in L(a, b)$  for  $n = 1, 2, \dots$  and that the sequence (9·12) converges for every bounded  $x(t)$ . Then the functions  $Y_n(t) = \int_a^t y_n(u) du$  are uniformly absolutely continuous.

(ii) The conclusion holds if (9·12) converges for those  $x(t)$  which are characteristic functions of measurable sets.

It is enough to prove (ii), and the proof will be similar to that of (9·5).

We show that  $\int_E y_n dt$  is small with  $|E|$ , uniformly in  $n$ . Let  $X$  be the set of all characteristic functions  $x(t)$  in  $(a, b)$ , and consider the integral (9·12), which we denote by  $I_n(x)$ . By hypothesis,  $I_n(x)$  converges for every  $x \in X$ . We have to show that  $|I_n(x)| \leq \epsilon$  for all  $n$ , if

$$\|x\| = \mathfrak{R}[x; a, b] \leq \delta = \delta(\epsilon).$$

Let  $I_{\mu, \nu}(x) = I_\mu(x) - I_\nu(x)$  and let  $X_n$  be the set of points  $x \in X$  such that  $|I_{\mu, \nu}(x)| \leq \frac{1}{2}\epsilon$  for all  $\mu, \nu \geq n$ . Then  $X = \Sigma X_n$ . The sets  $X_n$  are closed, and since  $X$ , being a complete metric space, is not of the first category, one of the  $X_n$ , say,  $X_{n_0}$ , contains a sphere  $S(x_0, \delta')$ . We now observe that  $X$ , though not a linear space, has a property which may be used instead of linearity: if  $x$  is any point of  $S(0, \delta')$ , we can find two points  $x_1$  and  $x_2$  in  $S(x_0, \delta')$  such that  $x = x_1 - x_2$ . It is enough to set

$$x_1(t) = x(t) + x_0(t) [1 - x(t)], \quad x_2(t) = x_0(t) [1 - x(t)].$$

Clearly  $|I_{\mu, \nu}(x)| = |I_{\mu, \nu}(x_1) - I_{\mu, \nu}(x_2)| \leq \frac{1}{2}\epsilon$  for  $\|x\| \leq \delta'$ ,  $\mu \geq n_0, \nu \geq n_0$ .

It follows that  $|I_\mu(x) - I_{n_0}(x)| = |I_{\mu, n_0}(x)| \leq \frac{1}{2}\epsilon$  for  $\|x\| \leq \delta'$ ,  $\mu \geq n_0$ . Since  $I_{n_0}(x)$  is small with  $\|x\|$ , there is a  $\delta'' > 0$  such that  $|I_{n_0}(x)| \leq \epsilon$  for  $\|x\| \leq \delta''$  and  $\mu \geq n_0$ . Finally, since  $I_1(x), I_2(x), \dots, I_{n_0-1}(x)$  are small with  $\|x\|$ , there is a  $\delta$  such that  $|I_n(x)| \leq \epsilon$  for  $\|x\| \leq \delta$  and all  $n$ .

We shall occasionally need the following analogue of (9·9) for series:

(9·14) THEOREM. Let  $a_1 + a_2 + \dots$  be a fixed series. Then

$$(i) \text{ if } \sum_1^\infty a_k b_k \tag{9·15}$$

converges for every bounded  $\{b_k\}$ , or even for every  $\{b_k\}$  tending to 0, we have  $\Sigma |a_k| < \infty$ ;

(ii) if (9·15) converges for every convergent  $\Sigma b_k$ , the sequence  $\{a_k\}$  is of bounded variation;

(iii) if (9·15) converges for every  $\{b_k\} \in l^r$ ,  $1 < r < \infty$ , we have  $\{a_k\} \in l^r$ .

(i) If  $\Sigma |a_k| = \infty$ , then (9·15) diverges for the bounded sequence  $b_n = \text{sign } a_n$ , or even for the convergent sequence  $b_n = \epsilon_n \text{sign } a_n$ , where  $\{\epsilon_n\}$  is a positive sequence tending to 0 sufficiently slowly.

(ii) It is not difficult to see that the hypotheses of (ii) imply that  $a_n = O(1)$ . Let  $\Sigma b_k$  be any series convergent to 0 and let  $t_k = b_1 + b_2 + \dots + b_k$ . Using summation by parts we can write the convergent series (9·15) in the form  $\sum_1^\infty t_k(a_k - a_{k+1})$ . The latter series converges for every sequence  $\{t_k\}$  tending to 0. Hence, by (i),  $\Sigma |a_k - a_{k+1}| < \infty$  (see also Chapter 1, (2·4)).

(iii) For a fixed  $n$ , the sum  $s_n = \sum_1^n a_k b_k$  is a linear functional in the space  $l^r$  of sequences  $\{b_k\}$ , and has norm  $N_n = (|a_1|^r + \dots + |a_n|^r)^{1/r}$ . By (9·5), the hypothesis that  $s_n = O(1)$  for every  $\{b_k\} \in l^r$  implies  $N_n = O(1)$ , that is,  $\{a_n\} \in l^r$ .

Consider a doubly infinite matrix  $\mathfrak{A} = \{\alpha_{nm}\}_{n,m=0,1,\dots}$ , defining a method of summation of sequences (Chapter III, § 1). For every sequence  $x = \{x_k\}$  of numbers we set

$$y_n = \alpha_{n0}x_0 + \alpha_{n1}x_1 + \dots + \alpha_{nk}x_k + \dots \tag{9-16}$$

Suppose  $x_k \rightarrow s$ ; then  $y_n \rightarrow s$ , provided  $\mathfrak{A}$  satisfies the three conditions (i), (ii), (iii) of regularity. We have already proved that conditions (i) and (iii) are also necessary, and we are now going to prove the necessity of (ii).

We slightly generalize the notion of summability  $\mathfrak{A}$  by assuming that the series (9-16) converge only for  $n$  sufficiently large, say for  $n \geq n_0$ , where  $n_0$  depends on  $\{x_k\}$ . We shall show that, if for every convergent sequence  $\{x_k\}$  the series (9-16) converge for all  $n$  sufficiently large, and if  $y_n = O(1)$  (we do not require the existence of  $\lim y_k$ , still less the relation  $\lim y_n = \lim x_k$ ), then the numbers

$$N_n = |\alpha_{n0}| + |\alpha_{n1}| + \dots + |\alpha_{nk}| + \dots$$

are finite and bounded for  $n$  sufficiently large. (If the matrix  $\mathfrak{A}$  is row-finite, the finiteness of all the  $N_n$  is obvious.)

Let  $E$  be the set of all convergent sequences  $x = \{x_k\}$ . It is a Banach space (see example (g) on p. 163). Write  $E = E_1 + E_2 + \dots + E_m + \dots$ , where  $E_m$  is the set of all the convergent sequences  $x$  such that the series (9-16) converge for all  $n \geq m$ . (If  $\mathfrak{A}$  is row-finite, then  $E_1 = E_2 = \dots = E$ .) Since  $E$  is not of the first category, some  $E_{m_0}$  is not of the first category. For every  $x = \{x_k\} \in E_{m_0}$ , the series (9-16) converge if  $n \geq m_0$ . This means that

$$y_{n,j} = \alpha_{n0}x_0 + \alpha_{n1}x_1 + \dots + \alpha_{nj}x_j$$

tends to a limit as  $j \rightarrow \infty$ , for fixed  $n \geq m_0$ . Here  $y_{n,j}$  is a linear functional in  $E$ . By (9-5), the norms of  $y_{n,j}$ , that is, the numbers  $|\alpha_{n0}| + |\alpha_{n1}| + \dots + |\alpha_{nj}|$ , are bounded as  $j \rightarrow \infty$ , which means that the numbers  $N_n$  are finite for  $n \geq m_0$ . It remains to show that the  $N_n$  are bounded. The finiteness of  $N_n$  implies that the  $y_n, n \geq n_0$ , in (9-16) are linear functionals in  $E$ , and, by hypothesis,  $y_n = O(1)$  for each  $x \in E$ . It is therefore enough to apply (9-5) once more.

We conclude this section by considering an application of Fourier series to a class of linear transformations of the space  $l^2$  of sequences  $x = \{x_n\}$  two-way infinite and with

$$\|x\| = (\sum |x_n|^2)^{\frac{1}{2}} < \infty. \text{ Series } \sum_{-\infty}^{+\infty} \text{ will be understood here to mean } \sum_{-\infty}^{-1} + \sum_0^{+\infty}.$$

Let  $a_{mn}$  be a two-way infinite matrix of numbers, with  $m, n$  ranging from  $-\infty$  to  $+\infty$ . With every  $x \in l^2$  we associate the point  $y = \{y_m\}$ , where

$$y_m = \sum_n a_{mn} x_n \quad (m = 0, \pm 1, \dots).$$

By (9-14),  $y$  is defined for every  $x \in l^2$  if and only if  $\sum_n |a_{mn}|^2 < \infty$  for each  $m$ . In investigating whether or not  $y \in l^2$ , we shall confine our attention to a special case, namely that of  $a_{mn} = a_{m-n}$ , where  $a = \{a_n\}$  is a two-way infinite sequence. This case is easily treated by means of Fourier series. Thus let

$$y_m = \sum_n a_{m-n} x_n \quad (m = 0, \pm 1, \dots). \tag{9-17}$$

If  $\sum a_k e^{ikt}$  is the Fourier series of a function  $a(t)$ , we shall call  $a(t)$  the characteristic of the transformation (9-17).

(9-18) THEOREM. (i) A necessary and sufficient condition that the  $y$  defined by (9-17) should satisfy  $y \in l^2$  for every  $x \in l^2$  is that the transformation should have a characteristic which is an essentially bounded function. If  $M$  is the essential upper bound of  $|a(t)|$ , then  $\|y\| \leq M \|x\|$  and  $M$  is the norm of the transformation.

(ii) Suppose that the transformation (9-17) has a characteristic  $a(t) \in L^1$ . A necessary and sufficient condition that for every  $y \in l^2$  there is an  $x \in l^2$  satisfying (9-17) is that  $1/a(t)$

should be essentially bounded. If  $1/a(t)$  is bounded, then the solution  $x$  is unique and is given by the formula

$$x_n = \sum_m a'_{n-m} y_m \quad (n=0, \pm 1, \dots), \quad (9-19)$$

where  $a'_k$  are the Fourier coefficients of  $1/a(t)$ .

We know that the condition  $\|a\| < \infty$  is both necessary and sufficient for the transformation (9-17) to be defined for all  $x \in l^2$ . If  $x(t) \sim \sum x_k e^{ikt}$ ,  $a(t) \sim \sum a_k e^{ikt}$ , then  $y_m = \frac{1}{2\pi} \int_0^{2\pi} a(t) x(t) e^{-imt} dt$  and, by Parseval's formula,

$$\|y\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |a(t) x(t)|^2 dt. \quad (9-20)$$

If  $\|y\|$  is to be finite for every  $x \in l^2$ , then  $a(t)$  must be essentially bounded (see (9-9)). Assuming this, let  $M$  be the essential upper bound of  $|a(t)|$ . By (9-20),  $\|y\| \leq M \|x\|$ , and  $M$  cannot be replaced by any smaller number, that is,  $M$  is the norm of the transformation (9-17). This proves (i).

Passing to (ii), let  $y(t) \sim \sum y_k e^{ikt} \in L^2$  be given. If (9-17) has a solution  $x = \{x_k\}$  such that  $x(t) \sim \sum x_k e^{ikt} \in L^2$ , then

$$y(t) = a(t) x(t). \quad (9-21)$$

Since  $y(t) \in L^2$  is arbitrary,  $a(t) \neq 0$  almost everywhere, and there is at most one  $x = \{x_k\}$  satisfying (9-17). If  $x(t) = y(t) a^{-1}(t)$  is to belong to  $L^2$  for every  $y(t) \in L^2$ , then  $1/a(t)$  must be essentially bounded. If the latter condition is satisfied, the point  $x = \{x_k\}$  defined by the formula  $y(t) a^{-1}(t) \sim \sum x_k e^{ikt}$  is the (unique) solution of (9-17) and is given by (9-19). This proves (ii).

Linear transformations in  $l^2$  which preserve the norm and have inverses are called unitary. As is seen from (9-21), the transformation (9-17) is unitary if and only if  $|a(t)| = 1$  almost everywhere. In this case,  $1/a(t) = \bar{a}(t)$ , and (9-19) may be written

$$x_n = \sum_m \bar{a}_{m-n} y_m. \quad (9-22)$$

Consider a product (i.e. a successive application) of two transformations of type (9-17), having respectively characteristics  $a(t)$  and  $b(t)$ . As is seen from (9-21), this product is of the same type, with characteristic  $a(t)b(t)$ .

If the characteristic  $a(t)$  depends on a parameter  $\alpha$  in accordance with the formula

$$a(t) = a_\alpha(t) = e^{z\phi(t)},$$

where  $\phi(t)$  is a measurable function, then the transformations (9-17), which we shall call  $T_\alpha$ , form a group with the property

$$T_{\alpha+\beta} = T_\alpha T_\beta, \quad T_\alpha^{-1} = T_{-\alpha}.$$

If  $\alpha$  is real and  $\phi(t)$  purely imaginary, the transformations are unitary.

*Examples.* (a) Suppose that  $a(t) = \sum' \frac{e^{int}}{n} = i(\pi - t)$  in  $(0, 2\pi)$  (see Chapter I, (4-12)).

Then (9-17) may be written

$$y_m = \sum' \frac{x_n}{m-n}. \quad (9-23)$$

This transformation, which has norm  $\pi$ , may be considered as a discrete analogue of the conjugate function. There exist  $y \in l^2$  such that (9-17) has no solution  $x \in l^2$ .

(b) If  $a_n(t) = e^{i\alpha(n-t)}$ , (9.17) reduces to

$$y_m = \frac{\sin \pi \alpha}{\pi} \sum_n \frac{x_n}{\alpha + m - n}$$

if  $\alpha$  is non-integral, and to

$$y_m = (-1)^\alpha x_{m+\alpha}$$

for  $\alpha$  integral. We have here a group of unitary transformations.

### 10. Classes $L_\Phi^*$

As before, let  $L_\Phi(a, b)$  denote the class of all functions  $f$  such that  $\Phi(|f|)$  is integrable over  $(a, b)$ . The class  $L^r$ , corresponding to  $\Phi(u) = u^r$ , is the most important special case, but occasionally quite natural problems lead to other classes. For example, the class  $L \log^+ L$  of functions  $f$  such that  $|f| \log^+ |f|$  is integrable, is of importance in several problems. This leads us to the question whether a class  $L_\Phi$  can be so modified as to give a normed linear space.

First of all we have to define a norm  $\|x\| = \|x\|_\Phi$ , and if the definition is to be useful, the finiteness of  $\|x\|$  and the integrability of  $\Phi(|x(t)|)$  should be to a reasonable extent equivalent. The idea (modelled on the case  $\Phi(u) = u^r$ ) of setting

$$\|x\| = \Phi_{-1} \left[ \int_a^b \Phi(|x|) dt \right],$$

where  $\Phi_{-1}$  is the function inverse to  $\Phi$ , must be discarded. First of all, the condition  $\|ax\| = |a| \|x\|$  would not, in general, be satisfied. Further, if  $\Phi(u)$  increases too rapidly the integrability of  $\Phi(|x|)$  need not imply that of  $\Phi(2|x|)$ . For these reasons we must proceed differently, and it turns out that a simple solution is possible if simultaneously with  $\Phi$  we consider another function  $\Psi$  such that the pair  $\Phi, \Psi$  is complementary in the sense of Young (Chapter I, §9). We have shown on p. 25 that for any function  $\Phi(u), u \geq 0$ , which is non-negative, convex, vanishing at the origin and such that  $\Phi(u)/u \rightarrow \infty$  with  $u$ , we can find such a  $\Psi$ . In the rest of this section we suppose that  $\Phi$  has all the properties just stated.

Consider the functions  $x(t), a \leq t \leq b$ , such that the product  $x(t)y(t)$  is integrable over  $(a, b)$  for every  $y(t) \in L_\Psi(a, b)$ , and set

$$\|x\| = \|x\|_\Phi = \sup_y \left| \int_a^b x(t)y(t) dt \right|, \tag{10.1}$$

the sup being with respect to all  $y$  with

$$\rho_y = \int_a^b \Psi(|y|) dt \leq 1.$$

The class of such  $x$  will be denoted by  $L_\Phi^*$ . It is not obvious that  $\|x\|$  must be finite. We prove this a little later; in the meanwhile, to avoid difficulty, we denote by  $L_\Phi^{*f}$  the class of functions  $x(t), a \leq t \leq b$ , such that the norm  $\|x\|_\Phi$  just defined is finite. Using Young's inequality, we see that  $L_\Phi \subset L_\Phi^{*f}$ . It is immediate that  $L_\Phi^{*f}$  is a normed linear space. Classes  $L_\Phi^*$  are often called *Orlicz spaces*.

We shall prove that  $L_\Phi^*$  is a complete space. Suppose that  $\|x_m - x_n\| \rightarrow 0$  for  $m, n \rightarrow \infty$ , so that  $\|x_m - x_n\| < \epsilon$  for  $m, n > \nu = \nu(\epsilon)$ . It follows that

$$\left| \int_a^b (x_m - x_n) y dt \right| \leq \epsilon, \tag{10.2}$$

and so also 
$$\int_a^b |x_m - x_n| |y| dt \leq \epsilon, \quad (10-3)$$

if  $\rho_\nu \leq 1$ , and  $m, n > \nu$ . Let  $\alpha$  be such that  $(b-a) \alpha = 1$ . Taking  $y(t) = \alpha \overline{\text{sign}}(x_m - x_n)$ , we get from (10-2) that  $\mathfrak{M}[x_m - x_n; a, b] \leq \epsilon/\alpha$ . Since  $\epsilon$  is arbitrary, there is a sequence  $\{x_{m_n}(t)\}$  converging almost everywhere to a function  $x(t)$  (Chapter I, (11-1)), and (10-3) shows that  $\mathfrak{M}[(x - x_n)y; a, b] \leq \epsilon$  if  $\rho_\nu \leq 1$ . Thus  $\|x - x_n\| \leq \epsilon$  for  $n > \nu$ , and the completeness of the space  $L_\phi^*$  follows.

We have tacitly assumed here that  $b - a < \infty$ , but the result holds if  $b - a = \infty$ . In fact, proceeding as before, we show that  $\mathfrak{M}[x_m - x_n; a', b'] \rightarrow 0$  for any interval  $(a', b')$  satisfying  $a < a' < b' < b$ . Thence we infer the existence of a subsequence of  $\{x_{m_n}(t)\}$  converging almost everywhere in  $(a, b)$ , and the rest of the proof is unchanged. In what follows, to shorten the exposition, we shall assume that  $b - a < \infty$ . The results, however, are also valid for  $b - a = \infty$ , as simple modifications of the proofs show.

We have already observed that if  $x \in L_\phi$ , then  $x \in L_\phi^*$ . More generally, if there is a number  $\theta > 0$  such that  $\theta x \in L_\phi$ , then  $x \in L_\phi^*$ . Conversely, we show that, if  $x \in L_\phi^*$ , then there is a constant  $\theta > 0$  such that  $\theta x \in L_\phi$ . More precisely,

(10-4) THEOREM. If  $x \in L_\phi^*$ ,  $\|x\| \neq 0$ , then

$$\int_a^b \Phi(|x|/\|x\|) dt \leq 1. \quad (10-5)$$

We begin by showing that

$$\left| \int_a^b xy dt \right| \begin{cases} \leq \|x\|, & \text{if } \rho_\nu \leq 1; \\ \leq \|x\| \rho_\nu, & \text{if } \rho_\nu > 1. \end{cases} \quad (10-6)$$

The first inequality is obvious. Suppose now that  $\rho_\nu > 1$  and replace  $y$  by  $y/\rho_\nu$  in the integral on the left. Since  $\Psi$  is convex and  $\Psi(0) = 0$ ,  $\Psi(|y|/\rho_\nu) \leq \Psi(|y|)/\rho_\nu$ , so that

$$\int_a^b \Psi(|y|/\rho_\nu) dt \leq 1, \quad \left| \int_a^b x \frac{y}{\rho_\nu} dt \right| \leq \|x\|,$$

which is the second inequality (10-6). It follows that the integral in (10-6) does not exceed  $\|x\| \rho'_\nu$ , where

$$\rho'_\nu = \max(\rho_\nu, 1).$$

We shall now prove (10-5) for  $x$  bounded. Let  $\phi = \Phi'$ . For the special case  $y = \phi(|x|/\|x\|)$  that of equality in Young's inequality (pp. 16, 25), we have

$$\rho'_\nu \geq \left| \int_a^b \frac{x}{\|x\|} y dt \right| = \int_a^b \Phi \left[ \frac{|x|}{\|x\|} \right] dt + \rho_\nu.$$

We may suppose that the integral preceding  $\rho_\nu$  is not 0. From this it follows, first, that (for the special  $y$ )  $\rho'_\nu > \rho_\nu$ , and so  $\rho'_\nu = 1$ , and then that

$$\int_a^b \Phi(|x|/\|x\|) dt \leq 1.$$

In the general case, let  $x_n(t)$  be equal to  $x(t)$  wherever  $|x(t)| \leq n$  and equal to 0 elsewhere. Then (10-5) holds if the integrand on the left is  $\Phi(|x_n|/\|x_n\|)$ ,  $n$  being so large that  $\|x_n\| \neq 0$ , hence also for  $\Phi(|x_n|/\|x\|)$ , and finally, making  $n \rightarrow \infty$ , for  $\Phi(|x|/\|x\|)$ .

- (10-7) THEOREM. (i) If the integral (9-8) exists for every  $x \in L_{\Phi}^*(a, b)$ , then  $y \in L_{\Psi}^*(a, b)$ .  
 (ii) If the sequence (9-12) is bounded for every  $x \in L_{\Phi}^*$ , then  $\|y_n\|_{\Psi} = O(1)$ .  
 (iii) If the sequence (9-12) is bounded for every  $x \in L_{\Phi}$  then there is a constant  $\theta > 0$  such that  $\mathfrak{M}[\Psi(\theta |y_n|)] = O(1)$ .

(i) Let  $y^n$  be the function  $y$  truncated by  $n$ , and let us consider the integral  $\int_a^b xy^n dt$ . Since each  $y^n$ , as a bounded function, belongs to  $L_{\Psi}$ , and since

$$|u_n(x)| \leq \|x\|_{\Phi} \rho'_{y^n},$$

$u_n(x)$  is a functional in  $L_{\Phi}^*$ . By hypothesis,  $\{u_n(x)\}$  converges for every  $x \in L_{\Phi}^*$ , so that there is a constant  $M$  such that  $\|u_n(x)\| \leq M \|x\|_{\Phi}$  for  $n = 1, 2, \dots$ . Now take any  $x$  such that  $\mathfrak{M}[\Phi(|x|)] \leq 1$ . Such an  $x$  belongs to  $L_{\Phi}^*$  and has norm  $\|x\|_{\Phi} \leq 2$ . But since the inequality  $\left| \int_a^b xy^n dt \right| \leq 2M$  is valid for every  $x$  with  $\mathfrak{M}[\Phi(|x|)] \leq 1$  we have  $\|y^n\|_{\Psi} \leq 2M$  for  $n = 1, 2, \dots$ , and so also

$$\|y\|_{\Psi} \leq 2M.$$

(ii) By virtue of (i), each  $y_n$  belongs to  $L_{\Psi}^*$ . It follows from the inequality

$$\lambda |u_n(x)| \leq \|x\|_{\Phi} \rho'_{\lambda y_n},$$

where  $\lambda = \lambda_n$  is a positive constant so small that  $\lambda y_n \in L_{\Psi}$ , that  $u_n(x)$  is a functional in  $L_{\Phi}^*$ . Thus, by (9-5),  $|u_n(x)| \leq M \|x\|_{\Phi}$ , for  $n = 1, 2, \dots$ . In particular,  $\left| \int_a^b xy_n dt \right| \leq 2M$  for  $x$  satisfying  $\mathfrak{M}[\Phi(|x|)] \leq 1$ . Thus

$$\|y_n\|_{\Psi} \leq 2M \quad \text{for } n = 1, 2, \dots$$

(iii) Suppose that  $x \in L_{\Phi}^*$ . Then  $x/\|x\|_{\Phi} \in L_{\Phi}$ , and the sequence  $\{u_n(x)/\|x\|_{\Phi}\}$  is bounded for every  $x \in L_{\Phi}^*$ . By (ii),  $\|y_n\|_{\Psi} \leq N$ , say, for  $n = 1, 2, \dots$ , so that

$$\int_a^b \Psi(\theta |y_n|) dt \leq 1 \quad \text{for } \theta = 1/N.$$

This completes the proof of (10-7).

We can now dispose of the superfluous restriction in the definition of  $L_{\Phi}^*$ , and show that if  $\int_a^b xy dt$  exists for every  $y$  with  $\mathfrak{M}[\Psi(|y|)] \leq 1$ , then the norm  $\|x\|$  defined by (10-1) is finite. Let  $x^n(t)$  be the function  $x$  truncated by  $n$ . By hypothesis, the sequence of integrals  $v_n(y) = \int_a^b x^n y dt$  converges for every  $y \in L_{\Phi}^*$ , for then

$$\int_a^b \Psi(|y|/\|y\|) dt \leq 1.$$

It follows, as in (i), that  $\|x^n\|_{\Phi} = O(1)$ , that is,  $\|x\|_{\Phi} < \infty$ .

It is useful to note that (10-7) holds if we consider the class  $L_{\Phi}^*(E)$ , where  $E$  is an arbitrary set.

We have already observed that a necessary and sufficient condition that  $x(t) \in L_{\Phi}^*$  is the existence of a constant  $\theta > 0$  such that  $\theta x \in L_{\Phi}$ . It follows that if there is a constant  $C$  such that

$$\Phi(2u) \leq C\Phi(u) \tag{10-8}$$

for  $u$  large enough, and if  $b - a < \infty$ , the classes  $L_{\Phi}$  and  $L_{\Phi}^*$  are identical.

A simple calculation shows that if  $\Phi(u) = u^r$ ,  $r > 1$ , then  $\|x\|_\Phi = r'^{1/r} r^{1/r} \mathfrak{M}_r[x]$ , so that, apart from a numerical constant factor, we have the same norm as in Example (d) on p. 163.

We may define the norm of  $x$  in a somewhat different way. We fix a  $\Phi(u)$  ( $0 \leq u < \infty$ ) convex, non-decreasing, and satisfying  $\Phi(0) = 0$  and  $\Phi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$ . Let  $\Psi(v)$  be complementary to  $\Phi(u)$  in the sense of Young, and let  $\phi = \Phi'$ ,  $\psi = \Psi'$ . It will be convenient to normalize  $\Phi$  by the condition

$$\Phi(1) + \Psi(1) = 1. \tag{10.9}$$

This can be done, for example, by replacing  $\phi(u)$  by  $\phi(ku)$ ,  $k$  being such that the area of the rectangle with opposite vertices at  $(0, 0)$  and  $(k, \phi(k))$  is 1 (this area is a continuous function of  $k$ , and increases from 0 to  $\infty$  with  $k$ ); the new  $\Phi(u)$  is then the old  $k^{-1}\Phi(ku)$ .

Consider the class of all functions  $x(t)$ ,  $a \leq t \leq b$ , such that  $\Phi(\theta |x|)$  is integrable over  $(a, b)$  for some  $\theta > 0$  (the normalization does not affect the class). We then have

$$\int_a^b \Phi(\lambda^{-1} |x|) dt \leq \Phi(1) \tag{10.10}$$

for all large enough positive  $\lambda$ . Let  $\lambda_0$  be the lower bound of all  $\lambda$ 's satisfying (10.10);  $\lambda_0$  is non-negative, and is 0 if and only if  $x \equiv 0$ . We call  $\lambda_0$  the norm of  $x$ , and denote it by  $Nx$  or  $N_\Phi x$ .  $Nx$  is non-negative, is 0 if and only if  $x \equiv 0$ , and  $N(\alpha x) = |\alpha| Nx$  for every scalar  $\alpha$ . We also have

$$N(x + y) \leq Nx + Ny. \tag{10.11}$$

For if  $\lambda > Nx$ ,  $\mu > Ny$ , then from (10.10) and the analogous inequality for  $|y|/\mu$  we obtain, using Jensen's inequality,

$$\begin{aligned} \int_a^b \Phi\left(\frac{|x+y|}{\lambda+\mu}\right) dt &\leq \frac{\lambda}{\lambda+\mu} \int_a^b \Phi\left(\frac{|x|}{\lambda}\right) dt + \frac{\mu}{\lambda+\mu} \int_a^b \Phi\left(\frac{|y|}{\mu}\right) dt \\ &\leq \frac{\lambda}{\lambda+\mu} \Phi(1) + \frac{\mu}{\lambda+\mu} \Phi(1) = \Phi(1). \end{aligned}$$

This implies that  $N(x + y) \leq \lambda + \mu$ , and making  $\lambda \rightarrow Nx$ ,  $\mu \rightarrow Ny$  we deduce (10.11). Thus  $Nx$  has all the properties of a norm.

If  $\Phi(u) = u^p/p$ , where  $1 < p < \infty$ , then  $\Psi(v) = v^{p'}/p'$ , (10.9) holds, and we immediately see that  $Nx = \|x\|_p$ .

The class of all functions  $x(t)$  such that  $\theta x$  is in  $L_\Phi(a, b)$  for some  $\theta > 0$  is, as we know, the class  $L_\Phi^*$  for which we have already defined a norm  $\|x\|_\Phi$ . Write  $\Phi(1) = \nu$ . Since  $0 < \nu < 1$  we deduce from (10.5) that

$$\int_a^b \Phi\left(\frac{\nu x}{\|x\|_\Phi}\right) dt \leq \nu \int_a^b \Phi\left(\frac{x}{\|x\|_\Phi}\right) dt \leq \nu = \Phi(1),$$

so that  $Nx \leq \|x\|_\Phi/\nu$ . On the other hand, if  $\lambda > Nx$  then for all  $y$  with  $\mathfrak{M}[\Psi(|y|)] \leq 1$  we have

$$\left| \int_a^b xy dt \right| \leq \lambda \int_a^b \left| \frac{x}{\lambda} \right| |y| dt \leq \lambda \left[ \int_a^b \Phi\left(\frac{|x|}{\lambda}\right) dt + \int_a^b \Psi(|y|) dt \right] \leq \lambda[\Phi(1) + 1] < 2\lambda,$$

since, by (10-9),  $\Phi(1) < 1$ . Taking the upper bound of the first integral with respect to all  $y$ , and making  $\lambda \rightarrow Nx$ , we see that  $\|x\|_{\Phi} \leq 2Nx$ . Thus

$$\Phi(1)Nx \leq \|x\|_{\Phi} \leq 2Nx. \tag{10-12}$$

These inequalities show that the norms  $\|x\|_{\Phi}$  and  $Nx$  are *equivalent*, in the sense that their ratio is contained between two positive absolute constants. They show that, since  $L_{\Phi}^*$  is complete with respect to the norm  $\|x\|_{\Phi}$ , it is also complete (and so is a Banach space) with respect to the norm  $N_{\Phi}x$ .

We apply the foregoing to homogenizing certain inequalities.

Suppose we have an operation  $y = Tx$  transforming functions  $x(s) \in L_{\Phi}(a, b)$  into functions  $y(t) \in L_{\Phi_1}(a_1, b_1)$ , and such that

$$\int_{a_1}^{b_1} \Phi_1(|y|) dt \leq A \int_a^b \Phi(|x|) ds + B, \tag{10-13}$$

where  $\Phi$  and  $\Phi_1$  are Young's functions, and  $A$  and  $B$  are constants independent of  $x$ . The operation  $T$  need not be linear; we assume only that it is *positively homogeneous*, by which we mean that, for each scalar  $\alpha$ , if  $Tx$  is defined so is  $T(\alpha x)$ , and

$$|T(\alpha x)| = |\alpha| |Tx|.$$

(10-14) **THEOREM.** *Under the hypotheses just stated,  $Tx$  is defined for all  $x \in L_{\Phi}^*(a, b)$ , and we have*

$$\|y\|_{\Phi_1} \leq C \|x\|_{\Phi}. \tag{10-15}$$

or, equivalently,

$$N_{\Phi_1}y \leq CN_{\Phi}x, \tag{10-16}$$

where the  $C$ 's are constants independent of  $x$ .

The advantage of (10-15) and (10-16) over (10-13) is the homogeneity of the relations.

That  $Tx$  is defined in  $L_{\Phi}^*(a, b)$  follows from the positive homogeneity of  $T$  and the fact that  $x \in L_{\Phi}^*$  implies  $x/\|x\|_{\Phi} \in L_{\Phi}$  (see (10-5)). Write  $\lambda = N_{\Phi}x$ ,  $\lambda_1 = N_{\Phi_1}y$ . From (10-13) applied to  $x/\|x\|_{\Phi}$  we see that  $\Phi_1(y/\|x\|_{\Phi})$  is integrable, so that  $y \in L_{\Phi_1}^*(a_1, b_1)$  and  $\lambda_1 = N_{\Phi_1}y$  is finite. We have to show that  $\lambda_1 \leq C\lambda$ .

If  $\lambda = 0$ , then  $x \equiv 0$  and (by the positive homogeneity of  $T$ )  $y \equiv 0$ ,  $\lambda_1 = 0$ , so that (10-16) holds. We may therefore suppose that  $\lambda > 0$ . We may also suppose that  $\lambda_1 > \lambda$ , for if  $\lambda_1 \leq \lambda$  we have (10-16) with  $C = 1$ .

Let  $\lambda < \lambda' < \lambda'_1 < \lambda_1$ . Applying (10-13) to  $x/\lambda'$  we obtain

$$\int_{a_1}^{b_1} \Phi_1\left(\frac{|y|}{\lambda'}\right) dt \leq A \int_a^b \Phi\left(\frac{|x|}{\lambda'}\right) ds + B \leq A\Phi(1) + B = C_1,$$

say, and since  $\Phi_1$  is convex and  $\Phi_1(0) = 0$ ,

$$\int_{a_1}^{b_1} \Phi_1\left(\frac{|y|}{\lambda'}\right) dt = \int_{a_1}^{b_1} \Phi_1\left(\frac{\lambda'_1}{\lambda'} \frac{|y|}{\lambda'_1}\right) dt \geq \frac{\lambda'_1}{\lambda'} \int_{a_1}^{b_1} \Phi_1\left(\frac{|y|}{\lambda'_1}\right) dt \geq \frac{\lambda'_1}{\lambda'} \Phi_1(1)$$

which, combined with the preceding inequalities, gives  $\lambda'_1 \leq C\lambda'$ , and so also (10-16), with  $C = C_1/\Phi_1(1)$ . This completes the proof of (10-14).

*Remark.* The argument holds in the extreme case when  $\Phi(u) = u$  or  $\Phi_1(u) = u$ .

As an illustration consider functions  $x(s)$  integrable over a finite interval  $(a, b)$  and the operation  $\Theta = Tx$  defined by

$$\Theta(t) = \sup_{\lambda} \frac{1}{\lambda} \int_t^{t+\lambda} |x(s)| ds.$$

Though not linear, the operation is positively homogeneous and satisfies (10.13) with  $(a_1, b_1) = (a, b)$ ,  $\Phi(u) = u \log^+ u$ ,  $\Phi_1(u) = u$  (Chapter I, (13.15) (iii)). It therefore also satisfies (10.15) and (10.16). A similar remark applies to the operation  $Tf = f$  (see Chapter VII, (2.9)).

The theorem which follows generalizes Hölder's inequality.

(10.17) **THEOREM.** *If  $x(t) \in L_{\Phi}^*(a, b)$ ,  $y(t) \in L_{\Psi}^*(a, b)$ , where  $\Phi$  and  $\Psi$  are complementary in the sense of Young, then  $xy$  is integrable over  $(a, b)$  and*

$$\left| \int_a^b xy dt \right| \leq N_{\Phi} x \cdot N_{\Psi} y. \quad (10.18)$$

This is immediate if either  $N_{\Phi} x$  or  $N_{\Psi} y$  is 0. If neither is 0, then for  $\lambda > N_{\Phi} x$ ,  $\mu > N_{\Psi} y$  we have

$$\left| \int_a^b \frac{x}{\lambda} \frac{y}{\mu} dt \right| \leq \int_a^b \left| \frac{x}{\lambda} \right| \left| \frac{y}{\mu} \right| dt \leq \int_a^b \Phi \left( \frac{|x|}{\lambda} \right) dt + \int_a^b \Psi \left( \frac{|y|}{\mu} \right) dt \leq \Phi(1) + \Psi(1) = 1, \quad (10.19)$$

and making  $\lambda \rightarrow N_{\Phi} x$ ,  $\mu \rightarrow N_{\Psi} y$  we obtain (10.18).

We now consider cases of equality in (10.18) and we may suppose  $\lambda_0 = N_{\Phi}(x)$  and  $\mu_0 = N_{\Psi} y$  are both positive, for otherwise we always have equality. Recalling the definition of  $\lambda_0$ , we observe that if (10.10) holds for all  $\lambda > \lambda_0$  then, by Fatou's lemma, it holds for  $\lambda = \lambda_0$ . Examples show that we can have strict inequality in (10.10) even if  $\lambda = \lambda_0$ , and we are interested in cases when

$$\int_a^b \Phi \left( \frac{|x|}{\lambda_0} \right) dt = \Phi(1). \quad (10.20)$$

If the integral (10.10) is finite for some  $\lambda < \lambda_0$ , then it is a continuous and strictly decreasing function of  $\lambda$  near  $\lambda_0$ , and (10.20) holds. Hence, for example, (10.20) holds if  $\phi(2u) \leq C\phi(u)$  for all  $u$ ; or if the inequality holds for  $u$  large enough and  $(a, b)$  is finite.

Suppose now that we have (10.20), and that the integral of  $\Psi(\mu_0^{-1}|y|)$  over  $(a, b)$  is also 1. If we substitute  $\lambda_0, \mu_0$  for  $\lambda, \mu$  in (10.19), the extreme terms are equal and, considering the cases of equality in Young's inequality  $\xi\eta \leq \Phi(\xi) + \Psi(\eta)$ , we come to the following conclusion: if  $\lambda_0 \neq 0, \mu_0 \neq 0$ , we have equality in (10.18) if and only if

- (i)  $\arg(xy)$  is constant almost everywhere in the set where  $xy \neq 0$ ;
- (ii) the point  $(x(t)\lambda_0^{-1}, y(t)\mu_0^{-1})$  is almost always on the continuous curve obtained from  $\eta = \phi(\xi)$  by adjoining vertical segments at the points of discontinuity of  $\phi$ .

The arguments of this section apply to Stieltjes integrals  $\int_a^b \Phi(|x(t)|) d\mu(t)$ , where  $\mu(t)$  is non-decreasing, and in particular to sums  $\Sigma \Phi(a_i)$ . We may define norms  $\|a\|_{\Phi}$  and  $N_{\Phi} a$  for sequences  $a = \{a_i\}$ , we have Hölder's inequality  $|\Sigma a_i b_i| \leq N_{\Phi} a \cdot N_{\Psi} b$ , an analogue of (10.14), etc.

## 11. Conversion factors for classes of Fourier series

Consider two trigonometric series

$$\sum_{-\infty}^{\infty} c_n e^{inx}, \quad (11.1)$$

$$\sum_{-\infty}^{\infty} \lambda_n e^{inx}, \quad (11.2)$$

and the associated series 
$$\sum_{-\infty}^{\infty} c_\nu \lambda_\nu e^{i\nu x}. \tag{11.3}$$

By  $\{\lambda_\nu\}$  we shall now understand the two-way infinite sequence  $\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots$ . Given two classes P and Q of trigonometric series we shall say that  $\{\lambda_\nu\}$  is of type (P, Q) if whenever (11.1) belongs to P, (11.3) belongs to Q.

(11.4) THEOREM. A necessary and sufficient condition for  $\{\lambda_\nu\}$  to be of any one of the types (B, B), (C, C), (L, L), (S, S) is that  $\sum \lambda_\nu e^{i\nu x}$  be a Fourier-Stieltjes series.

Let (11.1) be an  $S[f]$ , and let  $\sigma_n(x)$ ,  $l_n(x)$ ,  $\sigma_n^*(x)$  denote the (C, 1) means of the series (11.1), (11.2) and (11.3) respectively. Then

$$\sigma_n^*(x) = \sum_{\nu=-n}^n \left(1 - \frac{|\nu|}{n+1}\right) c_\nu \lambda_\nu e^{i\nu x} = \frac{1}{2\pi} \int_0^{2\pi} l_n(t) f(x-t) dt. \tag{11.5}$$

Let  $x=0$ . If  $\{\lambda_n\}$  is of type (C, C) or (B, B), the sequence  $\{\sigma_n^*(0)\}$  is bounded for every  $f \in C$ , and by (9.11) we have  $\mathfrak{M}[l_n(t)] = O(1)$ , whence (11.2) belongs to S. Conversely, if (11.2) is an  $S[dL]$ , we have

$$\sigma_n^*(x) = \frac{1}{2\pi} \int_0^{2\pi} \sigma_n(x-t) dL(t), \tag{11.6}$$

so that the uniform boundedness of  $\{\sigma_n(x)\}$  implies that of  $\{\sigma_n^*(x)\}$ . Similarly, if  $\sigma_m(x) - \sigma_n(x)$  tends uniformly to 0 as  $m, n \rightarrow \infty$ , so does  $\sigma_m^*(x) - \sigma_n^*(x)$ . This completes the proof of (11.4) for the types (B, B) and (C, C).

If  $\{\lambda_n\}$  is of type (S, S), it transforms, in particular, the series  $\sum_{-\infty}^{\infty} e^{i\nu x} \in S$  into the series (11.2), and the latter must therefore belong to S. Conversely, if (11.2) is an  $S[dL]$ , (11.6) gives

$$|\sigma_n^*(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\sigma_n(x-t)| |dL(t)|. \tag{11.7}$$

Integrating this over  $(0, 2\pi)$  and inverting the order of integration on the right, we get  $\mathfrak{M}[\sigma_n^*] \leq (v/2\pi) \mathfrak{M}[\sigma_n]$ , where  $v$  is the total variation of  $L(t)$  over  $(0, 2\pi)$ . Hence (11.3) belongs to S if (11.1) does.

It remains only to consider the case (L, L). Since

$$|\sigma_m^*(x) - \sigma_n^*(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} |\sigma_m(x-t) - \sigma_n(x-t)| |dL(t)|,$$

$$\mathfrak{M}[\sigma_m^* - \sigma_n^*] \leq (v/2\pi) \mathfrak{M}[\sigma_m - \sigma_n],$$

the sufficiency of the condition is obvious (see (5.5)). To prove the necessity, let us consider for each  $n$  a system  $I_n = (\alpha_1^n, \beta_1^n), (\alpha_2^n, \beta_2^n), \dots$  of non-overlapping intervals. It follows from (11.5) that

$$\int_{I_n} \sigma_n^*(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(t) \left\{ \int_{I_n} l_n(x-t) dx \right\} dt. \tag{11.8}$$

Suppose that (11.2) does not belong to S, so that the indefinite integrals of the functions  $l_n(x)$  are not of uniformly bounded variation. We can then find a sequence  $I_1, I_2, \dots$  such that the coefficient of  $f(t)$  in (11.8) is not bounded for  $t=0$ . Since it is continuous in  $t$  (for each  $n$ ), its essential upper bound is unbounded; thus by (9.11) there is an

$f \in L$  such that the right-hand side of (11.8) is unbounded, and a fortiori  $\mathfrak{M}[\sigma_n^*] \neq O(1)$ . Hence (11.3) does not belong to S, and  $\{\lambda_n\}$  is not of type (L, L).

Let  $\tilde{P}$  denote the class of trigonometric series conjugate to those belonging to P.

(11.9) THEOREM. A necessary and sufficient condition that  $\{\lambda_n\}$  should be of any one of the types  $(\tilde{B}, B)$ ,  $(\tilde{C}, C)$ ,  $(\tilde{L}, L)$ ,  $(\tilde{S}, S)$  is that the series conjugate to  $\Sigma \lambda_n e^{in x}$  should belong to S.

This follows from (11.4). For let  $\epsilon_\nu = -i \operatorname{sign} \nu$ . Saying that  $\{\lambda_n\}$  is of type  $(\tilde{B}, B)$  means that whenever  $\Sigma \epsilon_\nu e^{i\nu x}$  belongs to B, so does  $\Sigma \epsilon_\nu \lambda_\nu e^{i\nu x}$ . A necessary and sufficient condition for this is, by (11.4), that  $\Sigma \epsilon_\nu \lambda_\nu e^{i\nu x} \in S$ . Similarly for the remaining types.

(11.10) THEOREM. (i) A necessary and sufficient condition that  $\{\lambda_n\}$  should be of either of the types (B, C), (S, L) is that  $\Sigma \lambda_n e^{in x}$  should belong to L.

(ii) The types  $(\tilde{B}, C)$  and  $(\tilde{S}, L)$  are characterized by the fact that the series conjugate to  $\Sigma \lambda_n e^{in x}$  belongs to L.

It is enough to prove (i), the proof of (ii) being then analogous to that of (11.9).

Considering the series  $\sum_{-\infty}^{+\infty} e^{i\nu x} \in S$ , we see the necessity of the condition for type (S, L).

The sufficiency follows from the sufficiency for (L, L) in (11.4) on interchanging the roles of  $e_\nu$  and  $\lambda_\nu$  so that (11.3) belongs to L.

Let now  $f$  be any element from B. If  $\{\lambda_n\}$  is of type (B, C), we see, on taking  $x=0$  in (11.5) and using (9.13) (i), that the indefinite integrals of the  $l_n(x)$  must be uniformly absolutely continuous. Thus  $\Sigma \lambda_\nu e^{i\nu x} \in L$ . Conversely, if the latter condition is satisfied (11.5) implies that

$$2\pi | \sigma_m^*(x) - \sigma_n^*(x) | \leq \mathfrak{M}[l_m - l_n] \sup | f |.$$

Thus  $\{\sigma_m^*(x)\}$  converges uniformly and (11.3) belongs to C.

Let  $\chi(u)$ ,  $u \geq 0$ , be a function non-negative, non-decreasing, convex and such that  $\chi(u)/u \rightarrow \infty$  with  $u$ .

(11.11) THEOREM. If (11.1) is an  $S[f]$  such that  $\chi(|f|)$  is integrable, and if (11.2) is an  $S[dL]$ , then (11.3) is an  $S[g]$  such that  $\chi(2\pi |g|/v)$  is integrable,  $v$  denoting the total variation of  $L$  over  $(0, 2\pi)$ .

Jensen's inequality applied to (11.6) gives

$$\chi\{2\pi | \sigma_n^*(x) | / v\} \leq v^{-1} \int_0^{2\pi} \chi(|\sigma_n(x-t)|) |dL(t)|.$$

It is now sufficient to integrate this inequality over  $0 \leq x \leq 2\pi$ , invert the order of integration on the right, and apply (5.7). In particular, if (11.2) belongs to S,  $\{\lambda_n\}$  is of type  $(L^r, L^r)$  for every  $r \geq 1$ .

The condition imposed here on (11.2) is sufficient only. That it is not necessary is seen by the example  $\chi(u) = u^2$ , since by the Parseval formula and the Riesz-Fischer theorem  $\{\lambda_n\}$  is of type  $(L^2, L^2)$  if and only if  $\lambda_n = O(1)$ .

Let now  $\Phi, \Psi$  and  $\Phi_1, \Psi_1$  be two pairs of Young's complementary functions.

(11.12) THEOREM. The types  $(L_\Phi^*, L_{\Phi_1}^*)$  and  $(L_\Psi^*, L_{\Psi_1}^*)$  are identical.

(11·13) THEOREM. A necessary and sufficient condition that  $\Sigma c_n e^{inx}$  should belong to  $L^*_\Phi$  is that for every  $g \in L^*_\Psi$  with Fourier coefficients  $c'_n$ , the series

$$\sum_{-\infty}^{\infty} c_n c'_n \tag{11·14}$$

should be bounded (C, 1) (or, what is here equivalent, summable (C, 1)).

If  $f \in L^*_\Phi$ ,  $g \in L^*_\Psi$ , there exist positive constants  $\lambda, \mu$  such that  $\lambda f \in L_\Phi$ ,  $\mu g \in L_\Psi$ , and the necessity in (11·13) follows from (8·7). For the sufficiency, let  $\sigma_n(x)$  and  $\tau_n$  denote the (C, 1) means of  $\Sigma c_n e^{inx}$  and (11·14) respectively. Then

$$\tau_n = \frac{1}{2\pi} \int_0^{2\pi} g(-t) \sigma_n(t) dt.$$

Since  $\{\tau_n\}$  is assumed bounded for every  $g \in L^*_\Psi$ , it follows that  $\Sigma c_n e^{inx}$  belongs to  $L^*_\Phi$  (see (10·7)). This proves (11·13).

To prove (11·12), we now note that, by (11·13), if  $\{\lambda_n\}$  is of type  $(L^*_\Phi, L^*_{\Phi_1})$  then for every  $f \in L^*_\Phi$  with Fourier coefficients  $c_n$  and every  $g \in L^*_{\Phi_1}$  with coefficients  $c'_n$ , the series  $\sum_{-\infty}^{\infty} \lambda_n c_n c'_n$  is finite (C, 1). By (11·13) this also means that  $\Sigma \lambda_n c'_n e^{inx} \in L^*_\Psi$ , so that  $\{\lambda_n\}$  is of type  $(L^*_\Psi, L^*_{\Psi_1})$ .

As corollaries we get

- (i) if  $\Phi$  and  $\Psi$  are complementary functions, the types  $(L^*_\Phi, L^*_{\Phi_1})$  and  $(L^*_\Psi, L^*_{\Psi_1})$  are identical;
- (ii) if  $r > 1, s > 1$ , the types  $(L^r, L^s)$  and  $(L^s, L^r)$  are identical; in particular,

$$(L^r, L^r) = (L^r, L^r).$$

Suppose that (11·2) is  $S[dL]$  and (11·1) and (11·3) are respectively  $S[f]$  and  $S[g]$ , with  $f$  continuous. The formula

$$g(x) = \frac{1}{2\pi} \int_0^{2\pi} f(x-t) dL(t)$$

leads immediately to the following inequality for the moduli of continuity of  $f$  and  $g$ :

$$\omega(\delta; g) \leq \omega(\delta; f) \frac{1}{2\pi} \int_0^{2\pi} |dL|. \tag{11·15}$$

(11·16) THEOREM. Suppose that  $\Sigma \lambda_n e^{inx}$  belongs to  $S$ . Then  $\{\lambda_n\}$  is of type  $(\Lambda_\alpha, \Lambda_\alpha)$ ,  $0 < \alpha \leq 1$ , and of types  $(\Lambda_+, \Lambda_+)$  and  $(\lambda_+, \lambda_+)$ .

The assertion concerning the type  $(\Lambda_\alpha, \Lambda_\alpha)$  follows from (11·15), and the remainder from the similar inequality

$$\max_x |g(x+h) + g(x-h) - 2g(x)| \leq \max_x |f(x+h) + f(x-h) - 2f(x)| \frac{1}{2\pi} \int_0^{2\pi} |dL|.$$

The results obtained above may be stated in terms of 'real Fourier series'. If  $c_n = \frac{1}{2}(a_n - ib_n)$ ,  $c_{-n} = \bar{c}_n$ , (11·1) becomes

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{11·17}$$

If the  $\lambda_r$  are real and  $\lambda_{-r} = \lambda_r$  (11.2) and (11.3) become

$$2\left(\frac{1}{2}\lambda_0 + \sum_{r=1}^{\infty} \lambda_r \cos vx\right), \tag{11.18}$$

$$\frac{1}{2}a_0\lambda_0 + \sum_{r=1}^{\infty} (a_r \cos vx + b_r \sin vx) \lambda_r, \tag{11.19}$$

respectively. In translating the former results we have merely to substitute the new names (11.17), (11.18), (11.19) for the old (11.1), (11.2), (11.3).

The following illustration is useful.

Let  $\lambda_0, \lambda_1, \lambda_2, \dots$  be an arbitrary positive sequence tending to 0 and convex from some place on. For example, we may take

$$\lambda_n = \frac{1}{n^\alpha} \quad (\alpha > 0), \quad \lambda_n = \frac{1}{\log n}, \quad \lambda_n = \frac{1}{\log \log n}, \quad \dots$$

for  $n$  large enough. In Chapter V, § 1, we show that series  $\sum \lambda_r \cos vx$  belong to L. It follows that such sequences are of types (B, C) and (S, L).

It will also be shown (Chapter V, § 1) that the series  $2\sum \mu_r \sin vx$  conjugate to (11.18) belongs to L, if  $\mu_1, \mu_2, \dots$  are positive and monotonically decreasing and  $\sum \mu_r / \nu < \infty$ . Thus, in particular, the sequences

$$\mu_n = \frac{1}{(\log n)^{1+\epsilon}}, \quad \mu_n = \frac{1}{\log n (\log \log n)^{1+\epsilon}}, \quad \dots$$

are both of type (B, C) and (S, L), provided  $\epsilon > 0$ . For  $\epsilon = 0$  this is no longer true.

MISCELLANEOUS THEOREMS AND EXAMPLES

1. Let  $\phi_1(x), \phi_2(x), \dots$  be a system orthonormal over  $(0, 2\pi)$ . Then the system  $(2\pi)^{-1/2} \phi_1(x), \phi_2(x), \dots$  is also orthonormal there. If the first system is complete so is the second.

[Use the formula  $\int_0^{2\pi} fg dx = -\int_0^{2\pi} f'g dx$  for  $f \in L^1, g \in L^2$ .]

2. Let  $\phi_1, \phi_2, \dots$  be an orthonormal system of uniformly bounded functions in  $(a, b)$ . Let  $s_n$  be the partial sums of the series

$$a_1\phi_1 + a_2\phi_2 + \dots$$

and suppose that the functions  $S_n(x) = \int_a^x s_n(t) dt$  are uniformly absolutely continuous in  $(a, b)$ , which is certainly the case if all the  $|s_n|$  are majorized by an integrable function. Then the series is a Fourier series.

[By (4.6), a subsequence  $S_{n_k}(x)$  converges to an absolutely continuous function  $F(x) = \int_a^x f dt$ .

We show that for any bounded  $g$

$$\int_a^b g s_{n_k} dt \rightarrow \int_a^b g f dt.$$

This is obvious for any step function  $g$ , and any bounded  $g$  can be uniformly approximated by such step functions except in sets of arbitrarily small measure. For  $g = \bar{\phi}_m$  we get  $a_m = \int_a^b f \bar{\phi}_m dt$ .]

3. The integrals  $\int_0^\pi \frac{f(x+t) - f(x)}{t} dt, \int_0^\pi \frac{f(x+t) + f(x-t) - 2f(x)}{t} dt$

can diverge for all  $x$ , even if  $f$  is continuous (§ 3). Show that if  $f \in L$ , and if one of the integrals exists for  $x \in E$ , the other two exist almost everywhere in  $E$ .

[The integral  $\int_0^\pi \frac{f(x+t) - f(x-t)}{t} dt$  exists almost everywhere.]

4. If  $f \sim \frac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$ , then for almost all  $x$  we have the formula

$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \log \left| \frac{1}{2 \sin \frac{1}{2}t} \right| dt,$$

the series on the left being convergent almost everywhere.

5. Let  $f(x)$  be continuous and periodic. (a) A necessary and sufficient condition for  $f(x)$  to be continuous is that

$$f(x; h) = -\frac{1}{\pi} \int_h^{\pi} \psi_n(t) \cot \frac{1}{2}t dt$$

converges uniformly as  $h \rightarrow +0$ . (b) If  $f \in L$ , then a necessary and sufficient condition for  $f \in L$  is that  $f(x; h)$  tends to a limit in  $L$ . (For (a) see Zemanaky [3].)

[For (a) use uniformity in the relation (3.21) of Chapter III.]

6. Let  $P(r, x)$  be the Poisson kernel and  $q \geq 1$  be fixed. Then

$$\left\{ \int_0^{2\pi} P^q(r, x) dx \right\}^{1/q} \sim (1-r)^{-1/q},$$

which shows that, unlike (6.32), we cannot replace 'O' by 'o' in the conclusion of (6.34).

7. If the integral modulus of continuity  $\omega_1(\delta; f)$  is  $o((\log 1/\delta)^{-1})$ , then  $\mathfrak{M}[f - S_n[f]] \rightarrow 0$ . This is an integral analogue of the Dini-Lipschitz test (Chapter II, § 10).

8. A necessary and sufficient condition for a periodic  $f$  to belong to the class  $\Lambda_1^1$  is that  $f$  should coincide almost everywhere with a function of bounded variation. (Hardy and Littlewood [9].)

[Necessity: If  $\sigma_n = \sigma_n(x; f)$  then  $\mathfrak{M}[\sigma_n(x+h) - \sigma_n(x)] \leq \mathfrak{M}[f(x+h) - f(x)] \leq Ch$ ,  $\mathfrak{M}[\sigma'_n] \leq C$  and we apply (4.7). Sufficiency: If  $f$  is non-decreasing over  $0 < x < 2\pi$ ,  $\mathfrak{M}[f(x+h) - f(x)]$  is

$$\int_0^{2\pi-h} [f(x+h) - f(x)] dx + \int_{2\pi-h}^{2\pi} O(1) dx = \left( \int_{2\pi-h}^{2\pi} - \int_0^h \right) f dx + O(h) = O(h).$$

9. A necessary and sufficient condition for a periodic  $f$  to be in  $\Lambda_1^p$ ,  $p > 1$ , is that  $f$  should be equivalent to the integral of a function of  $L^p$ . (Hardy and Littlewood [9].)

[Necessity: Arguing as in Example 8 we get  $\mathfrak{M}_p[\sigma'_n] \leq C$  and apply (5.7). Sufficiency:

$$\mathfrak{M}_p^2[f(x+h) - f(x)] \leq \int_0^{2\pi} \left\{ \int_x^{x+h} |f'(t)|^p dt \right\}^2 dx \leq h^2 \int_0^{2\pi} |f'(t)|^p dt.]$$

10. A trigonometric series with (C, 1) means  $\sigma_n$  is a Fourier series if and only if there is a function  $\Phi(u)$ ,  $u \geq 0$ , non-negative, non-decreasing, satisfying the condition  $\Phi(u)/u \rightarrow \infty$  with  $u$ , and such that  $\int_0^{2\pi} \Phi(|\sigma_n|) dx = O(1)$ .

[If  $f$  is in  $L$ , so is  $\Phi(|f|)$  with a suitable  $\Phi$  satisfying the above conditions. For any such  $\Phi$  there is also a convex  $\Phi_1 \leq \Phi$  satisfying the conditions.]

11. Let  $\Phi(u)$ ,  $u \geq 0$ , be convex and non-negative, and suppose that  $\Phi(u)/u \rightarrow \infty$  as  $u \rightarrow \infty$ . A necessary and sufficient condition for a trigonometric series to be an  $S[dF]$ , with  $F$  having an absolutely continuous positive variation  $P(x)$  such that  $P'(x) \in L_\Phi$ , is that  $\int_0^{2\pi} \Phi(f^+(r e^{i\alpha})) dx = O(1)$  as  $r \rightarrow 1$  (see (6.13)). Similarly for  $\sigma_n$ . The case  $\Phi(u) = u^p$  is the most interesting one.

12. Given a periodic  $f \in L^p$ ,  $1 \leq p < \infty$ , consider the set  $U = U_f$  of all functions  $\phi$  which are finite linear combinations, with constant coefficients, of functions  $f(x+\lambda)$ , where  $-\infty < \lambda < +\infty$ . Show that a necessary and sufficient condition for  $U$  to be dense in  $L^p$  is that no Fourier coefficient of  $f$  should vanish. (This is an elementary analogue of a deeper theorem of Wiener [3], concerning Fourier transforms.)

[Necessity: If  $e^{ikx}$  is absent in  $S[f]$  it is absent in  $S[\phi]$ , and so

$$\mathfrak{M}_p[e^{ikx} - \phi] \geq \mathfrak{M}_1[e^{ikx} - \phi] = \mathfrak{M}_1[1 - \phi e^{-ikx}] \geq \left| \frac{1}{2\pi} \int_0^{2\pi} (1 - \phi e^{-ikx}) dx \right| = 1$$

for all  $\phi$ . *Sufficiency*: It is enough to show that if all terms in  $S[f]$  are non-zero, then for any integer  $k$  and any  $\epsilon > 0$  there is a  $\phi$  such that  $\mathfrak{R}_p[\phi - e^{i2\pi k}] < \epsilon$ . We may suppose that  $k=0$  and that the constant term of  $S[f]$  is 1. Let  $f_1 = \sigma_{n-1}[f]$  with  $n$  so large that  $\mathfrak{R}_p[f - f_1] < \epsilon$ . By Chapter II, (1.1),  $n^{-1} \sum_{j=1}^n f_1(x + 2\pi j/n) = 1$ , so that

$$\mathfrak{R}_p[n^{-1} \sum_j f(x + 2\pi j/n) - 1] < \epsilon.$$

The argument (suggested by R. Salem) holds for  $p = \infty$  if  $L^p$  is replaced by  $C$ .]

13. Let  $g = Tf$  be a linear operation, defined for real-valued  $f$ , with  $g$  real-valued and such that

$$(i) \|g\|_r \leq M \|f\|_r.$$

For every complex-valued  $f = f_1 + if_2 \in L^r$  let us set  $T[f] = T[f_1] + iT[f_2]$ . Then (i) still holds.

[Let  $g_r = T_r'$ . Integrating the inequality

$$\|g_1 \cos \alpha + g_2 \sin \alpha\|_r \leq M \|f_1 \cos \alpha + f_2 \sin \alpha\|_r$$

over  $0 \leq \alpha \leq 2\pi$ , interchanging the order of integration and observing that

$$\int_0^{2\pi} |a \cos \alpha + b \sin \alpha|^r d\alpha = (a^2 + b^2)^{r/2} C_r,$$

we get

$$\|(g_1^2 + g_2^2)^{1/2}\|_r \leq M \|(f_1^2 + f_2^2)^{1/2}\|_r.$$

14. Let  $1 \leq p \leq \infty$ ,  $n = 0, 1, \dots$ . Let  $L_n^p$  (the generalized Lebesgue constant) be defined as  $\sup_f \mathfrak{R}_p[S_n]$  for all  $f$  with  $\mathfrak{R}_p[f] \leq 1$ , where  $S_n = S_n[f]$ . Show that  $L_n^p = L_n^{p'}$ .

15. Let  $L^+$  denote the class of  $S[f]$  with  $f$  integrable and non-negative. We define  $B^+$ ,  $C^+$  correspondingly, and we denote by  $S^+$  the class of  $S[dF]$  with  $F$  non-decreasing. Then  $(\lambda_n)$  is of any one of the types  $(B^+, B^+)$ ,  $(C^+, C^+)$ ,  $(L^+, L^+)$ ,  $(S^+, S^+)$  if and only if  $\sum \lambda_n e^{in\pi} \in S^+$ ; and  $(\lambda_n)$  is of type  $(B^+, C^+)$  or of type  $(S^+, L^+)$  if and only if  $\sum \lambda_n e^{in\pi} \in L^+$ .

16. Let  $\sigma_n(x)$  and  $f(r, x)$  be the  $(C, 1)$  and Abel means of a trigonometric series  $T$ . Each of the conditions

$$\|\sigma_n\|_\Phi = O(1), \quad \|f(r, x)\|_\Phi = O(1)$$

(the norms being understood in the sense of § 10) is both necessary and sufficient for  $T$  to belong to  $L_\Phi^*$ .

17. If  $f \in L_\Phi^*$  and if  $\Phi$  satisfies the condition

$$(i) \Phi(2u)/\Phi(u) = O(1) \text{ as } u \rightarrow \infty,$$

then

$$\|f - \sigma_n\|_\Phi \rightarrow 0, \quad \|\sigma_n\|_\Phi \rightarrow \|f\|_\Phi.$$

Similarly for the Abel mean  $f(r, x)$ .

18. Condition (i) of Example 17 is both necessary and sufficient for the space  $L_\Phi^*$  to be separable. (Orlicz [2].) [*Sufficiency* follows from Example 17.]

19. Given a matrix  $\{a_{mn}\}$ ,  $m, n = 0, 1, \dots$ , consider the linear means

$$\sigma_m = a_{m0}s_0 + a_{m1}s_1 + \dots + a_{mn}s_n + \dots$$

of any sequence  $\{s_n\}$ . Show that a necessary and sufficient condition that any convergent  $\{s_n\}$  be transformed into a convergent  $\{\sigma_m\}$  (not necessarily with the same limit) is that

- (i)  $\lim_m a_{mn}$  exist for each  $n$ ;
- (ii)  $\sum_n |a_{mn}| \leq C$ , with  $C$  independent of  $m$ ;
- (iii)  $A_m = \sum_n a_{mn}$  tend to a limit as  $m \rightarrow \infty$ .

If these conditions are satisfied and if the limits in (i) and (iii) are  $\alpha_n$  and  $A$  respectively, then  $\sum |\alpha_n| \leq C$  and for any  $\{s_n\} \rightarrow s$  we have

$$\sigma_m \rightarrow As + \alpha_0(s_0 - s) + \alpha_1(s_1 - s) + \dots \quad (\text{Schur [1].})$$

[The proof is similar to the arguments on pp. 74-5 and 168.]

## SPECIAL TRIGONOMETRIC SERIES

In this chapter we study some special trigonometric series which not only are interesting in themselves but also illustrate the general theory.

## 1. Series with coefficients tending monotonically to zero

In Chapter I, § 1, we proved that if  $\{a_\nu\}$  is monotonically decreasing to zero, both the series

$$\frac{1}{2}a_0 + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \quad (1.1)$$

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu x, \quad (1.2)$$

converge uniformly outside an arbitrarily small neighbourhood of  $x=0$ . If  $a_\nu \geq 0$  for all  $\nu$  then obviously a necessary and sufficient condition for the uniform convergence everywhere of (1.1) is the convergence of  $\sum a_\nu$ . For (1.2), the situation is less obvious.

(1.3) THEOREM. Suppose that  $a_\nu \geq a_{\nu+1}$  and  $a_\nu \rightarrow 0$ . Then a necessary and sufficient condition for the uniform convergence of (1.2) is  $\nu a_\nu \rightarrow 0$ .

If (1.2) converges uniformly and if  $x = \pi/2n$ , then

$$\sum_{\{\frac{1}{2}n\}+1}^n a_\nu \sin \nu x \geq \sin \frac{1}{4}\pi \cdot a_n \sum_{\{\frac{1}{2}n\}+1}^n 1 \geq \sin \frac{1}{4}\pi \cdot a_n \cdot \frac{1}{2}n,$$

so that  $na_n \rightarrow 0$  as  $n \rightarrow \infty$ . This proves the necessity.

Conversely, let  $\nu a_\nu \rightarrow 0$ , so that  $\epsilon_k = \sup_{\nu > k} \nu a_\nu \rightarrow 0$ . Let  $0 < x \leq \pi$ , and let  $N = N_\epsilon$  be the integer satisfying

$$\pi/(N+1) < x \leq \pi/N.$$

We split the remainder  $R_m(x) = a_m \sin mx + \dots$  of (1.2) into two parts,  $R_m = R'_m + R''_m$ , where  $R'_m$  consists of the terms with indices  $\nu < m + N$ , and  $R''_m$  of those with  $\nu \geq m + N$ .

Then

$$|R'_m(x)| = \left| \sum_m^{m+N-1} a_\nu \sin \nu x \right| \leq x \sum_m^{m+N-1} \nu a_\nu \leq x N \epsilon_m \leq \pi \epsilon_m. \quad (1.4)$$

Summing by parts and using the inequality  $|\bar{D}_m(x)| \leq \pi/x$  (Chapter II, § 5), we find

$$|R''_m| = \left| \sum_{m+N}^{\infty} (a_\nu - a_{\nu-1}) \bar{D}_\nu(x) - a_{m+N} \bar{D}_{m+N-1}(x) \right| \leq 2a_{m+N} \pi/x \leq 2(N+1)a_{m+N} \leq 2\epsilon_m.$$

Hence  $|R_m| < 8\epsilon_m$ , and the uniform convergence of (1.2) follows.

Remarks. (a) The above proof of sufficiency could be simplified if we knew that  $\{\nu a_\nu\}$  decreased monotonically to 0. For then we could write (1.2) in the form  $\sum \nu a_\nu (\nu^{-1} \sin \nu x)$ ; and since the partial sums of the series  $\sum \nu^{-1} \sin \nu x$  are uniformly bounded it would be enough to apply Theorem (2.4) of Chapter I.

(b) If  $a_\nu \geq a_{\nu+1} \rightarrow 0$ , the condition  $\nu a_\nu = O(1)$  is both necessary and sufficient for the uniform boundedness of the partial sums of (1.2); the proof is an obvious adaptation of that just given. As the example  $\sum \nu^{-1} \sin \nu x$  shows, this condition does not imply uniform convergence.

(c) Under the hypotheses  $a_\nu \geq a_{\nu+1}$ ,  $a_\nu \rightarrow 0$ , the condition  $\nu a_\nu \rightarrow 0$  is both necessary and sufficient for  $\sum a_\nu \sin \nu x$  to be the Fourier series of a continuous function. It is enough to prove that the condition is necessary. Suppose then that the (C, 1) means  $\sigma_n(x)$  of  $\sum a_\nu \sin \nu x$  converge uniformly. In particular  $\sigma_n(\pi/2n) \rightarrow 0$ . Since  $\sin u \geq (2/\pi)u$  in  $(0, \frac{1}{2}\pi)$ , we have

$$\sum_{\nu=1}^n a_\nu \left(1 - \frac{\nu}{n+1}\right) \frac{2}{\pi} \left(\frac{\pi\nu}{2n}\right) \rightarrow 0.$$

Keeping  $m = [\frac{1}{2}n]$  terms on the left, we obtain successively

$$\frac{1}{n} \sum_{\nu=1}^m \nu a_\nu \rightarrow 0, \quad \frac{1}{n} a_m \sum_{\nu=1}^m \nu \rightarrow 0, \quad m a_m \rightarrow 0.$$

There is a corresponding modification of (b).

(1.5) THEOREM. If  $a_\nu \rightarrow 0$  and the sequence  $a_0, a_1, \dots$  is convex, the series (1.1) converges, save possibly at  $x = 0$ , to a non-negative and integrable sum  $f(x)$ , and is the Fourier series of  $f$ .

Summing twice by parts we have

$$s_n(x) = \sum_{\nu=0}^{n-2} (\nu+1) \Delta^2 a_\nu K_\nu(x) + n K_{n-1}(x) \Delta a_{n-1} + D_n(x) a_n, \quad (1.6)$$

where  $s_n$  is the partial sum of (1.1) and  $D_n$  and  $K_\nu$  are Dirichlet's and Fejér's kernels. If  $x \neq 0$ , the last two terms tend to 0 as  $n \rightarrow \infty$ . Thus  $s_n(x)$  tends to the limit

$$f(x) = \sum_{\nu=0}^{\infty} (\nu+1) \Delta^2 a_\nu K_\nu(x). \quad (1.7)$$

which is non-negative,  $\{a_\nu\}$  being convex. Also

$$\int_{-\pi}^{\pi} f(x) dx = \sum_{\nu=0}^{\infty} (\nu+1) \Delta^2 a_\nu \int_{-\pi}^{\pi} K_\nu(x) dx = \pi \sum_{\nu=0}^{\infty} (\nu+1) \Delta^2 a_\nu < +\infty$$

(see Chapter III, § 4) so that  $f$  is integrable.

In proving that (1.1) is  $S[f]$ , we may suppose that  $a_0 = 0$ . Since  $a_1, a_2, \dots$  monotonically decreases to 0 an application of (1.3) (see also Remark (a) above) shows that the series  $\sum \nu^{-1} a_\nu \sin \nu x$  obtained by termwise integration of (1.1) converges uniformly to a continuous function  $F(x)$ , and so is  $S[F]$ . Since  $F'(x)$  exists and is continuous for  $x \neq 0$ , and since  $F(x)$  is continuous everywhere,  $F$  is a primitive of  $F' = f$ . If we first integrate over the interval  $(\epsilon, \pi)$  and then make  $\epsilon \rightarrow 0$  we get, since  $F(0) = F(\pi) = 0$ ,

$$\frac{a_\nu}{\nu} = \frac{2}{\pi} \int_0^\pi F(x) \sin \nu x dx = \frac{2}{\pi \nu} \int_0^\pi f(x) \cos \nu x dx,$$

$$a_\nu = \frac{1}{\pi} \int_0^\pi f(x) \cos \nu x dx \quad (\nu > 0).$$

This is also true for  $\nu = 0$ , since,  $F$  being periodic,  $\int_{-\pi}^{\pi} f dt = 0 = \pi a_0$ . So (1.1) is  $S[f]$ .

In this proof that (1.1) is a Fourier series we actually used only the hypothesis that  $\{a_\nu\}$  decreases monotonically to 0. Hence we have:

(1·8) THEOREM. If  $a_n \rightarrow 0$ ,  $\Delta a_n \geq 0$ , the sum  $f(x)$  of (1·1) is continuous for  $x \neq 0$ , has a Riemann integral (in general improper) and is the Fourier-Riemann series of  $f$ .

If  $\{a_n\}$  is also convex, then, as we have just seen,  $f$  is non-negative and  $F$  is a Lebesgue integral of  $f$ . The mere fact that  $\{a_n\}$  is monotone, however, does not ensure the L-integrability of  $f$ ; we have, in fact,

(1·9) THEOREM. There is a series (1·1) with coefficients monotonically decreasing to 0 and sum  $f(x)$  not integrable L.

Suppose we have a sequence  $0 = \lambda_1 < \lambda_2 < \dots$  such that  $a_n$  is constant for  $\lambda_n < k \leq \lambda_{n+1}$ ,  $n = 1, 2, \dots$ . Summation by parts gives

$$f(x) = \sum_{\nu=0}^{\infty} \Delta a_{\nu} \cdot D_{\nu}(x) = \sum_{n=1}^{\infty} \alpha_n D_{\lambda_n}(x), \quad \alpha_n = \Delta a_{\lambda_n}. \quad (1\cdot10)$$

We now observe that

$$\int_0^{\pi} |D_n| dx \leq C \log n, \quad \int_{1/n}^{\pi} |D_n| dx \geq C_1 \log n.$$

These both follow from Chapter II, (12·1); for since  $D_n(x) = O(n)$ , the difference between the two integrals is  $O(1)$ . From (1·10), observing that  $|D_n(x)| < 2/x$  for  $0 < x \leq \pi$ , we get

$$\int_{1/\lambda_m}^{\pi} |f| dx \geq C_1 \alpha_m \log \lambda_m - C \sum_{n=1}^{m-1} \alpha_n \log \lambda_n - 2 \log(\pi \lambda_m) \sum_{n=m+1}^{\infty} \alpha_n.$$

Taking

$$\alpha_n = 1/n!, \quad \lambda_n = 2^{(n)!^2}$$

and arguing as on p. 134, we find that the last integral is unbounded as  $m \rightarrow \infty$ .

Given an arbitrary sequence of positive numbers  $\epsilon_n \rightarrow 0$ , we can easily construct, e.g. geometrically, a convex sequence  $\{a_n\}$  such that  $a_n \geq \epsilon_n$  and  $a_n \rightarrow 0$ . Thus there exist Fourier series with coefficients  $a_n$  tending to 0 arbitrarily slowly (see also Ex. 1 on p. 70).

If  $a_n, b_n$  are the Fourier coefficients of an integrable function, the series  $\sum b_n/n$  converges (Chapter II, § 8). The example of the Fourier series

$$\sum_{n=2}^{\infty} \frac{\cos nx}{\log n} \quad (1\cdot11)$$

shows that  $\sum a_n/n$  may be divergent.

Let  $s_n$  and  $\sigma_n$  be the partial sums and the (C, 1) means of (1·1). With the hypotheses of (1·5),  $\Re[f - \sigma_n] \rightarrow 0$  (cf. Chapter IV, (5·5)).

(1·12) THEOREM. With the hypotheses of (1·5),  $\Re[f - s_n]$  tends to 0 if and only if  $\alpha_n = o\{(\log n)^{-1}\}$ .

For, subtracting (1·7) from (1·6), we see that  $|f(x) - s_n(x)|$  is contained between

$$\alpha_n \{ |D_n(x)| \pm \left[ \sum_{\nu=n-1}^{\infty} (\nu+1) \Delta^2 a_{\nu} K_{\nu}(x) + \Delta a_{n-1} K_{n-1}(x) \right] \}.$$

If we integrate this over  $(-\pi, \pi)$  (the terms in the curly brackets are non-negative) we find that

$$\Re[f - s_n] = \pi \alpha_n L_n + o(1),$$

where  $L_n$  is the Lebesgue constant (Chapter II, § 12). Since  $L_n$  is exactly of order  $\log n$ , (1·12) follows.

If  $a_n \log n \rightarrow \infty$ , e.g. if  $a_n = (\log n)^{-\frac{1}{2}}$  for  $n > 1$ , then  $\mathfrak{M}[f - s_n] \rightarrow \infty$ , and so also  $\mathfrak{M}[s_n] \rightarrow \infty$ . The series (1.11), which is important for some problems, is a limiting case, since here  $\mathfrak{M}[f - s_n]$  is bounded and even tends to a limit, but the limit is not zero. (Also  $\mathfrak{M}[s_n]$  tends to a finite limit.)

We now pass to the series  $\sum a_\nu \sin \nu x$  with  $a_1 \geq a_2 \geq \dots \rightarrow 0$ . Summation by parts shows that for its partial sum  $t_n(x)$  we have

$$t_n(x) = \sum_{\nu=1}^{n-1} \tilde{D}_\nu(x) \Delta a_\nu + a_n \tilde{D}_n(x) \rightarrow \sum_{\nu=1}^{\infty} \tilde{D}_\nu(x) \Delta a_\nu = g(x). \quad (1.13)$$

say, as  $n \rightarrow \infty$ . If we substitute  $\tilde{D}_\nu^*$  for  $\tilde{D}_\nu$ , we get a function  $g^*(x)$ ,  $0 \leq x \leq \pi$ , differing from  $g(x)$  by the continuous function  $\frac{1}{2} \sum \Delta a_\nu \sin \nu x$ . The series defining  $g^*$  has non-negative terms, and since the integral of  $\tilde{D}_n^*(x)$  over  $(0, \pi)$  is exactly of order  $\log n$  (Chapter II, (12.3)), we conclude that  $g^*$ , and so also  $g$ , is integrable over  $(0, \pi)$  if and only if  $\sum \Delta a_\nu \log \nu$  converges. If we assume this convergence, and observe that

$$a_n \log n = \log n \sum_{\nu=n}^{\infty} \Delta a_\nu \leq \sum_{\nu=n}^{\infty} \Delta a_\nu \log \nu \rightarrow 0,$$

we see that  $\mathfrak{M}[g - t_n] \rightarrow 0$ . In particular,  $\sum a_\nu \sin \nu x$  is  $S[g]$ . Thus:

(1.14) THEOREM. Suppose that  $a_1 \geq a_2 \geq \dots \rightarrow 0$ . The sum  $g(x)$  of  $\sum a_\nu \sin \nu x$  is then integrable if and only if  $\sum \Delta a_\nu \log \nu < \infty$ . If this condition is satisfied, then  $\sum a_\nu \sin \nu x$  is  $S[g]$ , and  $\mathfrak{M}[g - t_n] \rightarrow 0$ .

Under the hypotheses of (1.14),  $g^*$  is non-negative in  $(0, \pi)$ , so that  $g$  is there bounded below. If the sequence  $a_1, a_2, \dots$  is also convex, then  $g(x)$  is positive in  $(0, \pi)$ , unless  $a_1 = a_2 = \dots = 0$ . To prove this we apply summation by parts to the series  $\sum \tilde{D}_\nu \Delta a_\nu$ , and use the fact that  $\tilde{K}_\nu \geq 0$  in  $(0, \pi)$  (Chapter III, (3.18)).

(1.15) THEOREM. If we assume only that  $a_\nu \rightarrow 0$  and  $\Delta a_\nu \geq 0$ , then  $\sum a_\nu \sin \nu x$  is a generalized Fourier sine series (Chapter II, § 4).

For a simple calculation shows that then

$$2g(x) \sin x = a_1 + a_2 \cos x + \sum_{\nu=2}^{\infty} (a_{\nu+1} - a_{\nu-1}) \cos \nu x.$$

The series on the right converges uniformly, and so is  $S[2g \sin x]$ . Writing the Fourier formulae for the coefficients  $a_1, a_2, a_3 - a_1, \dots$ , we get by addition the formula

$$a_\nu = \frac{2}{\pi} \int_0^\pi g(x) \sin \nu x dx \quad (\nu = 1, 2, \dots)$$

The integrand here is continuous, since  $g \sin x$  is continuous.

Let  $h_n$  be the partial sums of the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \dots$ , so that  $h_n \simeq \log n$ . Let  $a_n \rightarrow 0$ ,  $\Delta a_n > 0$ . From the formula

$$\sum_{\nu=1}^n \frac{a_\nu}{\nu} = \sum_{\nu=1}^{n-1} h_\nu \Delta a_\nu + a_n h_n \quad (1.16)$$

it follows that if  $\sum \nu^{-1} a_\nu$  is finite, so is  $\sum \Delta a_\nu \log \nu$ . Conversely, the finiteness of the latter series implies that  $a_n \log n \leq \sum_{\nu=n}^{\infty} \Delta a_\nu \log \nu = o(1)$ , so that, by (1.16),  $\sum \nu^{-1} a_\nu$  is finite. Thus,

if  $a_1 \geq a_2 \geq \dots \rightarrow 0$ , the conditions  $\sum \nu^{-1} a_n < \infty$  and  $\sum \Delta a_n \log \nu < \infty$  are equivalent. Hence in (1.14) we can replace the convergence of  $\sum \Delta a_n \log n$  by that of  $\sum n^{-1} a_n$ .

That the latter series is convergent if (1.2) is a Fourier series, no matter what the  $a_n$ , we know already. This implies in particular that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{\log n}, \quad (1.17)$$

conjugate to the Fourier series (1.11), is not a Fourier series.

## 2. The order of magnitude of functions represented by series with monotone coefficients

We begin by giving one more proof of the formulae

$$\left. \begin{aligned} \sum_{n=1}^{\infty} n^{-\beta} \cos nx &\simeq x^{\beta-1} \Gamma(1-\beta) \sin \frac{1}{2} \pi \beta \\ \sum_{n=1}^{\infty} n^{-\beta} \sin nx &\simeq x^{\beta-1} \Gamma(1-\beta) \cos \frac{1}{2} \pi \beta \end{aligned} \right\} (x \rightarrow +0, 0 < \beta < 1), \quad (2.1)$$

established in Chapter II, § 13.

By Chapter III, (1.9),

$$\sum_{n=0}^{\infty} A_n^{-\beta} r^n e^{inx} = (1 - r e^{ix})^{\beta-1} \quad (0 \leq r < 1). \quad (2.2)$$

Since the  $A_n^{-\beta}$  are positive and decreasing to 0,  $\sum A_n^{-\beta} e^{inx}$  converges for  $x \neq 0$ ; and making  $r \rightarrow 1$  we deduce from (2.2) that

$$\sum_{n=0}^{\infty} A_n^{-\beta} e^{inx} = (2 \sin \frac{1}{2} x)^{\beta-1} \exp \left\{ \frac{1}{2} i (\pi - x) (1 - \beta) \right\} \quad (0 < x < 2\pi). \quad (2.3)$$

Using the formula (1.18) of Chapter III (with  $\alpha = -\beta$ ), and the relation  $2 \sin \frac{1}{2} x \simeq x$ , we deduce (2.1) from (2.3) by separating the real and the imaginary parts there.

*Remark.* The argument shows that the second formula (2.1) holds for  $0 < \beta < 2$ .

We shall use the formulae (2.1) to obtain some more general ones. In dealing with series (1.1) it is often convenient to assume that  $a_n = a(n)$ , where  $a(u)$  is a function defined for all real  $u \geq 0$ . Usually, indeed,  $a_n$  is given as  $a(n)$ .

A positive  $b(u)$  defined for  $u > u_0$  will be called a *slowly varying function* if, for any  $\delta > 0$ ,  $b(u) u^\delta$  is an increasing, and  $b(u) u^{-\delta}$  a decreasing, function of  $u$  for  $u$  large enough.

If  $b(u)$  is slowly varying, then  $b(ku) \simeq b(u)$  as  $u \rightarrow \infty$ , (2.4)

for every fixed  $k > 0$ , and even uniformly in every interval  $\eta \leq k \leq 1/\eta$ ,  $0 < \eta < 1$ .

For if, for example,  $1 \leq k \leq 1/\eta$ , then

$$b(ku)/(ku)^\delta \leq b(u)/u^\delta, \quad b(ku) \leq k^\delta b(u) \leq \eta^{-\delta} b(u)$$

for large  $u$ . Similarly,  $b(ku) \geq \eta^\delta b(u)$ . Making  $\delta$  arbitrarily small, we prove (2.4) for  $1 \leq k \leq 1/\eta$ . The case  $\eta \leq k \leq 1$  is proved similarly.

If  $\beta_1, \beta_2, \dots$  are real, each of the functions

$$\log^\beta x, \quad \log \log^\beta x, \quad \dots, \quad (2.5)$$

and so also any product of any finite number of them, is slowly varying.

(2.6) THEOREM. Let  $a_n = n^{-\beta} b_n$ , where  $0 < \beta < 1$  and  $b(u)$ , with  $b(n) = b_n$ , is a slowly varying function. Let

$$f_\beta(x) = \sum_{n=1}^{\infty} n^{-\beta} b_n \cos nx, \quad g_\beta(x) = \sum_{n=1}^{\infty} n^{-\beta} b_n \sin nx. \quad (2.7)$$

Then, for  $x \rightarrow +0$ ,

$$f_\beta(x) \simeq x^{\beta-1} b(x^{-1}) \Gamma(1-\beta) \sin \frac{1}{2} \pi \beta, \quad g_\beta(x) \simeq x^{\beta-1} b(x^{-1}) \Gamma(1-\beta) \cos \frac{1}{2} \pi \beta. \quad (2.8)$$

It is enough to prove the formula for  $g_\beta$ , the proof for  $f_\beta$  being the same. Set

$$B = \Gamma(1-\beta) \cos \frac{1}{2} \pi \beta.$$

We note that, for all  $M, N > 0$ ,

$$\sum_{n < M} n^{-\beta} < C_\beta M^{1-\beta}, \quad \left| \sum_{n > N} n^{-\beta} \sin nx \right| < CN^{-\beta} x^{-1} \quad (0 < x \leq \pi), \quad (2.9)$$

where  $C$  is an absolute constant and  $C_\beta$  depends on  $\beta$  only. The first inequality follows from

$$\sum_{n < M} n^{-\beta} < \int_0^M t^{-\beta} dt = \frac{M^{1-\beta}}{1-\beta}$$

and the second from Chapter I, (2.2).

Let  $0 < \omega < \Omega < +\infty$ . Then

$$\sum n^{-\beta} \sin nx = \sum_{n < \omega/x} + \sum_{\omega/x < n < \Omega/x} + \sum_{n > \Omega/x} = S_1 + S_2 + S_3, \quad (2.10)$$

with a corresponding decomposition  $T_1 + T_2 + T_3$  of the series  $\sum b_n n^{-\beta} \sin nx$ . By (2.9),

$$|S_1| < C_\beta \omega^{1-\beta} x^{\beta-1}, \quad |S_3| < C \Omega^{-\beta} x^{\beta-1}. \quad (2.11)$$

We fix an  $\epsilon > 0$ . By virtue of (2.1) we have

$$(B - \epsilon) x^{\beta-1} \leq S_2 \leq (B + \epsilon) x^{\beta-1}, \quad (2.12)$$

provided  $\omega$  and  $1/\Omega$  are small enough (but fixed) and  $x$  is near enough to 0.

We fix  $\delta > 0$  such that  $\delta < \beta$  and  $\beta + \delta < 1$ . Since  $u^\delta b(u)$  tends monotonically to  $+\infty$  for large  $u$ , we have  $u^\delta b(u) \geq v^\delta b(v)$  for  $u$  large and all  $v \leq u$ . Hence, if  $x$  is small,

$$\begin{aligned} |T_1| &\leq \sum_{n < \omega/x} b_n n^{-\beta} n^{-\delta} \leq b(\omega/x) (\omega/x)^\delta \sum_{n < \omega/x} n^{-\beta-\delta} \\ &\leq b(\omega/x) (\omega/x)^\delta C_{\beta+\delta} (\omega/x)^{1-\beta-\delta} = C_{\beta+\delta} \omega^{1-\beta} x^{\beta-1} b(\omega/x). \end{aligned}$$

Since  $b(\omega/x) \simeq b(1/x)$ , the last product is less than  $\epsilon x^{\beta-1} b(1/x)$ , provided  $\omega$  is small enough and  $x$  is near 0.

By (2.11) with  $\beta - \delta$  for  $\beta$  and Chapter I, (2.1),

$$|T_2| = \left| \sum_{n > \Omega/x} b_n n^{-\beta} n^{-\delta} \sin nx \right| \leq 2b(\Omega/x) (\Omega/x)^{-\delta} C (\Omega/x)^{-\beta+\delta} x^{-1} = 2C \Omega^{-\beta} x^{\beta-1} b(\Omega/x),$$

and so

$$|T_3| < \epsilon b(1/x) x^{\beta-1}$$

for  $\Omega$  large enough and  $x$  near 0.

On the other hand,

$$T_2 = b(1/x) \sum_{\omega/x}^{\Omega/x} n^{-\beta} \sin nx + \sum_{\omega/x}^{\Omega/x} \{b(n) - b(1/x)\} n^{-\beta} \sin nx = T'_2 + T''_2.$$

Here  $T'_2 = b(1/x) S_2$  and so, by (2.12), is contained between  $(B \pm \epsilon) x^{\beta-1} b(1/x)$ . By (2.4),  $|T''_2|$  does not exceed

$$\max_{\omega/x < n \leq \Omega/x} |b(n) - b(1/x)| \sum_{\omega/x}^{\Omega/x} n^{-\beta} = o\{b(1/x)\} O(\Omega/x)^{1-\beta} = o\{x^{\beta-1} b(1/x)\}.$$

Collecting results we see that  $g_\beta = T_1 + T_2 + T_2' + T_2''$  is contained between

$$(B \pm 3\epsilon)x^{\beta-1}b(1/x)$$

for  $x$  small, and so  $g_\beta(x) \simeq Bx^{\beta-1}b(1/x)$  as  $x \rightarrow +0$ ,

which completes the proof of (2.6).

*Remark.* The second formula (2.8) holds for  $0 < \beta < 2$ . In particular

$$g_1(x) = \sum n^{-1}b_n \sin nx \simeq \frac{1}{2}\pi b(1/x) \quad (x \rightarrow +0). \quad (2.13)$$

It is enough to indicate modifications of the preceding proof. We easily verify that the second inequality (2.9) holds for any  $\beta > 0$ ; hence the inequality (2.11) for  $S_2$  and the estimate for  $T_2$  hold for  $\beta > 0$ . On the other hand, if  $\beta < 2$  we have

$$|S_1| \leq \sum_{n < \omega/x} n^{-\beta} \cdot nx = x \sum_{n < \omega/x} n^{1-\beta} \leq C_\beta \omega^{2-\beta} x^{\beta-1},$$

and instead of the previous inequality for  $T_1$  we have  $|T_1| \leq C_{\beta+\delta} \omega^{2-\delta} x^{\beta-1} b(\omega/x)$ . Using the fact that the second equation (2.1) holds for  $0 < \beta < 2$ , we see that the estimates for  $S_2$  and  $T_2$  remain unchanged, and the proof concludes as before.

If  $b(u)$  is slowly varying, then  $b(u)/u$  is ultimately decreasing and so the series  $\sum n^{-1}b_n$  and the integral  $\int_1^\infty u^{-1}b(u) du$  are simultaneously finite or infinite. Write

$$\left. \begin{aligned} B(u) &= \int_1^u t^{-1}b(t) dt, & R(u) &= \int_u^\infty t^{-1}b(t) dt, \\ B^*(u) &= \sum_{n < u} n^{-1}b(n), & R^*(u) &= \sum_{n > u} n^{-1}b_n. \end{aligned} \right\} \quad (2.14)$$

Then, as  $u \rightarrow \infty$ ,

- (i)  $b(u) = o\{B(u)\}$ ,  $B(u) \simeq B^*(u)$ , if  $B(u) \neq O(1)$ ;  
 (ii)  $b(u) = o\{R(u)\}$ ,  $R(u) \simeq R^*(u)$ , if  $B(u) = O(1)$ .

For let  $k > 1$ . For large  $u$

$$B(u) > \int_{u/k}^u t^{-1}b(t) dt \simeq b(u) \int_{u/k}^u t^{-1} dt = b(u) \log k.$$

Taking  $k$  large we obtain  $b(u) = o\{B(u)\}$ , whether  $B(u)$  is bounded or not. Similarly,  $b(u) = o\{R(u)\}$ . Since  $B(u) - B^*(u)$  tends to a finite limit, we have  $B(u) \simeq B^*(u)$  if either side is unbounded. Let now  $u \rightarrow \infty$  and let  $N$  be an integer satisfying  $N < u \leq N+1$ .

Then

$$R(N+1) \leq R^*(N+1) = R^*(u) < R(N), \quad R(N+1) \leq R(u) < R(N).$$

Since  $R(N+1) \simeq R(N)$  ( $b(u)$  being a slowly varying function), the second formula (ii) follows.

(2.15) THEOREM. According as  $\sum b_n/n$  diverges or converges,

$$\left. \begin{aligned} f_1(x) &= \sum n^{-1}b_n \cos nx \simeq B(1/x), \\ f_1(0) - f_1(x) &\simeq R(1/x), \end{aligned} \right\} \quad (x \rightarrow +0). \quad (2.16)$$

or

Considering the first formula (2.16), we write

$$B^*(1/x) - f_1(x) = \sum_{n < 1/x} b_n n^{-1} (1 - \cos nx) - \sum_{n > 1/x} b_n n^{-1} \cos nx = U_1 + U_2.$$

Then (writing  $b_n n^{-1}$  as  $b_n n^{-\delta} n^{-1+\delta}$  and arguing as for  $T_3$  in Theorem (2.6), but with  $\Omega = 1$ ) we have

$$U_2 = O\{b(1/x)\} = o\{B(1/x)\} = o\{B^*(1/x)\}.$$

Since  $b_n n^{-1}(1 - \cos nx) < \frac{1}{2} n b_n x^2$ , the familiar argument shows that

$$U_1 = O\{b(1/x)\} = o\{B^*(1/x)\}.$$

Hence

$$f_1(x) \simeq B^*(1/x) \simeq B(1/x).$$

If  $f_1(0) = \sum b_n/n < \infty$ , then  $f_1(0) - f_1(x)$  is

$$\sum_{n < 1/x} b_n n^{-1}(1 - \cos nx) + \sum_{n > 1/x} b_n n^{-1} - \sum_{n > 1/x} b_n n^{-1} \cos nx = V_1 + R^*(1/x) + V_2.$$

Here again both  $V_1$  and  $V_2$  are  $O\{b(1/x)\} = o\{R^*(1/x)\}$ , so that

$$f_1(0) - f_1(x) \simeq R^*(1/x) \simeq R(1/x).$$

We pass now to the limiting case  $\beta = 0$  of (2.6).

(2.17) THEOREM. If  $b(u)$  decreases to 0 and if  $-ub'(u)$  is slowly varying, then, for  $x \rightarrow +0$ ,

$$\left. \begin{aligned} f_0(x) &\simeq -\frac{1}{2}\pi x^{-2}b'(1/x), \\ g_0(x) &\simeq x^{-1}b(1/x). \end{aligned} \right\} \quad (2.18)$$

The hypotheses here are satisfied if, for instance,  $b(u)$  is for large  $u$  one of the functions (2.5), and so also if it is a product of a finite number of them. In particular,

$$\left. \begin{aligned} \sum_{n=2}^{\infty} \frac{\cos nx}{\log n} &\simeq \frac{1}{2}\pi x^{-1} \log^{-2}(1/x), \\ \sum_{n=2}^{\infty} \frac{\sin nx}{\log n} &\simeq x^{-1} \log^{-1}(1/x). \end{aligned} \right\} \quad (x \rightarrow +0). \quad (2.19)$$

Since  $x^{-1} \log^{-1}(1/x)$  is not integrable, the second formula shows that  $\sum(\sin nx)/\log n$  is not a Fourier series, a fact we already know (see p. 186).

We begin with  $f_0$ , and set

$$c(u) = u[b(u) - b(u+1)], \quad c_n = c(n) = n\Delta b_n.$$

We have  $c(u) = -ub'(u+\theta)$ ,  $0 < \theta < 1$ , and so  $c(u) \simeq -ub'(u)$ , since  $-ub'(u)$  is slowly varying. Clearly,

$$\sum_1^{\infty} b_n \cos nx = \sum_1^{\infty} \Delta b_n \{D_n(x) - \frac{1}{2}\} = (2 \tan \frac{1}{2}x)^{-1} \sum_1^{\infty} n^{-1} c_n \sin nx + O(1). \quad (2.20)$$

If the numbers  $c_n$  satisfy the conditions for the  $b_n$  in the proof of (2.15), that formula and (2.20) give

$$f_0(x) \simeq \frac{1}{2}\pi x^{-1}c(x^{-1}) + O(1) \simeq -\frac{1}{2}\pi x^{-2}b'(x^{-1}), \quad (2.21)$$

since  $x^{-1}c(x^{-1}) \rightarrow \infty$ .

On analysing the proof of (2.15) we see that it is sufficient that (ultimately)

- (i)  $c_n/n = \Delta b_n$  should decrease;
- (ii)  $c_n n^{\delta} = n^{1+\delta} \Delta b_n$  should increase for some  $0 < \delta < 1$ ;
- (iii)  $c(ku) \simeq c(u)$  uniformly in every interval  $\eta \leq k \leq 1/\eta$ .

Since  $c(u) \simeq -ub'(u)$ , and since  $-ub'(u)$  is slowly varying, condition (iii) holds.

Since  $-ub'(u)$  is slowly varying,  $-b'(u)$  is decreasing. Therefore  $b(u)$  is convex, and  $c_n/n = \Delta b_n$  is decreasing (condition (i)). Finally,

$$\frac{b(n) - b(n+1)}{b(n+1) - b(n+2)} = \frac{b'(n+\theta)}{b'(n+\theta+1)} \leq \left(\frac{n+1+\theta}{n+\theta}\right)^{1+\delta} \quad (0 < \theta < 1),$$

by Cauchy's mean-value theorem and by the fact that  $-u^{1+\delta}b'(u)$  increases. Since the last ratio is less than  $\{(n+1)/n\}^{1+\delta}$ , we see that  $n^{1+\delta}\Delta b_n$  increases if  $n$  is replaced by  $n+1$ , and (ii) follows.

The proof for  $g_0$  is similar. Since

$$g_0(x) = (2 \tan \frac{1}{2}x)^{-1} \sum n^{-1} c_n (1 - \cos nx) + O(1),$$

we apply the second formula (2.16) with  $c_n$  for  $b_n$ , so that the term  $R(1/x)$  is

$$\begin{aligned} \int_{x^{-1}}^{\infty} u^{-1} c(u) du &= \int_{x^{-1}}^{\infty} \{b(u) - b(u+1)\} du \\ &= \int_{x^{-1}}^{1+x^{-1}} b(u) du \simeq b(x^{-1}). \end{aligned}$$

Hence  $g_0(x) \simeq x^{-1}b(x^{-1})$ .

Theorems of this section have analogues in which the roles of the function  $f$  and of its coefficients are reversed. In these we assume that  $f(x)$ ,  $0 < x \leq \pi$ , is sufficiently 'well-behaved' in every interval  $(\epsilon, \pi)$ ,  $\epsilon > 0$ , but tends to  $\infty$  in a specific way as  $x \rightarrow +0$ , and we inquire about the behaviour of the cosine and sine coefficients  $a_n$ ,  $b_n$  of  $f$ . The two results that follow are analogues of (2.1) and of (2.6).

(2.22) THEOREM. Let  $0 < \beta < 1$ . For the coefficients of the function

$$f(x) = x^{-\beta} \quad (0 < x \leq \pi)$$

we have

$$\left. \begin{aligned} \frac{1}{2} \pi a_n &\simeq n^{\beta-1} \Gamma(1-\beta) \sin \frac{1}{2} \pi \beta, \\ \frac{1}{2} \pi b_n &\simeq n^{\beta-1} \Gamma(1-\beta) \cos \frac{1}{2} \pi \beta. \end{aligned} \right\} \quad (2.23)$$

(2.24) THEOREM. Let  $0 < \beta < 1$ , and let  $b(x)$  be a function of bounded variation in every interval  $(\epsilon, \pi)$ , slowly varying as  $x \rightarrow +0$ . Then for the coefficients of  $x^{-\beta}b(x)$ ,  $0 < x \leq \pi$ , we have the relations

$$\left. \begin{aligned} \frac{1}{2} \pi a_n &\simeq n^{\beta-1} b(1/n) \Gamma(1-\beta) \sin \frac{1}{2} \pi \beta, \\ \frac{1}{2} \pi b_n &\simeq n^{\beta-1} b(1/n) \Gamma(1-\beta) \cos \frac{1}{2} \pi \beta. \end{aligned} \right\} \quad (2.25)$$

For the  $a_n$  in (2.22) we have

$$a_n = 2\pi^{-1} \int_0^{\pi} t^{-\beta} \cos nt \, dt = 2\pi^{-1} n^{\beta-1} \int_0^{\pi/n} t^{-\beta} \cos t \, dt.$$

A similar formula holds for  $b_n$ , and the integrals on the right tend, for  $n \rightarrow \infty$ , to the real and imaginary parts respectively of the integral

$$\int_0^{\infty} t^{-\beta} e^{it} \, dt = \Gamma(1-\beta) \exp\{\frac{1}{2} \pi i(1-\beta)\}$$

(see Chapter II, (13.10)), whence (2.23) follows.

We omit the details of the proof of (2.24), which does not differ essentially from that of (2.6) (we split the interval  $(0, \pi)$  into  $(0, \omega/n)$ ,  $(\omega/n, \Omega/n)$  and  $(\Omega/n, \pi)$ ). The hypothesis

on  $b(x)$  means that for every  $\delta > 0$  the functions  $x^\delta b(x)$  and  $x^{-\delta} b(x)$  are respectively increasing and decreasing in some right-hand neighbourhood of  $x = 0$ . That  $b(x)$  is of bounded variation in every interval  $(\epsilon, \pi)$  guarantees that the contribution to  $a_n$ ,  $b_n$ , of the integrals extended over  $(\epsilon, \pi)$  are  $O(1/n)$  and so are small in comparison with the right-hand sides in (2.25).

It may be added that since periodic odd functions are usually discontinuous at the points  $\pm \pi$ , their Fourier coefficients cannot tend rapidly to 0; and thus we can often not obtain simple formulæ for the Fourier coefficients but only for the Fourier transforms, i.e. for the integrals of  $f \cos mx$  and of  $f \sin mx$  extended over the interval  $0 \leq x < \infty$  ( $f$  in general not being periodic).

For some problems it is of importance not only to estimate the sum of the series but also to find a common majorant for the partial sums (or the remainders) of it. For the series considered in this section such estimates are implicitly contained in the proofs given above. We shall be satisfied here with the following inequalities, in which  $0 < \beta < 1$  and  $0 < x \leq \pi$ :

$$\left| \sum_{n=1}^N n^{-\beta} \cos nx \right| \leq C_\beta x^{\beta-1}, \quad (2.26)$$

$$\left| \sum_{n=1}^N n^{-\beta} \sin nx \right| \leq C_\beta x^{\beta-1}. \quad (2.27)$$

$$\left| \sum_{n=1}^N n^{-1} \cos nx \right| \leq \log(1/x) + C. \quad (2.28)$$

Consider, for example, (2.27). If  $N < 1/x$ , the sum here is identical with the sum  $S_1$  in (2.10) corresponding to an  $\omega \leq 1$ , and so the inequality follows from the estimate (2.11) for  $S_1$ . If  $N > 1/x$ , the sum in (2.27) differs from  $\sum_1^\infty$  by the sum  $S_2$  in (2.10) corresponding to an  $\Omega \geq 1$ , so that (2.27) follows from the second formula (2.8) and second inequality (2.11). A similar proof holds for (2.26). Finally, (2.28) follows by combining an analogous argument with the proof of (2.16) for  $b_n = 1$ .

Since  $D_n(x)$  is bounded below on  $(0, \pi)$ , uniformly in  $n$ , summation by parts shows that, for each given  $\alpha > 0$ , the partial sums of  $\sum n^{-\alpha} \sin nx$  are uniformly bounded below on  $(0, \pi)$ . For  $\sum n^{-\alpha} \cos nx$  the situation is different.

(2.29) THEOREM. *There is an  $\alpha_0$ ,  $0 < \alpha_0 < 1$ , such that for each  $\alpha \geq \alpha_0$  the partial sums  $s_n$  of  $\sum n^{-\alpha} \cos nx$  are uniformly bounded below, and for each  $\alpha < \alpha_0$  they are not;  $\alpha_0$  is the (unique) root of the equation*

$$\int_0^{1/x} \frac{\cos u}{u^\alpha} du = 0 \quad (0 < \alpha < 1).$$

Summation by parts shows that if the  $s_n$  are uniformly bounded below for some  $\alpha$ , the same holds for any larger  $\alpha$ . It is therefore enough to consider the case  $0 < \alpha < 1$ ; we may also suppose that  $0 < x \leq \pi$ .

First we reduce the problem to one for integrals; we show that

$$\left| \int_0^x \frac{\cos ux}{u^\alpha} du - \frac{2 \sin \frac{1}{2}x}{x} \sum_{\nu=1}^n \frac{\cos \nu x}{\nu^\alpha} \right| \leq C_\alpha, \quad (2.30)$$

where  $C_\alpha$  depends on  $\alpha$  only. For

$$\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \cos ux \, du = \frac{2 \sin \frac{1}{2}x}{x} \cos \nu x,$$

$$\sum_{\nu=1}^n \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \frac{\cos ux}{u^\alpha} \, du = \sum_{\nu=1}^n \int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \cos ux \left\{ \frac{1}{u^\alpha} - \frac{1}{\nu^\alpha} \right\} \, du + \frac{2 \sin \frac{1}{2}x}{x} \sum_{\nu=1}^n \frac{\cos \nu x}{\nu^\alpha},$$

and since, by the mean-value theorem,  $u^{-\alpha} - \nu^{-\alpha} = O(\nu^{-\alpha-1})$  here, the first sum on the right is  $O(1)$  uniformly in  $n$  and  $x$ , and (2.30) follows:

It is therefore enough to consider the  $\alpha$ ,  $0 < \alpha < 1$ , for which

$$\int_0^n \frac{\cos ux}{u^\alpha} \, du = x^{\alpha-1} \int_0^v \frac{\cos u}{u^\alpha} \, du \quad (v = nx)$$

is bounded below. The expression is bounded below if and only if the last integral, *qua* function of  $v$ , has a non-negative minimum in  $(0, \infty)$ . This minimum is attained for  $\alpha = \frac{1}{2}\pi$  and is

$$m(\alpha) = \int_0^{\frac{1}{2}\pi} \frac{\cos u}{u^\alpha} \, du.$$

Since  $m(0) < 0$ ,  $m(1) = +\infty$ , it is enough to show that  $m(\alpha)$  increases with  $\alpha$ ,  $0 < \alpha < 1$ .

Now 
$$m'(\alpha) = \int_0^{\frac{1}{2}\pi} \log \frac{1}{u} \frac{\cos u}{u^\alpha} \, du > \int_0^{\frac{1}{2}\pi} \log \frac{1}{u} \frac{\cos u}{u^\alpha} \, du,$$

and since  $u^{-\alpha} \cos u$  decreases in  $0 \leq u \leq \frac{1}{2}\pi$ , the last integral exceeds

$$\cos 1 \left\{ \int_0^1 \log \frac{1}{u} \, du - \int_1^{\frac{1}{2}\pi} \log u \, du \right\} > \cos 1 \left\{ 1 - \int_1^{\frac{1}{2}\pi} 1 \, du \right\} > 0,$$

and the theorem follows.

Theorem (2.24) can be used to obtain asymptotic expressions for the coefficients of certain Taylor series.

(2.31) THEOREM. Let 
$$F(z) = \frac{1}{(1-z)^{\alpha+1}} \left\{ \log \frac{\alpha}{1-z} \right\}^\beta = \sum_{n=0}^{\infty} A_n^{\alpha, \beta} z^n, \tag{2.32}$$

where  $\alpha, \beta$  are real numbers and  $\alpha \geq 2$ . † Then

$$A_n^{\alpha, \beta} \approx \frac{n^\alpha}{\Gamma(\alpha+1)} (\log n)^\beta \quad \text{if } \alpha \neq -1, -2, \dots, \tag{2.33}$$

$$A_n^{\alpha, \beta} \approx (-1)^{\alpha-1} (|\alpha|-1)! \beta n^\alpha (\log n)^{\beta-1} \quad \text{if } \alpha = -1, -2, \dots \tag{2.34}$$

The  $A_n^{\alpha, \beta}$  are generalizations of the Cesàro numbers  $A_n^*$ , and (2.33) generalizes Chapter III, (1.15).

We first consider (2.33). For the function  $F(z) = F_{\alpha, \beta}(z)$  we have  $F'_{\alpha-1, \beta} = \alpha F_{\alpha, \beta} + \beta F_{\alpha, \beta-1}$ , so that

$$n A_n^{\alpha-1, \beta} = \alpha A_n^{\alpha, \beta} + \beta A_n^{\alpha, \beta-1}. \tag{2.35}$$

We deduce from this that if (2.33) is valid for some  $\alpha$ , it holds for  $\alpha-1$ . Hence if (2.33) is proved for  $-1 < \alpha < 0$ , it holds for all negative non-integral  $\alpha$ . On the other hand, suppose we have (2.33) for some  $\alpha > -1$ . Since  $F_{\alpha+1, \beta} = (1-z)^{-1} F_{\alpha, \beta}$  we have (see Chapter III, (1.7))

$$A_n^{\alpha+1, \beta} = \sum_{\nu=0}^n A_\nu^{\alpha, \beta} \approx \frac{1}{\Gamma(\alpha+1)} \sum_{\nu=2}^n \nu^\alpha (\log \nu)^\beta + O(1)$$

$$\approx \frac{1}{\Gamma(\alpha+1)} \frac{n^{\alpha+1}}{\alpha+1} (\log n)^\beta = \frac{n^{\alpha+1}}{\Gamma(\alpha+2)} (\log n)^\beta,$$

† If  $\alpha \geq 2$ ,  $\log \{ \alpha/(1-z) \}$  has no zero for  $|z| < 1$ .

‡ It is not difficult to see that if  $\alpha > -1$  and  $\phi(u)$  is slowly varying, then

$$\sum_{\nu=1}^n \nu^\alpha \phi(\nu) \approx \frac{n^{\alpha+1}}{\alpha+1} \phi(n).$$

and (2.33) holds with  $\alpha + 1$  for  $\alpha$ . It follows that if we prove (2.33) for  $-1 < \alpha < 0$ , we have it for all non-integral  $\alpha$ .

We now need Theorem (2.24). First we observe that if  $0 \leq \eta < \eta' \leq \pi$ , then under the hypotheses of (2.24) we have  $\int_{\eta}^{\eta'} x^{-\beta} b(x) \cos nx \, dx = O(n^{\beta-1} b(1/n))$ , uniformly in  $\eta$  and  $\eta'$ ; this is implicit in the proof of the theorem.

It follows from this that if  $\lambda(x)$  is of bounded variation in  $(0, \pi)$ , then under the hypotheses of (2.24) the Fourier coefficients of  $x^{-\beta} b(x) \lambda(x)$  are given by (2.25) with factor  $\lambda(+0)$  inserted on the right. It is enough to prove this for  $\lambda$  monotone. In computing the coefficients it is enough to integrate over an arbitrarily small interval  $(0, \epsilon)$  since the contribution of the remainder of  $(0, \pi)$  is  $O(1/n)$ . Writing  $\lambda(x) = \lambda(+0) + (\lambda(x) - \lambda(+0))$  and applying to the integral containing  $\lambda(x) - \lambda(+0)$  the second mean-value theorem and the remark of the preceding paragraph, we arrive at the conclusion.

We can now prove (2.33) for  $-1 < \alpha < 0$ . Since near  $x=0$  we have  $F(re^{ix}) = O(|x|^{-\alpha-1} \log^{\beta}(1/|x|))$  uniformly in  $r$ ,  $0 \leq r < 1$ , it follows that  $|F(re^{ix})|$  is majorized over  $(0, \pi)$  by an integrable function of  $x$ . Hence  $\sum A_n^{\alpha, \beta} e^{in\alpha} = S[F(e^{ix})]$  and  $\sum A_n^{\alpha, \beta} \cos nx = S[\mathcal{R}F(e^{ix})]$ . Now

$$\mathcal{R}F(e^{ix}) = \frac{1}{(2 \sin \frac{1}{2}x)^{\alpha+1}} \left\{ \log^{\beta} \frac{\alpha}{2 \sin \frac{1}{2}x} + \frac{1}{2}(\pi-x)^{\beta} \right\} \cos \Phi, \quad (2.36)$$

where

$$\Phi = \frac{1}{2}(\pi-x)(\alpha+1) + \beta \arctan \left\{ \frac{\frac{1}{2}(\pi-x)}{\log \frac{1}{2} \alpha \operatorname{cosec} \frac{1}{2}x} \right\}.$$

Since the factor  $\{\dots\}^{\beta}$  in (2.36) is slowly varying,  $\cos \Phi$  and  $(x/(2 \sin \frac{1}{2}x))^{\alpha+1}$  are of bounded variation and tend respectively to  $\cos \frac{1}{2}\pi(\alpha+1) = -\sin \frac{1}{2}\pi\alpha$  and 1 as  $x \rightarrow 0$ , we find from the previous remark and the first formula (2.25) that

$$A_n^{\alpha, \beta} \simeq -\frac{2}{\pi} \sin \frac{1}{2}\pi\alpha \cos \frac{1}{2}\pi\alpha \Gamma(-\alpha) n^{\alpha} \log^{\beta} n,$$

which is (2.33) in view of the equation  $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ .

Thus the theorem is proved for all non-integral  $\alpha$ . If we now prove (2.33) for  $\alpha=0$ , we prove the theorem also for  $\alpha=1, 2, \dots$  and, using (2.35), for  $\alpha=-1, -2, \dots$ . It remains therefore to show that  $A_n^{\alpha, \beta} \simeq (\log n)^{\beta}$ .

This can be deduced from (2.33) with, say,  $\alpha = -\frac{1}{2}$ . For  $F_{0, \beta} = F_{-\frac{1}{2}, \beta} F_{-\frac{1}{2}, 0}$  implies that

$$A_n^{\alpha, \beta} = \sum_{\nu=0}^n A_{\nu}^{-\frac{1}{2}, \beta} A_{n-\nu}^{-\frac{1}{2}, 0}.$$

Let  $\theta$  be a small positive number, say  $0 < \theta < \frac{1}{2}$ , and let  $n' = [n\theta]$ . We split the last sum into two,  $P_n$  and  $Q_n$ , extended respectively over  $\nu \leq n'$  and  $\nu > n'$ . Observing that  $A_{\nu}^{-\frac{1}{2}, \beta}$  is positive, decreasing and  $A_{\nu}^{-\frac{1}{2}, \beta} \leq C\nu^{-\frac{1}{2}}$  for  $\nu > 0$ , and also that the  $A_{\nu}^{-\frac{1}{2}, \beta}$  are all positive from some place on, we have

$$\begin{aligned} |P_n| &\leq A_{n-\frac{1}{2}}^{-\frac{1}{2}, \beta} \sum_{\nu=0}^{n'} |A_{\nu}^{-\frac{1}{2}, \beta}| = A_{n-\frac{1}{2}}^{-\frac{1}{2}, \beta} \left\{ \sum_{\nu=0}^{n'} A_{\nu}^{-\frac{1}{2}, \beta} + O(1) \right\} \\ &= A_{n-\frac{1}{2}}^{-\frac{1}{2}, \beta} (A_n^{\frac{1}{2}, \beta} + O(1)), \end{aligned} \quad (2.37)$$

and find that  $|P_n| \leq \frac{1}{2}\epsilon (\log n)^{\beta}$  if  $\theta$  is small enough (but fixed) and  $n$  large enough.

Since  $A_{\nu}^{-\frac{1}{2}, \beta} \simeq A_{\nu}^{-\frac{1}{2}, \beta} (\log \nu)^{\beta}$ , we have

$$Q_n = \sum_{\nu=n'+1}^n A_{\nu}^{-\frac{1}{2}, \beta} A_{n-\nu}^{-\frac{1}{2}, 0} \simeq (\log n)^{\beta} \sum_{\nu=n'+1}^n A_{\nu}^{-\frac{1}{2}, \beta} A_{n-\nu}^{-\frac{1}{2}, 0}.$$

Now the last sum is

$$\left( \sum_{\nu=0}^n - \sum_{\nu=0}^{n'} \right) A_{\nu}^{-\frac{1}{2}, \beta} A_{n-\nu}^{-\frac{1}{2}, 0} = 1 - \sum_{\nu=0}^{n'} A_{\nu}^{-\frac{1}{2}, \beta} A_{n-\nu}^{-\frac{1}{2}, 0},$$

(see Chapter III. (1.10) (i)), and so, arguing as in (2.37), is arbitrarily close to 1 if  $\theta$  is sufficiently small. Collecting results we see that  $A_n^{\alpha, \beta} = P_n + Q_n$  is contained between  $(1 \pm \epsilon) \log^{\beta} n$  for  $n$  sufficiently large, so that  $A_n^{\alpha, \beta} \simeq \log^{\beta} n$ . This completes the proof of (2.31).

A similar argument can be applied (though the details are a little awkward) to obtain the asymptotic values of the coefficients of

$$F(z) = \frac{1}{(1-z)^{\alpha+1}} \left\{ \log \frac{\alpha_1}{1-z} \right\}^{\beta_1} \left\{ \log \frac{\alpha_2}{1-z} \right\}^{\beta_2} \dots \left\{ \log \frac{\alpha_k}{1-z} \right\}^{\beta_k},$$

where  $\alpha$  and the  $\beta_j$  are real, the  $\alpha_j$  positive and so large that  $F$  is regular for  $|z| < 1$ .

### 3. A class of Fourier-Stieltjes series

We begin by constructing a class of perfect non-dense sets. Let  $OA$  be a segment of length  $l$  whose end-points have abscissae  $x$  and  $l+x$ . Let  $\alpha(1), \alpha(2), \dots, \alpha(d)$  be  $d$  numbers such that

$$0 \leq \alpha(1) < \alpha(2) < \dots < \alpha(d) < 1.$$

We consider  $d$  closed intervals with end-points  $l\alpha(j)+x$  and  $l\alpha(j)+l\eta+x$ , where  $\eta$  is a positive number so small that the intervals have no points in common and are all contained in  $OA$ . These intervals will be called 'white'. The complementary intervals with respect to the closed interval  $OA$  will be called 'black' intervals, and are removed. The dissection of  $OA$  thus obtained will be said to be of the type

$$[d; \alpha(1), \dots, \alpha(d); \eta].$$

Starting with the interval  $(0, 2\pi)$  we perform a dissection of the type

$$[d_1, \alpha_1(1), \dots, \alpha_1(d_1); \eta_1],$$

and remove the black intervals. On each white interval remaining we perform a dissection of the type  $[d_2; \alpha_2(1), \dots, \alpha_2(d_2); \eta_2]$ , and remove the black intervals...and so on. After  $p$  operations we have  $d_p$  white intervals, each of length  $2\pi\eta_1\eta_2\dots\eta_p$ . When  $p \rightarrow \infty$  we obtain a closed set  $P$  of measure

$$2\pi \lim d_1 \dots d_p \eta_1 \dots \eta_p.$$

(The limit exists, since  $\eta_p < 1/d_p$  for each  $p$ .) In subsequent applications we will have  $d_p \geq 2$  for each  $p$ . The set  $P$  will then be perfect and non-dense.

The abscissae of the left-hand ends of the white intervals of rank  $p$ , i.e. after the  $p$ th step in construction, are, as induction shows, given by the formula

$$x = 2\pi[\alpha_1(\theta_1) + \eta_1\alpha_2(\theta_2) + \eta_1\eta_2\alpha_3(\theta_3) + \dots + \eta_1\eta_2\dots\eta_{p-1}\alpha_p(\theta_p)], \quad (3.1)$$

where  $\theta_k$  takes the values  $1, 2, \dots, d_k$ . The abscissae of the points of  $P$  are given by the same series continued to infinity.

The structure of the set  $P$  displays a certain homogeneity. Consider a neighbourhood of a point of  $P$ . This neighbourhood contains a white interval  $I$  of a certain rank  $p$ . If  $I_1 = I, I_2, \dots, I_k$  are all the white intervals of rank  $p$ , then the sets  $I_1P, I_2P, \dots, I_kP$  are congruent and contained in intervals without points in common, and their union is  $P$ .

We now construct a non-decreasing function  $F(x)$  constant in the intervals contiguous to  $P$  and increasing at every point of  $P$ . For each  $k$  let  $\lambda_k(1), \lambda_k(2), \dots, \lambda_k(d_k)$  be  $d_k$  positive numbers of sum 1. We denote by  $\mu_k$  the largest of the  $\lambda_k(j)$ . Let  $F_p(x)$  be a continuous non-decreasing function defined by the conditions:

(a)  $F_p(0) = 0, F_p(2\pi) = 1;$

(b)  $F_p(x)$  increases linearly by  $\lambda_1(\theta_1)\lambda_2(\theta_2)\dots\lambda_p(\theta_p)$  in each of the white intervals with left-hand ends given by (3.1),

(c)  $F_p(x)$  is constant in every black interval of the  $p$ th stage of dissection.

We see at once, by considering  $|F_{p+1} - F_p|$ , that if the series

$$\Sigma(\mu_1 \mu_2 \dots \mu_p)$$

converges, which we shall always suppose, then  $F_p$  tends uniformly, as  $p \rightarrow \infty$ , to a function  $F(x)$  having the properties stated above and satisfying  $F(0) = 0$ ,  $F(2\pi) = 1$ .

The particular function  $F$  obtained by taking  $\lambda_k(j) = 1/d_k$  we will call the *Lebesgue function* constructed on the set.

We shall compute the Fourier-Stieltjes coefficient

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} dF(x).$$

Using the formula (3.1) for the abscissae of the left-hand ends of the white intervals, we find that  $c_n$  is the limit, for  $p \rightarrow \infty$ , of the sum

$$(2\pi)^{-1} \Sigma \lambda_1(\theta_1) \lambda_2(\theta_2) \dots \lambda_p(\theta_p) \exp \{ -2\pi i n [\alpha_1(\theta_1) + \eta_1 \alpha_2(\theta_2) + \dots + \eta_1 \dots \eta_{p-1} \alpha_p(\theta_p)] \}$$

extended over all possible combinations of  $\theta$ 's,  $\theta_k$  taking the values  $1, 2, \dots, d_k$ .

Writing

$$Q_k(\phi) = \lambda_k(1) e^{i\alpha_k(1)\phi} + \lambda_k(2) e^{i\alpha_k(2)\phi} + \dots + \lambda_k(d_k) e^{i\alpha_k(d_k)\phi},$$

we can write the preceding sum in the form  $(2\pi)^{-1} \prod_1^p \bar{Q}_k(2\pi n \eta_1 \dots \eta_{k-1})$  (with  $\eta_1 \dots \eta_{k-1} = 1$  for  $k=1$ ). Hence, passing to the limit,

$$c_n = \frac{1}{2\pi} \prod_{k=1}^{\infty} \bar{Q}_k(2\pi n \eta_1 \dots \eta_{k-1}). \quad (3.2)$$

We shall now consider a few examples.

(1) *Sets of the Cantor type.* These are obtained by successive dissections of the type

$$[2; 0, 1 - \xi_k; \xi_k],$$

where  $0 < \xi_k < \frac{1}{2}$ . The points of the set are of the form

$$x = 2\pi [\epsilon_1(1 - \xi_1) + \epsilon_2 \xi_1(1 - \xi_2) + \dots + \epsilon_p \xi_1 \dots \xi_{p-1}(1 - \xi_p) + \dots],$$

where  $\epsilon_p$  is 0 or 1.

The polynomial  $Q_k(\phi)$  corresponding to the Lebesgue function is

$$Q_k(\phi) = \frac{1}{2} (1 + e^{i(1-\xi_k)\phi}) = e^{i(1-\xi_k)\phi/2} \cos \frac{1}{2} (1 - \xi_k) \phi.$$

and hence the Fourier-Stieltjes coefficient of the Lebesgue function  $F$  is

$$\begin{aligned} c_n &= \frac{1}{2\pi} \prod_{k=1}^{\infty} \exp \{ -n\pi i (1 - \xi_k) \xi_1 \dots \xi_{k-1} \} \cos \pi n \xi_1 \dots \xi_{k-1} (1 - \xi_k) \\ &= \frac{(-1)^n}{2\pi} \prod_{k=1}^{\infty} \cos \pi n \xi_1 \dots \xi_{k-1} (1 - \xi_k). \end{aligned}$$

If we write  $\xi_1 \dots \xi_{k-1} (1 - \xi_k) = r_k$ , then

$$\sum_1^{\infty} r_k = 1, \quad r_p > \sum_{p+1}^{\infty} r_k. \quad (3.3)$$

Also 
$$x = 2\pi \sum_1^{\infty} \epsilon_k r_k, \quad c_n = (-1)^n (2\pi)^{-1} \prod_1^{\infty} \cos \pi n r_k. \quad (3.4)$$

A still more special case is obtained if all the  $\xi_k$  are the same number  $\xi$ ,  $0 < \xi < \frac{1}{2}$ . The set is then said to be of Cantor type with *constant ratio of dissection*, and

$$x = 2\pi(1 - \xi) \sum_1^{\infty} \epsilon_k \xi^{k-1}, \quad c_n = (-1)^n (2\pi)^{-1} \prod_{k=1}^{\infty} \cos \pi \xi^{k-1} (1 - \xi). \quad (3.5)$$

The classical Cantor case is  $\xi = \frac{1}{3}$ , when

$$x = 4\pi \sum_1^{\infty} \epsilon_k 3^{-k}, \quad c_n = (-1)^n (2\pi)^{-1} \prod_{k=1}^{\infty} \cos (2\pi n 3^{-k}),$$

the  $c_n$  again corresponding to the Lebesgue function.

Suppose that  $\xi = 1/q$ ,  $q = 3, 4, 5, \dots$ . For  $n = q^m$ , (3.5) gives

$$c_{q^m} = (-1)^{q^m} (2\pi)^{-1} \prod_{k=1}^m \cos \pi (q-1) q^{m-k} = - (2\pi)^{-1} \prod_{k=1}^m \cos \pi (q-1) q^{-k}.$$

The latter expression is independent of  $m$  and different from zero. Hence

(3.6) **THEOREM.** *If  $\xi = 1/q$ ,  $q = 3, 4, \dots$ , the Fourier-Stieltjes coefficients of the Lebesgue function  $F(x)$  do not tend to zero.*

Thus the periodic function  $F(x) - x/2\pi$  is of bounded variation and continuous, and yet its Fourier coefficients are not  $o(1/n)$  (compare Chapter II, p. 48).

(2) *Symmetrical perfect sets of order  $d$ .* These are obtained,  $d$  being an integer not less than 1, by successive dissections of the type

$$[d+1; 0, (1-\xi_2)d^{-1}, 2(1-\xi_2)d^{-1}, \dots, 1-\xi_2; \xi_2],$$

where  $0 < \xi_k < 1/(d+1)$ . The Cantor set corresponds to  $d = 1$ . The points  $\alpha_n(j)$ ,  $j = 1, 2, \dots, d+1$ , are in arithmetic progression, the first point being 0 and the last  $1 - \xi_2$ . The points of the set are

$$x = 2\pi d^{-1} [\epsilon_1(1-\xi_1) + \epsilon_2 \xi_1(1-\xi_2) + \dots + \epsilon_p \xi_1 \dots \xi_{p-1}(1-\xi_p) + \dots],$$

where  $\epsilon_p$  takes the values  $0, 1, \dots, d$ . On setting  $\gamma_k = (1-\xi_k)/d$ , we can write the polynomial  $Q_d$  corresponding to the Lebesgue function as

$$(d+1)^{-1} (1 + e^{i\gamma_1 \phi} + e^{2i\gamma_1 \phi} + \dots + e^{d i\gamma_1 \phi}) = (d+1)^{-1} \frac{e^{i(d+1)\gamma_1 \phi} - 1}{e^{i\gamma_1 \phi} - 1} \frac{\sin \{(d+1) \frac{1}{2} \gamma_1 \phi\}}{\sin \frac{1}{2} \gamma_1 \phi}.$$

Hence 
$$2\pi c_n = (-1)^n \prod_{k=1}^{\infty} \frac{\sin \{(d+1) \pi n d^{-1} \xi_1 \dots \xi_{k-1} (1-\xi_k)\}}{(d+1) \sin \{\pi n d^{-1} \xi_1 \dots \xi_{k-1} (1-\xi_k)\}}.$$

The set being always a symmetrical perfect set, let  $d = 2g$  be even, and let us construct a function such that for each  $k$  the number  $\lambda_k(j)$  is the coefficient  $\lambda(j)$  of  $z^{j-1}$  in the expansion of

$$(g+1)^{-1} (1 + z + z^2 + \dots + z^g).$$

Then  $\lambda(1) + \lambda(2) + \dots + \lambda(2g+1) = 1$ , as required; and, with  $\gamma_k = (1-\xi_k)/2g$ ,

$$Q_d(\phi) = \left( \frac{1 + e^{i\gamma_1 \phi} + \dots + e^{g i\gamma_1 \phi}}{g+1} \right)^2 = \left[ \frac{e^{i(g+1)\gamma_1 \phi} - 1}{(g+1) \sin \frac{1}{2} \gamma_1 \phi} \right]^2,$$

$$2\pi c_n = (-1)^n \prod_{k=1}^{\infty} \left[ \frac{\sin \left\{ (g+1) \frac{\pi n}{2g} \xi_1 \dots \xi_{k-1} (1-\xi_k) \right\}}{(g+1) \sin \left\{ \frac{\pi n}{2g} \xi_1 \dots \xi_{k-1} (1-\xi_k) \right\}} \right]^2.$$

(3) Consider a symmetrical perfect set of order  $d$ , and let  $\lambda_k(j)$  be equal for all  $k$  to the coefficient  $\lambda(j)$  of  $z^{j-1}$  in the expansion of  $2^{-d}(1+z)^d$ . Then

$$Q_d(\phi) = \left[ \frac{1}{2} (1 + e^{i\gamma_k \phi}) \right]^d \quad (\gamma_k = (1-\xi_k) d^{-1}),$$

$$2\pi c_n = (-1)^n \prod_{k=1}^{\infty} [\cos \pi n d^{-1} \xi_1 \dots \xi_{k-1} (1-\xi_k)]^d.$$

We shall now consider the modulus of continuity of the function  $F$ , confining our attention to the case where  $d_p = d$ ,  $\eta_p = \eta$  and  $\mu_k = \max[\lambda_k(1), \lambda_k(2), \dots, \lambda_k(d)] = \mu$  are all constant. We shall show that then  $F$  satisfies a Lipschitz condition of order  $|\log \mu| / |\log \eta|$ .

Let  $x$  and  $x' > x$  be two points of  $P$ . If  $x$  and  $x'$  are end-points of the same interval contiguous to  $P$ , then

$$F(x') - F(x) = 0.$$

If not, let  $p$  be the order of the dissection when for the first time appear at least two black intervals in  $(x, x')$ . Thus there is at least one white interval of rank  $p$  included in  $(x, x')$ , and so

$$x' - x \geq 2\pi\eta^p.$$

On the other hand, at the dissection of order  $p - 1$  there is at most one black interval  $(\beta, \beta')$  in  $(x, x')$ . It follows that

$$F(x') - F(x) = F(x') - F(\beta') + F(\beta) - F(x) \leq 2\mu^p.$$

Thus

$$F(x') - F(x) \leq A(x' - x)^{|\log \mu| / |\log \eta|},$$

$A$  being independent of  $x$ .

The extension to the case in which  $x$ , or  $x'$ , or both, are outside  $P$  is immediate, since we apply the preceding inequality to the interval  $(x_1, x'_1)$ , where  $x_1$  and  $x'_1$  are the first and last points of  $P$  in  $(x, x')$ .

*Example.* The Lebesgue function constructed on a symmetrical perfect set of order  $d$  and of constant ratio of dissection  $\xi$ , belongs to  $\Lambda_\alpha$  with  $\alpha = \log(d+1) / |\log \xi|$ .

#### 4. The series $\sum_{n=1}^{\infty} e^{icn \log n} e^{inx}$

The power series 
$$\sum_{n=1}^{\infty} e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2} + \alpha}}, \quad (4.1)$$

which was first studied by Hardy and Littlewood, possesses many interesting properties. We suppose that  $\alpha$  is real and  $c$  positive.

(4.2) THEOREM. If  $0 < \alpha < 1$ , the series (4.1) converges uniformly in the interval  $0 \leq x \leq 2\pi$  to a function  $\phi_\alpha(x) \in \Lambda_\alpha$ .

The theorem is a consequence of certain lemmas, due to van der Corput, of considerable interest in themselves.

Given a real-valued function  $f(u)$  and numbers  $a < b$ , we set

$$F(u) = e^{2\pi i f(u)},$$

$$I(F; a, b) = \int_a^b F(u) du, \quad S(F; a, b) = \sum_{a < n \leq b} F(n),$$

$$D(F; a, b) = I(F; a, b) - S(F; a, b).$$

(4.3) LEMMA. (i) If  $f(u)$ ,  $a \leq u \leq b$ , has a monotone derivative  $f'(u)$ , and if there is a positive  $\lambda$  such that  $f' \geq \lambda$  or  $f' \leq -\lambda$  in  $(a, b)$ , then  $|I(F; a, b)| < 1/\lambda$ .

(ii) If  $f''(u) \geq \rho > 0$  or  $f''(u) \leq -\rho < 0$ , then  $|I(F; a, b)| \leq 4\rho^{-\frac{1}{2}}$ .

(i) Since  $I(F; a, b) = (2\pi i)^{-1} \int_a^b dF(u)/f'(u)$ , the second mean-value theorem, applied to the real and imaginary parts of the last integral, shows that  $|I| \leq 2/\pi\lambda < 1/\lambda$ .

(ii) We may suppose that  $f' \geq \rho$ . (Otherwise replace  $f$  by  $-f$  and  $I$  by  $J$ .) Then  $f'$  is increasing. Suppose for the moment that  $f'$  is of constant sign in  $(a, b)$ , say  $f' \geq 0$ . If  $a < \gamma < b$ , then  $f' \geq (\gamma - a)\rho$  in  $(\gamma, b)$ . Therefore

$$|I(F; a, b)| \leq |I(F; a, \gamma)| + |I(F; \gamma, b)| \leq (\gamma - a) + 1/(\gamma - a)\rho,$$

and, choosing  $\gamma$  so as to make the last sum a minimum, we find that  $|I(F; a, b)| \leq 2\rho^{-\frac{1}{2}}$ . In the general case  $(a, b)$  is a sum of two intervals in each of which  $f'$  is of constant sign, and (ii) follows by adding the inequalities for these two intervals.

(4.4) LEMMA. If  $f'(u)$  is monotone and  $|f'| \leq \frac{1}{2}$  in  $(a, b)$ , then

$$|D(F; a, b)| \leq A,$$

where  $A$  is an absolute constant.

Suppose first that  $a$  and  $b$  are not integers. The sum  $S$  is then  $\int_a^b F(u) d\psi(u)$ , where  $\psi(u)$  is a function constant in the intervals  $n < u < n + 1$  and having jumps 1 at the points  $n$ . If we take  $\psi(u) = [u] + \frac{1}{2}$  for  $u$  non-integral ( $[u]$  being the integral part of  $u$ ) and  $\psi(n) = n$ , then

$$D(F; a, b) = \int_a^b F(u) d\chi(u), \text{ where } \chi(u) = u - [u] - \frac{1}{2} \quad (u \neq 0, \pm 1, \dots).$$

The function  $\chi$  has period 1, and integration by parts gives

$$D(F; a, b) = -I(F'\chi; a, b) + R, \quad |R| \leq 1.$$

The partial sums of  $S[\chi] = -\sum (\sin 2\pi nu)/\pi n$  are uniformly bounded. Multiplying  $S[\chi]$  by  $F'$  and integrating over  $(a, b)$  we find that  $D - R$  is equal to the sum of the expressions

$$\frac{1}{2\pi i n} \left\{ \int_a^b \frac{f'(u)}{f'(u) + n} d e^{2\pi i(f(u) + nu)} - \int_a^b \frac{f'(u)}{f'(u) - n} d e^{2\pi i(f(u) - nu)} \right\} \quad (4.5)$$

for  $n = 1, 2, \dots$ . The ratios  $f'/(f' \pm n)$  being monotone, the second mean-value theorem shows that (4.5) numerically does not exceed  $2/\pi n(n - \frac{1}{2})$ , and so the series with terms (4.5) converges absolutely and uniformly. This completes the proof if  $a$  and  $b$  are not integers. If  $a$  or  $b$  is an integer, it is enough to observe that  $D(F; a, b)$  differs from  $\lim_{\epsilon \rightarrow 0} D(F; a + \epsilon; b - \epsilon)$  by 1 at most.

Remark. The condition  $|f'| \leq \frac{1}{2}$  can be replaced by  $|f'| \leq 1 - \epsilon$ , with  $\epsilon > 0$ , if we simultaneously replace  $A$  by  $A_\epsilon$ .

(4.6) LEMMA. If  $f''(u) \geq \rho > 0$  or  $f''(u) \leq -\rho < 0$ , then

$$|S(F; a, b)| \leq [ |f'(b) - f'(a)| + 2 ] (4\rho^{-\frac{1}{2}} + A).$$

We may suppose that  $f'' \geq \rho$ . Let  $\alpha_p$  be the point (if any) where  $f'(\alpha_p) = p - \frac{1}{2}$ , and let

$$F_p(u) = e^{2\pi i(f(u) - pu)},$$

for  $p = 0, \pm 1, \pm 2, \dots$ . Then  $|f'(u) - p| \leq \frac{1}{2}$  in  $(\alpha_p, \alpha_{p+1})$ . Let  $\alpha_r, \alpha_{r+1}, \dots, \alpha_{r+\epsilon}$  be the points  $\alpha$ , if any such exist, belonging to the interval  $a \leq u \leq b$ . From (4.3) and (4.4) it follows that

$$S(F; \alpha_p, \alpha_{p+1}) = S(F_p; \alpha_p, \alpha_{p+1}) = I(F_p; \alpha_p, \alpha_{p+1}) - D(F_p; \alpha_p, \alpha_{p+1})$$

does not exceed  $4\rho^{-\frac{1}{2}} + A$  in absolute value. The same holds for  $S(F; a, \alpha_r)$  and  $S(F; \alpha_{r+s}, b)$ . Since  $S(F; a, b)$  is the sum of these expressions for the intervals  $(a, \alpha_r)$ ,  $(\alpha_r, \alpha_{r+1})$ , ...,  $(\alpha_{r+s}, b)$ , whose number is  $s + 2 = f'(\alpha_{r+s}) - f'(\alpha_r) + 2$ , the lemma follows.

To complete the proof of (4.2) we need the following result:

(4.7) THEOREM. The partial sums  $s_N(x)$  of the series  $\sum c^{cn \log n} e^{inz}$  are  $O(N^{\frac{1}{2}})$ , uniformly in  $x$ .

The function  $f(u) = (2\pi)^{-1}(cu \log u + ux)$  has an increasing derivative. If  $\nu \geq 0$  is an integer and  $a = 2^\nu$ ,  $b = 2^{\nu+1}$ , a simple application of Lemma (4.6) shows that  $|S(F; a, b)| \leq C2^{2\nu}$ , with  $C$  depending on  $c$  only. The same holds if  $2^\nu = a < b < 2^{\nu+1}$ . If  $2^\nu < N \leq 2^{\nu+1}$ , then

$$\begin{aligned} |s_N(x)| &\leq 1 + |S(F; 1, 2)| + |S(F; 2, 4)| + \dots + |S(F; 2^\nu, N)| \\ &\leq 1 + C(1 + 2^{\frac{1}{2}} + \dots + 2^{2\nu}) \leq C_1 2^{2\nu} \leq C_1 N^{\frac{1}{2}}, \end{aligned}$$

with  $C_1$  depending on  $c$  only, and (4.7) is established.

Return to (4.2). Summation by parts gives for the  $N$ th partial sum of (4.1) the value

$$\sum_{\nu=1}^{N-1} s_\nu(x) \Delta \nu^{-\frac{1}{2}-\alpha} + s_N(x) N^{-\frac{1}{2}-\alpha}. \tag{4.8}$$

Since  $\Delta \nu^{-\frac{1}{2}-\alpha} = O(\nu^{-\frac{1}{2}-\alpha})$ , we conclude from (4.8) and from the estimate  $s_\nu(x) = O(\nu^{\frac{1}{2}})$  that the partial sums of (4.1) are

- (i) uniformly convergent if  $\alpha > 0$ ;
- (ii) uniformly  $O(\log N)$  if  $\alpha = 0$ ;
- (iii) uniformly  $O(N^{-\alpha})$  if  $\alpha < 0$ .

Let  $0 < x < 1$ . Making  $N \rightarrow \infty$  in (4.8) we obtain

$$\phi_\alpha(x+h) - \phi_\alpha(x) = \sum_{\nu=1}^{\infty} \{s_\nu(x+h) - s_\nu(x)\} \Delta \nu^{-\frac{1}{2}-\alpha} = \sum_{\nu=1}^N + \sum_{\nu=N+1}^{\infty} = P + Q,$$

say. Let  $0 < h \leq 1$ ,  $N = [1/h]$ . The terms of  $Q$  are  $O(\nu^{\frac{1}{2}}) \Delta \nu^{-\frac{1}{2}-\alpha} = O(\nu^{-1-\alpha})$  so that

$$Q = O(N^{-\alpha}) = O(h^\alpha).$$

On the other hand, since  $s'_\nu(x)$ —apart from a numerical factor—is the partial sum of (4.1) with  $\alpha = -\frac{1}{2}$ , we have  $s'_\nu(x) = O(\nu^{\frac{1}{2}})$ , by case (iii) above. Therefore, applying the mean-value theorem to the real and imaginary parts of  $s_\nu(x+h) - s_\nu(x)$ , we get

$$|P| \leq \sum_{\nu=1}^N O(h\nu^{\frac{1}{2}}) \Delta \nu^{-\frac{1}{2}-\alpha} = O(h) \sum_1^N \nu^{-\alpha} = O(hN^{1-\alpha}) = O(h^\alpha).$$

Since  $P$  and  $Q$  are  $O(h^\alpha)$ , so is  $\phi_\alpha(x+h) - \phi_\alpha(x)$ ; and thus  $\phi_\alpha \in \Lambda_\alpha$ .

Theorem (4.2) ceases to be true when  $\alpha = 0$ . It may be shown that in this case (4.1) is nowhere summable A and so certainly is not a Fourier series. (Another interesting consequence is that the function

$$\sum_1^{\infty} \frac{e^{icn \log n}}{n^\alpha} z^n,$$

which is regular for  $|z| < 1$ , cannot be continued across  $|z| = 1$  for any  $\alpha$ .) However, we have

(4.9) THEOREM. If  $\beta > 1$  and  $c$  is positive, the series

$$\sum_{n=2}^{\infty} \frac{e^{icn \log n}}{n^{\frac{1}{2}} (\log n)^\beta} e^{inz} \tag{4.10}$$

converges uniformly for  $0 \leq x \leq 2\pi$ .

We replace  $\Delta v^{-1-\alpha}$  by  $\Delta v^{-1} \log^{-\beta} v = O(v^{-1} \log^{-\beta} v)$ ,  $N^{-1-\alpha}$  by  $N^{-1} \log^{-\beta} N$  in (4.8), and observe that a series with terms  $O(v^{-1} \log^{-\beta} v)$  converges.

(4.11) THEOREM. *There is a continuous function  $f(x)$  such that, if  $a_n, b_n$  are the Fourier coefficients of  $f$ . The series  $\sum (|a_n|^{2-\epsilon} + |b_n|^{2-\epsilon})$  diverges for every  $\epsilon > 0$ .*

For if  $f(x)$  is either the real or imaginary part of the function (4.10), with  $\beta > 1$ , and if  $\rho_n = (a_n^2 + b_n^2)^{1/2}$ , then  $\rho_n = n^{-1} \log^{-\beta} n$ ,  $\sum \rho_n^{2-\epsilon}$  diverges, and this is equivalent to the divergence of  $\sum (|a_n|^{2-\epsilon} + |b_n|^{2-\epsilon})$ .

5. The series  $\sum v^{-\beta} e^{i\alpha v} e^{ivx}$

We shall now discuss the series

$$\sum_{n=1}^{\infty} v^{-\beta} e^{i\alpha v} e^{ivx} \tag{5.1}$$

Here, once for all,  $0 < \alpha < 1$ ,  $-\pi \leq x \leq \pi$ .

(5.2) THEOREM (i) *If  $\beta > 1 - \frac{1}{2}\alpha (> \frac{1}{2})$ , the series (5.1) converges uniformly to a continuous sum  $\psi_{\alpha, \beta}(x)$ .*

(ii) *If, in addition,  $\frac{1}{2}\alpha + \beta < 2$ , then  $\psi_{\alpha, \beta}(x) \in \Lambda_{\frac{1}{2}\alpha + \beta - 1}$ .*

For fixed  $x$  and for  $u > 0$ , the function  $f(u) = (2\pi)^{-1}(u^\alpha + ux)$  has a decreasing derivative

$$f'(u) = (2\pi)^{-1}(\alpha u^{\alpha-1} + x). \tag{5.3}$$

Hence,  $n_0$  being any positive integer, (4.3) (ii) gives

$$\left| \int_{n_0}^n e^{i\alpha u} e^{iux} du \right| \leq 4(2n)^{\frac{1}{2}} (\alpha(1-\alpha))^{-\frac{1}{2}} n^{-\frac{1}{2} + \epsilon} = A_\alpha n^{1-\frac{1}{2} + \epsilon} \quad (n \geq n_0). \tag{5.4}$$

Since  $f'(u) \rightarrow x/2\pi$  as  $u \rightarrow \infty$ , we have  $|f'(u)| \leq \frac{1}{2}$  for  $u \geq n_0$  and  $n_0$  large enough. By Lemma (4.4), and the Remark following it,  $|D(F; n_0, n)| \leq A$ . Combining this with (5.4), we get

$$\left| \sum_{v=1}^n e^{i\alpha v} e^{ivx} \right| \leq \left| \sum_1^{n_0} \right| + \left| \sum_{n_0+1}^n \right| < n_0 + O(n^{1-\frac{1}{2} + \epsilon}) + A = O(n^{1-\frac{1}{2} + \epsilon}).$$

Let  $s_n(x)$  be the sum on the left. Then, summing by parts, we get for the  $N$ th partial sum of (5.1),

$$\sum_{n=1}^{N-1} s_n(x) \Delta n^{-\beta} + s_N(x) N^{-\beta} \tag{5.5}$$

The terms of the sum here are  $O(n^{1-\frac{1}{2} + \epsilon}) \cdot O(n^{-\beta-1}) = O(n^{-\frac{1}{2} + \epsilon - \beta})$ , and  $s_N N^{-\beta} = O(N^{1-\frac{1}{2} + \epsilon - \beta})$ . It follows that, under the hypotheses of (i), (5.5) tends uniformly to a limit as  $N \rightarrow \infty$ .

We also observe that if  $\frac{1}{2}\alpha + \beta = 1$  then (5.5) is uniformly  $O(\log N)$ , and if  $\frac{1}{2}\alpha + \beta < 1$  it is uniformly  $O(N^{1-\frac{1}{2} + \epsilon - \beta})$ .

To prove (ii), we make  $N \rightarrow \infty$  in (5.5). We have

$$\psi_{\alpha, \beta}(x) = \sum_1^{\infty} s_n(x) \Delta n^{-\beta}.$$

$$\psi_{\alpha, \beta}(x+h) - \psi_{\alpha, \beta}(x) = \sum_1^{\infty} \{s_n(x+h) - s_n(x)\} \Delta n^{-\beta} = \sum_1^N + \sum_{N+1}^{\infty} = P + Q,$$

say, where  $0 < h \leq 1$  and  $N = [1/h]$ . The terms of  $Q$  are  $O(n^{\alpha - \frac{1}{2}}), O(n^{-\beta - 1}), \dots, O(n^{-\frac{1}{2} - \beta})$ . Hence†

$$Q = O(N^{1 - \frac{1}{2} - \beta}) = O(h^{\frac{1}{2} + \beta - 1}).$$

Applying the mean-value theorem to the real and imaginary parts of  $s_n$ , and using the remark just made (for  $\beta = -1$ ), we find that the terms of  $P$  are  $O(hn^{\alpha - \frac{1}{2}}), O(n^{-\beta - 1})$ . Hence

$$P = O(hN^{\alpha - \frac{1}{2} - \beta}) = O(h^{\frac{1}{2} + \beta - \alpha}).$$

It follows that  $\psi_{\alpha, \beta} \in \Lambda_{\frac{1}{2} + \beta - \alpha}$ .

(5.6) THEOREM. Let  $\beta > 0$ . Then

(i) the series  $\sum v^{\beta} n^{\alpha - 2} e^{i\alpha v}$  converges uniformly for  $\epsilon \leq |x| \leq \pi - \epsilon > 0$ , and, in particular, converges for  $x \neq 0$ ,

(ii) if  $\frac{1}{2}\alpha + \beta < 1$ , the sum  $\psi_{\alpha, \beta}(x)$  of the series is

$$O(x^{-\alpha - \alpha - \beta(\alpha - \alpha)}), \quad O(\log |x|), \quad O(1) \quad \text{for } x \rightarrow +0, \text{ according as } \alpha + \beta - 1 = 1 > 1,$$

and is

$$O(|x|^{\alpha - \frac{1}{2} - \beta(\alpha - \alpha)}) \quad \text{for } x \rightarrow -0,$$

(iii) if  $\frac{1}{2}\alpha + \beta = 1$ , then

$$\psi_{\alpha, \beta}(x) = O(1) \quad \text{for } x \rightarrow +0, \quad \psi_{\alpha, \beta}(x) = O(\log |x|) \quad \text{for } x \rightarrow -0;$$

(iv) if  $\beta > \frac{1}{2}\alpha$ , (5.1) is  $S[\psi_{\alpha, \beta}]$

(i) For  $\epsilon \leq |x| \leq \pi$  and  $u \geq n_0 = n_0(\epsilon)$ ,  $|f'(u)|$  has a positive lower bound. By Lemma (4.3) (i), the left-hand side of (5.4) is uniformly bounded. Using Lemma (4.4) and the Remark to it, we see that the partial sums  $s_n(x)$  of  $\sum e^{i\alpha v + i\beta x}$  are uniformly bounded for  $\epsilon \leq |x| \leq \pi$ . An application of partial summation completes the proof of (i).

(ii) By  $C$  we shall here denote a positive constant independent of  $x$  and  $n$ . First, let  $0 < x \leq \pi$ . We shall show that

$$|s_n(x)| \leq Cn^{1-\alpha} \quad |s_n(x)| \leq C/x \quad (0 < x \leq \pi). \tag{5.7}$$

In virtue of (4.4) and (i), it is enough to prove that, for  $n$  small enough, these inequalities are satisfied by the integrals  $I_n(x) = \int_1^n e^{i\alpha v} e^{i\beta x} dv$ . The new inequalities follow immediately, if we observe (see (5.2)) that  $f'(u)$  exceeds both  $Cu^{\alpha-1}$  and  $C/x$ , and apply (4.3).

For fixed  $x$ , the first inequality (5.7) is more advantageous if  $n$  is small, the second if  $n$  is large. For  $n \sim x^{1/(\alpha-1)}$ , the right hand sides in (5.7) are of the same order. Hence, setting  $M = [x^{1/(\alpha-1)}]$ , we have

$$\psi_{\alpha, \beta}(x) = \sum_{n=1}^{\infty} s_n(x) \Delta n^{-\beta} = \sum_1^M + \sum_{M+1}^{\infty} = A + B.$$

The terms of  $A$  are  $O(n^{1-\alpha}), O(n^{-\beta-1})$ , and the terms of  $B$  are  $O(x^{-1}\Delta n^{-\beta})$ . It follows that, if  $\alpha + \beta < 1$ , then

$$\psi_{\alpha, \beta}(x) = O(M^{1-\alpha-\beta}) + O(x^{-1}M^{-\beta}) = O(x^{-1-\alpha-\beta(\alpha-1)}).$$

Similarly we get the other estimates in (ii) for  $x \rightarrow +0$ .

The case  $-\pi \leq x < 0$  is slightly less simple, since then  $f'(u)$  is not of constant sign. The single zero of  $f'$  is

$$u_0 = u_0(x) = \frac{1}{2} |x|^{1/(\alpha-1)}.$$

† The interval  $(x, x+h)$  may be partly outside  $(-\pi, \pi)$ , but since it is interior to  $(-\pi, 2\pi)$  it is easy to see that the conclusion holds.

It tends to  $\infty$  as  $x \rightarrow -0$ , and we need only consider small  $x$ . Clearly  $|f'| \geq C|x|$  for  $u \geq 2u_0$ . Set  $N = [2u_0]$  and split the series (5.1) into two parts,  $\Sigma_1$  and  $\Sigma_2$ , corresponding to  $n \leq N$  and  $n > N$ . Since in any case  $s_n(x) = O(n^{1-\frac{1}{2}\alpha})$  uniformly in  $x$ , we have, by (5.5),

$$\Sigma_1 = O((2u_0)^{1-\frac{1}{2}\alpha-\beta}) = O(|x|^{-1-\frac{1}{2}\alpha-\beta/(1-\alpha)}).$$

If we can get the same estimate for  $\Sigma_2$ , the proof of (ii) in the case  $x < 0$  will be complete.

Now, summing by parts,

$$\Sigma_2 = \sum_{N+1}^{\infty} \{s_n(x) - s_N(x)\} \Delta n^{-\beta}.$$

Using the fact that  $|f'| \geq C|x|$  for  $u \geq N + 1$  and applying Lemmas (4.4) and (4.3) (i), we obtain  $s_n(x) - s_N(x) = O(x^{-1})$  for  $n > N$ , which leads to

$$\Sigma_2 = O(x^{-1}N^{-\beta}) = O(|x|^{-1-\alpha-\beta/(1-\alpha)}) = O(|x|^{-1-\frac{1}{2}\alpha-\beta/(1-\alpha)}).$$

(iii) The proof is contained in that of (ii).

(iv) It follows from (i), (ii) and (iii) that the function  $\psi_{\alpha,\beta}(x)$  is always integrable  $L$  over  $(0, \pi)$ . The estimates for  $x \rightarrow -0$  involve a much larger order of magnitude, since the exponent  $(1 - \frac{1}{2}\alpha - \beta)/(1 - \alpha)$  can be arbitrarily large if  $1 - \alpha$  and  $\beta$  are sufficiently small. However,  $\psi_{\alpha,\beta}$  is  $L$ -integrable over  $(-\pi, 0)$  if  $\beta > \frac{1}{2}\alpha$ .

On the other hand, it is easy to see that (5.1) is a Fourier-Riemann series if merely  $0 < \alpha < 1, \beta > 0$ . For, when it is integrated termwise with respect to  $x$ , it converges absolutely and uniformly to a continuous function  $\Psi(x)$ , such that  $\Psi'(x) = \psi_{\alpha,\beta}(x)$  for  $x \neq 0$  (see (i)). Thus  $\Psi(x)$  is the Riemann integral of  $\psi_{\alpha,\beta}(x)$ , and (5.1) a Fourier-Riemann series of  $\psi_{\alpha,\beta}$ . Hence, for  $\beta > \frac{1}{2}\alpha$ , we have a Fourier-Lebesgue series.

The special case  $\beta = \frac{1}{2}\alpha$  in (5.6) (ii) leads to the estimate  $O(1/x)$  for  $x \rightarrow -0$ . For a later application we shall need the following result:

(5.8) THEOREM. If  $0 < \alpha < 1$  and  $\gamma$  is real, the function

$$\chi_{\alpha,\gamma}(x) = \sum_2^{\infty} n^{-\frac{1}{2}\alpha} (\log n)^{-\gamma} e^{in\alpha} e^{inx} \tag{5.9}$$

is of the form

$$O\left(x^{-(1-\frac{1}{2}\alpha)/(1-\alpha)} \log^{-\gamma} \frac{1}{x}\right) \text{ and } O\left(x^{-1} \log^{-\gamma} \frac{1}{|x|}\right) \text{ for } x \rightarrow +0 \text{ and } x \rightarrow -0,$$

respectively. Moreover, if  $\gamma > 1, \chi_{\alpha,\gamma}$  is integrable and (5.9) is  $S[\chi_{\alpha,\gamma}]$ .

The proof is essentially the same as that of parts (ii) and (iv) of (5.6).

### 6. Lacunary series

Lacunary trigonometric series are series in which the terms that differ from zero are 'very sparse'. Such series may be written in the form

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) = \sum_{k=1}^{\infty} A_{n_k}(x), \tag{6.1}$$

supposing for simplicity that the constant term also vanishes. We define a lacunary series more specifically as one for which the  $n_k$  satisfy for all  $k$  an inequality

$$n_{k+1}/n_k > q > 1,$$

that is, for which they increase at least as rapidly as a geometric progression with ratio greater than 1.

Given a lacunary series (6.1), consider the sum

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \sum_{k=1}^{\infty} \rho_k^2. \quad (6.2)$$

(6.3) THEOREM. *If  $\sum \rho_k^2$  is finite, the series  $\sum A_{n_k}(x)$  converges almost everywhere.*

Let  $s_m(x)$  and  $\sigma_m(x)$  denote the partial sums and the arithmetic means of  $\sum A_{n_k}(x)$  with the vacant terms replaced by 0's. The sequence  $\sigma_m(x)$  converges almost everywhere and (6.3) follows from the fact that  $s_m(x) - \sigma_m(x) \rightarrow 0$  for every  $x$  (Chapter III, (1.27)).

The converse of 6.3 is also true and lies deeper. *If  $\sum A_{n_k}(x)$  converges in a set of points of positive measure, then  $\sum \rho_k^2$  is finite.* We shall prove an even more general theorem. Let  $T^*$  be any linear method of summation satisfying the first and third conditions of regularity (Chapter III, § 1); the second need not be satisfied. All linear methods of summation used in analysis are  $T^*$  methods.

(6.4) THEOREM. *If  $\sum A_{n_k}(x)$  is summable  $T^*$  in a set  $E$  of positive measure, then  $\sum \rho_k^2$  converges.*

We need the following lemma:

(6.5)\* LEMMA. *Suppose we are given a set  $\mathcal{E} \subset (0, 2\pi)$  of positive measure, and numbers  $\lambda > 1$ ,  $q > 1$ . Then there exists an integer  $h_0 = h_0(\mathcal{E}, \lambda, q)$  such that for any trigonometric polynomial  $P(x) = \sum (a_n \cos n_j x + b_n \sin n_j x)$  with  $n_{j+1}/n_j > q > 1$  and  $n_1 \geq h_0$  we have*

$$\lambda^{-1} |\mathcal{E}|^{\frac{1}{2}} \sum (a_j^2 + b_j^2) \leq \int_{\mathcal{E}} P^2(x) dx \leq \lambda |\mathcal{E}|^{\frac{1}{2}} \sum (a_j^2 + b_j^2). \quad (6.6)$$

The inequalities hold also if  $P(x)$  is an infinite series with  $\sum (a_j^2 + b_j^2) < \infty$ .

Write the polynomial  $P$  in the complex form  $\sum c_\nu e^{in_\nu x}$ , with  $n_\nu = -n_\mu$ . Then

$$\sum |c_\nu|^2 = \frac{1}{2} \sum (a_j^2 + b_j^2).$$

We have

$$\begin{aligned} \int_{\mathcal{E}} P^2(x) dx &= \int_{\mathcal{E}} (\sum c_\mu e^{in_\mu x}) (\sum c_\nu e^{-in_\nu x}) dx \\ &= |\mathcal{E}| \sum |c_\nu|^2 + \sum_{\mu \neq \nu} c_\mu \bar{c}_\nu \int_{\mathcal{E}} e^{i(n_\mu - n_\nu)x} dx. \end{aligned} \quad (6.7)$$

Let  $\gamma_n$  denote the complex coefficients of the characteristic function of  $\mathcal{E}$ . The last integral is then  $2\pi \gamma_{n_\nu - n_\mu}$ . By Schwarz's inequality, the modulus of the last sum does not exceed

$$2\pi \left( \sum_{\mu, \nu} |c_\mu c_\nu|^2 \right)^{\frac{1}{2}} \left( \sum_{\mu \neq \nu} |\gamma_{n_\nu - n_\mu}|^2 \right)^{\frac{1}{2}} = 2\pi \left( \sum_{\nu} |c_\nu|^2 \right) \left( \sum_{\mu \neq \nu} |\gamma_{n_\nu - n_\mu}|^2 \right)^{\frac{1}{2}}. \quad (6.8)$$

We assert that there exists a number  $\Delta = \Delta(q)$  such that no integer  $N$  can be represented more than  $\Delta$  times in the form  $n_\nu - n_\mu$ , with  $\mu \neq \nu$ .

It is enough to assume that  $0 < \mu < \nu$  and consider the two cases.

- (i)  $N = n_\nu + n_\mu$
- (ii)  $N = n_\nu - n_\mu$ .

In case (i) we have  $\frac{1}{2}N < n_\nu < N$ , and since the  $n_\nu$  increase at least as rapidly as  $q^\nu$ , the number of  $n_\nu$  satisfying this inequality does not exceed  $y + 1$ , where  $q^y = 2$ . In case (ii), since  $n_\mu < n_\nu/q$ , we have  $n_\nu - n_\nu/q < N$ , that is  $n_\nu < Nq/(q - 1)$ . On the other hand,  $n_\nu > N$ . Since the number of the  $n_\nu$  between  $N$  and  $Nq/(q - 1)$  is bounded (the bound depending on  $q$ ), the existence of  $\Delta(q)$  follows.

Thus the last factor on the right of (6.8) does not exceed  $\{2\Delta(|\gamma_h|^2 + |\gamma_{h+1}|^2 + \dots)\}^{\frac{1}{2}}$ , where  $h$  is the least integer representable in the form  $n_\nu - n_\mu$  with  $1 \leq \mu < \nu$ . But

$$n_\nu - n_\mu \geq n_\nu - n_{\nu-1} \geq n_\nu(1 - q^{-1}) \geq n_1(1 - q^{-1}).$$

This shows that  $h$  is large with  $n_1$ .

The  $\gamma$ 's depend only on  $\mathcal{E}$ , and  $\Sigma |\gamma_\nu|^2 = |\mathcal{E}| (2\pi)^{-1} \leq 1$ . Hence if  $n_1$  is large enough,  $n_1 \geq h_0(\mathcal{E}, \lambda, q)$ , we can make the right-hand side of (6.8) less than

$$(1 - \lambda^{-1})|\mathcal{E}| < (\lambda - 1)|\mathcal{E}|,$$

and we obtain (6.6) in virtue of (6.7).

If  $P$  is an infinite series with  $\Sigma(a_k^2 + b_k^2)$  finite, we first apply (6.6) to the partial sums  $P_l$  of  $P$ . Then making  $l \rightarrow \infty$  and observing that

$$\int_{\mathcal{E}} P_l^2 dx \rightarrow \int_{\mathcal{E}} P^2 dx,$$

we get the required result.

Passing to the proof of (6.4), we denote by  $\beta_{mn}$  the elements of the matrix  $T^*$  considered. The hypothesis is that for every  $x \in E$  each of the series  $\sum_n \beta_{mn} s_n(x)$ ,  $m = 0, 1, 2, \dots$ , converges to a sum  $\tau_m(x)$ , which tends to a finite limit as  $m \rightarrow \infty$ . We begin with the case when the matrix is row-finite. If we set  $\beta_{mn} + \beta_{m,n+1} + \dots = R_{mn}$ , then

$$\tau_m(x) = \sum_{k=1}^{\infty} A_{n_k}(x) R_{mn_k}, \tag{6.9}$$

where  $A_{n_k}(x) = a_k \cos n_k x + b_k \sin n_k x$ . The sum here has only a finite number of terms different from zero. Since  $\tau_m(x)$  converges in  $E$ , we can find a subset  $\mathcal{E}$  of  $E$  with  $|\mathcal{E}| > 0$  and a number  $M$  such that  $|\tau_m(x)| \leq M$  for all  $x \in \mathcal{E}$  and all  $m$ . (For  $E = E_1 + E_2 + \dots$ , where  $E_p$  is the set of points  $x \in E$  such that  $|\tau_m(x)| \leq p$  for all  $m$ . At least one set  $E_p$ , say  $E_M$ , is of positive measure and may be taken for  $\mathcal{E}$ .)

We now apply (6.5) with  $\lambda = 2$ . The set  $\mathcal{E}$  and the numbers  $q, \lambda$  determine an integer  $h_0$  such that (6.6) holds for  $n_1 > h_0$ . The latter condition may be assumed satisfied here, since we may always reject a finite number of terms from  $\Sigma A_{n_k}(x)$  without influencing its summability  $T^*$  (although this can affect the value of the constant  $M$ ).

Thus

$$\begin{aligned} \frac{1}{2}|\mathcal{E}| \sum_k (a_k^2 + b_k^2) R_{mn_k}^2 &\leq \int_{\mathcal{E}} \tau_m^2(x) dx \leq M^2 |\mathcal{E}|, \\ \Sigma (a_k^2 + b_k^2) R_{mn_k}^2 &\leq 4M^2. \end{aligned}$$

Let now  $K > 0$  be any fixed integer. Since  $\lim_{m \rightarrow \infty} R_{mn_k} = 1, k = 1, 2, \dots$ , the last inequality gives

$$\sum_{k=1}^K (a_k^2 + b_k^2) R_{mn_k}^2 \leq 4M^2, \quad \sum_{k=1}^K (a_k^2 + b_k^2) \leq 4M^2,$$

and the convergence of (6.2) follows.

We can remove the restriction on  $\{\beta_{\nu q}\}$  to be row-finite, as follows. Let  $\tau_{m_i}^*(x)$  be an expression analogous to  $\tau_m(x)$  (cf. (6.9)), except that the upper limit of summation is not  $+\infty$  but a number  $N = N(m)$ . We take  $N$  so large that the following conditions are satisfied:

- (a)  $|\tau_m(x) - \tau_m^*(x)| < 1/m$  for  $x \in E - E^m$ ,  $|E^m| < |E| 2^{-m-1}$ ;
- (b)  $\lim_{m \rightarrow \infty} (\beta_{m0} + \beta_{m1} + \dots + \beta_{m,\nu}) = 1$ .

If  $E^* = E^1 + E^2 + \dots$ , then  $|E^*| < |E|$ , and in the set  $E - E^*$ , which is of positive measure, the linear means  $\tau_m^*(x)$  tend to a finite limit. But condition (b) ensures that the  $\tau_m^*(x)$  are  $T^*$  means corresponding to a matrix with only a finite number of terms different from zero in each row. Thus the general case is reduced to the special one already dealt with.

*Remarks.* (a) If the  $\Sigma \rho_k^2$  is infinite, (6.4) implies that  $\Sigma A_{n_k}(x)$  is non-summable almost everywhere by any method of summation. Considering, in particular, the method (C, 1) we get: *If  $\Sigma \rho_k^2$  diverges,  $\Sigma A_{n_k}(x)$  is not a Fourier series.*

(b) If  $\Sigma \rho_k^2$  is infinite, then not only does the sequence of the partial sums of  $\Sigma A_{n_k}(x)$  diverge almost everywhere, but so does every subsequence of this sequence. For selecting such a subsequence amounts to an application of a linear method of summation, in whose matrix each row consists entirely of zeros except for a single element 1.

(d) The proof of (6.4) holds if we assume that  $\Sigma A_{n_k}(x)$  is merely bounded  $T^*$  at every point of  $E$ ,  $|E| > 0$ . For some problems it is desirable to have a similar result for one-sided boundedness.

(6.10) THEOREM. *Suppose that  $\Sigma \rho_k^2$  diverges, and let  $\tau_m(x)$  be the  $T^*$  means of  $\Sigma A_{n_k}(x)$ . Then the set of points  $x$  at which*

$$\tau_m^+(x) = o\left\{\sum_{k=1}^{\infty} \rho_k^2 R_{mn_k}^2\right\}^{\frac{1}{2}} \quad (\rho_k^2 = a_k^2 + b_k^2) \tag{6.11}$$

is of measure zero.

Here  $\tau_m^-(x) = \max\{0, \tau_m(x)\}$ . The sum in curly brackets, which we shall denote by  $\Gamma_m^2$ , tends to  $+\infty$  with  $m$ , since  $R_{mn_k} \rightarrow 1$  for fixed  $k$ . Hence (6.10) implies that *if the  $T^*$  means of  $\Sigma A_{n_k}(x)$  are bounded above (or below) at every point of a set of positive measure, the series  $\Sigma \rho_k^2$  converges.*

Suppose that we have (6.11) for every  $x \in E$ ,  $|E| > 0$ , and that  $\Sigma \rho_k^2$  diverges. Given any  $\epsilon > 0$ , there is a set  $\mathcal{E} \subset E$  with  $|\mathcal{E}| > \frac{1}{2}|E|$  such that  $\tau_m(x), \Gamma_m \leq \epsilon$  in  $\mathcal{E}$ , for  $m > m_0$ . By dropping the few first terms of  $\Sigma A_{n_k}(x)$ , we may, without changing  $\mathcal{E}$ , suppose  $n_1$  as large as we please. Let  $\alpha_n, \beta_n$  be the Fourier coefficients of the characteristic function of the set  $\mathcal{E}$ . Then

$$\begin{aligned} \int_{\mathcal{E}} |\tau_m(x)| dx &\leq \int_{\mathcal{E}} \{|\tau_m(x) - \epsilon \Gamma_m| + \epsilon \Gamma_m\} dx = \int_{\mathcal{E}} \{2\epsilon \Gamma_m - \tau_m(x)\} dx \\ &= 2\epsilon \Gamma_m |\mathcal{E}| - \pi \sum_{k=1}^{\infty} (\alpha_{n_k} a_k + \beta_{n_k} b_k) R_{mn_k} \leq 2\epsilon \Gamma_m |\mathcal{E}| + \pi \Gamma_m \{\Sigma (\alpha_{n_k}^2 + \beta_{n_k}^2)\}^{\frac{1}{2}}. \end{aligned}$$

The right-hand side here is less than  $\epsilon \Gamma_m (2|\mathcal{E}| + \pi)$  if  $n_1$  is large enough. This shows that

$$\int_{\mathcal{E}} |\tau_m| dx = o(\Gamma_m). \tag{6.12}$$

By (6.5), the left-hand side of

$$\int_E \tau_m^2 dx \leq \left( \int_E |\tau_m| dx \right)^{\frac{1}{2}} \left( \int_E \tau_m^4 dx \right)^{\frac{1}{2}}$$

(an immediate consequence of Hölder's inequality) exceeds some fixed multiple of  $\Gamma_m^{\frac{1}{2}}$  for  $n_1$  large enough. By Theorem (8.20), which will be proved below, the integral  $\int_E \tau_m^4 dx$  ( $\leq \mathfrak{M}_4^4[\tau_m]$ ) does not exceed a fixed multiple of  $\Gamma_m^2$ . Thus,  $\int_E |\tau_m|$  exceeds some fixed multiple of  $\Gamma_m$ . This contradicts (6.12) and proves Theorem (6.10).

In this argument we tacitly assumed that the  $\Gamma_m$  were finite. This follows from the hypothesis that the series defining  $\tau_m(\tau)$  converges in a set of positive measure.

Consider the two lacunary series

$$\sum b^{-n\alpha} \cos b^n x = f_\alpha(x), \quad \sum b^{-n\alpha} \epsilon_n \cos b^n x = g_\alpha(x)$$

(already discussed in Chapter II, § 4), where  $\alpha$  is positive,  $b$  is an integer not less than 2, and  $\epsilon_n \rightarrow 0$ . From (6.4) we deduce that, if  $0 < \alpha \leq 1$ , then the continuous function  $f_\alpha(x)$  is differentiable at most in a set of measure zero. For

$$\frac{f_\alpha(x+h) - f_\alpha(x-h)}{2h} = -\sum b^{n(1-\alpha)} \sin b^n x \left( \frac{\sin b^n h}{b^n h} \right).$$

At every point of differentiability of  $f_\alpha$  the left-hand side tends to  $f'_\alpha(x)$  as  $h \rightarrow 0$ , which means that  $S[f_\alpha] = -\sum b^{n(1-\alpha)} \sin b^n x$  is summable by a linear method of summation to a finite limit. Hence, if  $f'_\alpha$  existed in a set of positive measure, we should have  $\sum b^{2n(1-\alpha)} < \infty$ , which is false.

This result asserts less than the classical result of Weierstrass-Hurdy (see p. 48) that  $f_\alpha$  is *nowhere* differentiable if  $0 < \alpha \leq 1$ . The proof of the latter result, however, uses the special structure of the coefficients and exponents in  $S[f_\alpha]$ , while the proof given above is valid for general lacunary series for which no such results are possible (see Example 17, p. 230). For example, the above proof shows that  $g_1(x)$  is almost nowhere differentiable if  $\sum \epsilon_n^2 = \infty$ . On the other hand, we know that  $g_1(x)$  is smooth and so certainly differentiable in a set of points having the power of the continuum (Chapter II, § 3).

Theorem (6.4) shows that if a lacunary series 'behaves well' on a set  $E$  of positive measure, then it 'behaves well' in  $(0, 2\pi)$ . We shall now give another example of this principle.

**(6.13) THEOREM.** (i) Suppose that  $\sum A_{n_k}(x)$  converges on a set  $E$ ,  $|E| > 0$ , to a function  $f(x)$  which coincides on  $E$  with another function  $g(x)$  defined over an interval  $I = (\alpha, \beta) \supset E$  and analytic on  $I$ . Then the series

$$\sum (a_k \cos n_k x + b_k \sin n_k x) \rho^{n_k}$$

converges in some circle  $|z| < 1 + c$  ( $z = \rho e^{ix}$ ,  $c > 0$ ).

(ii) If  $\sum A_{n_k}(x)$  converges to zero on a set  $E$  of positive measure, then the series vanishes identically†.

The hypothesis concerning  $g$  means that in the neighbourhood of every point  $x \in (\alpha, \beta)$ ,  $g$  is represented by a power series.

† The result holds if the series contains a constant term.

(6.14) LEMMA. If  $H$  is any measurable set in  $(0, 2\pi)$ , then we can find a sequence of numbers  $h_m \rightarrow 0$  such that for almost all  $x \in H$  and for  $m > m_0(x)$  the points  $x \pm h_m$  are in  $H$ .

Let  $\chi(x)$  be the characteristic function of  $H$ . Then

$$I(t) = \int_0^{2\pi} |\chi(x+t) - \chi(x)| dx \rightarrow 0$$

as  $t \rightarrow 0$ . By Theorem (11.8) of Chapter I there is a sequence  $k_m \rightarrow 0$  such that  $\chi(x+k_m) - \chi(x) \rightarrow 0$  almost everywhere, and so also almost everywhere in  $E$ . Since  $\chi$  only takes the values 0 and 1, we have  $\chi(x+k_m) = \chi(x)$  for almost all  $x \in E$  and  $m > m_0(x)$ . Moreover,  $\{k_m\}$  may be a subsequence of any sequence tending to 0. Therefore,

repeating the argument with the integral  $J(k_m) = \int_0^{2\pi} |\chi(x-k_m) - \chi(x)| dx$  we obtain an  $\{h_m\}$  with the required properties.

We apply this to  $H = E$  in (6.13) (i). For almost all  $x \in E$  and sufficiently large  $m$ ,

$$\frac{g(x+h_m) - g(x-h_m)}{2h_m} = \frac{f(x+h_m) - f(x-h_m)}{2h_m} = \sum n_k (b_k \cos n_k x - a_k \sin n_k x) \frac{\sin n_k h_m}{n_k h_m}.$$

As  $m \rightarrow \infty$ , the left-hand side here tends to  $g'(x)$ . It follows that the series

$$\sum n_k (b_k \cos n_k x - a_k \sin n_k x)$$

is summable by a linear method of summation almost everywhere in  $E$ . Let  $S, S', S'', \dots$  denote respectively the series  $\sum A_{n_k}(x)$  and the series obtained from it by successive termwise differentiations. By (6.4),  $\sum n_k^2 (a_k^2 + b_k^2)$  converges. Hence there is a subset  $E_1 \subset E$ ,  $|E_1| = |E|$ , such that  $S'$  converges in  $E_1$  to sum  $g'(x)$ . Similarly, repeating the argument, there is a set  $E_2 \subset E_1$ ,  $|E_2| = |E_1|$ , such that  $S''$  converges in  $E_2$  to sum  $g''(x)$ , and so on.

All the  $S^{(\nu)}$  converge in the set  $E^* = E E_1 E_2 \dots$ . Clearly  $|E^*| = |E|$ . We apply Lemma (6.5) with  $\lambda = 2$ , to  $P = S^{(\nu)}$ ,  $\mathcal{E} = E^*$ . We may suppose  $n_1$  so large that (6.6) holds. Thus, with  $\gamma_k^2 = a_k^2 + b_k^2$ ,

$$\sum \gamma_k^2 n_k^{2\nu} \leq \frac{4}{|E^*|} \int_{E^*} |g^{(\nu)}(x)|^2 dx. \quad (6.15)$$

We may further suppose that the interval  $(\alpha, \beta)$  is closed. The classical inequality of Cauchy for the coefficients of power series then gives

$$|g^{(\nu)}(x)| \leq M \nu! \delta^{-\nu} \quad (\alpha \leq x \leq \beta, \nu = 1, 2, \dots)$$

with suitable  $M$  and  $\delta$ . Applying this to (6.15), and keeping only one term on the left, we get

$$\gamma_k^2 n_k^{2\nu} \leq (2M\nu! \delta^{-\nu})^2 \leq (2M\nu \delta^{-\nu})^2,$$

$$\gamma_k^{1/n_k} \leq (2M)^{1/n_k} \left( \frac{\nu}{\delta n_k} \right)^{\nu/n_k}.$$

If we set  $\nu = [\frac{1}{2} \delta n_k] =$  integral part of  $\frac{1}{2} \delta n_k$ , we obtain

$$\limsup \gamma_k^{1/n_k} \leq 2^{-1/2} < 1,$$

and part (i) of (6.13) is established.

As a corollary of this we have the following classical theorem.

**(6.15) THEOREM OF HADAMARD.** *If a power series*

$$\sum c_k z^{n_k}, \quad n_{k-1}, n_k \geq q > 1,$$

converges for  $|z| < 1$  and is analytically continuable across an arc of  $|z| = 1$ , then the radius of convergence of (6.17) exceeds 1.

For  $\sum c_k e^{in_k x}$  is, by hypothesis, Abel summable to a function  $g(x)$  analytic on an arc  $(\alpha, \beta)$  and is also, by (6.4) and (6.3), convergent almost everywhere in  $(\alpha, \beta)$ .

In case (ii),  $g \equiv 0$ . The rejection of the first few terms of (6.1), so as to make  $n_1$  large enough and (6.6) applicable, amounts to making  $g$  a polynomial of order  $m < n_1$ . Clearly  $|g^{(p)}|$  is majorized by  $Mm^p$ , where  $M$  is now the sum of the moduli of the coefficients of  $g$ , and (6.15) leads successively to

$$\sum \gamma_k^2 n_k^{2p} \leq 4M^2 m^{2p}, \quad \gamma_1 n_1^p \leq 2Mm^p.$$

The last inequality is impossible for  $p$  large enough, unless  $\gamma_1 = 0$ . Similarly  $\gamma_2 = \gamma_3 = \dots = 0$ , and case (ii) is established.

*Remark.* It follows from (6.13) (ii) that if two lacunary series  $S_1$  and  $S_2$  have the same exponents (or, what amounts to the same thing, if the joint sequence of the exponents in  $S_1$  and  $S_2$  is still lacunary), and if they converge to the same sum on a set of positive measure, then  $S_1 \equiv S_2$ . The result holds for any two lacunary series, but the proof is then more difficult.

**7. Riesz products**

Consider the infinite product

$$\prod_{v=1}^{\infty} (1 + \alpha_v \cos n_v x), \tag{7.1}$$

where the positive integers  $n_v$  satisfy a condition

$$n_{v-1} / n_v \geq q > 1,$$

and  $-1 \leq \alpha_v \leq 1$ ,  $\alpha_v \neq 0$  for all  $v$ . Let

$$\mu_j = n_k - n_{k-1} + \dots + n_1, \quad \mu'_k = n_{k-1} - n_k - \dots - n_1 \quad (k = 1, 2, \dots).$$

Then

$$\begin{aligned} \mu_k &< n_k(1 + q^{-1} + q^{-2} + \dots) = n_k q / (q - 1) \\ \mu'_k &> n_{k-1}(1 - q^{-1} - q^{-2} - \dots) \geq n_k q (q - 2) / (q - 1). \end{aligned}$$

Thus  $\mu'_k > \mu_k$  if  $q - 2 \geq 1$ , that is, if  $q \geq 3$  which we assume henceforth.

The  $k$ th partial product of (7.1) is a non-negative trigonometric polynomial

$$p_k(x) = 1 + \sum_{v=1}^{\mu_k} \gamma_v \cos vx = \prod_{i=1}^k (1 + \alpha_i \cos n_i x), \tag{7.2}$$

where  $\gamma_i = 0$  if  $i$  is not of the form  $n_p + n_{p-1} + \dots$ , with  $k > i > i'' > \dots$ . The difference

$$p_{k-1} - p_k = \beta_k \gamma_{\mu_{k-1}} \cos n_{\mu_{k-1}} x$$

is a polynomial whose lowest term is of rank  $\mu'_k > \mu_k$ . Hence the passage from  $p_k$  to  $p_{k-1}$  consists in adding to (7.2) a group of terms whose ranks all exceed  $\mu_k$ . Making  $k \rightarrow \infty$ , we obtain from (7.2) an infinite series

$$1 + \sum_{v=1}^{\infty} \gamma_v \cos vx \tag{7.3}$$

in which  $\gamma_i = 0$  if  $\nu \neq n_i \pm n_{i'} \pm n_{i''} \pm \dots$ ,  $i > i' > \dots$ . We shall say that (7.3) represents the product (7.1). The partial sums  $s_{\mu_i}(x)$  of (7.3) have the property  $s_{\mu_i}(x) = p_{\mu_i}(x) \geq 0$ . It follows from Theorem (5.29) of Chapter IV that (7.3) is the Fourier-Stieltjes series of a non-decreasing continuous function  $F(x)$ . This function is obtained by integrating (7.3) termwise. In particular

$$F(x) - F(0) = \lim_{k \rightarrow \infty} \int_0^x p_k(t) dt. \quad (7.4)$$

Thus

(7.5) THEOREM. *The series (7.3) representing the product (7.1), with  $n_{k+1}/n_k \geq 3$ ,  $-1 \leq \alpha_i \leq \pm 1$ , is the Fourier-Stieltjes series of a non-decreasing continuous function  $F$  defined by (7.4).*

The series (7.3) is formally obtained by multiplying out (7.1) and replacing the products of cosines by linear combinations of cosines. No two terms thus obtained are of the same rank, since every integer  $N$  can be represented in the form  $n_i \pm n_{i'} \pm n_{i''} \pm \dots$ , with  $i > i' > i'' > \dots$ , at most once. (Such sums, being greater than  $\mu_{i-1}$ , must be positive.) For suppose we have another representation  $N = n_k \pm n_{k'} \pm \dots$ , with  $k \neq i$ , say  $k < i$ . Then  $n_i = an_{i-1} + bn_{i-2} + cn_{i-3} + \dots$ , where  $a, b, c, \dots$  take only the values  $0, \pm 1$  and  $\pm 2$ . The right-hand side of this equation is less than

$$2n_{i-1}(1 + 3^{-1} + 3^{-2} + \dots) = 3n_{i-1},$$

and so cannot be equal to  $n_i$ . Hence  $k = i$ , and we have  $n_i \pm n_{i'} \pm \dots = n_k \pm n_{k'} \pm \dots$ . This gives  $i' = k'$ , and so on.

In particular,  $\gamma_{n_i} = \alpha_i$ . If  $\alpha_i$  does not tend to 0 (e.g. if  $\alpha_i = 1$ ,  $n_i = 3^i$ ) we obtain, with F. Riesz, a new example (historically the first) of a continuous function of bounded variation whose Fourier coefficients are not  $o(1/n)$ .

The products (7.1) are called *Riesz products*.

(7.6) THEOREM. *If  $-1 \leq \alpha_\nu \leq 1$ ,  $n_{\nu+1}/n_\nu \geq q > 3$ , and  $\Sigma \alpha_\nu^2 = \infty$ , then the function  $F$  of (7.4) has a derivative 0 almost everywhere.*

By Chapter III, (8.1), the series (7.3) is almost everywhere summable ( $C-1$ ) to sum  $F'(x)$ . The series has infinitely many gaps  $(\mu_i, \mu_k)$ , and since

$$\mu_k/\mu_i \geq n_{k-1}(1 - q^{-1} - q^{-2} - \dots) \cdot n_i(1 + q^{-1} + q^{-2} + \dots) \geq q - 2 > 1,$$

Chapter III, (i.27) shows that the partial products  $p_k(x)$  converge to  $F'(x)$  almost everywhere. The inequality  $1 + u \leq e^u$  gives

$$0 \leq p_k(x) \leq \exp\left(\sum_{\nu=1}^k \alpha_\nu \cos n_\nu x\right).$$

Since  $\Sigma \alpha_\nu^2 = \infty$ , the partial sums of  $\Sigma \alpha_\nu \cos n_\nu x$  take arbitrarily large negative values at almost all points (see (6.10)). Thus  $\liminf p_k(x) = 0$ , that is,  $F'(x) = 0$  almost everywhere. Incidentally we have proved that the product (7.1) converges to 0 almost everywhere.

*Remark.* Using Theorem (6.3) we easily prove that if in (7.5) we assume that  $\Sigma \alpha_\nu^2 < \infty$ , then (7.1) converges almost everywhere to finite values different from 0.

(7.7) THEOREM. *If  $\alpha_i \rightarrow 0$ ,  $\Sigma \alpha_i^2 = \infty$ , and  $n_{i+1}/n_i \geq q > 3$ , then both the series (7.3) representing (7.1) and its conjugate series converge almost everywhere, the former to zero.*

The partial sums  $s_{\mu_k}(x)$  of (7.3) converge almost everywhere. The same holds for the partial sums  $\bar{s}_{\mu_k}(x)$  of the conjugate series since the latter is summable (C, 1) almost everywhere and has the same gaps as (7.3). If  $t_n = s_n + i\bar{s}_n$  are the partial sums of  $1 + \sum_1^n \gamma_\nu e^{i\nu x}$ , then  $M(x) = \sup_k |t_{\mu_k}'(x)|$  is finite almost everywhere.

Take any point  $x$  at which  $t_{\mu_k}$  converges, so that  $M = M(x)$  is finite, fix  $k$  and let  $A$  be so large that

$$\left| \sum_\nu^{\mu_{k-1}} \gamma_\nu e^{i\nu x} \right| \leq A \quad (1 \leq \mu \leq \mu_{k-1}), \tag{7.8_{k-1}}$$

$$\left| \sum_\nu^{\mu_k} \gamma_\nu e^{i\nu x} \right| \leq A - 2M \quad (\mu_{i-1} < \mu \leq \mu_i; i = 1, 2, \dots, k-1; \mu_0 = 0). \tag{7.9_{k-1}}$$

The number  $A$  *prima facie* depends on  $k$ , but we give an inductive proof that the inequalities are true for all  $k$  and an  $A$  independent of  $k$ .

We have 
$$s_{\mu_k} = \left( 1 + \sum_{\nu=1}^{\mu_{k-1}} \gamma_\nu \cos \nu x \right) (1 + \alpha_k \cos n_k x)$$

$$= s_{\mu_{k-1}} + \alpha_k \cos n_k x + \frac{1}{2} \alpha_k \sum_{\nu=1}^{\mu_{k-1}} \gamma_\nu [\cos (n_k - \nu) x + \cos (n_k + \nu) x]. \tag{7.10}$$

Since  $n_k \pm \nu > 0$ , the passage from  $s_{\nu}$  to  $t_{\mu_k}$  consists of replacing cosines by exponentials. We shall now estimate  $t_\lambda$  for  $\mu_{k-1} < \lambda \leq \mu_k$ .

Consider separately the cases

$$(a) \quad \mu_{k-1} < \lambda < n_k, \quad (b) \quad n_k \leq \lambda \leq \mu_k.$$

In case (a), as we see from (7.10),

$$t_\lambda = t_{\mu_{k-1}} \quad \text{or} \quad t_\lambda = t_{\mu_{k-1}} + \frac{1}{2} \alpha_k \sum_{\nu=n_k-\lambda}^{\mu_{k-1}} \gamma_\nu e^{i(n_k-\nu)x},$$

according as  $\lambda < n_k - \mu_{k-1}$  or  $\lambda \geq n_k - \mu_{k-1}$ . In the latter case, the last term on the right is absolutely  $\leq \frac{1}{2} |\alpha_k| \left| \sum_{n_k-\lambda}^{\mu_{k-1}} \gamma_\nu e^{i\nu x} \right| \leq \frac{1}{2} |\alpha_k| A$ , by (7.8\_{k-1}). In case (b),

$$t_{n_k} = t_{\mu_{k-1}} + \alpha_k e^{in_k x}, \quad t_\lambda = t_{\mu_k} - \frac{1}{2} \alpha_k \sum_{\lambda-n_k+1}^{\mu_{k-1}} \gamma_\nu e^{i(n_k+\nu)x} \quad \text{for } \lambda > n_k,$$

and the last term is again absolutely  $\leq \frac{1}{2} |\alpha_k| A$ . Hence

$$|t_\lambda - t_{\mu_{k-1}}| \leq \frac{1}{2} |\alpha_k| (A + 2) \quad \text{or} \quad |t_\lambda - t_{\mu_k}| \leq \frac{1}{2} |\alpha_k| A \tag{7.11}$$

for  $\mu_{k-1} < \lambda \leq n_k$  or  $n_k < \lambda \leq \mu_k$  respectively (the additional  $|\alpha_k|$  on the right of the first inequality being actually needed for  $\lambda = n_k$  only). In particular,

$$|t_\lambda| \leq M + \frac{1}{2} |\alpha_k| (A + 2) \leq M + \frac{1}{2} A + 1$$

for all  $\lambda$  in the range  $(\mu_{k-1} + 1, \mu_k)$ .

Suppose now that  $A$  is so large that

$$2M + (\frac{1}{2}A + 1) \leq A - 2M.$$

Then, if  $\mu_{k-1} < \mu \leq \mu_k$ ,

$$\left| \sum_\nu^\mu \gamma_\nu e^{i\nu x} \right| \leq |t_{\mu_k}| + |t_{\mu_{k-1}}| \leq M + M + \frac{1}{2} A + 1 \leq A - 2M.$$

If  $\mu_{j-1} < \mu \leq \mu_j$ ,  $j < k$ , we have

$$\left| \sum_{\mu}^{\mu_k} \gamma_{\nu} e^{i\nu x} \right| \leq \left| \sum_{\mu}^{\mu_j} \right| + \left| \sum_{\mu_{j+1}}^{\mu_k} \right| \leq A - 2M + 2M = A.$$

Thus (7·9<sub>k-1</sub>) and (7·8<sub>k-1</sub>) imply (7·9<sub>k</sub>) and (7·8<sub>k</sub>), which shows that these hold for all  $k$ . In particular, the  $t_{\lambda}(x)$  are bounded, even if  $\alpha_{\nu}$  does not tend to 0. If  $\alpha_{\nu} \rightarrow 0$  then, by (7·11),  $t_{\lambda}$  converges almost everywhere. Since  $s_{\mu_k}$  converges to 0 almost everywhere, so does  $s_{\lambda}$ . This completes the proof of (7·7).

(7·12) THEOREM. Let  $n_{\nu+1}/n_{\nu} \geq 3$  for all  $\nu$ , and let  $\Sigma \alpha_{\nu}^2 < \infty$ . Then (a) the (complex-valued) series

$$1 + \sum_1^{\infty} \delta_{\nu} \cos \nu x \quad (7·13)$$

representing the product 
$$\prod_1^{\infty} (1 + i\alpha_{\nu} \cos n_{\nu} x) \quad (7·14)$$

is the Fourier series of a bounded function; (b) if  $n_{\nu+1}/n_{\nu} \geq q > 3$ , the series (7·13) and its conjugate converge almost everywhere.

Here the  $\delta_{\nu}$  are obtained from the  $i\alpha_{\nu}$  in the same way as the  $\gamma_{\nu}$  were obtained from the  $\alpha_{\nu}$ . The  $\delta_{\nu}$  are either real or purely imaginary. Obviously

$$\left| \prod_{\nu=1}^k (1 + i\alpha_{\nu} \cos n_{\nu} x) \right| \leq \prod_1^k (1 + \alpha_{\nu}^2)^{\frac{1}{2}} < \prod_1^{\infty} (1 + \alpha_{\nu}^2)^{\frac{1}{2}} < \infty,$$

and so there is a subsequence of the partial sums of (7·13) which is uniformly bounded. This proves (a). (See Chapter IV, p. 148).

If  $t_n$  denotes the partial sums of  $1 + \sum_1^{\infty} \delta_{\nu} e^{i\nu x}$ , the proof of (b) is a repetition of that of Theorem (7·7) with minor modifications caused by the terms of (7·13) being imaginary. For clearly the partial sums of order  $\mu_k$  of both (7·13) and its conjugate  $\Sigma \delta_{\nu} \sin \nu x$  converge almost everywhere. Hence  $t_{\mu_k}$  converges almost everywhere, and the proof analogous to that of (7·7) shows that  $t_{\lambda}$  converges almost everywhere. It is now enough to observe that at each  $x$  where both  $t_{\lambda}(x)$  and  $t_{\lambda}(-x)$  converge, we have the convergence of  $\Sigma \delta_{\nu} \cos \nu x$  and  $\Sigma \delta_{\nu} \sin \nu x$ .

*Remarks.* (a) Theorems (7·6), (7·7) and (7·12) remain valid if in (7·1) and (7·14)  $\alpha_{\nu} \cos n_{\nu} x$  is replaced by  $\alpha_{\nu} \cos n_{\nu} x + \beta_{\nu} \sin n_{\nu} x = \rho_{\nu} \cos(n_{\nu} x + \theta_{\nu})$ , with obvious conditions on the  $\rho_{\nu}$ . The proofs remain the same.

(b) The indices of the non-zero terms of (7·3) and (7·13) are confined to the intervals  $(\mu'_{k-1}, \mu_k)$ . Since the latter interval contains  $n_k$ , and since

$$\mu_k / \mu'_{k-1} \leq n_k (1 + q^{-1} + \dots) / n_k (1 - q^{-1} - \dots) = q / (q - 2),$$

we see that no matter how small  $\epsilon$  is,  $\epsilon > 0$ , the indices of the non-zero terms of these series will lie in the intervals  $(n_k(1 - \epsilon), n_k(1 + \epsilon))$ , provided  $q$  is large enough,  $q > q_0(\epsilon)$ . We shall use this remark later (Chapter VI, § 6).

(c) Theorems (7·6), (7·7) and (7·12) (b) remain valid for  $n_{\nu+1}/n_{\nu} \geq 3$ .

For (7·6) this is proved by splitting (7·1) into two subproducts, corresponding respectively to even or odd  $\nu$ . At least one subproduct satisfies the hypotheses of (7·6), with  $q = 9$ . Hence in virtue of the remark to Theorem (7·6)  $p_k(x)$  converges to 0 almost everywhere. Using the fact (to be proved in Chapter XIII, (5·13)) that if a series

$\Sigma A_k(x)$  is summable (C, 1) almost everywhere to sum  $\sigma(x)$  and if a sequence  $\{s_{n_k}\}$  of its partial sums converges almost everywhere to limit  $s(x)$ , then  $s(x) = \sigma(x)$  almost everywhere, we see that  $F'(x) = 0$  almost everywhere.

The extensions of (7.7) and (7.12) (b) are based on a theorem (see Chapter XIII, p. 176, Remark (i)) that if a sequence of the partial sums of an  $S[f]$  or  $S[dF]$  converges almost everywhere, so does the same sequence of the partial sums of the conjugate series. In our case,  $s_{\mu_k}(x)$  converges almost everywhere, and so the same holds for  $\tilde{s}_{\mu_k}(x)$ . From this point on, the proofs remain unchanged.

## 8. Rademacher series and their applications

Several properties of lacunary trigonometric series are shared by *Rademacher series*

$$\sum_{\nu=0}^{\infty} c_{\nu} \phi_{\nu}(t), \quad (8.1)$$

the functions  $\phi_{\nu}$  being those defined in Chapter I, § 3. This is not entirely surprising in view of the definition

$$\phi_{\nu}(t) = \text{sign} \sin 2^{\nu+1}\pi t.$$

Rademacher series have a close connexion with the calculus of probabilities and are typical of a very large class of series arising there. We shall need only simple properties of (8.1) which can be proved directly.

We suppose that the  $c_{\nu}$  in (8.1) may be complex numbers

**(8.2) THEOREM.** *The series (8.1) converges almost everywhere if  $\Sigma |c_{\nu}|^2 < \infty$ . If  $\Sigma |c_{\nu}|^2 = \infty$ , then, whatever the method  $T^*$  of summation, (8.1) is almost everywhere non-summable  $T^*$ .*

The proof of the second part of (8.2) follows the same line as that of (6.4), and may be left to the reader. We need only observe that the system of functions

$$\phi_{j,k}(t) = \phi_j(t) \phi_k(t) \quad (0 \leq j < k < \infty),$$

is orthonormal over  $(0, 1)$ . (Similarly we can prove an analogue of (6.10).)

If  $\Sigma |c_{\nu}|^2 \neq \infty$ , the series (8.1), with partial sums  $s_n(t)$ , is the Fourier series of a function  $f \in L^2$ . Moreover (see Chapter IV, (1.1))

$$\int_0^1 |f - s_n|^2 dt \rightarrow 0, \quad \int_0^1 |f - s_r| dt \rightarrow 0, \quad \int_a^b (s_n - f) dt \rightarrow 0,$$

where  $0 \leq a < b \leq 1$ . The third relation, which holds uniformly in  $a$  and  $b$ , is a consequence of the second, and the second follows from the first by an application of Schwarz's inequality.

Let  $F(t)$  be the indefinite integral of  $f(t)$ , and let  $E$ ,  $|E| = 1$ , be the set of points where  $F'(t)$  exists and is finite. We have just proved that the integral of  $s_n$  over any interval  $I$  tends to the integral of  $f$  over  $I$ . Therefore the integral of  $s_n - s_{k-1}$  over  $I$  tends to the integral of  $f - s_{k-1}$ , as  $n \rightarrow \infty$ . Let  $I$  be of the form  $(l2^{-k}, (l+1)2^{-k})$ ,  $l = 0, 1, \dots, 2^k - 1$ . Since the integral of  $\phi_j(t)$  over  $I$  is zero for  $j \geq k$ , the integral of  $s_n - s_{k-1}$  over  $I$  is zero. It follows that for the intervals  $I$  just mentioned the integral of  $f$  over  $I$  equals the integral of  $s_{k-1}$  over  $I$ .

Let now  $t_0 \neq p^{2q}$ ,  $t_0 \in E$ ,  $t_0 \in I_k = (l2^{-k}, (l+1)2^{-k})$ . Since  $s_{k-1}$  is constant over  $I_k$ ,

$$s_{k-1}(t_0) = \frac{1}{|I_k|} \int_{I_k} s_{k-1}(t) dt = \frac{1}{|I_k|} \int_{I_k} f(t) dt = P(t_0) \quad \text{as } k \rightarrow \infty,$$

which completes the proof of (8.2).

The analogue of Lemma (6.5) will be needed later and so we state it separately.

**(8.3) LEMMA.** *Given any set  $\mathcal{E} \subset (0, 1)$  and any number  $\lambda > 1$ , there is an integer  $h_0 = h_0(\mathcal{E}, \lambda)$  such that for any finite sum  $P(t) = \sum_{h_0}^N c_k \phi_k(t)$ ,*

$$\lambda^{-1} |\mathcal{E}| \sum |c_k|^2 \leq \int_{\mathcal{E}} |P(t)|^2 dt \leq \lambda |\mathcal{E}| \sum |c_k|^2.$$

The result holds for  $N = \infty$  provided  $\sum |c_k|^2 < \infty$ .

The proof is similar to that of Lemma (6.5) and we leave it to the reader.

**(8.4) THEOREM.** *If  $\sum |c_n|^2 < \infty$ , the sum  $f(t)$  of (8.1) belongs to  $L^r$  for all  $r > 0$ . More precisely,*

$$A_r (\sum |c_n|^2)^{\frac{1}{2}} \leq \left| \int_0^1 |f|^r dt \right|^{\frac{1}{r}} \leq B_r (\sum |c_n|^2)^{\frac{1}{2}} \quad (r > 0), \tag{8.5}$$

where  $A_r, B_r$  are positive and finite, and depend only on  $r$ . Moreover  $B_r \leq 2k^{\frac{1}{2}}$ , where  $2k$  is the least even integer not less than  $r$ .

We suppose first that the  $c_n$  are real and that  $r = 2k$  is an even integer. Then

$$\int_0^1 s_n^{2k}(t) dt = \sum A_{\alpha_1 \alpha_2 \dots \alpha_j} c_{m_1}^{2\alpha_1} c_{m_2}^{2\alpha_2} \dots c_{m_j}^{2\alpha_j} \int_0^1 \phi_{m_1}^{\alpha_1} \dots \phi_{m_j}^{\alpha_j} dt, \tag{8.6}$$

where  $A_{\alpha_1 \alpha_2 \dots \alpha_j} = (\alpha_1 + \alpha_2 + \dots + \alpha_j)! / \alpha_1! \alpha_2! \dots \alpha_j!$ , and  $\alpha_1, \alpha_2, \dots, \alpha_j$  are any positive integers whose sum is  $2k$ . The indices  $m_1, m_2, \dots, m_j$  vary between 0 and  $n$ . It is easily verified that the integrals on the right vanish unless  $\alpha_1, \alpha_2, \dots, \alpha_j$  are all even, in which case the integrals are equal to 1. Observing that

$$\sum A_{\beta_1 \beta_2 \dots \beta_j} c_{m_1}^{2\beta_1} c_{m_2}^{2\beta_2} \dots c_{m_j}^{2\beta_j} = (c_1^2 + c_2^2 + \dots + c_n^2)^k \quad (\beta_1 + \beta_2 + \dots + \beta_j = k)$$

we obtain the second inequality (8.5) with  $f = s_n$ ,  $r = 2k$ , and  $B_{2k}^{\frac{1}{2}}$  equal to the upper bound of the ratio  $A_{2\beta_1 \dots 2\beta_j} / A_{\beta_1 \dots \beta_j}$ . Since  $s_n(t) \rightarrow f(t)$  almost everywhere, the inequality for  $f$  follows.

If we observe that

$$\frac{A_{2\beta_1 \dots 2\beta_j}}{A_{\beta_1 \dots \beta_j}} = \frac{(k+1)(k+2) \dots 2k}{1! (\beta_1+1) \dots 2\beta_j}$$

we see that

$$B_{2k}^{\frac{1}{2}} \leq (k+1)(k+2) \dots 2k, \quad 2k \leq k^k, \quad B_{2k} \leq k^{\frac{1}{2}}.$$

The second inequality (8.5), being true for  $r = 2, 4, \dots$ , must hold for any  $r > 0$ , since  $\mathfrak{M}_r[f; 0, 1]$  is a non-decreasing function of  $r$  (Chapter I, (10.12) (i)). Clearly  $B_r \leq k^{\frac{1}{2}}$ , where  $2k$  is the least even integer not less than  $r$ .

The first inequality (8.5) is immediate for  $r \geq 2$ , for then

$$\mathfrak{M}_r[f] \geq \mathfrak{M}_2[f] = (\sum c_n^2)^{\frac{1}{2}} = \gamma,$$

say. If  $0 < r < 2 < 4$ , let  $t_1$  and  $t_2$  be positive and such that  $t_1 + t_2 = 1$ ,  $2 = rt_1 + 4t_2$ . The function  $\mathfrak{M}_r^2[f]$  being logarithmically convex in  $\alpha$  (Chapter I, (10.12) (ii)),

$$\gamma^2 = \mathfrak{M}_r^2[f] \leq \mathfrak{M}_r^{2t_1} \mathfrak{M}_4^{2t_2} \leq \mathfrak{M}_r^{2t_1} [f] (2^{\frac{1}{2}} \gamma)^{4t_2},$$

which gives  $\mathfrak{M}_r[f] \geq \gamma 2^{-(2-t_1)r}$ .

If  $c_\nu = c'_\nu + ic''_\nu$  and  $f = f' + if''$  are complex, then

$$\mathfrak{M}_r[f] \leq \mathfrak{M}_r[f'] + \mathfrak{M}_r[f''] \leq B_r \{ (\Sigma c'_\nu{}^2)^{\frac{1}{2}} + (\Sigma c''_\nu{}^2)^{\frac{1}{2}} \} \leq 2B_r (\Sigma |c_\nu|^2)^{\frac{1}{2}},$$

and the second inequality (8.5) follows with  $B_r$  doubled. Also if, for example,  $(\Sigma c'_\nu{}^2)^{\frac{1}{2}} \geq (\Sigma c''_\nu{}^2)^{\frac{1}{2}}$ , then

$$\mathfrak{M}_r[f] \geq \mathfrak{M}_r[f'] \geq A_r (\Sigma c'_\nu{}^2)^{\frac{1}{2}} \geq \frac{1}{2} A_r (\Sigma |c_\nu|^2)^{\frac{1}{2}},$$

which gives the first inequality (8.5) with half the previous  $A_r$ .

The estimate  $B_{2k} \leq 2k^{\frac{1}{2}}$  enables us to strengthen the second inequality (8.5).

(8.7) THEOREM. If  $\Sigma |c_\nu|^2 < \infty$ ,  $\exp \{ \mu |f(t)|^2 \}$  is integrable for every  $\mu > 0$ .

For

$$\int_0^1 \exp(\mu |f|^2) dt = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \int_0^1 |f|^{2k} dt \leq \sum_{k=0}^{\infty} \frac{k^k}{k!} (4\mu\gamma^2)^k. \quad (8.8)$$

Since  $k^k/k! < \Sigma k^n/n! = e^k$ , the series on the right converges if  $4e\mu\gamma^2 < 1$ , that is if  $\gamma$  is small enough. It follows that for every  $\mu > 0$  the function  $\exp(\mu |f - s_n|^2)$  is integrable if only  $n$  is large enough. Since  $|f|^2 \leq 2[|f - s_n|^2 + |s_n|^2]$ , and  $s_n(t)$  is bounded, the integrability of  $\exp(\mu |f|^2)$  follows.

Theorems on Rademacher series enable us to prove some results about the series

$$\pm \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \pm (a_n \cos nx + b_n \sin nx), \quad (8.9)$$

which we obtain from the standard series

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_0^{\infty} A_n(x) \quad (8.10)$$

by changing the signs of the terms of the latter in an arbitrary way. Neglecting the sequences  $\pm 1$  containing only a finite number of  $+1$  or of  $-1$ , we may write (8.9) in the form

$$\sum_{n=0}^{\infty} A_n(x) \phi_n(t), \quad (8.11)$$

where the  $\phi_n$  are the Rademacher functions and the parameter  $t$ ,  $t \neq p/2^q$ , runs through the interval  $(0, 1)$ . If the values of  $t$  for which the series (8.11) has some property P form a set of measure 1, we shall say that *almost all* the series (8.9) possess the property P.

(8.12) THEOREM. If

$$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \quad (8.13)$$

is finite, then almost all series (8.9) converge almost everywhere in the interval  $0 \leq x \leq 2\pi$ . If (8.13) is infinite, then, whatever method  $T^*$  of summability we consider, almost all series (8.9) are almost everywhere non-summable  $T^*$ .

Let  $S_t(x)$  denote the series (8.11), and if the latter converges let  $S_t(x)$  also denote its sum. Let  $E$  be the set of points  $(x, t)$  of the rectangle  $0 \leq x \leq 2\pi$ ,  $0 \leq t \leq 1$  where the series converges. If (8.13) is finite then, by (8.2), the intersection of  $E$  with every line  $x = x_0$  is of measure 1. Since  $E$  is measurable, its plane measure is  $2\pi$ , and therefore the intersection of  $E$  with almost every line  $t = t_0$  is of measure  $2\pi$ . This is just the first part of (8.12). The second part follows by the same argument provided we can show that the divergence of (8.13) implies the divergence of  $A_1^2(x) + A_2^2(x) + \dots$  for almost all  $x$ .

To establish this, suppose that  $A_1^2(x) + A_2^2(x) + \dots$  converges in a set  $H$ ,  $|H| > 0$ . Then there is a subset  $H'$  of  $H$ ,  $|H'| > 0$ , and a constant  $M$  such that in  $H'$  the sum of our series does not exceed  $M$ . Let  $A_n(x) = \rho_n \cos(nx + \xi_n)$ ,  $\rho_n \geq 0$ . Integrating the series over  $H'$  we get

$$\sum_{n=1}^{\infty} \rho_n^2 \int_{H'} \cos^2(nx + \xi_n) dx \leq M |H'|.$$

Since the coefficient of  $\rho_n^2$  tends to  $\frac{1}{2} |H'| > 0$  (see Chapter II, (4.5)), the convergence of  $\sum \rho_n^2$  follows, contrary to hypothesis. Thus (8.12) is proved. As a corollary, taking for example  $T^* = (C, 1)$ , we get

(8.14) THEOREM. *If (8.13) diverges, almost all the series (8.9) are not Fourier series.*

The theorem of Riesz-Fischer asserts that if (8.13) is finite, (8.10) is a Fourier series. We now see that the Riesz-Fischer theorem is in a way the best possible, since:

(8.15) THEOREM. *No condition on the moduli of the numbers  $a_n, b_n$  which permits (8.13) to diverge can possibly be a sufficient condition for (8.10) to be a Fourier series.*

(8.16) THEOREM. *If (8.13) is finite then for almost all  $t$  the sum  $S_t(x)$  of (8.9) belongs to every  $L^r$ ,  $r > 0$ . More generally, for any  $\mu$ ,  $\exp\{\mu S_t^2(x)\}$  is integrable over  $0 \leq x \leq 2\pi$  for almost all  $t$ .*

Let  $\gamma^2$  denote the sum of (8.13), and let  $\mu$  be so small that the right-hand side of (8.8) converges. If  $K = K(\mu, \gamma)$  is the sum of the latter series, we have as in (8.7)

$$\int_0^1 \exp\{\mu S_t^2(x)\} dt \leq K.$$

Integrate this over  $0 \leq x \leq 2\pi$  and interchange the order of integration; then

$$\int_0^1 dt \int_0^{2\pi} \exp\{\mu S_t^2(x)\} dx \leq 2\pi K. \quad (8.17)$$

The inner integral here is finite for almost all  $t$ . To remove the assumption that  $\mu$  is small we argue as in the proof of (8.7).

We shall now consider lacunary series

$$\Sigma(a_k \cos n_k x + b_k \sin n_k x) \quad (n_{k+1}/n_k \geq q > 1) \quad (8.18)$$

and the sums

$$\gamma^2 = \Sigma(a_k^2 + b_k^2). \quad (8.19)$$

(8.20) THEOREM. *Suppose that  $n_{k+1}/n_k \geq q > 1$  for all  $k$  and that (8.19) is finite, so that (8.18) is an  $S[f]$ . Then*

$$A_{r,q} \{\Sigma(a_k^2 + b_k^2)\}^{\frac{1}{2}} \leq \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f|^r dx \right\}^{1/r} \leq B_{r,q} \{\Sigma(a_k^2 + b_k^2)\}^{\frac{1}{2}} \quad (8.21)$$

for every  $r > 0$ , where  $A_{r,q}$  and  $B_{r,q}$  depend on  $r$  and  $q$  only. If  $\gamma \leq 1$ , then also

$$\int_0^{2\pi} \exp \mu f^2 dx \leq C, \quad (8.22)$$

provided  $\mu \leq \mu_0(q)$ , with  $C$  an absolute constant.

It is enough to prove (8.22), since then the second inequality (8.21) follows. The first inequality (8.21) follows from the second by the convexity argument used in the proof of (8.4).

We first suppose that  $q \geq 3$ , and consider the series

$$S_t(x) = \sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x) \phi_{n_\nu}(t).$$

Then (8.17) is valid, provided  $\mu$  is small enough, with  $K$  an absolute constant. It follows that there is a  $t_0 \neq p/2^q$  such that

$$\int_0^{2\pi} \exp\{\mu S_{t_0}^2(x)\} dx \leq 2\pi K. \tag{8.23}$$

Consider the Riesz product (§ 7)

$$p_k(x) = \prod_{\nu=1}^k (1 + \phi_{n_\nu}(t_0) \cos n_\nu x) = 1 + \sum \gamma_\nu \cos \nu x.$$

We have  $\gamma_{n_\nu} = \phi_{n_\nu}(t_0)$  for  $\nu = 1, 2, \dots, k$ , and

$$s_{n_k}(x, f) = \sum_{\nu=1}^k A_{n_\nu}(x) = \frac{1}{\pi} \int_0^{2\pi} S_{t_0}(x+u) p_k(u) du.$$

The function  $\chi(v) = \exp(\mu v^2)$  is increasing and convex for  $v \geq 0$ . The function  $(2\pi)^{-1} p_k(x)$  is non-negative, and its integral over  $(0, 2\pi)$  is 1. Jensen's inequality therefore gives

$$\begin{aligned} \chi\left(\frac{1}{2} |s_{n_k}(x, f)|\right) &\leq \chi\left(\frac{1}{2\pi} \int_0^{2\pi} |S_{t_0}(x+u)| p_k(u) du\right) \leq \frac{1}{2\pi} \int_0^{2\pi} \chi(|S_{t_0}(x+u)|) p_k(u) du, \\ \int_0^{2\pi} \chi\left(\frac{1}{2} |s_{n_k}(x, f)|\right) dx &\leq \int_0^{2\pi} \chi(|S_{t_0}(x)|) dx \leq 2\pi K, \end{aligned}$$

by (8.23). So, making  $k \rightarrow \infty$ ,

$$\int_0^{2\pi} \exp\left(\frac{1}{4} \mu f^2\right) dx \leq 2\pi K. \tag{8.24}$$

This is just (8.22) except that  $\mu$  is replaced by  $\frac{1}{4}\mu$ . The right-hand side here,  $2\pi K$ , is an absolute constant, since  $\gamma \leq 1$  and  $\mu$  is small enough.

For general  $q > 1$  we decompose (8.18) into  $Q$  lacunary series for each of which  $q \geq 3$ . (For  $Q$  we may take the least integer  $y$  such that  $q^y \geq 3$ ). Correspondingly,  $f = f_1 + f_2 + \dots + f_Q$ . By Jensen's inequality, and by (8.22) in the case  $q \geq 3$ .

$$\int_0^{2\pi} \exp\{\mu(f/Q)^2\} dx \leq Q^{-1} \sum_{k=1}^Q \int_0^{2\pi} \exp\{\mu f_k^2\} dx \leq C,$$

since the  $\gamma$ 's corresponding to the  $f_k$  are not greater than 1. This proves (8.22) in the general case.

The device used in the proof of (8.7) shows that, under the hypotheses of (8.20), the left-hand side of (8.22) is finite for every  $\mu > 0$ .

In what follows  $f^+ = \max\{f, 0\}$ ,  $f^- = \max\{-f, 0\}$ .

**(8.25) THEOREM.** *Suppose that  $\gamma^2 = \sum(a_k^2 + b_k^2) < \infty$  and write  $f(x) = \sum A_{n_k}(x)$ . Then both  $f^+$  and  $f^-$ , and so also  $|f|$ , are not less than  $\gamma \lambda_q$  in sets of points of measure not less than  $2\pi \mu_q$ , where  $\lambda_q$  and  $\mu_q$  are positive numbers depending on  $q$  only.*

The proof is based on the following lemma, useful also in other problems:

**(8.26) LEMMA.** *Suppose that  $g(x) \geq 0$  is defined in a set  $E$ ,  $|E| > 0$ , and that*

$$(i) \quad \frac{1}{|E|} \int_E g dx \geq A > 0, \quad (ii) \quad \frac{1}{|E|} \int_E g^2 dx \leq B.$$

Then for any  $\delta < \delta < 1$  the subset  $E_\delta$  of  $E$  in which  $g(x) \geq \delta A$  is of measure not less than  $|E| (1 - \delta)^2 (A^2/B)$ .

The integral of  $g$  over the set  $E - E_\delta$ , in which  $g < \delta A$ , is less than  $\delta A |E|$ . Hence the integral of  $g$  over  $E_\delta$  exceeds  $(1 - \delta) A |E|$ , by (i). On the other hand,

$$\int_{E_\delta} g dx \leq \left( \int_{E_\delta} g^2 dx \right)^{\frac{1}{2}} |E_\delta|^{\frac{1}{2}} \leq (B |E|)^{\frac{1}{2}} |E_\delta|^{\frac{1}{2}}$$

by (ii). Hence

$$A |E| (1 - \delta) \leq (B |E|)^{\frac{1}{2}} |E_\delta|^{\frac{1}{2}}$$

that is,

$$|E_\delta| \geq |E| (A^2/B) (1 - \delta)^2$$

Return to (8.25). It is enough to prove the result for  $f^+$ . Since the integral of  $f$  over  $(0, 2\pi)$  is 0,

$$\frac{1}{2\pi} \int_0^{2\pi} f^+ dx = \frac{1}{4\pi} \int_0^{2\pi} |f| dx \geq \frac{1}{2} A_{1,q} \gamma$$

(see (8.21)). Since

$$\frac{1}{2\pi} \int_0^{2\pi} (f^+)^2 dx \leq \frac{1}{2\pi} \int_0^{2\pi} f^2 dx = \frac{1}{2} \gamma^2,$$

an application of (8.26) with  $\delta = \frac{1}{2}$  shows that  $f^+$  exceeds  $\frac{1}{4} \gamma A_{1,q} = \gamma \lambda_q$  in a set of measure not less than  $2\pi \frac{1}{8} A_{1,q}^2 = 2\pi \mu_q$ .

The following analogue of (8.25) for Rademacher functions will be needed later:

**(8.27) THEOREM.** Let  $f(x) = \sum c_n \phi_n(x)$ ,  $0 \leq x \leq 1$ , where the  $c_n$  are real and  $\gamma^2 = \sum c_n^2 < \infty$ . There exist two positive absolute constants  $\epsilon, \eta$  such that both  $f^+$  and  $f^-$  (and so also  $|f|$ ) are not less than  $\eta \gamma$  in sets of measure not less than  $\epsilon$ .

This is a consequence of Lemma (8.26) and of the inequalities  $\mathfrak{M}_2[f] \leq B_2 \gamma$ ,  $\mathfrak{M}_1[f] \geq A_1 \gamma$  (see (8.5)).

**(8.28) THEOREM.** Let  $f(r, x) = \sum (a_k \cos n_k x + b_k \sin n_k x) r^{n_k}$  be the function associated with (8.18), harmonic for  $r < 1$ . If  $\sum (a_k^2 + b_k^2) = \infty$ , and if  $\omega(u)$  is any function defined for  $u \geq 0$  and monotonically tending to  $+\infty$  with  $u$ , then

$$\int_0^{2\pi} \omega(|f(r, x)|) dx \rightarrow +\infty \text{ as } r \rightarrow 1.$$

This follows from (8.25), since the integrand is not less than  $\omega\{\lambda_q [\sum (a_k^2 + b_k^2) r^{2n_k}]^{\frac{1}{2}}\}$  in a set of measure not less than  $2\pi \mu_q$ .

**(8.29) THEOREM.** Let  $f_i(r, x)$  be the harmonic function  $\sum A_n(x) \phi_n(t) r^n$ . If  $\sum (a_k^2 + b_k^2) = \infty$ , and if  $\omega(u)$  is as in (8.28), then for almost all  $t$  the integral  $\int_0^{2\pi} \omega(|f_i(r, x)|) dx$  is unbounded as  $r \rightarrow 1$ .

It is enough to show the existence of sequences  $\{r_k\} \rightarrow 1$  and  $\{M_k\} \rightarrow +\infty$  with the following properties: for almost all  $t$  we have

$$|f_i(r_k, x)| \geq M_k \tag{8.30}$$

for infinitely many  $k$  and for  $x \in X = X_{t,k}$ , with  $|X| \geq \sigma > 0$ , where  $\sigma$  is an absolute constant. For then our integral exceeds  $\sigma \omega(M_k)$  for  $r = r_k$  and infinitely many  $k$ .

Applying (8.27) to  $f_i(r, x)$  we see that for every  $r < 1$  the set  $E = E_r$  of those points of the rectangle  $0 \leq x \leq 2\pi, 0 \leq t \leq 1$  at which

$$|f_i(r, x)| \geq \eta \{\sum A_n^2(x) r^{2n}\}^{\frac{1}{2}} \tag{8.31}$$

has an intersection of measure not less than  $\epsilon$  with every line  $x = \text{const}$ . Hence  $|E| \geq 2\pi\epsilon$ . For  $\delta < 1$ , let  $H_\delta$  denote the set of the numbers  $t_0$  such that the intersection of  $E$  with the line  $t = t_0$  is of measure not greater than  $2\pi\delta$ . Then

$$2\pi\delta |H_\delta| + 2\pi(1 - |H_\delta|) \geq |E| \geq 2\pi\epsilon,$$

which gives  $|H_\delta| \leq (1 - \epsilon)/(1 - \delta)$ . For  $\delta = \epsilon^2$  we get  $|H_\delta| \leq 1/(1 + \epsilon)$ . Hence the set of the numbers  $t_0$  such that the intersection of  $E$  with the line  $t = t_0$  is of measure greater than  $2\pi\epsilon^2$  has measure  $\geq \epsilon/(1 + \epsilon) > \frac{1}{2}\epsilon$  (supposing, as we may, that  $\epsilon < 1$ ).

Clearly we must show that  $\eta\{\sum A_n^2(x)r_n^{2n}\}^{\frac{1}{2}}$  becomes large, as  $r \rightarrow 1$ , outside a set of  $x$  of small measure. More precisely, we will show that there exist sequences  $r_k \rightarrow 1$ ,  $M'_k \rightarrow \infty$  such that

$$\eta\{\sum A_n^2(x)r_n^{2n}\}^{\frac{1}{2}} \geq M'_k \tag{8.32}$$

outside a set of measure at most  $\pi\epsilon^2$ .

Suppose we have already proved the last statement. What we have then shown is the following: there exist sequences  $r_k \rightarrow 1$  and  $M'_k \rightarrow \infty$ , and a sequence of sets  $T_k$ ,  $|T_k| > \frac{1}{2}\epsilon$ , such that for each  $t$  in  $T_k$ ,

$$|f_t(r_k, x)| \geq M'_k$$

for all  $x$  in a set  $X_{t,k}$  of measure at least  $\pi\epsilon^2$ .

Let 
$$T_0 = \limsup T_k.$$

Clearly  $|T_0| \geq \frac{1}{2}\epsilon$ , and the assertion containing (8.30) is true in  $T_0$  with  $M_k = M'_k$ . Since the replacement of  $t$  by  $t + p2^{-q}$  affects only a finite number of terms of  $f_t(x)$ , (8.30) is valid in the union of all translations of  $T_0$  by  $p2^{-q}$ , provided we set, e.g.  $M_k = \frac{1}{2}M'_k$ . This union is of measure 1 and the theorem follows.

We now prove the assertion containing (8.32). We set  $A_n(x) = \rho_n \cos(\pi x + x_n)$  and distinguish two cases:

(i)  $\rho_n = O(1)$ ,    (ii)  $\rho_n = O(1)$ .

In case (i) there is a sequence  $n_1 < n_2 < \dots$  such that  $\rho_{n_i} \rightarrow \infty$ . Let  $r_k = 1 - 1/n_k$ . We have

$$\eta\{\sum A_n^2(x)r_n^{2n}\}^{\frac{1}{2}} \geq \eta |A_{n_k}(x)| \geq \eta \rho_{n_k} |\cos(\pi x + x_{n_k})| e^{-1}$$

for any  $k$ . The set of points where  $|\cos(\pi x + x_j)| \geq \delta$  does not hold is of measure less than  $\pi\epsilon^2$  provided  $\delta$  is small enough (the limitation imposed on  $\delta$  is independent of  $j$ ). Taking  $M'_k = \eta\delta e^{-1}\rho_{n_k}$ , (8.32) follows for case (i).

Let us now pass to case (ii). We may assume that  $\rho_n \leq 1$  for all  $n$ . Given any  $M > 0$ , we shall show that the measure of the set  $B = B_M$  of points  $x$  for which

$$\sum A_n^2(x)r_n^{2n} \leq M$$

tends to 0 as  $r \rightarrow 1$ . Set  $h(r) = \sum \rho_n^2 r_n^{2n}$  and integrate the last inequality over  $B$ . We get

$$\frac{1}{2} |B| \sum \rho_n^2 r_n^{2n} + \frac{1}{2} \pi \sum \rho_n^2 r_n^{2n} (\alpha_{2n} \cos 2\alpha_{2n} - \beta_{2n} \sin 2\alpha_{2n}) \leq M |B|, \tag{8.33}$$

where  $\alpha_j, \beta_j$  are the Fourier coefficients of the characteristic function of  $B$ . Schwarz's inequality shows that the second term on the left numerically does not exceed

$$\frac{1}{2} \pi \{(\sum (\alpha_{2n}^2 + \beta_{2n}^2))\}^{\frac{1}{2}} (\sum \rho_n^4 r_n^{4n})^{\frac{1}{2}} \leq \frac{1}{2} \pi (|B| \pi^{-1})^{\frac{1}{2}} h^{\frac{1}{2}}(r)$$

by Bessel's inequality, since  $\rho_n \leq 1, r < 1$ . If for a sequence of  $r$ 's tending to 1 we had  $|B|$  greater than a positive constant, then, since  $h^{\frac{1}{2}}(r) = o\{h(r)\}$ , (8.33) would give  $\frac{1}{2} |B| h(r) \leq M |B|$  for such  $r$ 's and  $1 - r$  small enough which is false.

This shows that  $|B_r| \rightarrow 0$  as  $r \rightarrow 1$ . Then taking, for example  $M'_k = k$  we find an  $r_k < 1$  such that

$$q(\sum A_n^2(x) r_k^{2n})^{\frac{1}{2}} > M'_k$$

outside a set of  $x$  of measure not exceeding  $\pi\epsilon^2$ . Hence  $|f_t(r, x)| \geq M'_k$  for  $t$  in a set  $T'_k$  of measure not less than  $\frac{1}{2}\epsilon$  and for  $x$  in a set  $X_{i,k}$  of measure not less than  $\pi\epsilon^2$ ; the same conclusion (with a different  $M'_k$ ) which we reached in the case (i). This completes the proof of (8.29).

The series  $\sum \pm A_n(x)$  will be called *randomly continuous* if almost all of them are Fourier series of continuous functions.

Let  $\sum(a_k^2 + b_k^2)$  be finite. Then for almost all  $t$  the sum  $S_t(x)$  of (3.9) belongs to every  $\mathcal{L}$ . It is natural to ask whether the  $\sum \pm A_k(x)$  are randomly continuous. That this is not so follows from the fact (see Chapter VI, (6.1)) that if a lacunary series is the Fourier series of a bounded function, then the sum of the moduli of the terms of the series is finite. Thus for no sequence of signs is

$$\pm \sin 10x \pm 2^{-1} \sin 10^2x \pm \dots \pm n^{-1} \sin 10^n x \pm \dots$$

the Fourier series of a bounded (still less of a continuous) function. We have, however, the following theorem:

(8.34) THEOREM. Let  $s_{n,t}(x)$  denote the partial sums of the series (8.9).

- (i) If  $\gamma^2 = \sum(a_k^2 + b_k^2) < \infty$ , then for almost all  $t$  we have  $s_{n,t}(x) = o\{(\log n)^{\frac{1}{2}}\}$ , uniformly in  $x$ .
- (ii) If  $\sum(a_k^2 + b_k^2)(\log k)^{1-\epsilon} < \infty$  for some  $\epsilon > 0$ , then almost all series (8.9) converge uniformly and so are Fourier series of continuous functions.

As the lacunary series

$$\sum \pm (n \log n)^{-1} \sin 10^n x$$

shows, (ii) is false for  $\epsilon = 0$ .

(i) Consider the inequality (8.17). As its proof shows, it holds for arbitrarily large  $\mu$ , provided  $\gamma$  is small enough. It then also holds for the partial sums  $s_{n,t}(x)$  (for which  $\gamma$  is decreased):

$$\int_0^1 dt \int_0^{2\pi} \exp\{\mu s_{n,t}^2(x)\} dx \leq 2\pi K, \tag{8.35}$$

with  $K$  independent of  $n$ . Fix  $t$  and let  $M_n(t)$  be the maximum of  $|s_{n,t}(x)|$ . Let  $x_0$  be a point at which this maximum is attained. Since the derivative of  $s_{n,t}$  does not exceed  $2nM_n(t)$  (Chapter III, (13.17)),  $s_{n,t}$  cannot change by more than  $\frac{1}{2}M_n(t)$  over any interval of length  $1/4n$ . In particular,  $|s_{n,t}|$  exceeds  $\frac{1}{2}M_n(t)$  for  $x_0 \leq x \leq x_0 + 1/4n$ , and

$$\int_{x_0}^{x_0+1/4n} \exp\{\mu s_{n,t}^2(x)\} dx \geq \frac{1}{4n} \exp\{\frac{1}{4}\mu M_n^2(t)\}.$$

The integral on the left is increased if it is taken the whole interval  $(0, 2\pi)$ . By (8.35), we have

$$\int_0^1 \exp\{\frac{1}{4}\mu M_n^2(t)\} dt \leq 8K\pi n.$$

Hence  $\int_0^1 \exp\{\frac{1}{4}\mu(M_n^2(t) - \alpha \log n)\} dt \leq 8K\pi n^{\alpha} - \frac{1}{4}\mu\alpha$

for any  $\alpha > 0$ . Take  $\alpha\mu = 12$ . Then the right-hand sides, being  $O(n^{-2})$ , form a convergent series. By Chapter I, (11.5), the series  $\sum \exp\{\frac{1}{4}\mu M_n^2 - 3 \log n\}$  converges almost every-

where, and so  $M_n^2(t) \leq 12\mu^{-1} \log n$ , for almost all  $t$  and large enough  $n$ . Since by dropping the first few terms we can make  $\gamma$  arbitrarily small, and so  $\mu$  arbitrarily large. we have

$$M_n(t) = o\{(\log n)\}^{\frac{1}{2}}$$

for almost all  $t$ , and (i) follows.

(ii) Let

$$S_{n,t}(x) = \sum_1^n A_k(x) \phi_k(t) (\log k)^{\frac{1}{2} + \frac{1}{2}\epsilon}.$$

By (i),  $S_{n,t}(x) = o\{(\log n)\}^{\frac{1}{2}}$  for almost all  $t$ , uniformly in  $x$ . We fix such a  $t$  and suppose for simplicity that  $a_1 = b_1 = 0$ . Then summation by parts gives

$$s_{n,t}(x) = \sum_2^{n-1} S_{k,t}(x) \Delta \frac{1}{(\log k)^{\frac{1}{2} + \frac{1}{2}\epsilon}} + S_{n,t}(x) \frac{1}{(\log n)^{\frac{1}{2} + \frac{1}{2}\epsilon}} = \sum_2^n o\{(\log k)\}^{\frac{1}{2}} O\left\{\frac{1}{k(\log k)^{\frac{1}{2} + \frac{1}{2}\epsilon}}\right\} + o(1).$$

The terms of the last series being  $o\{k^{-1}(\log k)^{-1 - \frac{1}{2}\epsilon}\}$ ,  $s_{n,t}(x)$  converges uniformly as  $n \rightarrow \infty$ .

We may ask if the random continuity of the series  $\sum \pm A_n(x)$  implies that almost all the series converge uniformly. This is an open problem, but we can prove the following result:

**(8·36) THEOREM.** *Let  $\{n_k\}$  be any lacunary sequence of indices ( $n_{k+1}/n_k > q > 1$ ). If  $\sum \pm A_n(x)$  is randomly continuous, the sequence  $\{s_{n_k,t}(x)\}$  converges uniformly in  $x$  for almost all  $t$ .*

Let  $t_0$  be any fixed number, not a diadic fraction. We first note that almost all the series

$$\sum \phi_m(t_0) \phi_m(t) A_m(x)$$

are of the class C (i.e. are Fourier series of continuous functions). For let  $E \subset (0, 1)$ ,  $|E| = 1$ , be such that  $\sum A_m(x) \phi_m(t)$  is in C for  $t \in E$ . For each  $t \in E$  we define  $t'$  by

$$\phi_m(t') \phi_m(t_0) = \phi_m(t) \quad (m = 0, 1, 2, \dots).$$

This transformation merely interchanges diadic intervals of the same rank. Since any open set can be covered by non-overlapping diadic intervals, it follows that the transformation preserves the measure of any open set, and so also (passing to the complements) of any closed set and, finally, of any measurable set. In particular, the set of the  $t'$  is also of measure 1.

We now split the series  $\sum A_m(x) \phi_m(t)$  into blocks  $P_k = \sum_{n_k+1}^{n_{k+1}} A_m(x) \phi_m(t)$ . By the remark just made, the two series

$$P_0 + P_1 + P_2 + P_3 + \dots, \quad P_0 - P_1 + P_2 - P_3 + \dots$$

are in C for almost all  $t$  (there is a  $t_0$  such that  $\{\phi_m(t_0)\}$  takes the necessary sequence of values  $\pm 1$ ). Hence the series  $P_0 + P_2 + P_4 + \dots$  and  $P_1 + P_3 + P_5 + \dots$  are in C for almost every  $t$ . But both series have gaps and Theorem (1·27) of Chapter III shows that  $\{s_{n_k,t}(x)\}$  converges uniformly in  $x$  for almost all  $t$ .

We now prove a theorem of a slightly different character.

**(8·37) THEOREM.** *If the power series  $\sum a_n z^n$  has radius of convergence 1, then almost all the functions*

$$f_t(z) = \sum_0^\infty a_n z^n \phi_n(t), \quad z = r e^{i\theta}$$

are not continuable across  $|z| = 1$ .

Suppose that for every  $t$  in a set  $E$  of positive outer measure there is an arc  $(\alpha, \beta)$  on  $|z| = 1$  and two positive numbers  $\delta, M$  such that in the domain

$$\Delta: 1 - 2\delta \leq r \leq 1 + 2\delta, \quad \alpha - \delta \leq \theta \leq \beta + \delta$$

$f_t(z)$  is regular and numerically not greater than  $M$ . The numbers  $\alpha, \beta, M, \delta$  depend on  $t$ , but taking them rational we can select a subset of  $E$ —call it  $E$  again—also of positive outer measure such that they are independent of  $t \in E$ .

Let  $\Delta^*$  denote the domain  $1 - \delta \leq r \leq 1, \alpha \leq \theta \leq \beta$ , and let  $\epsilon > 0$  be so small that every circle with centre  $z \in \Delta^*$  and radius  $\epsilon$  is contained in  $\Delta$ . By Cauchy's theorem,

$$|f_t^{(p)}(z)| \leq \frac{Mp!}{\epsilon^p} \leq Cp^p \quad \text{for } p = 1, 2, \dots, t \in E, z \in \Delta^*.$$

Let  $\mathcal{E} \supset E$  be the set of all  $t$  for which these inequalities are satisfied. Clearly,  $\mathcal{E}$  is measurable (even measurable B) and  $|\mathcal{E}| > 0$ . But, for  $|z| < 1$ ,

$$f_t^{(p)}(z) = \sum_0^{\infty} b_n \phi_n(t), \quad \text{with } b_n = a_n n(n-1) \dots (n-p+1) r^{n-p} e^{i(n-p)\theta},$$

and so, applying (8.3) with  $\lambda = 2$  and supposing  $p$  large enough, we get

$$\sum |b_n|^2 \leq 2C^2 p^{2p}.$$

In particular (making  $r \rightarrow 1$ )

$$a_n n(n-1) \dots (n-p+1) \leq 2^4 C^2 p^{2p}.$$

Set  $p = [\eta n] + 1$ , where  $0 < \eta < 1$ . For  $n$  large enough,

$$|a_n| \{n(1-\eta)\}^{\eta} \leq C^{2\eta} (2\eta n)^{\eta} \eta^{+1},$$

and so,

$$\limsup |a_n|^{1/n} \leq \{2C^2 \eta / (1-\eta)\}^{\eta} < 1,$$

for  $\eta$  fixed and sufficiently small. Hence the radius of convergence of  $\sum a_n z^n$  is greater than 1, contrary to hypothesis, and this contradiction proves the theorem.

Random insertion of the signs  $\pm 1$  into a trigonometric series has a close connexion with the random insertion of the factors 0, 1, that is, with the random suppression of terms. It is enough to replace the  $\phi_n(t)$  in (8.11) by

$$\phi_n^*(t) = \frac{1}{2}(1 + \phi_n(t)). \tag{8.38}$$

The functions  $\phi_n^*(t)$  take the values 0, 1, each in sets of measure  $\frac{1}{2}$ .

In the two theorems that follow,  $T$  is any linear method of summation which satisfies conditions (i), (ii), (iii) of regularity (Ch. III, §1), and  $T^*$  is any linear method which satisfies conditions (i) and (iii).

(8.39) THEOREM. (i) If  $\sum c_n$  is summable by a method  $T$ , and if  $\sum |c_n|^2 < \infty$ , then

$$\sum c_n \phi_n^*(t) \quad (0 \leq t \leq 1) \tag{8.40}$$

is summable  $T$  almost everywhere.

(ii) Conversely, if (8.40) is summable in a set  $E$  of positive measure by a method  $T^*$ , then  $\sum c_n$  is summable  $T^*$  and  $\sum |c_n|^2 < \infty$ .

Case (i) is immediate since, by (8.2),  $\sum c_n \phi_n(t)$  is convergent, and so also summable  $T$ , almost everywhere, and the same holds for (8.40). In case (ii), if  $E$  is the set of all points where  $\sum c_n \phi_n^*$  is summable  $T^*$ , the measure of  $E$  must be 1. For the replacement of  $t$  by  $t + p/2^q$  changes only a finite number of terms in (8.40), and so  $E$  is invariant under translations by  $p/2^q$ . This means that the average density of  $E$  in each of the intervals

$$I_{p,\epsilon} = (p/2^q, (p+1)/2^q)$$

is the same, and so equal to  $|E|$ . Since  $|E| > 0$ , the density theorem for measurable sets asserts that the relative density of  $E$  in some of the intervals  $I_{p,\epsilon}$  must be arbitrarily close to 1. Hence  $|E| = 1$ . Let  $E^*$  be the reflexion of  $E$  in the point  $t = \frac{1}{2}$ . Since  $|E^*| = 1$ , there is a  $t_0 \in EE^*$ . Adding the series (8.40) for  $t = t_0$  and  $t = 1 - t_0$ , and observing that  $\phi_n(1 - t_0) = -\phi_n(t_0)$  for all  $n$ , we obtain the summability  $T^*$  of  $\sum c_n$ . This shows that  $\sum c_n \phi_n(t)$  is summable  $T^*$  in  $E$ , and the finiteness of  $\sum |c_n|^2$  follows from (8.2).

(8.41) THEOREM. Suppose that  $\Sigma(a_n^2 + b_n^2) = \infty$ . Then for every method of summation  $T^*$ , and for almost all  $t$ , the series

$$\Sigma(a_n \cos nx + b_n \sin nx) \phi_n^*(t) = \Sigma A_n(x) \phi_n^*(t) \quad (8.42)$$

is summable  $T^*$  for almost no  $x$ . In particular, almost no series (8.42) is a Fourier series.

If the first conclusion were false, there would exist a set  $X$ ,  $|X| > 0$ , such that for each  $x_0 \in X$  the series  $\Sigma A_n(x_0) \phi_n^*(t)$  is summable  $T^*$  for all  $t$  in a set of positive measure. That, by (8.39) (ii), would imply that  $\Sigma A_n^2(x_0) < \infty$ , and so also  $\Sigma(a_n^2 + b_n^2) < \infty$ , contrary to hypothesis.

In the case  $\Sigma(a_n^2 + b_n^2) < \infty$ , there seems to be less analogy between the series (8.11) and (8.42). Thus, by (8.34), almost all series  $\Sigma \phi_n(t) n^{-1} \sin nx$  are Fourier series of continuous functions, whereas, because of the function  $\Sigma n^{-1} \sin nx$ , almost all series  $\Sigma \phi_n^*(t) n^{-1} \sin nx$  are Fourier series of discontinuous functions.

## 9. Series with 'small' gaps

This name will be given to the series

$$\Sigma(a_k \cos n_k \theta + b_k \sin n_k \theta),$$

where the indices  $n_1 < n_2 < \dots$  satisfy an inequality

$$n_{k+1} - n_k \geq q > 0 \quad \text{for } k = 1, 2, \dots,$$

that is, increase at least as rapidly as an arithmetic progression with difference  $q$ . Only the case  $q > 1$  need be considered. Every lacunary series (see § 6) is a member of this class (at least after the rejection of the first few terms), but not conversely.

Theorems about lacunary series proved in § 6 show that if they 'behave well' on a set  $E$  of positive measure, they 'behave well' in  $(0, 2\pi)$ . It will now be shown that if  $E$  is a large enough interval, somewhat similar conclusions hold for series with 'small' gaps. It will be convenient to write the series in the complex form.

(9.1) THEOREM. Let

$$P(\theta) = \sum_{k=-N}^N c_k e^{in_k \theta} \quad (n_{-k} = -n_k) \quad (9.2)$$

be a finite sum with

$$n_{k+1} - n_k \geq q > 0 \quad (k = 0, 1, \dots), \quad (9.3)$$

and let  $I$  be any interval of length greater than  $2\pi/q$ , so that

$$|I| = 2\pi(1 + \delta)/q \quad (\delta > 0).$$

Then

$$\Sigma |c_k|^2 \leq A_\delta \frac{1}{|I|} \int_I |P(\theta)|^2 d\theta, \quad (9.4)$$

$$|c_k| \leq A_\delta \frac{1}{|I|} \int_I |P(\theta)| d\theta, \quad (9.5)$$

where  $A_\delta$  depends only on  $\delta$ .

The results hold for infinite sums if the series (9.2) converges uniformly.

The inequality (9.4) somewhat resembles the first inequality (6.6), and there is also a resemblance between the proofs. The proof of (9.4) consists in showing that, for a suitable function  $\chi$ , the integral  $\int_I |P|^2 \chi d\theta$  majorizes a fixed multiple of  $\Sigma |c_k|^2$ . In the lacunary case we had  $\chi \equiv 1$  on  $I$ . The sparsity of terms in a lacunary series made it possible to base the proof only on the most obvious properties of the coefficients  $\gamma_k$  of the function  $\chi$  (completed by 0 outside  $I$ ), namely, on  $\Sigma |\gamma_k|^2 < \infty$ . In the present

case we need, as we shall see below, at least  $\sum |\gamma_k| < \infty$ , a condition which does not hold for a discontinuous characteristic function and which therefore requires a different choice of  $\chi$ .

Though we are not interested in the generalization for its own sake, the proof of (9.1) runs more smoothly if we do not require the  $n_k$  to be integers. The simultaneous transformations  $\theta \rightarrow c\theta$ ,  $n_k \rightarrow n_k/c$  change neither  $P$  nor the right-hand sides of (9.4) and (9.5). Since we may also assume that  $I$  is symmetric with respect to  $\theta = 0$  (if  $\theta_0$  is the midpoint of  $I$ , transformation to  $\theta - \theta_0$  does not alter the  $|c_n|$ ), it is enough to take

$$I = (-\pi, \pi), \quad q = 1 + \delta.$$

Let  $\chi$  be any real-valued function vanishing outside  $I$  and  $\gamma(u)$  its Fourier transform (p. 8). Then

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi(x) |P(x)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\sum c_k e^{in_k x}) (\sum \bar{c}_l e^{-in_l x}) \chi(x) dx = \sum c_k \bar{c}_l \gamma(n_l - n_k) \\ &\geq \gamma(0) \sum |c_k|^2 - \sum_{k \neq l} \frac{1}{2} (|c_k|^2 + |c_l|^2) |\gamma(n_l - n_k)| \\ &= \sum_k |c_k|^2 \{ \gamma(0) - \sum_{l \neq k} |\gamma(n_l - n_k)| \}, \end{aligned} \quad (9.6)$$

where the dash indicates that  $l \neq k$  in the summation. If  $\chi$  is bounded, say not greater than  $M$ , and the expression in curly brackets exceeds a positive number  $\Gamma$  depending on  $\delta$  only, a comparison of the extreme terms of (9.6) gives

$$\sum |c_k|^2 \leq \frac{M}{\Gamma} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P|^2 dx,$$

which is (9.4) with the simplifications adopted.

To show that this hypothetical situation can be realized, let

$$\chi(x) = 2\pi \cos \frac{1}{2}x \quad \text{for } |x| \leq \pi, \quad \chi(x) = 0 \text{ elsewhere.} \quad (9.7)$$

Then

$$\gamma(u) = \frac{4 \cos \pi u}{1 - 4u^2},$$

and since  $|n_k - n_l| \geq |k - l|q$ ,

$$\begin{aligned} \sum_l' |\gamma(n_l - n_k)| &\leq \sum_l' \frac{4}{4(k-l)^2 q^2 - 1} < \frac{8}{q^2} \sum_{l=1}^{\infty} \frac{1}{4l^2 - 1} \\ &= \frac{4}{q^2} \sum_{l=1}^{\infty} \left( \frac{1}{2l-1} - \frac{1}{2l+1} \right) = \frac{4}{q^2} = \frac{\gamma(0)}{(1+\delta)^2}. \end{aligned} \quad (9.8)$$

This gives (9.4) with  $A_\delta = 2\pi(1+\delta)^2/4\delta(2+\delta) < A(1+\delta^{-1})$ .

To prove (9.5), let  $|c_j|$  be the largest of the  $|c_k|$ . Then

$$\begin{aligned} \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(x) P(x) e^{-in_j x} dx \right| &= \left| \sum_k c_k \gamma(n_j - n_k) \right| \\ &\geq |c_j| \gamma(0) - \sum_{k \neq j} |c_k| |\gamma(n_j - n_k)| \geq |c_j| \left( 4 - \frac{4}{(1+\delta)^2} \right), \end{aligned}$$

using (9.8). Since the left-hand side here does not exceed  $\int_{-\pi}^{\pi} |P| dx$ , (9.5) follows, with the same  $A_\delta$  as before.

The inequality opposite to (9.4) is also true. It is easier and is valid under more general conditions.

(9.9) THEOREM. Let  $P$  and  $\{n_k\}$  be the same as in (9.1) and let  $J$  be any interval of length  $2\pi\eta/q$ , where  $\eta > 0$ . Then

$$\frac{1}{|J|} \int_J |P(x)|^2 dx \leq B_\eta \sum |c_k|^2. \quad (9.10)$$

We may suppose that  $q = 1$ . The inequality (9.10) follows from Parseval's formula if the  $n_k$  are integers and, say,  $|J| \leq 2\pi$ . For the left-hand side is then not greater than

$$\frac{2\pi}{|J|} \frac{1}{2\pi} \int_0^{2\pi} |P|^2 dx = \eta^{-1} \sum |c_k|^2.$$

To prove (9.10) in the general case, we note that in the last term of (9.6) we have  $\Sigma' \leq \gamma(0)$ . Hence

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi |P|^2 dx &\leq \sum |c_k|^2 \{\gamma(0) + \Sigma'\} \leq 2\gamma(0) \sum |c_k|^2, \\ 2^{-\frac{1}{2}} \int_{-1^n}^{1^n} |P|^2 dx &\leq 8 \sum |c_k|^2. \end{aligned}$$

This estimate holds for the integral of  $|P|^2$  over any interval of length  $\pi$  and leads to (9.10) if  $\eta = 1$ , and so also if  $\eta < 1$  (with  $B_\eta = \eta^{-1} B_1$ ).

If  $\eta > 1$ , we split  $J$  into a finite number of intervals  $J_k$  of lengths contained between  $\pi$  and  $2\pi$  and observe that the left-hand side of (9.10) does not exceed the largest of the ratios  $|J_k|^{-1} \int_{J_k} |P|^2 dx$ . Thus (9.9) is established, with  $B_\eta < A(1 + \eta^{-1})$ .

The following generalization of (6.16) follows easily from (9.1).

(9.11) THEOREM. If the radius of convergence  $R$  of the power series

$$\sum c_k z^{r_k} = f(z), \quad n_{k+1} - n_k \rightarrow \infty, \quad (9.12)$$

is 1, the function  $f$  is not continuable across  $|z| = 1$ .

For suppose that a closed arc  $I$  on  $|z| = 1$  is one of regularity for  $f$ . There are then constants  $C, \delta$  such that (compare a similar argument on p. 221)

$$|f^{(p)}(z)| \leq C^p p! \leq C^p p^p \quad \text{for } p = 1, 2, \dots, \theta \in I, 1 - \delta \leq r < 1.$$

We reject the first few terms of the series (9.12), possibly altering the value of  $C$ , so as to make (9.5) applicable to  $P(\theta) = f^{(p)}(r e^{i\theta})$ . Making  $r \rightarrow 1$  we get

$$|c_k| n_k(n_k - 1) \dots (n_k - p + 1) \leq A \limsup_{r \rightarrow 1} \frac{1}{|I|} \int_I |f^{(p)}(r e^{i\theta})| d\theta \leq A C^p p^p.$$

For  $p = [n_k \epsilon] + 1$  and  $k$  large enough we therefore have

$$|c_k| \{n_k(1 - \epsilon)\}^{n_k \epsilon} \leq (2C n_k \epsilon)^{n_k \epsilon},$$

$$\limsup |c_k|^{1/n_k} \leq \left( \frac{2C\epsilon}{1 - \epsilon} \right)^\epsilon < 1,$$

provided  $\epsilon$  is small enough. This gives  $R > 1$ , and the contradiction proves (9.11).

The proof actually gives more than is explicitly stated, namely, if

$$\liminf (n_{k+1} - n_k) \geq \gamma,$$

then every arc of length greater than  $2\pi/\gamma$ , and so also every arc of length  $2\pi/\gamma$ , on  $|z| = 1$  contains at least one singular point of  $f$ .

### 10. A power series of Salem

We return to Theorem (8·34). It holds of course for power series. Following Salem we complete it as follows:

(10·1) THEOREM. Let  $r_1, r_2, \dots$  be a sequence of positive numbers tending monotonically to 0, such that  $\sum r_n^2$  converges and that  $\{1/r_n\}$  is concave. Then there is a sequence of numbers  $\epsilon_n$ ,  $|\epsilon_n| = 1$ , such that  $\sum \epsilon_n r_n e^{in\pi x}$  converges uniformly.†

Examples of sequences  $\{r_n\}$  satisfying the hypothesis are

$$n^{-\frac{1}{2}(\log n)^{-\frac{1}{2}-\epsilon}}, \quad n^{-\frac{1}{2}(\log n)^{-\frac{1}{2}}(\log \log n)^{-\frac{1}{2}-\epsilon}}, \quad \dots,$$

etc., for  $\epsilon > 0$  and  $n$  large enough. The factors  $\epsilon_n$  are not  $\pm 1$  and we know nothing about the set of admissible  $\{\epsilon_n\}$ , except for the obvious fact that it is of the power of the continuum.

We may suppose that  $r_n = r(n)$ , where  $r(u)$  is monotonically decreasing and differentiable, and  $1/r(u)$  is concave. The convergence of  $\sum r_n^2$  is equivalent to that of  $\int_1^\infty r^2(u) du$ .

The proof of (10·1) is based on certain extensions of the lemmas of van der Corput proved in § 4 to the expressions

$$I(F; a, b) = \int_a^b r(u) F(u) du, \quad S(F; a, b) = \sum_{a < n \leq b} r(n) F(n),$$

where  $r(u)$  is a positive decreasing function and  $F(u) = \exp 2\pi i f(u)$ . Some of these extensions are immediate consequences of the case  $r \equiv 1$ , others are less so.

We take for the variable of integration the primitive  $R = R(u)$  of  $r(u)$ . It is an increasing function of  $u$ , so that  $u = u(R)$  and  $I = \int e^{2\pi i f} dR$ , with

$$f_R = f'(u)/r(u), \quad f_{RR} = f''(u)/r^2(u) - f'(u)r'(u)/r^3(u).$$

Here  $r' \leq 0$ . Hence  $f_{RR} \geq f''(u)/r^2(u)$ , if only  $f'(u) \geq 0$ . Applying Lemma (4·3), we get the following result:

(10·2) LEMMA. If  $f''(u) > 0$  and  $f'(u) \geq 0$ , then

$$|I(F; a, b)| \leq 4 \max(r(u)/\{f''(u)\}^{\frac{1}{2}}).$$

The case  $f'' < 0$ ,  $f' \geq 0$  is slightly less simple, and we shall have to introduce the additional hypothesis that  $r'/f''$  is monotone.

(10·3) LEMMA. If  $f'' < 0$ ,  $f' \geq 0$ , and  $r'/f''$  is monotone, then

$$|I| \leq 4 \max \frac{r}{|f''|^{\frac{1}{2}}} + \max \left| \frac{r'}{f''} \right|.$$

Write  $I = \int_a^b [r(u) - r(b)] F(u) du + r(b) \int_a^b F(u) du = P + Q$ . (10·4)

† For  $r_n$  monotonically decreasing to 0 and  $\{1/r_n\}$  convex, the theorem is an immediate consequence of (8·34). For then  $1/r_n$  exceeds a fixed positive multiple of  $n$ ,  $r_n = O(1/n)$ , the hypotheses of (8·34) are satisfied and we can take for  $\{\epsilon_n\}$  almost any sequence of  $\pm 1$ .

Then  $|Q| \leq 4r(b) \max \{1/|f''(u)|^{\frac{1}{2}}\} \leq 4 \max \{r(u)/|f''(u)|^{\frac{1}{2}}\},$

$$P = \frac{1}{2\pi i} \int_a^b \frac{r(u) - r(b)}{f'(u) - f'(b)} \frac{f'(u) - f'(b)}{f'(u)} d e^{2\pi i f(u)}.$$

The factor  $(f'(u) - f'(b))/f'(u)$  is decreasing and contained between 0 and 1. The derivative of the preceding factor can be written

$$\frac{f''(u)}{f'(u) - f'(b)} \left[ \frac{r'(u)}{f''(u)} - \frac{r(u) - r(b)}{f'(u) - f'(b)} \right] = \frac{f''(u)}{f'(u) - f'(b)} \left[ \frac{r'(u)}{f''(u)} - \frac{r'(v)}{f''(v)} \right] \quad (u < v < b),$$

and so is of constant sign. Hence, applying the second mean-value theorem to the two monotone factors we obtain

$$|P| \leq \frac{2}{\pi} \max \frac{r(u) - r(b)}{f'(u) - f'(b)} \cdot \max \frac{f'(u) - f'(b)}{f'(u)} \leq \max \frac{r'}{f''} \cdot 1,$$

and collecting results we get (10.4).

Changing  $f$  into  $-f$  (which does not affect  $|I|$ ), we may replace the hypotheses of (10.3) by  $f'' > 0, f' \leq 0$ . If  $f'' > 0$ , but nothing is assumed about the sign of  $f'$ , we split  $(a, b)$  into subintervals in which the sign of  $f'$  is constant, and deduce from (10.2) and (10.3):

(10.5) LEMMA. *If  $f''(u)$  is of constant sign and  $r'/f''$  is monotone, then*

$$|I(F; a, b)| \leq 8 \max (r; |f''|^{\frac{1}{2}}) + \max |r'/f''|.$$

*Remark.* The term  $\max |r'/f''|$  is necessary here. Take, for instance,

$$r = f', \quad I = (2\pi i)^{-1} (e^{2\pi i f(b)} - e^{2\pi i f(a)}),$$

and supposing that  $f(u)$  increases indefinitely with  $u$ , take  $f(b) = f(a) + \frac{1}{2}$ . Then  $|I| = 1/\pi$ . But choosing, for example,  $f = \log \log u$ , we see that  $\max (r; |f''|^{\frac{1}{2}})$  in  $(a, b)$  tends to 0 as  $u \rightarrow \infty$ .

(10.6) LEMMA. *If  $f'(u)$  is monotone and  $|f'| \leq \frac{1}{2}$ , then*

$$|I(F; a, b) - S(F; a, b)| \leq A \max r(u).$$

Here  $A$  is an absolute constant. For  $r(u) = 1$  this is (4.4). The general case is reduced to this by applying the second mean-value theorem to the equation

$$I - S = \int_a^b r(u) F(u) d\chi(u),$$

with  $\chi$  defined as in §4.

(10.7) LEMMA. *Suppose that  $f''(u)$  is of constant sign, and  $r'/f''$  and  $|f''|^{\frac{1}{2}}$  are monotone. Then*

$$|S| \leq 16 \max (r; |f''|^{\frac{1}{2}}) + 2 \max |r'/f''| + 2A \max r + \int_a^b (8r|f''|^{\frac{1}{2}} + |r'| + Ar|f''|) du, \tag{10.8}$$

where  $A$  is the constant in (10.6).

The proof is similar to that of (4.6). By hypothesis,  $f'$  is monotone, say increasing. Let  $\alpha_k$  be defined by the condition  $f'(\alpha_k) = k - \frac{1}{2}$ , for  $k$  integral, and let  $\alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}$

be the points  $\alpha$ , if such exist, in the interval  $a \leq u \leq b$ . We consider the values  $a, \alpha_r, \alpha_{r+1}, \dots, \alpha_{r+s}, b$  of  $u$ , to which correspond the values

$$f'(a), r - \frac{1}{2}, r + \frac{1}{2}, \dots, r + s - \frac{1}{2}, f'(b)$$

of  $v = f'(u)$ . In the interval  $(\alpha_k, \alpha_{k+1})$  we have  $|f' - k| \leq \frac{1}{2}$ . Let

$$S_k = \sum_{\alpha_k < n \leq \alpha_{k+1}} r(n) e^{2\pi i f(n)} = \sum_{\alpha_k < n \leq \alpha_{k+1}} r(n) e^{2\pi i (f(n) - kn)}$$

Since  $(f - uk)' = f' - k, (f - uk)'' = f''$ , we get from (10.5) and (10.6)

$$|S_k| \leq 8 \max (r' |f''|^{-\frac{1}{2}}) + \max |r' / f''| + A \max r, \tag{10.9}$$

where the max are taken over  $\alpha_k \leq u \leq \alpha_{k+1}$ . (10.9) holds also for the incomplete intervals  $(a, \alpha_r)$  and  $(\alpha_{r+s}, b)$ .

Let now  $\phi(u)$  be any positive monotone function in  $(a, b)$ , say decreasing, and consider the sum

$$\sigma(a, b) = \sum \max_{\alpha_k < u \leq \alpha_{k+1}} \phi(u) = \phi(a) + \phi(\alpha_r) + \dots + \phi(\alpha_{r+s}),$$

which takes into account also the intervals  $(a, \alpha_r)$  and  $(\alpha_{r+s}, b)$ . If we introduce the new variable  $v = f'(u)$ ,  $\phi(v)$  becomes a decreasing function  $\Phi(v)$ , and  $\sigma$  is

$$\begin{aligned} \Phi[f'(a)] + \Phi(r - \frac{1}{2}) + \dots + \Phi(r + s - \frac{1}{2}) &\leq \Phi[f'(a)] + \Phi(r - \frac{1}{2}) + \int_{r - \frac{1}{2}}^{r + s - \frac{1}{2}} \Phi(v) dv \\ &\leq 2\Phi[f'(a)] + \int_{f'(a)}^{f'(b)} \Phi(v) dv \\ &= 2\phi(a) + \int_a^b \phi(u) f''(u) du. \end{aligned}$$

Since  $\phi(u)$  is positive and monotone,

$$\sigma \leq 2 \max \phi + \int_a^b \phi f'' du.$$

Thus and the inequalities (10.9) give (10.8) (of course, we can omit the term  $|r'|$  on the right and increase the value of  $A$  by 1).

We now pass to the proof of Theorem (10.1). We consider the series

$$\sum_1^\infty r(n) e^{2\pi i (g(n) + nx)}, \tag{10.10}$$

where  $g''(u) = r^2(u) / \int_u^\infty r^2(t) dt, g'(u) = \int_1^u \log \left( \int_v^\infty r^2 dt \right)^{-1} dv \quad (u \geq 1),$

and apply to it the estimate (10.8) with  $f(u) = g(u) + ux$ . Since  $f''(u) = g''(u)$ , the estimate will be valid uniformly in  $x$ . We shall show that  $S$  is small for  $a$  large and  $b > a$ .

Now  $r(u) \{f''(u)\}^{-\frac{1}{2}} = \left( \int_u^\infty r^2 dt \right)^{\frac{1}{2}}$  is decreasing and tends to 0. Also

$$|r' / f''| = |r'| r^{-2} \int_u^\infty r^2 dt$$

decreases monotonically to 0, since  $|r'| r^{-2} = (1/r)'$  is positive and decreases,  $1/r$  being concave. Thus the first three terms on the right in (10.8) are small for large  $a$ . Finally,  $1/r$  being concave,

$$\int_u^\infty r^2 dt = \int_u^\infty \frac{r^2}{|r'|} |r'| dt \geq \frac{r^2(u)}{|r'(u)|} \int_u^\infty |r'| dt = \frac{r^2(u)}{|r'(u)|}.$$

Hence  $rf'' = r^2 \int_a^\infty r^2 dt \leq |r'|$ , and the integral in (10.8) does not exceed

$$\begin{aligned} 8 \int_a^b r(f'')^{\frac{1}{2}} du + (A+1) \int_a^b |r'| du &\leq 8 \int_a^\infty \frac{r^2(u) du}{\left( \int_a^\infty r^2 dt \right)^{\frac{1}{2}}} + (A+1)r(a) \\ &= 16 \left( \int_a^\infty r^2(u) du \right)^{\frac{1}{2}} + (A+1)r(a), \end{aligned}$$

which is small for large  $a$ . Hence (10.10) converges uniformly, and (10.1) follows, with  $\epsilon_n = e^{2\pi t(n)}$ .

#### MISCELLANEOUS THEOREMS AND EXAMPLES

1. If  $a_n = \epsilon_n/n$  and  $\{\epsilon_n\}$  is a positive decreasing sequence, the partial sums  $t_n$  of the series  $\sum a_n \sin nx$  are positive for  $0 < x < \pi$ .

[Sum by parts and use Chapter II, (9.4).]

2. If  $\Sigma(a_k \cos kx + b_k \sin kx) = A_k(x)$  is a Fourier series, the series  $\Sigma A_k(x) \log k$  converges in the metric L. So does  $\Sigma B_k(x) / (\log k)^{1+\epsilon}$  for  $\epsilon > 0$ , though not for  $\epsilon = 0$ .

[Consider the series  $\Sigma \cos nx / \log n$  and  $\Sigma \sin nx / (\log n)^{1+\epsilon}$ .]

3. Suppose that  $\{a_n\}$  is positive, convex and monotonically decreasing. Then the modified partial sums  $t_n^* = \frac{1}{2}(t_n + t_{n-1})$  of  $\Sigma a_n \sin nx$  are positive for  $0 < x < \pi$ . This need not be true for the  $t_n$ .

[Sum by parts twice and use the fact that  $R_n^* > 0$ ,  $D_n^* > 0$  inside  $(0, \pi)$ . For the negative assertion, consider  $\Sigma r^n \sin nx$ ,  $n = 2$ ,  $r > \frac{1}{2}$ .]

4. Let 
$$R_k(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{\sin nh}{nh} \right)^k \cos nx,$$

where  $k = 1, 2, \dots$  and  $h$  is a constant such that  $0 < kh \leq \pi$ . Show that  $R_k$  vanishes in  $(kh, \pi)$  and is a polynomial of degree  $k-1$  in each of the intervals  $((k-2)h, kh)$ ,  $((k-4)h, (k-2)h)$ ,  $\dots$  (For  $k = 1, 2$ , see p. 10.)

[Consider the function  $B_k(x)$  from p. 42 and the  $k$ th difference

$$B_k(x+kh) - \binom{k}{1} B_k(x+(k-2)h) + \dots \pm B_k(x-kh).$$

The result can also be obtained by repeated application of Theorem (1.5) of Chapter II to  $R_1(x)$ .]

5. Let  $h_1, h_2, \dots, h_s, \dots$  be positive numbers with  $\Sigma h_s < \infty$ . Let

$$R(x) = \frac{1}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{where} \quad a_n = \prod_{s=1}^{\infty} \left( \frac{\sin nh_s}{nh_s} \right).$$

The function  $R(x)$  has derivatives of all orders and is not constant. If  $d = h_1 - (h_2 + h_3 + \dots)$  is positive,  $R(x)$  is constant in  $(0, d)$ . If  $h = h_1 + h_2 + \dots < \pi$ , then  $R(x) = 0$  in  $(h, \pi)$ . Cf. Mandelbrojt [2].

[Observe that  $a_1 \neq 0$ ,  $a_n = O(n^{-k})$  for each fixed  $k$ .]

6. If  $a_n$  decreases monotonically to 0 and if  $\Sigma a_n \sin nx \in L$ , then  $\Sigma a_n \cos nx \in L$ .

[ $\Sigma a_n/n < \infty$ .]

7. If  $\{a_n\}$  is positive, convex, and tends to 0, then the sums of both the series  $\Sigma a_n \cos nx$  and  $\Sigma a_n \sin nx$  have continuous derivatives in the interior of  $(0, 2\pi)$ .

[The series  $\Sigma a_n e^{inx}$  differentiated termwise is uniformly summable (C, 1) in every closed interval interior to  $(0, 2\pi)$ .]

8. Let  $a_n$  be the cosine coefficients of a function  $f(x)$  such that  $f(x) \log 1/|x|$  is integrable over  $(-\pi, \pi)$ . Then the series  $\Sigma a_n/n$  converges and

$$\sum_{n=1}^{\infty} a_n/n = -\frac{1}{\pi} \int_0^{2\pi} f(x) \log(2 \sin \frac{1}{2}x) dx.$$

In particular, this holds for functions  $f$  such that  $f \log^+ |f|$  is integrable. (Hardy and Littlewood [12].)

[Multiply both sides of the first formula (2.8) of Chapter I by  $f(x)$  and integrate over  $(-\pi, \pi)$ . The argument is justified because the partial sums of  $\Sigma(\cos nx)/n$  are uniformly  $O(\log |x|)$  near  $x=0$  (see (2.28)).]

9. Let  $a_1 \geq a_2 \geq \dots \rightarrow 0$ ,  $a_1 > 0$ , and let  $t_n(x)$  be the partial sums of  $\Sigma a_n \sin \nu x$ ,  $g(x) = \lim t_n(x)$ . Then

$$(i) \quad \lim g(x)/x = \Sigma \nu a_n$$

(even if the series on the right diverges). In particular  $g(x)$  is strictly positive in some interval  $0 < x \leq \delta$ .

(ii) There is an interval  $0 < x \leq \delta$  in which all the  $t_n$  are strictly positive. Hartman and Wintner [1].

[(i) Using (1.13) we have

$$(*) \quad g(x) = -\frac{1}{2}a_1 \tan \frac{1}{2}x + \sum_{\nu=1}^{\infty} \Delta a_{\nu} \cdot \frac{1 - \cos(\nu + \frac{1}{2})x}{2 \sin \frac{1}{2}x}.$$

Since the terms of the last series are non-negative and the cofactor of  $\Delta a_{\nu}$  is asymptotically equal to  $\frac{1}{2}(\nu + \frac{1}{2})^2 x$  as  $x \rightarrow +0$ , we obtain  $\lim g(x)/x = -\frac{1}{2}a_1 + \frac{1}{2}\Sigma(\nu + \frac{1}{2})^2 \Delta a_{\nu}$ , and summation by parts shows that this expression is  $\Sigma \nu a_{\nu}$ .]

10. If  $a_1 \geq a_2 \geq \dots \rightarrow 0$ ,  $a_1 > 0$ ,  $g(x) \sim \Sigma a_n \sin \nu x$ ,  $0 < \gamma < 2$ , then  $x^{-\gamma}g(x) \in L(0, \pi)$  if and only if  $\Sigma \nu^{\gamma-1}a_n$  converges. (Boas [I<sub>III</sub>], Heywood [2]; for generalizations see Aljančić, Bojunić and Tomić [1].)

[Consider the formula (\*) in Example 9. The integral of  $x^{-\gamma}[1 - \cos(\nu + \frac{1}{2})x](2 \sin \frac{1}{2}x)^{-1}$  over  $(0, \pi)$  being exactly of the order  $\nu^{\gamma}$  (the proof is the same as that of Chapter II, (12.1)),  $x^{-\gamma}g(x)$  is in  $L(0, \pi)$  if and only if  $\Sigma \nu^{\gamma} \Delta a_{\nu} < \infty$ , or, what is the same,  $\Sigma \nu^{\gamma-1}a_n < \infty$ .]

11. Let  $b(u)$  be a slowly varying function. Then, with the notation (2.7), and assuming  $1 < \gamma < 2$ , we have the formulae

$$f_{\gamma}(x) - f_{\gamma}(0) \simeq x^{\gamma-1}b(x^{-1}) \Gamma(1-\gamma) \sin \frac{1}{2}\pi\gamma,$$

$$g_{\gamma}(x) \simeq x^{\gamma-1}b(x^{-1}) \Gamma(1-\gamma) \cos \frac{1}{2}\pi\gamma,$$

analogous to (2.8) (see also the remark on p. 188).

[Set  $\gamma = \beta + 1$  and integrate the relations (2.8) near  $x=0$ . By repeating this procedure we obtain analogous relations for  $k < \gamma < k+1$ ,  $k=2, 3, \dots$ .]

12. If  $b(u)$  is slowly varying then, with the same notation as above,

$$\left. \begin{aligned} f_2(x) - f_2(0) &\simeq -\frac{1}{2}\pi x b(1/x), \\ g_2(x) &\simeq x B(1/x), \quad g_2(x) - x f_1(0) \simeq x R(1/x), \end{aligned} \right\} \quad (i)$$

the last two relations being respectively valid according as  $\Sigma b_n/n$  diverges or converges.

[Integrate (2.13) and (2.16). In the proof of (i) we need the fact—easily obtainable by differentiation—that if  $b(u)$  is slowly varying so are  $B(u)$  and  $R(u)$ , for  $\Sigma b_n/n$  divergent and convergent respectively. A similar argument gives formulae for  $f_k$  and  $g_k$  if  $k=3, 4, \dots$ .]

13. For  $\alpha=1$ , the sum  $\phi_{\alpha}(x)$  of (4.1) belongs to  $\Lambda_*$ . (It has been already pointed out that it does not belong to  $\Lambda_1$ .) For  $\frac{1}{2}\alpha + \beta = 2$  the sum  $\psi_{\alpha, \beta}(x)$  of (5.1) belongs to  $\Lambda_*$ .

14. If  $\gamma > 0$ ,  $\delta > \frac{1}{2}(1+\gamma)$ , the series  $\Sigma \frac{\exp[2\pi i n(\log n)^{\gamma}]}{n^{\delta}(\log n)^{\delta}} z^n$  converges uniformly on  $|z|=1$ . (Ingham [2].)

[The proof follows the same line as in § 4. We have  $\sum_{\nu=2}^n \nu^{-\delta} \exp 2\pi i(\nu(\log \nu)^{\gamma} + i\nu x)] \ll A(\log n)^{\delta(1+\gamma)}$ .]

A periodic and integrable function  $f(x)$  is said to be of *bounded deviation* (Hadamard) if for each fixed interval  $(a, b)$  we have  $\int_a^b f e^{-inx} dx = O(1/n)$ . In particular, the Fourier coefficients of such functions are  $O(1/n)$ . Every function of bounded variation is of bounded deviation. Examples of functions of bounded deviation but not of bounded variation have been given by Aloxits [2], Bray (see Mandelbrojt [1]) and Hille [3].

15. The function  $\tau \sin(1/x)$  ( $|x| \leq \pi$ ) is of bounded deviation. (Bray; see Mandelbrojt [1].)

[It is enough to show that the integral of  $x \exp(ix) \pm 1/x$  over any subinterval of  $(0, \pi)$  is uniformly  $O(1/n)$ . The substitution  $x^2 = u$  reduces the problem to the function  $\exp(nu^{\frac{1}{2}} \pm u^{-\frac{1}{2}})$  and any subinterval  $(c, d)$  of  $(0, \pi^2)$ . We may suppose that  $c \geq C/n$ , since the integral over a subinterval of  $(0, C/n)$  is  $O(1/n)$ . For a suitable  $C$  the derivative of  $nu^{\frac{1}{2}} \pm u^{-\frac{1}{2}}$  in  $(c, n, \pi^2)$  is monotone and greater than  $C/n$ . Apply (4.3) (i).]

16. Show that if in the lacunary series  $\Sigma(a_k \cos n_k x + b_k \sin n_k x)$  we have  $|a_{k+1}| + |b_{k+1}| = O(1/n_k)$ , then the sum  $f(x)$  of the series is of bounded deviation. It need not be of bounded variation, as the example of the function  $f(x) = \Sigma 2^{-k} \cos 2^k x$ , differentiable almost nowhere, shows. (Hille [3].)

17. Suppose that for the lacunary series  $\Sigma(a_k \cos n_k x + b_k \sin n_k x) = f(x)$  the sum  $\Sigma(|a_n| + |b_n|)$  is finite but  $\Sigma(n_k^2(a_k^2 + b_k^2))$  infinite. Then  $f(x)$  is differentiable almost nowhere. [Compare p. 206.]

18. Suppose that  $\Sigma(|a_n| + |b_n|)/n$  converges but  $\Sigma(a_n^2 + b_n^2)$  diverges. (Take, for example,  $a_n = n^{-1}$ ,  $b_n = 0$ .) Then almost all continuous functions

$$\Sigma \pm n^{-1}(a_n \cos nx + b_n \sin nx)$$

are differentiable almost nowhere.

19. The conclusion in Example 7 is false if  $\{a_n\}$  is merely monotonically decreasing to zero. The sums of the series  $\Sigma a_n \cos nx$  and  $\Sigma a_n \sin nx$  may be then differentiable almost nowhere.

[Summation by parts gives

$$\Sigma a_n \cos nx = (2 \sin y)^{-1} \Sigma \Delta a_n \sin(2n+1)y,$$

with  $y = \frac{1}{2}x$ . Consider the case when the last series is lacunary and apply Example 17.]

20. If  $n_{k+1}/n_k \geq 3$  and  $\Sigma \alpha_k^2 < \infty$ , then the Riesz product  $\Pi(1 + \alpha_k \cos n_k x) = 1 + \Sigma \gamma_n \cos nx$  is the Fourier series of a function  $f$  such that  $\exp(\lambda(\log |f|)^2)$  is integrable for every  $\lambda > 0$ . In particular,  $f \in L^p$  for every  $p > 0$ .

21. Consider the product  $\Pi(1 + \alpha_k \cos n_k x)$ , where  $n_{k+1}/n_k > q > 2$ . It is then no longer true that the partial product  $p_k$  is a partial sum of  $p_{k+1}$ , but the rank of the lowest term in  $p_{k+1} - p_k$  tends to infinity with  $k$ . In particular,  $p_k$  tends termwise to a trigonometric series. The latter is the Fourier-Stieltjes series of a continuous, non-decreasing function. (Wiener and Wintner [1].)

22. Let  $K_n$  denote Fejér's kernel. The  $N$ th partial product of the Riesz product  $\prod_0^N (1 + \cos 2^k x)$  is  $2K_{2^N}(\cdot)$  and so tends termwise to  $1 + 2 \cos x + 2 \cos 2x + \dots$ , the Fourier-Stieltjes series of a discontinuous function. (Wiener and Wintner [1].)

23. Suppose that  $n_{k+1}/n_k \geq q > 2$ ,  $|\alpha_k| \leq 1$ ,  $dF \sim \Pi(1 + \alpha_k \cos n_k x)$ . Show that  $\alpha \in \Lambda_2$ , where  $\alpha = 1 - \log 2/\log q$ .

24. Let  $-1 \leq \alpha_k \leq 1$ ,  $\Sigma \alpha_k^2 = \infty$ . Show that for almost all sequences of signs  $\pm 1$  the product  $\Pi(1 \pm \alpha_k \cos kx)$  diverges to 0 almost everywhere in  $\pi$ .

25. Suppose that  $-1 \leq \alpha_k \leq +1$ ,  $\Sigma \alpha_k^2 = \infty$ ,  $n_{k+1}/n_k \geq 3$ . Show that the trigonometric series representing  $\Pi(1 + i\alpha_k \cos n_k x)$  diverges almost everywhere (cf. (7.12)).

26. Let  $M$  be any (not necessarily linear) method of summation of series  $u_0 + u_1 + \dots$  which has the following properties: (a) if  $\Sigma u_n$  converges to sum  $s$ , it is also summable  $M$  to  $s$ ; (b) if  $\Sigma u_n$  and  $\Sigma v_n$  are summable  $M$  to sums  $s$  and  $t$  respectively, then  $\Sigma(mu_n + bv_n)$  is summable  $M$  to sum  $cs + bt$ ; (c) if  $u_0 + u_1 + u_2 + \dots$  is summable  $M$  to  $s$ , the series  $u_1 + u_2 + \dots$  is summable  $M$  to  $s - u_0$ . Show that the series

$$(*) \quad \cos x + \cos 2x + \dots + \cos 2^k x + \dots$$

cannot be summable  $M$  on a measurable set  $E \subset (0, 2\pi)$ ,  $|E| > 0$ , to a finite measurable sum. (Kolmogorov [4].)

[Suppose  $(*)$  is summable  $M$  on  $E$ ,  $|E| > 0$ , to a measurable sum  $f(x)$ . Since summability of  $(*)$  at  $x$  implies summability at  $2^k x$ , we have  $|E| = 2\pi$ . In particular,  $f(x)$  is bounded on a set  $H \subset E$ , of measure arbitrarily close to  $2\pi$ . Let  $\chi(x)$  be the characteristic function of  $H$ , and let  $H_N$  be the

set whose characteristic function is  $\chi(2^N x)$ . By Chapter II, (4.15),  $|HH_N| \rightarrow |H|^2/2\pi$ , and so, for suitable  $H$ , and for  $N$  large enough,  $|HH_N|$  is arbitrarily close to  $2\pi$ . For  $x \in HH_N$  we have

$$f(x) = \cos x + \dots + \cos 2^{N-1}x + f(2^N x),$$

and we arrive at a contradiction since  $f(x)$  and  $f(2^N x)$  are bounded on  $HH_N$ , while, by (8.25), the sum  $\cos x + \dots + \cos 2^{N-1}x$  is large with  $N$  on a substantial subset of  $(0, 2\pi)$ , and so also at some points of  $HH_N$ .]

27. Let  $E \subset (0, 2\pi)$ ,  $|E| > 0$ ,  $n_{k+1}/n_k \geq q > 1$ ,

$$\gamma^2 = \Sigma(a_k^2 + b_k^2) < \infty, \quad f = \Sigma(a_k \cos n_k x + b_k \sin n_k x).$$

Show that there are two positive constants  $\lambda_q$  and  $\mu_q$ , depending on  $q$  only, such that

$$f^+ \geq \lambda_q \gamma, \quad f^- \geq \lambda_q \gamma$$

in subsets of  $E$  of measure not less than  $\mu_q |E|$ , provided  $n_1$  is large enough.

[The proof is analogous to that of (8.25). We show first that

$$\int_E f^4 dx \leq A_q \gamma^4 |E|,$$

provided  $n_1$  is large enough. This, together with the first inequality (6.6) (valid with  $P$  replaced by  $f$ ), gives  $\int_E |f| dx \geq B_q \gamma |E|$ . Since

$$\int_E f^+ dx + \int_E f^- dx = \int_E |f| dx, \quad \int_E f^+ dx - \int_E f^- dx = \int_0^{2\pi} f \chi dx,$$

where  $\chi$  is the characteristic function of  $E$ , and since the last integral is small in comparison with  $\gamma$  if  $n_1$  is large enough, we see that  $\int_E f^+ dx \geq \frac{1}{2} B_q \gamma |E|$ . Combining this with

$$\int_E (f^+)^2 dx \leq \int_E f^2 dx \leq \frac{1}{2} |E| \gamma^2$$

(see (6.6)) and applying (8.26), we obtain the conclusion for  $f^+$ .]

## THE ABSOLUTE CONVERGENCE OF TRIGONOMETRIC SERIES

### 1. General series

The absolute convergence of the complex trigonometric series  $\sum c_k e^{ikx}$  (in particular, of a power series in  $e^{ix}$ ) at a single point  $x_0$  implies the convergence of  $\sum |c_k|$ , and so the absolute (and uniform) convergence of  $\sum c_k e^{ikx}$  for all  $x$ . For the series

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{\infty} A_k(x) \quad (1.1)$$

the situation is less simple. The convergence of

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) \quad (1.2)$$

naturally implies the absolute (and uniform) convergence of (1.1) for all  $x$ ; on the other hand, (1.1) may converge absolutely at an infinite set of points without (1.2) converging. An example is given by

$$\sum \sin n! x,$$

whose terms vanish from some point onwards for every  $x$  commensurable with  $\pi$ .

(1.3) **THEOREM OF DENJOY-LUSIN.** *If (1.1) converges absolutely for  $x$  belonging to a set  $A$  of positive measure, (1.2) converges.*

Suppose for simplicity that  $a_0 = 0$ , and let the  $k$ th term of (1.1) be  $\rho_k \sin(kx + x_k)$ , with  $\rho_k = (a_k^2 + b_k^2)^{\frac{1}{2}} \geq 0$ . The function

$$\alpha(x) = \sum_{k=1}^{\infty} \rho_k |\sin(kx + x_k)| \quad (1.4)$$

is finite at every  $x \in A$ . Hence there is a set  $E \subset A$ ,  $|E| > 0$ , such that  $\alpha(x)$  is bounded on  $E$ ,  $\alpha(x) < M$  say. Since the partial sums of (1.4) are uniformly bounded on  $E$ , the series may be termwise integrated over  $E$ :

$$\sum_{k=1}^{\infty} \rho_k \int_E |\sin(kx + x_k)| dx = \int_E \alpha(x) dx \leq M |E|. \quad (1.5)$$

To prove the convergence of  $\rho_1 + \rho_2 + \dots$ , which is equivalent to that of (1.2), it is enough to show that the coefficients of  $\rho_k$  in (1.5) all exceed an  $\epsilon > 0$ . This is immediate, if we observe that by replacing the integrand by  $\sin^2(kx + x_k)$  we do not increase the integral and that the new integral tends to  $\frac{1}{2} |E|$  (Chapter II, (4.5)).

(1.6) **THEOREM.** *Suppose that  $|a_1| \geq |a_2| \geq \dots$*

- (i) *If  $\sum a_n \cos nx$  converges absolutely at a point  $x_0$ , then  $\sum |a_n| < \infty$ .*
- (ii) *The same holds for the series  $\sum a_n \sin nx$ , if  $x_0 \not\equiv 0 \pmod{\pi}$ .*

In (i) we may suppose that  $0 < x_0 < \pi$ . The hypothesis implies that  $\sum |a_n| \cos^2 nx_0$  is finite. Since  $2 \cos^2 nx_0 = 1 + \cos ny_0$ , with  $y_0 = 2x_0$ , and since the partial sums of  $\sum |a_n| \cos ny_0$  are bounded, the result follows. Part (ii) is proved similarly.

(1·7) THEOREM. Let  $\rho_n = (a_n^2 + b_n^2)^{\frac{1}{2}}$ . If  $\rho_1 \geq \rho_2 \geq \dots$  and if  $\Sigma A_n(x)$  converges absolutely at two points  $x', x''$  with  $|x' - x''| < \pi$ , then  $\Sigma \rho_n < \infty$ .

Write (1·1) in the form  $\Sigma \rho_n \sin(nx + x_n)$ , and let  $t = x' - x''$ . Since

$$nt = (nx' + x_n) - (nx'' + x_n),$$

it follows that  $|\sin nt| \leq |\sin(nx' + x_n)| + |\sin(nx'' + x_n)|$ ; (1·8)

thus the series  $\Sigma \rho_n |\sin nt|$  converges, and (1·7) follows from (1·6) (ii).

It is obvious that (1·6) and (1·7) hold if we suppose only that  $\{a_n\}$  and  $\{\rho_n\}$  respectively are of bounded variation.

The set  $A$  in Theorem (1·3) is of positive measure. This, while sufficient to ensure the convergence of (1·2), is not necessary. The problem of characterizing those sets  $A$  for which the absolute convergence of (1·1) in  $A$  implies the finiteness of (1·2) is still unsolved. The results that follow, however, throw some light on the situation.

Suppose that  $\rho_1 + \rho_2 + \dots = \infty$  for (1·1) and let  $A$  be the set of points at which  $\alpha(x)$  is finite. The complementary set is a product of a sequence of open sets (see the proof of Theorem (12·2) of Chapter I). Hence  $A$  is the sum of a sequence of closed sets. None of these closed sets contains an interval, for otherwise we should have  $|A| > 0$ , and  $\rho_1 + \rho_2 + \dots < \infty$ . It follows that all of them are non-dense, and  $A$  is of the first category. Thus:

(1·9) THEOREM. If  $\Sigma A_n(x)$  converges absolutely in a set of the second category (even one of measure 0), the series (1·2) converges.

The set  $A$  of points where  $\Sigma A_n(x)$  converges absolutely has curious properties. Let  $\bar{A}$  be the set of points of absolute convergence of the series  $\Sigma B_n(x)$  conjugate to  $\Sigma A_n(x)$  and let  $C$  and  $\bar{C}$  respectively be the sets of points where  $\Sigma A_n(x)$  and  $\Sigma B_n(x)$  converge, not necessarily absolutely. It will be convenient to place all these sets on the circumference of the unit circle.

(1·10) THEOREM. Every point of  $A$  is a point of symmetry of the sets  $A, \bar{A}, C, \bar{C}$ .

We have

$$A_n(x+h) + A_n(x-h) = 2A_n(x) \cos nh,$$

$$B_n(x+h) - B_n(x-h) = 2A_n(x) \sin nh.$$

The first formula implies that if  $\Sigma |A_n(x)| < \infty$  and if  $\Sigma A_n(x+h)$  converges, or converges absolutely, so does  $\Sigma A_n(x-h)$ . This proves the symmetry property for  $A$  and  $C$ . The second formula gives the proof for  $\bar{A}, \bar{C}$ .

By (1·3), the set  $A$  (and similarly  $\bar{A}$ ) has measure either 0 or  $2\pi$ .

(1·11) THEOREM. If  $A$  is infinite, then  $C$ , and similarly  $\bar{C}$ , has measure either 0 or  $2\pi$ .

By (1·10), if  $x \in A$ , and  $x+h \in A$ , then all the points  $x+h, x+2h, x+3h, \dots$  belong to  $A$ . Since  $A$  is infinite,  $h$  may be arbitrarily small, so that  $A$  is everywhere dense. Suppose that  $C$  and its complement  $C'$  are both of positive measure, and let  $c, c'$  be points of density for  $C$  and  $C'$  respectively. There is an  $\epsilon > 0$  such that if any interval  $I$  of length  $\leq 2\epsilon$  contains  $c$ , then  $|IC| > \frac{1}{2}|I|$ , and if any interval  $J$  of length  $\leq 2\epsilon$  contains  $c'$ , then  $|JC'| > \frac{1}{2}|J|$ . Let  $I = (c-\epsilon, c+\epsilon)$ , and take an  $x_0$  belonging to  $A$

and distant less than  $\frac{1}{2}\epsilon$  from the midpoint of the arc  $(c, c')$ . Reflexion in  $x_0$  takes  $C$  into itself and  $I$  into an interval  $J$ ,  $|J| = 2\epsilon$ , containing  $c'$ . The inequalities

$$|JC| > \frac{1}{2}|J|, \quad |JC'| > \frac{1}{2}|J|$$

being incompatible, we have a contradiction. The argument for  $\bar{C}$  is identical.

There exist trigonometric series absolutely convergent in a perfect set but not everywhere (see Example 1 at the end of the chapter). On the other hand, we shall prove the existence of perfect sets  $P$  of measure zero which as regards the absolute convergence of trigonometric series resemble the sets of positive measure: if (1.1) is absolutely convergent in  $P$ , (1.2) is finite.

A point set  $S$  will be called a *basis*, if every real  $x$  can be represented in the form  $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m$ , where  $\alpha_1, \alpha_2, \dots$  are integers,  $x_1, x_2, \dots$  belong to  $S$ , and  $m$  depends on  $x$ . We may also write

$$x = \epsilon_1 x_1 + \dots + \epsilon_n x_n,$$

where  $\epsilon_j = \pm 1$  and the  $x_j$  are not necessarily different.

(1.12) THEOREM. *If  $S$  is a basis, and if  $\Sigma A_n(x)$  is absolutely convergent in  $S$ , then  $\Sigma \rho_n$  is finite.*

We shall reduce the general case to that of a purely sine series, which is immediate. In fact the inequality

$$|\sin n(\epsilon_1 x_1 + \epsilon_2 x_2 + \dots + \epsilon_m x_m)| \leq |\sin nx_1| + \dots + |\sin nx_m| \quad (\epsilon_j = \pm 1), \quad (1.13)$$

which it is easy to prove by induction, shows that if  $\Sigma |b_n \sin nx|$  converges in  $S$ , it converges everywhere, and so  $\Sigma |b_n| < \infty$ . (The argument works also for purely cosine series  $\Sigma a_n \cos nx$ , since the inequality

$$|\sin n(\epsilon_1 x_1 + \dots + \epsilon_m x_m)| \leq |\cos nx_1| + \dots + |\cos nx_m|$$

shows that if  $\Sigma |a_n \cos nx|$  converges in  $S$ , then  $\Sigma |a_n \sin nx|$  converges everywhere.) In the general case, we need the following lemma:

(1.14) LEMMA. *Let  $S$  be a basis, and let  $S^* = S_u$  be the set  $S$  translated by  $u$ . There is then a set  $T$  of the second category such that for every  $y \in T$  we have*

$$y = \alpha_1 x_1^* + \alpha_2 x_2^* + \dots + \alpha_m x_m^*,$$

with  $\alpha_j$  integral and  $x_j^* \in S^*$  for all  $j$ .

By hypothesis, for every  $x$  we have  $x = \alpha_1(x_1^* - u) + \alpha_2(x_2^* - u) + \dots$ , that is,

$$x + ku = \alpha_1 x_1^* + \dots + \alpha_m x_m^*,$$

where  $k = k(x)$  is an integer. Let  $E_n$  be the set of  $x$  for which  $k(x) = n$ . At least one of the  $E_n$ , say  $E_{n_0}$ , is of the second category, and we may take for  $T$  the set  $E_{n_0}$  translated by  $n_0 u$ . We say that  $S^*$  is a *basis* for  $T$ . (Incidentally it is not difficult to deduce that  $S^*$  is a basis (cf. Example 2 at the end of the chapter), but we do not need this.)

Passing to the proof of (1.12), let  $v$  be any point of  $S$ , and let  $x = y + v$ . Then

$$A_n(x) = A_n(v) \cos ny - B_n(v) \sin ny.$$

By hypothesis  $\Sigma |A_n(v)| < \infty$ , and therefore  $\Sigma B_n(v) \sin ny$  converges absolutely in a set  $S^*$  obtained from  $S$  by a translation  $-v$ . By (1.14),  $S^*$  is a basis for a set  $T$  of

the second category. The argument which we applied to sine series shows that  $\sum B_n(v) \sin ny$  is absolutely convergent in  $T$ , and so for all  $y$  (see (1.9)). The same holds for the series  $\sum \{A_n(v) \cos ny - B_n(v) \sin ny\} = \sum A_n(x)$ . Thus (1.2) is finite.

The above proof may be modified slightly, by basing it on Theorem (1.3) instead of (1.9). It is enough to observe that (1.14) and its proof remain valid if we replace there the words 'second category' by 'positive outer measure'. Since the set of points  $y$  for which  $\sum |B_n(v) \sin ny|$  converges is measurable, it is of positive measure, and therefore is the whole interval  $(0, 2\pi)$ .

To give an example of (1.12), we shall show that *Cantor's ternary set  $C$  constructed on  $(0, 1)$  (or on any other interval) is a basis*. More precisely, we shall show that *the set of all sums  $x + y$ , with  $x \in C, y \in C$ , fills the whole interval  $(0, 2)$* .

Consider the set  $K$  of all points  $(x, y)$  of the plane such that  $x \in C, y \in C$ . The set  $K$  may be also obtained as follows. Divide the square  $0 \leq x \leq 1, 0 \leq y \leq 1$ , which we call  $Q_0$ , into nine equal parts, and let  $Q_1$  be the sum of the four closed corner squares. Repeat the procedure for each of these corner squares, denoting the sum of the new corner squares by  $Q_2$ , and so on. Plainly,  $K = Q_0 Q_1 Q_2 \dots$ . The projection of any  $Q_j$  on the diagonal  $y = x$  of  $Q_0$  fills up that diagonal. In other words, any straight line  $L_h$  with equation  $x + y = h, 0 \leq h \leq 2$ , meets every  $Q_j$  at one point at least. Since the  $Q_j$  are closed and form a decreasing sequence, it follows that  $KL_h \neq \emptyset$ , and this is just what we wanted to prove. Similarly we can show that *the set of all differences  $x - y$  ( $x \in C, y \in C$ ) fills the interval  $(-1, 1)$* .

## 2. Sets N

In what follows we shall repeatedly use the following classical result of Dirichlet:

(2.1) LEMMA. *Let  $\alpha_1, \alpha_2, \dots, \alpha_k$  be any  $k$  real numbers, and let  $Q$  be any positive integer. Then we can find an integer  $q$  with  $1 \leq q \leq Q^k$  and integers  $p_1, p_2, \dots, p_k$  such that*

$$\left| \alpha_j - \frac{p_j}{q} \right| < \frac{1}{Qq} \leq \frac{1}{q^{1+1/k}} \quad (j = 1, 2, \dots, k). \quad (2.2)$$

Let  $\langle x \rangle = x - [x]$  be the fractional part of  $x$ . Consider the  $k$ -dimensional half-closed unit cube  $I$  defined by the inequalities  $0 \leq x_j < 1$  for  $j = 1, \dots, k$ , and divide  $I$  into  $Q^k$  congruent half-closed sub-cubes by drawing hyperplanes parallel to the faces of  $I$  at distances  $1/Q$ . Of the  $Q^k + 1$  points with co-ordinates

$$\langle n\alpha_1 \rangle, \langle n\alpha_2 \rangle, \dots, \langle n\alpha_k \rangle \quad (n = 0, 1, \dots, Q^k),$$

at least two, say those corresponding to  $n = q_1$  and  $n = q_2 > q_1$ , are in the same sub-cube of  $I$ . Hence  $|\langle q_2 \alpha_j \rangle - \langle q_1 \alpha_j \rangle| < 1/Q$  for all  $j$ , or, setting  $[q_1 \alpha_j] = p_j', [q_2 \alpha_j] = p_j''$ ,

$$|(q_2 - q_1) \alpha_j - (p_j'' - p_j')| < 1/Q,$$

which is the first inequality (2.2) with  $q = q_2 - q_1 \geq 1, p_j = p_j'' - p_j'$ .

*Remarks.* (a) The fractions  $p_j/q$  may not all be irreducible.

(b) The first inequality (2.2) shows that given any  $\epsilon > 0$  we can find an integer  $q > 0$  such that all the products  $q\alpha_1, q\alpha_2, \dots, q\alpha_k$  differ from integers by less than  $\epsilon$ . It is enough to take  $Q \geq 1/\epsilon$ .

(c) For any system  $\alpha_1, \alpha_2, \dots, \alpha_k$  there exist fractions  $p_j/q$  with  $q$  arbitrarily large, satisfying

$$|\alpha_j - p_j/q| < q^{-1+k} \quad (j=1, \dots, k). \quad (2.3)$$

For we may take, successively,  $Q = 1, 2, \dots$ . If the corresponding  $q$ 's are bounded, or even have a bounded subsequence only, we may suppose that  $q$  is the same for infinitely many  $Q$ 's. Making a further selection, we may suppose (since the fractions  $p_j/q$  are bounded) that the  $p_j$  also stay constant. For  $Q \rightarrow \infty$ , the first inequality (2.2) then gives  $\alpha_j = p_j/q$ . Thus the  $q$  in (2.2) can be arbitrarily large, unless all the  $\alpha_j$  are rational. In the latter case, however, we even have  $\alpha_j - p_j/q = 0$  for infinitely many  $q$ 's.

In the preceding section we investigated properties of sets  $E$  such that whenever the series  $\sum A_k(x)$  converges absolutely in  $E$  it converges everywhere. We now prove a number of results about those sets which do not have this property.

A set  $E$  will be called of *type N* if there is a series  $\sum A_k(x)$  which converges absolutely in  $E$  but not everywhere (and so has  $\sum \rho_k = \infty$ ). If the series in question is a sine series, we shall call  $E$  of *type N<sub>s</sub>*. Every set of type  $N_s$  is also of type  $N$ . The converse will be proved below.

(2.4) LEMMA. *Every denumerable set is of type N.*

Let  $E$  consist of the numbers  $\pi x_1, \pi x_2, \dots$ . By Remark (b), for each  $k$  there exists an integer  $q = n_k$  such that

$$|\sin \pi n_k \alpha_j| < 1/k^2 \quad \text{for } j=1, 2, \dots, k.$$

We may suppose that  $n_{k+1} > n_k$ . Then the terms of the series  $\sum |\sin \pi n_k x|$  are less than  $1/k^2$  for  $x = \pi x_j$  and  $k \geq j$ , and the series converges in  $E$ , though not everywhere.

(2.5) THEOREM. *Every set E of type N is also of type N<sub>s</sub>.*

The proof is in two parts.

(2.6) LEMMA. *Let E be of type N, and let  $x_0$  be a fixed point of E. Let  $E_1$  denote the set E translated by  $-x_0$ , that is, the set of all points  $x - x_0$  with  $x \in E$ . Then  $E_1$  is of type N<sub>s</sub>.*

For using the inequality (1.8) with  $x' = x$ ,  $x'' = x_0$  we see that if the series

$$\sum \rho_n \sin(\pi x + \alpha_n), \quad \text{with } \sum \rho_n = \infty,$$

converges absolutely in  $E$ , so does the series  $\sum \rho_n \sin \pi(x - x_0)$ , that is,  $\sum \rho_n \sin \pi x$  converges absolutely in  $E_{x_0}$ .

(2.7) LEMMA. *If a set E is of type N<sub>s</sub>, the set  $E_0$  obtained by the addition to E of any point  $x_0$  outside E is also of type N<sub>s</sub>.*

For suppose that  $\sum \rho_n |\sin \pi x|$  converges for  $x \in E$  and that  $\sum \rho_n = \infty$ . Let  $\omega_n$  be numbers not less than 1, monotonically increasing to  $+\infty$ , and such that

$$\sum \rho_n \omega_n = \infty, \quad \sum \rho_n / \omega_n^2 < \infty.$$

(We may take, for example,  $\omega_n = \sum_1^n \rho_k$  for  $\rho$  large.) By (2.4) with  $k=1$  there exist for each  $n$  integers  $q_n > 0$  and  $p_n$  such that

$$1 \leq q_n \leq \omega_n, \quad \left| q_n \frac{\pi x_0}{n} - p_n \right| < \frac{1}{[\omega_n]}.$$

Hence

$$\left| q_n n x_0 - p_n \pi \right| < \frac{\pi}{[\omega_n]} < \frac{2\pi}{\omega_n}, \quad \left| \sin n q_n x_0 \right| < \frac{2\pi}{\omega_n},$$

$$\sum \frac{\rho_n}{\omega_n} \left| \sin n q_n x_0 \right| < \sum \frac{2\pi \rho_n}{\omega_n^2} < \infty.$$

For  $x \in E$  we have

$$\sum \frac{\rho_n}{\omega_n} \left| \sin n q_n x \right| \leq \sum \frac{\rho_n q_n}{\omega_n} \left| \sin nx \right| < \sum \rho_n \left| \sin nx \right| < \infty.$$

Thus  $\sum (\rho_n / \omega_n) \sin n q_n x$  converges absolutely for  $x = x_0$  and for  $x \in E$ , though  $\sum (\rho_n / \omega_n)$  diverges.

The integers  $nq_n$  are not necessarily increasing, but obviously only a finite number of them can be equal to any given integer. Hence, rearranging  $\sum (\rho_n / \omega_n) \sin n q_n x$ , we obtain a trigonometric sine series converging absolutely in  $E_0$  but not everywhere.

Return to (2.5) and let  $x_0 \in E$ . By (2.6), there is a series  $\sum \rho_n \left| \sin nx \right|$  with  $\sum \rho_n = \infty$  converging in  $E_{x_0}$ . By (2.7) there is a series  $\sum r_n \left| \sin nx \right|$  with  $\sum r_n = \infty$  converging in  $E_{x_0}$  and at  $x_0$ . Thus

$$\sum r_n \left| \sin n(x - x_0) \right| < \infty \quad (x \in E); \quad \sum r_n \left| \sin n x_0 \right| < \infty,$$

and so

$$\sum r_n \left| \sin nx \right| < \infty \quad \text{for } x \in E,$$

which proves (2.5).

Theorem (2.5) gives a new proof of (1.12), since the latter, as we observed, is immediate for sine series.

The properties N and  $N_s$  being equivalent, it follows from (2.7) that property N is not affected if we add to the set one point, and so if we add any finite number of points.

More generally

(2.8) THEOREM. *If E is of type N and D is denumerable, then E + D is of type N.*

We know that E is of type  $N_s$ . Let the points of D be  $x_1, x_2, \dots$  and let  $E_k$  be the set E augmented by the points  $x_1, x_2, \dots, x_k$ .

As the proof of (2.7) shows, given any series  $\sum r_n \left| \sin nx \right|$  convergent in E, with  $\sum r_n = \infty$ , we can construct a series  $\sum r'_n \left| \sin nx \right|$  convergent in  $E_k$ . Moreover,

$$\sum r'_n \left| \sin nx \right| \leq \sum r_n \left| \sin nx \right| \quad \text{for all } x.$$

More than this is true; for a moment's consideration shows that, given any  $N \geq 1$ , we can find an  $N' \geq N$  such that

$$\sum_1^{N'} r'_n \left| \sin nx \right| \leq \sum_1^N r_n \left| \sin nx \right|.$$

Multiplying  $\sum r'_n \left| \sin nx \right|$  by a sufficiently small number, we may also suppose that the sum of the series at the points  $x_1, x_2, \dots, x_k$  is less than a given  $\epsilon > 0$ .

Consider now a series  $\sum \rho'_n \sin nx$  with  $\sum \rho'_n = \infty$  and

$$\sum \rho'_n \left| \sin nx \right| < \infty \quad \text{for } x \in E_1. \quad (2.9)$$

Take  $N_1$  so large that

$$\sum_1^{N_1} \rho'_n \geq 1, \quad \sum_1^{N_1} \rho'_n \left| \sin nx \right| \leq \frac{1}{2} \quad \text{for } x = x_1.$$

Starting with the remainder  $\sum_{N_1+1}^{\infty} \rho'_n \sin nx$ , we construct a series  $\sum \rho''_n \left| \sin nx \right|$  convergent in  $E_2$ , with  $\sum \rho''_n = \infty$ . Let  $N_2 > N_1$  be such that

$$\sum_{N_1+1}^{N_2} \rho''_n \geq 1, \quad \sum_{N_1+1}^{N_2} \rho''_n \left| \sin nx \right| < \frac{1}{2^2} \quad \text{for } x = x_1, x_2.$$

and  $\bar{N}_2 \geq N_2$  such that

$$\sum_{N_1+1}^{N_2} \rho_n' |\sin nx| \leq \sum_{N_1+1}^{\bar{N}_2} \rho_n' |\sin nx|.$$

Generally, having defined  $N_{k-1}$  and  $\bar{N}_{k-1} \geq N_{k-1}$ , we consider  $\sum_{\bar{N}_{k-1}+1}^{\infty} \rho_n' \sin nx$  and construct a series

$\sum_{\bar{N}_{k-1}+1}^{\infty} \rho_n^{(k)} \sin nx$ , with  $\sum \rho_n^{(k)} = \infty$ . We take numbers  $N_k > \bar{N}_{k-1}$  and  $\bar{N}_k \geq N_k$  such that

$$\sum_{\bar{N}_{k-1}+1}^{N_k} \rho_n^{(k)} \geq 1, \quad \sum_{\bar{N}_{k-1}+1}^{N_k} \rho_n^{(k)} |\sin nx| \leq \frac{1}{2^k} \quad \text{for } x = x_1, x_2, \dots, x_k, \tag{2.10}$$

$$\sum_{\bar{N}_{k-1}+1}^{N_k} \rho_n^{(k)} |\sin nx| \leq \sum_{\bar{N}_{k-1}+1}^{\bar{N}_k} \rho_n' |\sin nx| \quad \text{for all } x, \tag{2.11}$$

and so on.

The series

$$\sum \rho_n \sin nx = \sum_{k=1}^{\infty} \sum_{n=\bar{N}_{k-1}+1}^{N_k} \rho_n^{(k)} \sin nx,$$

where  $N_0 = 0, \bar{N}_1 = N_1$ , is the required series. For

$$\sum_{k=1}^{\infty} \sum_{n=\bar{N}_{k-1}+1}^{N_k} \rho_n^{(k)} |\sin nx|$$

converges in  $E$ , as may be seen from (2.11) and (2.9). It converges in  $D$  by virtue of the second inequality (2.10). Finally, the first inequality (2.10) shows that  $\sum \rho_n = \infty$ .

The following theorem shows that we cannot replace  $D$  in (2.8) by an arbitrary set of type  $N$ :

(2.12) THEOREM. *There exist two sets  $A$  and  $B$  of type  $N$  such that  $A + B$  is not of type  $N$ .*

Every  $x, 0 \leq x \leq 2\pi$ , may be written in the form

$$2\pi \sum_{k=1}^{\infty} \epsilon_k 2^{-k} \quad (\epsilon_k = 0, 1).$$

Let  $k_1 = 1 < k_2 < \dots < k_p < \dots$  be given, and let

$$\Delta_p = 2\pi \sum_{k_p}^{k_{p+1}-1} \epsilon_k 2^{-k} \quad (\epsilon_k = 0, 1).$$

The expansions

$$\Delta_1 + \Delta_3 + \Delta_5 + \dots, \quad \Delta_2 + \Delta_4 + \Delta_6 + \dots$$

represent two perfect and non-dense sets  $A$  and  $B$  in  $(0, 2\pi)$ . Since all points in  $(0, 2\pi)$  can be written in the form  $x + y$ , with  $x \in A, y \in B$ , it follows that  $A + B$  is not of type  $N$ . But for a suitable choice of the  $k_p$ , both  $A$  and  $B$  will be of type  $N$ .

For

$$2^{k_{2p}}(\Delta_1 + \Delta_3 + \dots + \Delta_{2p-1}) \equiv 0 \pmod{2\pi},$$

$$2^{k_{2p}}(\Delta_{2p+1} + \Delta_{2p+3} + \dots) < 4\pi(2^{-(k_{2p}+1-k_{2p})} + \dots) < 8\pi 2^{-(k_{2p}+1-k_{2p})}.$$

Take, for example,  $k_p = p^2$ , and consider the series  $\sum |\sin 2^{k_p} x|$ . The preceding inequalities immediately show that it converges in  $A$ . Thus  $A$  is of type  $N$ . Likewise  $B$  is of type  $N$ .

It is obvious that any translation of a set of type  $N$  is of type  $N$ . We have the following deeper result, in which  $E_\lambda$  denotes the set of points  $x\lambda$  with  $x \in E$ .

(2.13) THEOREM. *If  $E$  is of type  $N$ , so is  $E_\lambda$  for every  $\lambda$ .*

The case  $\lambda = 0$  is trivial. We are considering  $E$  as a periodic set of period  $2\pi$ . Hence  $E_\lambda$  has period  $2\pi |\lambda|$ , and the set  $E_\lambda$  reduced mod  $2\pi$  is, in general, 'richer' in points than a portion of  $E_\lambda$  situated in an interval of length  $2\pi$ .

Since  $E$  is of type  $N_s$ , there is a series  $\sum \rho_n |\sin(nx/\lambda)|$  converging in  $E_\lambda$ , with  $\sum \rho_n = \infty$ . Let  $\{\omega_n\}$  be, as in the proof of (2.7), an increasing sequence of numbers satisfying  $\sum \rho_n \omega_n^{-1} = \infty$ ,  $\sum \rho_n \omega_n^{-2} < \infty$ . By means of (2.1), we choose integers  $q_n, p_n$  such that

$$1 \leq q_n \leq \omega_n, \quad \left| q_n \frac{n}{\lambda} - p_n \right| < \frac{1}{[\omega_n]} < \frac{2}{\omega_n}.$$

Then

$$|\sin p_n x| < \left| \sin \frac{2x}{\omega_n} \right| + \left| \sin q_n \frac{n}{\lambda} x \right| < \frac{2|x|}{\omega_n} + q_n \left| \sin \frac{n}{\lambda} x \right|.$$

$$\frac{\rho_n}{\omega_n} |\sin p_n x| < 2|x| \frac{\rho_n}{\omega_n^2} + \rho_n \left| \sin \frac{n}{\lambda} x \right|.$$

Hence the series  $\sum \rho_n \omega_n^{-1} |\sin p_n x|$  converges in  $E_\lambda$ , and since  $\sum \rho_n \omega_n^{-1} = \infty$ ,  $E_\lambda$  is of type N (only a finite number of the  $p_n$  can take a given value).

We shall now obtain a necessary condition for a set  $E \subset (0, 2\pi)$  to be of type N. Let  $F(x)$ ,  $0 \leq x \leq 2\pi$ , represent a mass distribution of total mass 1, concentrated on  $E$ ; that is,

$$\int_0^{2\pi} dF = \int_E dF = 1. \quad (2.14)$$

If  $E$  is closed,  $F$  is a non-decreasing function constant in the intervals contiguous to  $E$ .

By hypothesis, there is a series  $\sum \rho_n |\sin nx|$  convergent in  $E$ , with  $\sum \rho_n = \infty$ . In particular,  $\sum \rho_n \sin^2 nx < \infty$  in  $E$ , and so also

$$\frac{\sum_1^N \rho_n \sin^2 nx}{\sum_1^N \rho_n} \rightarrow 0 \quad (x \in E).$$

The ratio here does not exceed 1. Multiplying it by  $dF$  and integrating over  $(0, 2\pi)$ , we get

$$\frac{\sum_1^N \rho_n \int_0^{2\pi} \sin^2 nx dF}{\sum_1^N \rho_n} \rightarrow 0.$$

Hence

$$\liminf_{n \rightarrow \infty} \int_0^{2\pi} \sin^2 nx dF = 0,$$

or

$$\limsup_{n \rightarrow \infty} \int_0^{2\pi} \cos 2nx dF = 1. \quad (2.15)$$

Thus

(2.16) THEOREM. *If  $E$  is of type N, then for any positive mass distribution  $dF$  satisfying (2.14) we have (2.15).*

This means that not only do the Fourier-Stieltjes coefficients of  $dF$  not tend to zero, but their upper limit is as large as possible. The fact that for any distribution of mass 1 over  $E$  we have (2.15), might be interpreted as indicating the 'smallness' of  $E$  (see also Example 10, p. 215).

Remarks. (a) It seems not to be known whether the condition (2.15), which is fulfilled for all functions of the type described above, is sufficient for  $E$  to be of type N. It is almost immediate, however, that if (2.15) is satisfied for a given  $F'$ , then there is a subset  $E''$  of  $E$  of type N such that

$$\int_{E-E''} dF = 0.$$

For (2.15) implies the existence of a sequence  $\{n_k\}$  such that  $\int_0^{2\pi} \sin^2 n_k x dF \leq 2^{-k}$ . Hence

$$\int_0^{2\pi} \left\{ \sum_1^{\infty} |\sin n_k x| \right\} dF \leq \sum 2^{-k/2} < \infty,$$

so that  $\sum |\sin n_k x| < \infty$  for  $x \in E' \subset E$ , where  $E'$  is such that the variation of  $F$  over  $E - E'$  is zero.

(b) We shall apply (2.16) to symmetrical perfect sets of the Cantor type (Chapter V, § 3). We know that the points of such a set  $P$  are:

$$x = 2\pi(\epsilon_1 r_1 + \epsilon_2 r_2 + \dots) \quad (\epsilon_k = 0, 1), \tag{2.17}$$

where  $r_1 + r_2 + \dots = 1$  and  $r_p > r_{p+1} + r_{p+2} + \dots$ , or alternatively, putting in evidence the successive ratios of dissection,  $r_k = \xi_1 \dots \xi_{k-1} (1 - \xi_k)$ .

Let  $F$  be the Lebesgue function for  $P$ . Then (Chapter V, (3.4))

$$\int_0^{2\pi} \cos 2nx dF = \prod_{k=1}^{\infty} \cos 2\pi n r_k.$$

If  $P$  is of type N, we have the necessary condition

$$\limsup_{n \rightarrow \infty} \prod_{k=1}^{\infty} \cos^2 2\pi n r_k = 1,$$

or 
$$\limsup_{n \rightarrow \infty} \prod_{k=1}^{\infty} (1 - \sin^2 2\pi n r_k) = 1. \tag{2.18}$$

The latter condition is equivalent to

$$\liminf_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sin^2 2\pi n r_k = 0. \tag{2.19}$$

That (2.18) implies (2.19) follows from the inequality  $1 + x \leq e^x$  and the fact that the product in (2.18) does not exceed 1. The converse is a consequence of the inequality

$$(1 - \epsilon_1)(1 - \epsilon_2) \dots (1 - \epsilon_N) \geq 1 - (\epsilon_1 + \dots + \epsilon_N),$$

valid for  $\epsilon_k \leq 1$  and easily verifiable by induction.

We now show that, if the sequence  $r_k$  (which is decreasing) is such that  $r_k/r_{k+1} = O(1)$ , then  $P$  is not of type N.

Suppose that  $r_k/r_{k+1} \leq M$ , and take any  $\alpha$ ,  $0 < \alpha < 1/M$ . Then, if  $n$  is large enough, there is at least one  $r_k$  in the interval  $(\alpha/4n, 1/4n)$ , and for this  $r_k$

$$\sin^2 2\pi n r_k \geq \sin^2 \left( 2\pi n \frac{\alpha}{4n} \right) = \sin^2 \frac{1}{2} \pi \alpha,$$

which makes (2.19) impossible.

In particular, if  $\xi_k \geq \delta > 0$  for all  $k$ ,  $P$  is not of type N.

(c) The condition  $\liminf_{n \rightarrow \infty} \sum_{k=1}^{\infty} |\sin 2\pi n r_k| = 0$ , very similar to (2.19), is sufficient for  $P$  to be of type N. For let  $\{n_p\}$  be such that

$$\sum_{k=1}^{\infty} |\sin 2\pi n_p r_k| < \eta_p,$$

with  $\sum \eta_p < \infty$ . The relation (2.17) implies

$$|\sin n_p x| \leq \sum_{k=1}^{\infty} |\sin 2\pi n_p r_k| \leq \eta_p$$

and gives  $\sum |\sin n_p x| < \infty$ , for  $x \in P$ .

### 3. The absolute convergence of Fourier series

(3.1) THEOREM OF S. BERNSTEIN. *If  $f \in \Lambda_{\alpha}$ ,  $\alpha > \frac{1}{2}$ , then  $S[f]$  converges absolutely. For  $\alpha = \frac{1}{2}$  this is not necessarily true.*

Suppose that  $\Sigma A_n(x)$  is  $S[f]$ . Then

$$\begin{aligned} f(x+h) - f(x-h) &\sim -2 \sum_{n=1}^{\infty} B_n(x) \sin nh, \\ \frac{1}{\pi} \int_0^{2\pi} [f(x+h) - f(x-h)]^2 dx &= 4 \sum_{n=1}^{\infty} \rho_n^2 \sin^2 nh. \end{aligned} \quad (3.2)$$

If  $\omega(\delta)$  is the modulus of continuity of  $f$ , the left-hand side of (3.2) does not exceed  $2\omega^2(2h)$ . On setting  $h = \pi/2^{\nu+1}$ ,  $\nu = 1, 2, \dots$ , we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{\pi n}{2^{\nu+1}} &\leq \frac{1}{2} \omega^2 \left( \frac{\pi}{2^{\nu}} \right), \\ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^2 \sin^2 \frac{\pi n}{2^{\nu+1}} &\leq \frac{1}{2} \omega^2 \left( \frac{\pi}{2^{\nu}} \right), \\ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^2 &\leq \omega^2 \left( \frac{\pi}{2^{\nu}} \right), \end{aligned} \quad (3.3)$$

since in the preceding sum the co-factors of the  $\rho_n^2$  all exceed  $\frac{1}{2}$ . By Schwarz's inequality.

$$\sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n \leq \left( \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n^2 \right)^{\frac{1}{2}} \left( \sum_{n=2^{\nu-1}+1}^{2^{\nu}} 1^2 \right)^{\frac{1}{2}} \leq 2^{\frac{1}{2}\nu} \omega \left( \frac{\pi}{2^{\nu}} \right), \quad (3.4)$$

and finally, 
$$\sum_{n=2}^{\infty} \rho_n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \rho_n \leq \sum_{\nu=1}^{\infty} 2^{\frac{1}{2}\nu} \omega \left( \frac{\pi}{2^{\nu}} \right). \quad (3.5)$$

If  $\omega(\delta) \leq C\delta^{\alpha}$ ,  $\alpha > \frac{1}{2}$ , the last series converges and the first part of (3.1) is established. The proof of the second part is contained in that of (3.10).

*Remark.* The above proof gives slightly more than is actually stated in (3.1). For  $S[f]$  to converge absolutely it is enough that  $\Sigma 2^{\frac{1}{2}\nu} \omega(\pi/2^{\nu})$  converges. This condition is readily seen to be equivalent to the convergence of the series  $\Sigma n^{-\frac{1}{2}} \omega(\pi/n)$ , or of the integral

$$\int_0^1 \delta^{-\frac{1}{2}} \omega(\delta) d\delta.$$

**(3.6) THEOREM.** *If  $f(x)$  is of bounded variation and of the class  $\Lambda_{\alpha}$ , for some  $\alpha > 0$ ,  $S[f]$  converges absolutely.*

The second condition imposed on  $f$  is not superfluous, as the example

$$f(x) \sim \sum_{n=2}^{\infty} \frac{\sin nx}{n \log n} \quad (3.7)$$

shows. Here  $f(x)$  is of bounded variation, indeed absolutely continuous (Chapter V, (1.5)), but  $S[f]$  is not absolutely convergent.

Let  $\omega(\delta)$  be the modulus of continuity of  $f$ , and  $V$  the total variation of  $f$  over  $(0, 2\pi)$ . Obviously

$$\sum_{k=1}^{2N} \left[ f\left(x + \frac{k\pi}{N}\right) - f\left(x + \frac{(k-1)\pi}{N}\right) \right]^2 \leq \omega\left(\frac{\pi}{N}\right) \sum_{k=1}^{2N} \left| f\left(x + \frac{k\pi}{N}\right) - f\left(x + \frac{(k-1)\pi}{N}\right) \right| \leq \omega\left(\frac{\pi}{N}\right) V.$$

We integrate this over  $(0, 2\pi)$  and observe that all the integrals on the left are equal. This gives successively, with  $N = 2^v$ ,

$$2N \int_0^{2\pi} \left[ f\left(x + \frac{\pi}{2N}\right) - f\left(x - \frac{\pi}{2N}\right) \right]^2 dx \leq 2\pi V\omega(\pi/N),$$

$$\sum_{n=1}^{\infty} \rho_n^2 \sin^2 \frac{\pi n}{2N} \leq \frac{1}{2} N^{-1} V\omega\left(\frac{\pi}{N}\right),$$

$$\sum_{n=2^{v-1}+1}^{2^v} \rho_n^2 \leq \frac{1}{2} 2^{-v} V\omega(\pi 2^{-v}),$$

$$\sum_{n=2^{v-1}+1}^{2^v} \rho_n \leq \frac{1}{2} V^{\frac{1}{2}} \omega^{\frac{1}{2}}(\pi 2^{-v}),$$

$$\sum_{n=2}^{\infty} \rho_n \leq \frac{1}{2} V^{\frac{1}{2}} \sum_{v=1}^{\infty} \omega^{\frac{1}{2}}(\pi 2^{-v}).$$

If  $\omega(\delta) \leq C\delta^\alpha$ ,  $\alpha > 0$ , the series on the right converges and (3.6) follows. The convergence of  $\sum \omega^{\frac{1}{2}}(\pi 2^{-v})$  is equivalent to that of  $\sum n^{-1} \omega^{\frac{1}{2}}(\pi/n)$ , or of

$$\int_0^1 \delta^{-1} \omega^{\frac{1}{2}}(\delta) d\delta.$$

**(3.8) THEOREM.** *If  $f$  is absolutely continuous and if  $f' \in L^p$ ,  $p > 1$ , then  $S[f]$  converges absolutely.*

This follows from (3.6); for if  $f' \in L^p$ ,  $p > 1$ , then  $f \in \Lambda_{1/p}$ , since, if  $0 < h \leq 2\pi$ ,

$$|f(x+h) - f(x)| \leq \int_x^{x+h} |f'(t)| dt \leq \left( \int_x^{x+h} |f'|^p dt \right)^{1/p} h^{1/p'} \leq \left( \int_0^{2\pi} |f'|^p dt \right)^{1/p} h^{1/p'}.$$

For  $p = 2$  (and so  $p \geq 2$ ), (3.8) is immediate. For if  $a_n, b_n$  are the coefficients of  $f'$ , those of  $f$  are  $-b_n/n, a_n/n$ , and the inequalities

$$|a_n| n^{-1} \leq \frac{1}{2}(a_n^2 + n^{-2}), \quad |b_n| n^{-1} \leq \frac{1}{2}(b_n^2 + n^{-2}),$$

coupled with Bessel's inequality  $\Sigma(a_n^2 + b_n^2) < \infty$ , imply the absolute convergence of  $S[f]$ . (The argument can be extended to general  $p > 1$  if instead of Bessel's inequality we use its extension for  $1 < p \leq 2$ , the Hausdorff-Young theorem (2.3) of Chapter XII.)

**(3.9) THEOREM.** *The conclusion of (3.8) remains valid if instead of the integrability of  $|f'|^p$  we assume that of  $|f'| \log^+ |f'|$ .*

The proof is postponed to Chapter VII, p. 287, where the result will be obtained as a corollary of the following theorem:

*If  $S[f]$  and  $\tilde{S}[f]$  are both Fourier series of functions of bounded variation,  $S[f]$  converges absolutely.*

Here we only observe that the integrability of  $|f'| (\log^+ |f'|)^{1-\epsilon}$ ,  $\epsilon > 0$ , would not be enough. For if  $f$  is given by (3.7), then  $S[f]$  converges absolutely only for  $x = 0$  and  $x = \pi$ . But  $f'(x) \sim 1/x \log^2 x$  as  $x \rightarrow +0$  (Chapter V, (2.19)), so that  $|f'| (\log^+ |f'|)^{1-\epsilon}$  is integrable for every  $0 < \epsilon < 1$ .

The problem of absolute convergence of Fourier series may be generalized as follows. Given a series  $\Sigma A_n(x)$ , we ask about the values of the exponent  $\beta$  which make

$$\Sigma(|a_n|^\beta + |b_n|^\beta)$$

convergent. Theorem (3.1) is a special case of the following result:

(3.10) THEOREM. *If  $f \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , then  $\Sigma(|a_n|^\beta + |b_n|^\beta)$  converges for  $\beta > 2/(2\alpha + 1)$ , but not necessarily for  $\beta = 2/(2\alpha + 1)$ .*

The proof of the first part is like that of the first part of (3.1). Let  $\gamma = 2/(2\alpha + 1)$ . Since  $0 < \gamma < 2$ , we may suppose further that  $0 < \beta < 2$ . By Hölder's inequality and (3.3), we have

$$\begin{aligned} \sum_{2^{\nu-1}+1}^{2^\nu} \rho_n^\beta &\leq \left( \sum_{2^{\nu-1}+1}^{2^\nu} \rho_n^2 \right)^{\frac{1}{2}\beta} \left( \sum_{2^{\nu-1}+1}^{2^\nu} 1 \right)^{1-\frac{1}{2}\beta} \leq 2^{\nu(1-\frac{1}{2}\beta)} \omega^\beta(\pi 2^{-\nu}), \\ \sum_{n=2}^{\infty} \rho_n^\beta &\leq \sum_{\nu=1}^{\infty} 2^{\nu(1-\frac{1}{2}\beta)} \omega^\beta(\pi 2^{-\nu}), \end{aligned}$$

and the last series converges if  $\omega(\delta) \leq C\delta^\alpha$ , since  $1 - \beta(\frac{1}{2} + \alpha) < 0$ . This gives the finiteness of  $\Sigma \rho_n^\beta$ , and so also of  $\Sigma(|a_n|^\beta + |b_n|^\beta)$ .†

The second parts of (3.1) and of (3.10) are corollaries of the results obtained in Chapter V, § 4. It was proved there that the real and imaginary parts of the series

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{n^{\frac{1}{2}+\alpha}} e^{inz} \quad (0 < \alpha < 1) \quad (3.11)$$

belong to  $\Lambda_\alpha$ , and it is easy to see that, for both,  $\Sigma \rho_n^{2(2\alpha+1)} = \infty$ . The series

$$\sum_{n=1}^{\infty} \frac{e^{in \log n}}{(n \log n)^{\frac{1}{2}}} e^{inz} \quad (3.12)$$

belongs to  $\Lambda_1$  (Chapter V, (4.9)), but  $\Sigma \rho_n^2 = \infty$ .

(3.13) THEOREM. *If  $f$  is of bounded variation, and if  $f \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , then  $\Sigma(|a_n|^\beta + |b_n|^\beta)$  converges for  $\beta > 2/(\alpha + 2)$ , though not necessarily for  $\beta = 2/(\alpha + 2)$ .*

The proof of the convergence for  $\beta > 2/(2 + \alpha)$  is analogous to the proofs of Theorems (3.6) and (3.10), and is left to the reader.

In proving that  $\Sigma(|a_n|^\beta + |b_n|^\beta)$  can diverge for  $\beta = 2/(2 + \alpha)$ , we may suppose that  $0 < \alpha < 1$ , since the case  $\alpha = 1$  is taken care of by (3.12). We start from the series

$$\sum_{n=2}^{\infty} n^{-\frac{1}{2}\alpha} (\log n)^{-\gamma} e^{in^2} e^{inz} \quad (0 < \alpha < 1).$$

It was proved in Chapter V, (5.8), that this is a Fourier series if  $\gamma > 1$ . So the integrated series

$$-i \Sigma n^{-\frac{1}{2}\alpha-1} (\log n)^{-\gamma} e^{in^2} e^{inz} \quad (3.14)$$

is the Fourier series of a function of bounded variation (indeed absolutely continuous). By Chapter V, (5.2), the sum of (3.14) without the logarithmic factor is a function of the class  $\Lambda_\alpha$ , and by Chapter IV, (11.16), the sum of (3.14) is in  $\Lambda_\alpha$ . On the other hand,

$$\Sigma |c_n|^{2(2+\alpha)} = \Sigma n^{-1} (\log n)^{-\gamma 2(2+\alpha)} = \infty,$$

if  $\gamma$  is close enough to 1. It follows that for both the real and the imaginary parts of (3.14) the series  $\Sigma(|a_n|^\beta + |b_n|^\beta)$  diverges if  $\beta = 2/(2 + \alpha)$ .

† If  $A_n(x) = \rho_n \cos(nx + \phi_n)$ , then  $(|a_n|^\beta + |b_n|^\beta)/\rho_n^\beta = |\cos \phi_n|^\beta + |\sin \phi_n|^\beta$  is contained between  $2^{-1/\beta}$  and 2, so that the series  $\Sigma(|a_n|^\beta + |b_n|^\beta)$  and  $\Sigma \rho_n^\beta$  both converge or both diverge.

**4. Inequalities for polynomials**

The problem of the absolute convergence of Fourier series has close connexion with the following one, for trigonometric polynomials.

Consider all polynomials

$$T(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx), \tag{4.1}$$

of fixed order  $n \geq 1$ , such that  $|T(x)| \leq 1$ . How large can the sum

$$\Gamma(T) = \frac{1}{2}|a_0| + \sum_{k=1}^n (|a_k| + |b_k|)$$

be? The answer is given by the following theorem, in which  $A$  and  $B$  denote positive absolute constants.

- (4.2) THEOREM. (i) If  $|T(x)| \leq 1$ , then  $\Gamma(T) \leq An^{\frac{1}{2}}$ ;
- (ii) Conversely, for every  $n$  there is a polynomial (4.1) such that  $|T(x)| \leq 1$ ,  $\Gamma(T) \geq Bn^{\frac{1}{2}}$ .

Since  $|a_k| + |b_k|$  is contained between  $\rho_k$  and  $2^{\frac{1}{2}}\rho_k$ , (4.2) is not affected if we replace  $\Gamma(T)$  by  $\Gamma^*(T) = \frac{1}{2}\rho_0 + \sum_1^n \rho_k$ .

Part (i) is immediate, since

$$\begin{aligned} \Gamma^*(T) &= \frac{1}{2}\rho_0 + \sum_1^n \rho_k \leq \left( \frac{1}{2}\rho_0^2 + \sum_1^n \rho_k^2 \right)^{\frac{1}{2}} \left( \frac{1}{2} + \sum_1^n 1^2 \right)^{\frac{1}{2}} \\ &= \left( \frac{1}{\pi} \int_0^{2\pi} T^2 dx \right)^{\frac{1}{2}} (n + \frac{1}{2})^{\frac{1}{2}} \leq 2^{\frac{1}{2}}(n + \frac{1}{2})^{\frac{1}{2}} = (2n + 1)^{\frac{1}{2}} \leq 3^{\frac{1}{2}}n^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

To prove (ii), consider the polynomial

$$g(x) = g_l(x) = \sum_{k=1}^N \{ \phi_{\pi^{k-1}}(t) \cos kx + \phi_{2k}(t) \sin kx \},$$

where  $\phi_1, \phi_2, \dots$  are Rademacher's functions (Chapter I, § 3). By Chapter V, (8.4)

$$\begin{aligned} \int_0^1 dt \int_0^{2\pi} |g_l(x)| dx &= \int_0^{2\pi} dx \int_0^1 |g_l(x)| dt \geq A_1 \int_0^{2\pi} (\cos^2 x + \sin^2 x + \dots + \sin^2 Nx)^{\frac{1}{2}} dx \\ &= 2\pi A_1 N^{\frac{1}{2}}. \end{aligned}$$

It follows that there is a  $t = t_0$  such that  $\int_0^{2\pi} |g_{l_0}(x)| dx \geq 2\pi A_1 N^{\frac{1}{2}}$ .

Let  $a_k, b_k$  be the Fourier coefficients of the function  $f(x) = \text{sign } g_{l_0}(x)$ . Let  $\sigma_\nu$  be the (C, 1) means of  $S[f]$ , and let

$$\tau_N = 2\sigma_{2N-1} - \sigma_{N-1} = \frac{1}{2}a_0 + \sum_{k=1}^N A_k(x) + \sum_{k=N+1}^{2N-1} \left( 1 - \frac{k-N}{N} \right) A_k(x)$$

be the delayed means of  $S[f]$  (Chapter III, § 1); for  $\nu = 2N - 1$  or  $n = 2N$ ,  $\frac{1}{3}\tau_N$  will be the required polynomial  $T$ .

Clearly,  $|\frac{1}{3}\tau_N| \leq 1$ , since  $|f| \leq 1$  and so  $|\sigma_\nu| \leq 1$ . It is therefore enough to show that  $\Gamma(\tau_N) \geq CN^{\frac{1}{2}}$  for some fixed  $C$ . Now

$$\Gamma(\tau_N) \geq \sum_1^N (|a_k| + |b_k|) \geq \left| \sum_1^N a_k \phi_{2k-1}(t_0) + b_k \phi_{2k}(t_0) \right| = \left| \frac{1}{\pi} \int_0^{2\pi} g_{l_0} \tau_N dx \right|.$$

Since  $g_{l_0}(x)$  is a polynomial of order  $N$ , and the  $N$ th partial sums of  $\tau_N$  and of  $S[f]$  coincide, this last expression is

$$\left| \frac{1}{\pi} \int_0^{2\pi} g_{l_0} f dx \right| = \frac{1}{\pi} \int_0^{2\pi} |g_{l_0}| dx = 2A_1 N^{\frac{1}{2}}.$$

This completes the proof of (ii).

(4.4) THEOREM. For any  $r > 0$  and any polynomial (4.1), let

$$\Gamma_r(T) = \left\{ \frac{1}{2} |a_0|^r + \sum (|a_k|^r + |b_k|^r) \right\}^{1/r}.$$

Then, (i) for any  $1 \leq r \leq 2$  and any  $T(x)$  with  $|T| \leq 1$ , we have  $\Gamma_r(T) \leq An^{-\frac{1}{2}+1/r}$ , and (ii) for any  $1 \leq r \leq 2$  and any  $n$ , there is a polynomial (4.1) satisfying  $|T| \leq 1$  with  $\Gamma_r(T) \geq Bn^{-\frac{1}{2}+1/r}$ .

The restriction on  $r$  is essential: for  $r > 2$ , part (i) is false and part (ii) trivial.

Part (i) is proved here as in (4.2), except that we use Hölder's inequality instead of Schwarz's. For (ii) we have, by Hölder's inequality,

$$\Gamma(T) \leq \Gamma_r(T) (2n+1)^{(r-1)/r} \leq 3\Gamma_r(T) n^{(r-1)/r}$$

for any  $T$  of order  $n$ ; and the  $T$  for which we have  $\Gamma(T) \geq Bn^{\frac{1}{2}}$  satisfies also the inequality  $\Gamma_r(T) \geq \frac{1}{3} Bn^{-\frac{1}{2}+1/r}$ .

## 5. Theorems of Wiener and Lévy

It is obvious that the absolute convergence of  $S[f]$  at a point  $x_0$  is not a local property, but depends on the behaviour of  $f$  in the whole interval  $(0, 2\pi)$ . However,

(5.1) THEOREM. If to every point  $x_0$  there corresponds a neighbourhood  $I_{x_0}$  of  $x_0$  and a function  $g(x) = g_{x_0}(x)$  such that (i)  $S[g]$  converges absolutely and (ii)  $g(x) = f(x)$  in  $I_{x_0}$ , then  $S[f]$  converges absolutely.

By the Heine-Borel theorem we can find a finite number of points  $x_1 < x_2 < \dots < x_m$  such that the intervals  $I_{x_1}, I_{x_2}, \dots, I_{x_m}$  cover  $(0, 2\pi)$ . Let  $I_{x_k} = (u_k, v_k)$ . We may suppose that the successive intervals overlap and even that  $u_k < v_{k-1} < u_{k+1} < v_k$ ,  $k = 1, 2, \dots, m$ , where  $u_{m+1} = u_1 + 2\pi$ ,  $v_0 = v_m - 2\pi$ . Let  $\lambda_k(x)$  be the periodic and continuous function equal to 1 in  $(v_{k-1}, u_{k+1})$ , vanishing outside  $(u_k, v_k)$ , and linear in  $(u_k, v_{k-1})$  and  $(u_{k+1}, v_k)$ . It is readily seen that  $\lambda_1(x) + \lambda_2(x) + \dots + \lambda_m(x) = 1$ . Since  $\lambda'_k(x)$  is of bounded variation, the Fourier coefficients of  $\lambda_k$  are  $O(n^{-2})$ , and  $S[\lambda_k]$  converges absolutely.

Since  $S[f\lambda_k] = S[g_{x_k}\lambda_k] = S[g_{x_k}]S[\lambda_k]$ , it follows that  $S[f\lambda_k]$  converges absolutely (Chapter IV, §8). To complete the proof of (5.1), we observe that

$$S[f] = S[f \cdot (\lambda_1 + \dots + \lambda_m)] = S[f\lambda_1] + \dots + S[f\lambda_m].$$

(5.2) THEOREM. (i) Suppose that  $S[f]$  converges absolutely, that the values (in general, complex) of  $f(t)$  lie on a curve  $C$ , and that  $\phi(z)$  is an analytic (not necessarily single-valued) function of a complex variable regular at every point of  $C$ . Then  $S[\phi(f)]$  converges absolutely.

(ii) In particular, if  $f(t) \neq 0$ , and if  $S[f]$  converges absolutely, so does  $S[1/f]$ .

For every  $g(x) = \sum a_k e^{ikx}$  we write

$$\|g\| = \sum |a_k|, \quad Mg = \max_x |g(x)|. \quad (5.3)$$

Clearly,

$$\|\sum g_i\| \leq \sum \|g_i\|, \quad \|g_1 g_2\| \leq \|g_1\| \|g_2\|. \quad (5.4)$$

(5.5) LEMMA. (a) If  $g(x) = \sum a_n e^{inx}$  is twice continuously differentiable, then

$$\|g\| \leq 4(Mg + Mg''). \quad (5.6)$$

(b) If  $g(x, \theta)$  is periodic in  $x$ , and if for each value of the parameter  $\theta$ ,  $0 \leq \theta \leq 2\pi$ , we have  $\|g(x, \theta)\| \leq A$ , then

$$\left\| \int_0^{2\pi} g(x, \theta) d\theta \right\| \leq 2\pi A.$$

(a) We have  $|a_n| \leq Mg$ , and if  $n \neq 0$  integration by parts gives  $|a_n| \leq n^{-2}Mg''$ . Since  $\sum n^{-2} = \frac{1}{3}\pi^2 < 4$ , (5.6) follows. Part (b) is immediate.

Return to (5.2) (i). Since  $\phi$  is analytic for  $z = f(x)$ , there is a  $\rho > 0$  such that  $\phi(z)$  is regular in each circle  $|z - f(x)| \leq 2\rho$ . Let  $s(x)$  be a partial sum of  $S[f]$  such that

$$M(s - f) \leq \|s - f\| \leq \frac{1}{2}\rho.$$

Then  $\phi[s(x) + \rho e^{i\theta}]$  is twice continuously differentiable in  $x$  and  $\theta$ . Hence, for each  $\theta$ ,  $\|\phi[s(x) + \rho e^{i\theta}]\|$  is finite, and this norm is a bounded function of  $\theta$ .

On the other hand, we have

$$(s + \rho e^{i\theta} - f)^{-1} = \rho^{-1} e^{-i\theta} \left\{ 1 + \sum_1^{\infty} (f - s)^n \rho^{-n} e^{-in\theta} \right\}$$

and therefore, by (5.4),

$$\|(s + \rho e^{i\theta} - f)^{-1}\| \leq \rho^{-1} \left\{ 1 + \sum_1^{\infty} (\frac{1}{2}\rho)^n \rho^{-n} \right\} = 2\rho^{-1}.$$

Since, by Cauchy's formula,

$$\phi[f(x)] = \frac{1}{2\pi} \int_0^{2\pi} \frac{\phi[s(x) + \rho e^{i\theta}]}{s(x) + \rho e^{i\theta} - f(x)} \rho e^{i\theta} d\theta,$$

part (b) of Lemma (5.5) implies that  $\|\phi[f(x)]\|$  is finite. This completes the proof of (5.2).

Theorems (5.1) and (5.2) (ii) are due to Wiener, (5.2) (i) to Lévy. A corollary of (5.2) (ii) is that, if  $F(z)$  is regular for  $|z| < 1$ , continuous and distinct from 0 in  $|z| \leq 1$ , and if the Taylor series of  $F$  converges absolutely on  $|z| = 1$ , then the Taylor series of  $1/F$  also converges absolutely on  $|z| = 1$ .

We verify without difficulty that the argument of the preceding proof also yields the following result:

(5.7) THEOREM. Let  $f(x)$  have an absolutely convergent Fourier series, and let  $F_n(x)$  be a sequence of analytic functions converging uniformly to 0 in a neighbourhood of the range of  $f(x)$ . Then  $\|F_n[f(x)]\| \rightarrow 0$ .

From this we deduce the following theorem:

(5.8) THEOREM. If  $f(x)$  has an absolutely convergent Fourier series, then

$$\lim_{n \rightarrow \infty} \|f^n\|^{1/n} = Mf.$$

Let  $R > Mf$ ,  $F_n(z) = (z/R)^n$ . By (5.7),

$$\lim_{n \rightarrow \infty} \|F_n[f(x)]\| = \lim_{n \rightarrow \infty} \frac{\|f^n\|}{R^n} = 0,$$

which implies that

$$\limsup_{n \rightarrow \infty} \frac{\|f^n\|^{1/n}}{R} \leq 1.$$

Since this holds for every  $R > Mf$ , it follows that

$$\limsup \|f^n\|^{1/n} \leq Mf.$$

On the other hand, we have

$$\|f^n\| \geq Mf^n = (Mf)^n,$$

whence  $\liminf \|f^n\|^{1/n} \geq Mf$ . This completes the proof of (5.8).

## 6. The absolute convergence of lacunary series

(6.1) THEOREM. *If a lacunary series*

$$\Sigma(a_j \cos n_j x + b_j \sin n_j x) = \Sigma \rho_j \cos(n_j x + x_j) \quad (n_{j+1}/n_j > q > 1) \quad (6.2)$$

is an  $S[f]$  with  $f$  bounded above (or below), then  $\Sigma \rho_j < \infty$ .

(6.3) THEOREM. *If the partial sums  $s_m$  of any lacunary series (6.2) satisfy the condition*

$$\limsup_{m \rightarrow \infty} s_m(x) < +\infty \quad (6.4)$$

(or  $\liminf s_m(x) > -\infty$ ) at every point of an interval  $(\alpha, \beta)$ , then  $\Sigma \rho_j < \infty$ .

Theorem (6.1) is a corollary of (6.3). For the hypothesis of (6.1) implies that the  $(C, 1)$  means  $\sigma_m(x)$  of (6.2) are bounded above. Since  $s_m(x) - \sigma_m(x) \rightarrow 0$  (owing to the lacunary character of the series and the relations  $a_k, b_k \rightarrow 0$ ; see p. 79), the  $s_m(x)$  are bounded above and  $\Sigma \rho_j < \infty$  by (6.3).

Nevertheless, it is more convenient to begin by proving (6.1), which is the easier of the two; the more so since part of the argument will be used later in the proof of (6.3).

We first recall a few properties of the Riesz products, defined in § 7 of Chapter V.

Let  $m_1, m_2, \dots, m_k$  be a sequence of positive integers satisfying  $m_{j+1}/m_j \geq Q \geq 3$  for all  $j$ , and let

$$R(x) = \prod_{j=1}^k \{1 + \cos(m_j x + \xi_j)\}. \quad (6.5)$$

$R(x)$  is a non-negative polynomial with constant term 1,

$$R(x) = 1 + \Sigma \gamma_\nu \cos(\nu x + \eta_\nu), \quad (6.6)$$

so that

$$\frac{1}{2\pi} \int_0^{2\pi} R(x) dx = 1. \quad (6.7)$$

Moreover,  $0 \leq \gamma_\nu \leq 1$  for all  $\nu$ , and  $\gamma_\nu = 0$  unless

$$\nu = m_j \pm m_{j'} \pm m_{j''} \pm \dots \quad (k \geq j > j' > j'' > \dots). \quad (6.8)$$

There is at most one such representation of any given  $\nu$ , and so in particular the  $m_j$ th term of (6.6) is  $\cos(m_j x + \xi_j)$ .

Let  $q > 1$  be given. Since the sum in (6.8) is contained between

$$m_j(1 - Q^{-1} - Q^{-2} - \dots) = m_j(Q - 2)/(Q - 1) \quad \text{and} \quad m_j(1 + Q^{-1} + Q^{-2} + \dots) = m_j Q/(Q - 1),$$

the  $\nu$  with  $\gamma_\nu \neq 0$  are confined to the intervals  $(m_j/q, m_j q)$  provided  $Q$  is large enough, say  $Q > Q_0(q)$ . We shall call the interval  $(m_j/q, m_j q)$  the  $q$ -neighbourhood of  $m_j$ .

Return now to (6.1), and split  $\{n_j\}$  into  $r$  subsequences

$$\{n_{jr+s}\}_{j=0,1,\dots}, \quad \text{where } s = 1, 2, \dots, r.$$

We take  $r$  so large that  $Q = q^r$  exceeds both 3 and the number  $Q_0(q)$ . Fix  $s$ , and set

$$R^s(x) = \prod_{j=0}^k \{1 + \cos(n_{j_r+s}x + x_{j_r+s})\}. \tag{6.9}$$

Since the ranks of the (non-zero) terms of  $R^s$  are in the  $q$ -neighbourhoods of the  $n_{j_r+s}$  ( $j = 0, 1, \dots, k$ ), and the only non-zero term of  $S[f]$  in such a neighbourhood has rank  $n_{j_r+s}$ , it follows that

$$\frac{1}{\pi} \int_0^{2\pi} f(x) R^s(x) dx = \frac{1}{\pi} \int_0^{2\pi} f(x) \left\{ \sum_{j=0}^k \cos(n_{j_r+s}x + x_{j_r+s}) \right\} dx = \sum_{j=0}^k \rho_{j_r+s}. \tag{6.10}$$

But if  $M$  is the upper bound of  $f$  the left-hand side here does not exceed

$$M \frac{1}{\pi} \int_0^{2\pi} R^s(x) dx = 2M,$$

by (6.7). It follows that the right-hand side of (6.10) does not exceed  $2M$ . Making  $k \rightarrow \infty$  and summing the inequalities over  $s = 1, 2, \dots, r$ , we see that  $\sum \rho_j \leq 2Mr$ , and (6.1) is established.

Under the hypothesis of (6.3) there is a subinterval  $I$  of  $(\alpha, \beta)$  such that the  $s_m$  are uniformly bounded above, say  $s_m \leq M$ , in  $I$  (see Chapter I, (12.3)). Hence (6.3) is a corollary of the following lemma about polynomials.

(6.11) LEMMA. *Given any interval  $I$  and any  $q > 1$ , we can find an integer  $n_0 = n_0(q, |I|)$  and a constant  $A = A(q, |I|)$  such that for every lacunary polynomial*

$$T(x) = \sum(a_j \cos n_j x + b_j \sin n_j x) = \sum \rho_j \cos(n_j x + x_j) \quad (n_{j+1}/n_j > q > 1) \tag{6.12}$$

for which  $n_1 \geq n_0$  and  $T(x) \leq M$  in  $I$ , we have

$$\sum \rho_j \leq AM.$$

We need more information about the  $\gamma_n$  in (6.6).

Suppose  $\nu$  is given by (6.8) but is not equal to  $m_j$ . Then

$$|\nu - m_j| \geq m_j(1 - Q^{-1} - Q^{-2} - \dots) = m_j(Q - 2)/(Q - 1) \geq \frac{1}{2}m_j.$$

We have already observed that  $\nu$  is contained between  $m_j(Q - 2)/(Q - 1)$  and  $m_j Q/(Q - 1)$ , and so between  $\frac{1}{2}m_j$  and  $\frac{3}{2}m_j$ . It follows that  $\nu$  must differ from  $m_{j+1}$  (if such an  $m$  exists) by at least  $\frac{1}{2}m_j \geq \frac{1}{3}m_1$ . Similarly it must differ from  $m_{j-1}$  (if such an  $m$  exists) by at least

$$\frac{1}{2}m_j - \frac{1}{3}m_j = \frac{1}{6}m_j \geq \frac{1}{2}m_{j-1} \geq \frac{1}{2}m_1.$$

Collecting these estimates we see that each  $\nu$  with  $\gamma_n \neq 0$  in (6.6) either is an  $m_j$  or differs from all  $m_j$  by at least  $\frac{1}{2}m_1$ .

Return to  $T(x)$ . As before, we split  $\{n_j\}$  into  $r$  subsequences  $\{n_{j_r+s}\}$ , where  $s = 1, 2, \dots, r$ , and  $r$  satisfies the conditions imposed above. Let  $T^s$  consist of the terms of  $T$  which have rank  $n_{j_r+s}$  for some  $j$  (so that  $T$  is the sum of all  $T^s$ ) and let  $R^s$  be defined by (6.9). We set

$$U = R^1 + R^2 + \dots + R^r.$$

An argument similar to (6.10) gives

$$\frac{1}{\pi} \int_0^{2\pi} T U = \sum_s \frac{1}{\pi} \int_0^{2\pi} T R^s dx = \sum_s \frac{1}{\pi} \int_0^{2\pi} T^s R^s dx = \sum \rho_j. \tag{6.13}$$

To estimate the left-hand side here we write

$$T = \sum c_j e^{in_j x}, \quad U = \sum \delta_\nu e^{i\nu x}, \quad \mu = \sum \rho_j,$$

where  $n_{-j} = -n_j$ . Since the only overlapping terms in different  $R^s$  are the constant ones, we have  $\delta_0 = r$ ,  $|\delta_\nu| \leq \frac{1}{2}$  for  $\nu \neq 0$ . Hence

$$|c_j| = \frac{1}{2} \rho_{|j|}, \quad |\delta_\nu| \leq r \quad \text{for all } j \text{ and } \nu. \quad (6.14)$$

$$\text{Let} \quad TU = \sum C_m e^{imx}, \quad \text{where} \quad C_m = \sum_{n_j - \nu = m} c_j \bar{\delta}_\nu, \quad (6.15)$$

and let  $N = [\frac{1}{2}n_1]$ . We observe that

- (i)  $C_0 = \frac{1}{2}\mu$  (since the left-hand side of (6.13) is  $2C_0$ );
- (ii)  $|C_m| \leq r \sum |c_j| = r\mu$  (by (6.15) and (6.14));
- (iii)  $C_m = 0$  for  $0 < |m| < N$  (since  $\delta_\nu = 0$  in  $R^s$  if  $\nu$  differs from any  $n_j$  by less than  $N$ ).

Let now  $\lambda(x)$  be the periodic and continuous function equal to 0 outside  $I$ , equal to 1 at the midpoint of  $I$ , and linear in each half of  $I$ . The series  $S[\lambda] = \sum \lambda_\nu e^{i\nu x}$  converges absolutely. The actual values of the  $\lambda_\nu$  (see Chapter I, (4.16)) are not needed. Consider the formula

$$\frac{1}{2\pi} \int_0^{2\pi} \lambda T U dx = \sum \lambda_m \bar{C}_m.$$

On the one hand, by (i), (ii) and (iii), the sum here exceeds

$$\lambda_0 C_0 - \sum_{|m| \geq N} |\lambda_m C_m| \geq \frac{1}{2}\mu(\lambda_0 - 2r \sum_{|m| \geq N} |\lambda_m|); \quad (6.16)$$

on the other hand, the integral is

$$\frac{1}{2\pi} \int_I \lambda T U dx \leq \frac{M}{2\pi} \int_I \lambda U dx \leq \frac{M}{2\pi} \int_0^{2\pi} U dx = Mr. \quad (6.17)$$

If  $N = [\frac{1}{2}n_1]$  is so large that the expression in brackets on the right of (6.16) exceeds  $\frac{1}{4}\lambda_0$ , a comparison of (6.17) and (6.16) shows that  $\frac{1}{4}\lambda_0\mu \leq Mr$ , so that  $\mu \leq AM$  where  $A = 4r/\lambda_0$ , and (6.11) is proved.

*Remarks.* (a) Theorem (6.3) holds if instead of the partial sums  $s_m$  of (6.2) we consider linear means. It is easier to deal with means applicable to the terms rather than to the partial sums of series (see p. 84).

Suppose we have a matrix  $\{\alpha_{mn}\}_{m, n=0, 1, \dots}$  and a series  $u_0 + u_1 + \dots$ . Consider the expression

$$\sigma_m = \sum_n \alpha_{mn} u_n.$$

The only thing we assume about the  $\alpha_{mn}$  is that  $\lim_m \alpha_{mn} = 1$  for each  $n$ . Suppose now that the means

$$\sigma_m(x) = \sum \alpha_{mn} \rho_j \cos(n_j x + x_j) \quad (6.18)$$

of (6.2) satisfy the following conditions for  $x \in (\alpha, \beta)$ :

- (i) they exist (i.e. the series (6.18) converge);
- (ii) they are continuous;
- (iii)  $\limsup_m \sigma_m(x) < +\infty$ .

Then, as before, there exists a subinterval  $I$  of  $(\alpha, \beta)$  in which the  $\sigma_m$  are uniformly bounded above, say by  $M$ . Observing that (6.11) holds not only for polynomials but also for infinite series (as is seen by first considering the (C, 1) means of  $T$  and then making a passage to the limit), we have

$$\sum_j |\alpha_{mn_j}| \rho_j \leq AM.$$

If we retain only a fixed finite number of terms on the left and then make  $m \rightarrow \infty$ , we see that each partial sum of  $\Sigma \rho_j$  is  $\leq AM$ , and so also  $\Sigma \rho_j \leq AM$ .

Condition (ii) is certainly satisfied if the matrix is row-finite. It is also satisfied in some other cases, for example for summability A.

(b) The conclusion of (6.3) holds if instead of (6.4) we suppose that for each  $x \in (\alpha, \beta)$  we have at least one of the inequalities

$$\limsup s_m(x) < +\infty, \quad \liminf s_m(x) > -\infty. \quad (6.19)$$

It is enough to show that the new hypothesis implies that at least one of the inequalities (6.19) is satisfied in a subinterval of  $(\alpha, \beta)$ . For suppose this is not the case, and let  $E^+$  and  $E^-$  denote respectively the subsets of  $(\alpha, \beta)$  in which  $\limsup s_m = +\infty$  and  $\liminf s_m = -\infty$ . By Chapter I. (12.2), the set  $E^+$  (being a set of points at which the sequence of continuous functions

$$s_m^+(x) = \max\{0, s_m(x)\}$$

is unbounded) is the complement of a set of the first category in  $(\alpha, \beta)$ . Similarly  $E^-$  is the complement of a set of the first category in  $(\alpha, \beta)$ . It follows that  $E^+E^-$  is the complement of a set of the first category in  $(\alpha, \beta)$  and, in particular, is not empty, contrary to hypothesis.

The conclusion may be stated slightly differently: if  $\Sigma \rho_j = +\infty$ , then in a set of points which is dense and of the second category in every interval we have simultaneously

$$\liminf s_m(x) = -\infty, \quad \limsup s_m(x) = +\infty. \quad (6.20)$$

There is a corresponding result for the linear means discussed in (a).

(c) If  $q$  is large, Theorem (6.3) can be obtained by the following simple geometrical argument.

Let  $I$  be a subinterval of  $(\alpha, \beta)$  in which  $s_m(x) \leq M$  for all  $m$ . Consider the curves  $y = \cos(n_j x + x_j)$  for  $j = 1, 2, \dots$ , and denote, generally, by  $I_j$  any of the intervals in which  $\cos(n_j x + x_j) \geq \frac{1}{2}$ . Assuming, as we may, that  $n_1$  is large enough, we select an interval  $I_1$  totally included in  $I$ . Since  $n_2/n_1$  is large we can find an interval  $I_2$  totally included in the  $I_1$  just considered, and so on. Let  $x^*$  be the point common to  $I, I_1, I_2, \dots$ . Since

$$\sum_{j=1}^m \rho_j \cos(n_j x^* + x_j) \leq M$$

for all  $m$  and since  $\cos(n_j x^* + x_j) \geq \frac{1}{2}$ , it follows that  $\Sigma \rho_j$  converges.

This argument is valid for  $q \geq 4$ . For let  $d_j$  and  $\delta_j$  denote respectively the length of  $I_j$  and the distance between two consecutive  $I_j$ . We can find an  $I_{j+1} \subset I_j$  if

$$2d_{j+1} + \delta_{j+1} \leq d_j. \quad (6.21)$$

Observing that  $d_j = 2\pi/3n_j$ ,  $\delta_j = 4\pi/3n_j$ , we find that (6.21) is equivalent to  $n_{j+1}/n_j \geq 4$ .

By considering the inequalities  $\cos(n_j x + x_j) \geq \epsilon > 0$ , where  $\epsilon$  is arbitrarily small but fixed, we can extend the preceding argument to the case  $q > 3$ . The argument does not work, however, for general  $q > 1$ .

## MISCELLANEOUS THEOREMS AND EXAMPLES

1. The set of points where  $\Sigma n^{-1} \sin n!x$  converges absolutely contains a perfect subset. [Consider the graphs of the curves  $y = \sin n!x$ .]

2. (i) Every measurable set of positive measure is a basis. (Steinhaus [5].)

(ii) Every set of the second category is a basis. (Niemytzki [1].)

[Suppose  $|E| > 0$ ,  $x \in E$ ,  $y \in E$ . To prove (i), it is enough to show that the set of differences  $x - y$  contains an interval. Let  $E_h$  denote the set  $E$  translated by  $h$ . Considering the neighbourhood of a point of density of  $E$ , we see without difficulty that  $EE_h \neq 0$  for all small enough  $h$ . The proof of (ii) is similar.]

3. Let  $C$  be Cantor's ternary set constructed on  $(0, 2\pi)$ . Every  $x \in C$  can be written in the form  $2\pi(\alpha_1 3^{-1} + \alpha_2 3^{-2} + \dots)$ , where  $\alpha_i$  is either 0 or 2. Using this, show that every  $z \in (0, 4\pi)$  can be written in the form  $x + y$  with  $x \in C$ ,  $y \in C$  (which shows that  $C$  is a basis).

4. Let  $A$  be the set of points of absolute convergence of  $\sum A_n(x)$ . Then  $A$  is invariant under

- (i) the translation, (ii) the symmetry,

making any two points  $a, b$  of  $A$  correspond. (Arbault [1].)

[If [1.7] is written  $\sum \rho_n \sin(nx + x_n)$ , and if  $a, b, c \in A$ , an argument similar to the proof of (1.7) shows that the series converges absolutely at the points (i)  $a - b + c$ , (ii)  $a + b - c$ .]

5. If a trigonometric series converges absolutely on a perfect set  $E$  of Cantor type (Chapter V, § 3), then the series converges uniformly on  $E$ . (Arbault [1], Malliavin [1].)

[A consequence of Chapter I, (12.3) (ii), of the homogeneous character of  $E$ , and of the proof of (1.10).]

6. A necessary and sufficient condition for  $S[h]$  to converge absolutely is that  $h$  be the convolution (Chapter II, § 1) of two functions  $f$  and  $g$  of the class  $L^2$ . (M. Riesz; see Hardy and Littlewood [12].)

[The sufficiency of the condition follows from Chapter II, (1.7); if  $\sum c_n e^{inx}$  converges absolutely, consider the functions with Fourier coefficients  $|c_n|^{\frac{1}{2}}$  and  $|c_n|^{\frac{1}{2}} \text{sign } c_n$ .]

7. The hypotheses of theorems (3.1) and (3.10) are unnecessarily stringent. The conclusions hold if the condition  $f \in \Lambda_\alpha$  is replaced by  $f \in \Lambda_\alpha^*$ .

8. Let  $0 < \alpha \leq 1$ ,  $1 \leq p \leq 2$ . If  $a_n, b_n$  are the coefficients of an  $f \in \Lambda_\alpha^*$ , then  $\sum (|a_n|^\beta + |b_n|^\beta)$  converges if  $\beta > p/(p(1+\alpha) - 1)$ . (Szász [2].)

[The proof is similar to that of (3.10) if instead of Parseval's formula we use the inequality of Hausdorff-Young, which will be established in Chapter XII, § 2.]

9. (i) If  $f \in \Lambda_\alpha$ ,  $0 < \alpha \leq 1$ , then  $\sum n^{\beta-1} (|a_n| + |b_n|)$  converges for  $\beta < \alpha$ . (Weyl [1], Hardy [1].)

(ii) If  $f$  is, in addition, of bounded variation, then  $\sum n^{\beta/\alpha} (|a_n| + |b_n|) < \infty$ .

(iii) If  $f \in \Lambda_\alpha^*$  for  $0 < \alpha \leq 1$ ,  $1 \leq p \leq 2$ , then  $\sum n^\gamma (|a_n| + |b_n|) < \infty$  for  $\gamma < \alpha - 1/p$ .

10. Let  $F(x)$ ,  $0 \leq x \leq 2\pi$ , be a non-decreasing function of jumps; that is, the increment of  $F$  over any interval is equal to the sum of all the jumps of  $F$  in that interval. Let  $d_1, d_2, \dots$  be all the jumps of  $F$ . Show that

$$\limsup_{n \rightarrow \infty} \int_0^{2\pi} \cos 2nx dF(x) = \sum d_i.$$

[Compare (2.16).]

11. If a lacunary series  $\sum (a_k \cos n_k x + b_k \sin n_k x)$ ,  $n_{k+1}/n_k > q > 1$ , is absolutely summable  $A$  (see p. 83) in a set  $E$  of positive measure, then  $\sum (|a_k| + |b_k|)$  converges.

[Supposing for simplicity that the series is purely cosine, write  $f(r, x) = \sum a_k r^{n_k} \cos n_k x$ . By reducing  $E$  we may suppose that  $\int_0^1 \left| \frac{\partial}{\partial r} f(r, x) \right| dr$  is uniformly bounded for  $x \in E$ . Integrating the integral with respect to  $x$ , changing the order of integration, and applying Chapter V, (6.5), we deduce that  $\int_0^1 (\sum a_k^2 n_k^2 r^{2n_k})^{\frac{1}{2}} dr$  is finite. If we consider the integral over  $1 - q/n_k \leq r \leq 1 - 1/n_k$  and in the sum  $\sum$  keep only the  $k$ th term we deduce that  $\sum |a_k|$  converges.]