4

Vector-Valued Functions

... who by vigor of mind almost divine, the motions and figures of the planets, the paths of comets, and the tides of the seas first demonstrated.

Newton's Epitaph

Chapters 2 and 3 focused on *real*-valued functions. This chapter is largely concerned with *vector*-valued functions. We begin in the first section with a continuation of our study of paths, adding applications of Newton's second law. Then we study arc length of paths. Following this, we introduce the divergence and curl of a vector field which, in addition to the gradient, are basic operations in vector *differential* calculus. The basic geometry and calculus of the divergence and curl are studied. The associated *integral* calculus will be given in Chapter 8.

4.1 Acceleration and Newton's Second Law

In Section 2.4, we studied the basic geometry of paths, learning how to sketch curves (the images of paths) and compute tangent lines. We also learned to think of, as the name suggests, a path as the trajectory of a particle and to regard the derivative of the path as its velocity vector. In this section, we continue our study of paths, including additional topics, especially acceleration and Newton's second law.

Differentiation of Paths

Recall that a path in $\mathbb{R}^n$ is a map $\mathbf{c}$ of $\mathbb{R}$ or an interval in $\mathbb{R}$ to $\mathbb{R}^n$. If the path is differentiable, its derivative at each time t is an $n \times 1$ matrix. Specifically, if $x_1(t), \dots, x_n(t)$ are the component functions of $\mathbf{c}$, the derivative matrix is

$$\mathbf{c}'(t) = \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix},$$

which can also be written in vector form as

$$(dx_1/dt, \ldots, dx_n/dt) \qquad \text{or as} \qquad (x_1'(t), \ldots, x_n'(t)).$$

Recall from Section 2.4 that $\mathbf{c}'(t)$ is the *tangent vector* to the path at the point $\mathbf{c}(t)$. Also recall that if $\mathbf{c}$ represents the path of a moving particle, then its *velocity vector* is

$$\mathbf{v} = \mathbf{c}'(t),$$

and its *speed* is $s = \|\mathbf{v}\|$.

The differentiation of paths is facilitated by the following rules.

Differentiation Rules Let $\mathbf{b}(t)$ and $\mathbf{c}(t)$ be differentiable paths in $\mathbb{R}^3$ and $p(t)$ and $q(t)$ be differentiable scalar functions:

Sum Rule: $$\frac{d}{dt}[\mathbf{b}(t) + \mathbf{c}(t)] = \mathbf{b}'(t) + \mathbf{c}'(t)$$

Scalar Multiplication Rule: $$\frac{d}{dt}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$$

Dot Product Rule: $$\frac{d}{dt}[\mathbf{b}(t) \cdot \mathbf{c}(t)] = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$$

Cross Product Rule: $$\frac{d}{dt}[\mathbf{b}(t) \times \mathbf{c}(t)] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t)$$

Chain Rule: $$\frac{d}{dt}[\mathbf{c}(q(t))] = q'(t)\mathbf{c}'(q(t)).$$

These rules follow by applying the usual differentiation rules to the components.

EXAMPLE 1 Show that if $\mathbf{c}(t)$ is a vector function such that $\|\mathbf{c}(t)\|$ is constant, then $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$ for all t.

SOLUTION Because $\|\mathbf{c}(t)\|$ is constant, so is its square $\|\mathbf{c}(t)\|^2 = \mathbf{c}(t) \cdot \mathbf{c}(t)$. The derivative of this constant is zero, so by the dot product rule,

$$0 = \frac{d}{dt}[\mathbf{c}(t) \cdot \mathbf{c}(t)] = \mathbf{c}'(t) \cdot \mathbf{c}(t) + \mathbf{c}(t) \cdot \mathbf{c}'(t) = 2\mathbf{c}(t) \cdot \mathbf{c}'(t);$$

thus, $\mathbf{c}(t) \cdot \mathbf{c}'(t) = 0$; that is, $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$. ▲

For a path describing uniform rectilinear motion, the velocity vector is constant. In general, the velocity vector is a vector function $\mathbf{v} = \mathbf{c}'(t)$ that depends on t. The derivative $\mathbf{a} = d\mathbf{v}/dt = \mathbf{c}''(t)$ is called the ***acceleration*** of the curve. If the curve is

$(x(t), y(t), z(t))$, then the acceleration at time t is given by

$$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}.$$

EXAMPLE 2 A particle moves in such a way that its acceleration is constantly equal to $-\mathbf{k}$. If the position when $t = 0$ is $(0, 0, 1)$ and the velocity at $t = 0$ is $\mathbf{i} + \mathbf{j}$, when and where does the particle fall below the plane $z = 0$? Describe the path traveled by the particle (assume $t \geq 0$).

SOLUTION Let $(x(t), y(t), z(t))$ be the path traced out by the particle, so that the velocity vector is $\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. The acceleration $\mathbf{c}''(t)$ is $-\mathbf{k}$, so $x''(t) = 0$, $y''(t) = 0$, and $z''(t) = -1$. It follows that $x'(t)$ and $y'(t)$ are constant functions, and $z'(t)$ is a linear function with slope -1. Because $\mathbf{c}'(0) = \mathbf{i} + \mathbf{j}$, we get $\mathbf{c}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$. Integrating again and using the initial position $(0, 0, 1)$, we find that $(x(t), y(t), z(t)) = (t, t, 1 - \frac{1}{2}t^2)$. The particle drops below the plane $z = 0$ when $1 - \frac{1}{2}t^2 = 0$; that is, $t = \sqrt{2}$ (because $t \geq 0$). At that instant, the position is $(\sqrt{2}, \sqrt{2}, 0)$. The path traveled by the particle is a parabola in the plane $y = x$ (see Figure 4.1.1), because in this plane the equation is described by $z = 1 - \frac{1}{2}x^2$. ▲

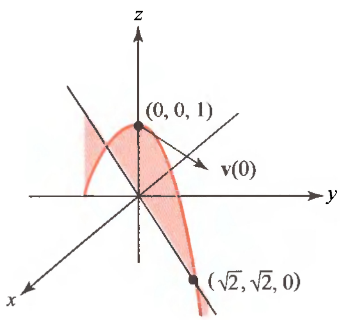

Figure 4.1.1 The path of the particle with initial position $(0, 0, 1)$, initial velocity $\mathbf{i} + \mathbf{j}$, and constant acceleration $-\mathbf{k}$ is a parabola in the plane $y = x$.

The image of a C^1 path is not necessarily "very smooth"; indeed, it may have sharp bends or changes of direction. For instance, the cycloid $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ shown in Figure 2.4.6 has cusps at all points where $\mathbf{c}$ touches the x axis (that is, when $1 - \cos t = 0$, which happens when $t = 2\pi n$, $n = 0, \pm 1, \ldots$). Another example is the ***hypocycloid of four cusps***, $\mathbf{c}\colon [0, 2\pi] \to \mathbb{R}^2$, $t \mapsto (\cos^3 t, \sin^3 t)$, which has cusps at four points (Figure 4.1.2). At all such points, however, $\mathbf{c}'(t) = \mathbf{0}$, and the tangent line is not well defined. Evidently, the direction of $\mathbf{c}'(t)$ may change abruptly at points where it slows to rest.

A differentiable path $\mathbf{c}$ is said to be ***regular*** at $t = t_0$ if $\mathbf{c}'(t_0) \neq \mathbf{0}$. If $\mathbf{c}'(t) \neq \mathbf{0}$ for all t, we say that c is a regular path. In this case, the image curve looks smooth.

EXAMPLE 3 A particle moves along a hypocycloid according to the equations

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad a \leq t \leq b.$$

What are the velocity and speed of the particle?

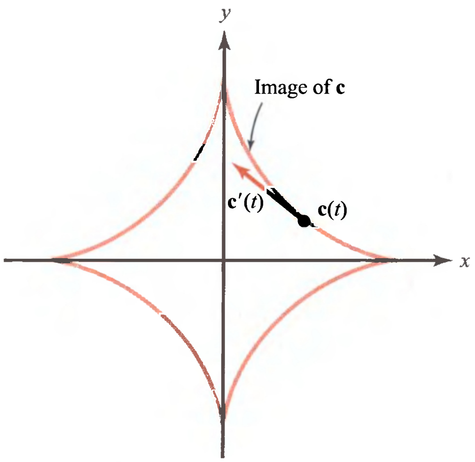

Figure 4.1.2 The image of the smooth path $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$, a hypocycloid, does not "look smooth."

SOLUTION The velocity vector of the particle is

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = -(3 \sin t \cos^2 t)\mathbf{i} + (3 \cos t \sin^2 t)\mathbf{j},$$

and its speed is

$$s = \|\mathbf{v}\| = (9 \sin^2 t \cos^4 t + 9 \cos^2 t \sin^4 t)^{1/2} = 3\,|\sin t|\ |\cos t|. \quad \blacktriangle$$

Newton's Second Law

If a particle of mass m moves in $\mathbb{R}^3$, the force $\mathbf{F}$ acting on it at the point $\mathbf{c}(t)$ is related to the acceleration $\mathbf{a}(t)$ by ***Newton's second law***:[1]

$$\mathbf{F}(\mathbf{c}(t)) = m\mathbf{a}(t).$$

In particular, if no forces act on a particle, then $\mathbf{a}(t) = \mathbf{0}$, so $\mathbf{c}'(t)$ is constant and the particle follows a straight line.

Acceleration and Newton's Second Law The ***acceleration*** of a path $\mathbf{c}(t)$ is

$$\mathbf{a}(t) = \mathbf{c}''(t).$$

If $\mathbf{F}$ is the force acting and m is the mass of the particle, then

$$\mathbf{F} = m\mathbf{a}.$$

[1] Most scientists acknowledge that $\mathbf{F} = m\mathbf{a}$ is the single most important equation in all of science and engineering.

In the problem of determining the path $\mathbf{c}(t)$ of a particle under the influence of a given force field, $\mathbf{F}$, Newton's law becomes a differential equation (i.e., an equation involving derivatives) for $\mathbf{c}(t)$.

For example, the motion of a planet moving along a path $\mathbf{r}(t)$ around the sun (considered to be located at the origin in $\mathbb{R}^3$) obeys the law

$$m\mathbf{r}'' = -\frac{GmM}{r^3}\mathbf{r},$$

where M is the mass of the sun, m that of the planet, $r = \|\mathbf{r}\|$, and G is the gravitational constant. The relation used in determining the force, $\mathbf{F} = -GmM\mathbf{r}/r^3$, is called ***Newton's law of gravitation*** (see Figure 4.1.3). We shall not make a general study of such equations in this book, but content ourselves with the special case of circular orbits. (More general orbits—the conic sections—are discussed in the Internet supplement.)

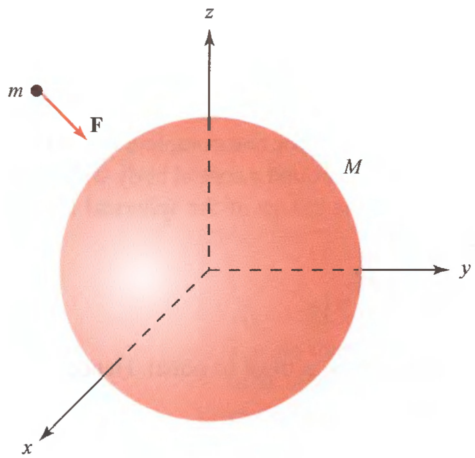

Figure 4.1.3 A mass M attracts a mass m with a force $\mathbf{F}$ given by Newton's law of gravitation: $\mathbf{F} = -GmM\mathbf{r}/r^3$.

Circular Orbits

Consider a particle of mass m moving at constant speed s in a circular path of radius r_0. Supposing that it moves in the xy plane, we can suppress the third component and write its location as

$$\mathbf{r}(t) = \left(r_0 \cos\frac{st}{r_0}, r_0 \sin\frac{st}{r_0}\right).$$

Note that this is a circle of radius r_0 and that its speed is given by $\|\mathbf{r}'(t)\| = s$. The quantity s/r_0 is called the ***frequency*** and is denoted ω. Thus,

$$\mathbf{r}(t) = (r_0 \cos \omega t,\ r_0 \sin \omega t).$$

The acceleration is given by

$$\mathbf{a}(t) = \mathbf{r}''(t) = \left(-\frac{s^2}{r_0}\cos\frac{st}{r_0},\ -\frac{s^2}{r_0}\sin\frac{st}{r_0}\right) = -\frac{s^2}{r_0^2}\mathbf{r}(t) = -\omega^2\mathbf{r}(t).$$

Thus, the acceleration is in a direction opposite to $\mathbf{r}(t)$; that is, it is directed toward the center of the circle (see Figure 4.1.4). This acceleration multiplied by the mass of the particle is called the ***centripetal force***. Even though the speed is constant, the direction of the velocity is continuously changing and therefore the acceleration, which is a rate of change in either speed or direction or both, is nonzero.

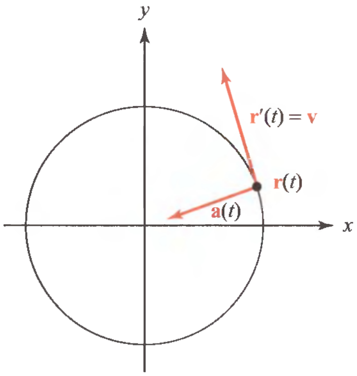

Figure 4.1.4 The position, velocity, and acceleration of a particle in circular motion.

Newton's law helps us discover a relationship between the radius of the orbit of a revolving body and the period, that is, the time it takes for one complete revolution. Consider a satellite of mass m moving with a speed s around a central body with mass M in a *circular* orbit of radius r_0 (distance from the *center* of the spherical central body). By Newton's second law $\mathbf{F} = m\mathbf{a}$, we get

$$-\frac{s^2 m}{r_0^2}\mathbf{r}(t) = -\frac{GmM}{r_0^3}\mathbf{r}(t).$$

The lengths of the vectors on both sides of this equation must be equal. Hence,

$$s^2 = \frac{GM}{r_0}.$$

If T denotes the period, then $s = 2\pi r_0/T$; substituting this value for s in the preceding equation and solving for T, we obtain the following:

Kepler's Law

$$T^2 = r_0^3\frac{(2\pi)^2}{GM}.$$

Thus, the *square of the period is proportional to the cube of the radius.*

We have defined two basic concepts associated with a path; its velocity and its acceleration. Both involve *differential* calculus. The basic concept of the length of a path, which involves *integral* calculus, will be taken up in the next section.

EXAMPLE 4 Suppose that a satellite is to be in a circular orbit about the earth such that it stays fixed in the sky over one point on the equator. What is the radius of such a *geosynchronous* orbit? (The mass of the earth is 5.98×10^{24} kilograms and $G = 6.67 \times 10^{-11}$ in the meter-kilogram-second system of units.)

SOLUTION The period of the satellite should be 1 day, so $T = 60 \times 60 \times 24 = 86{,}400$ seconds. From the formula $T^2 = r_0^3(2\pi)^2/GM$, we get $r_0^3 = T^2GM/(2\pi)^2$, and so

$$r_0^3 = \frac{T^2GM}{(2\pi)^2} = \frac{(86{,}400)^2 \times (6.67 \times 10^{-11}) \times (5.98 \times 10^{24})}{(2\pi)^2} \approx 7.54 \times 10^{22}\,\mathrm{m}^3.$$

Thus, $r_0 = 4.23 \times 10^7\,\mathrm{m} = 42{,}300\,\mathrm{km} \approx 26{,}200\,\mathrm{mi}$. ▲

Supplement to Section 4.1: Planetary Orbits, Hamilton's Principle, and Spacecraft Trajectories

In this section, we have been studying paths in space and Newton's second law. Hopefully, the student realizes that these ideas apply to the real world—the motion of our earth around the sun, for example, is governed by these laws. But there is more to the story, and we will try to convey some of it here.

Historical Note

Kepler, Newton, and Hamilton

As we discussed in the historical introduction, the law of planetary motion stating that the square of the period is proportional to the cube of the radius of an orbit is one of the three that Kepler observed before Newton formulated his laws of motion, known more generally as Newton's mechanics. These mechanics enable one to compute the period of a satellite about the earth or a planet about the sun (when the radius of its orbit is given), and, as we will indicate shortly, trajectories of space missions.

Kepler discovered and used results like this not only for circular orbits but more generally for elliptical orbits. Newton was able to derive Kepler's three celestial laws from his own law of gravitation. The neat mathematical order of the universe that these laws provided had a great impact on eighteenth-century thought.

Newton never wrote down his laws of mechanics as differential equations. This was first done by Euler around 1730. Newton made most of his deductions (at least those in published form) by geometric methods. Euler also showed how Newton's equations followed from Maupertuis's action principle. The clearest version of the action principle in mechanics, now known as ***Hamilton's principle***, is due to William Rowan Hamilton around 1830, who, as we all should now know, happens to also be the father of vector calculus. Hamilton's version of Maupertuis's principle was elegantly presented by Richard Feynman, as we discuss next.

Feynman and Hamilton's Principle

In his legendary Caltech *Lectures on Physics*, Nobel Prize–winning physicist Richard Phillips Feynman (see Figure 4.1.5) included what he called a "Special Lecture" on a topic clearly very close to his heart—one that he first heard about from his New York high school teacher, Mr. Bader. Mr. Bader told his (apparently bored) student Feynman how principles of maxima and minima apply to the trajectories of moving objects and in particular how the action principle of Maupertuis, Leibniz, and Hamilton (discussed in Section 3.3) applies to Newton's mechanics, governed by $\mathbf{F} = m\mathbf{a}$.

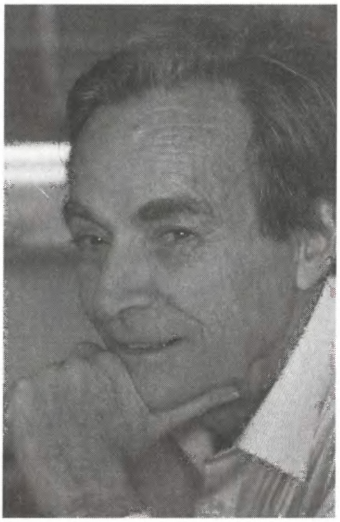

Figure 4.1.5 Richard P. Feynman (1918–1988).

Professor Feynman, at the end of his lecture, notes that "a physicist, a student of Mr. Bader, in 1942 showed how this action principle applied to quantum mechanics." That student was Feynman himself, who received the Nobel Prize for his insights, which also included the discovery of *Feynman integrals*. The moral here is *pay attention to your teachers—especially the best ones*!

We include the first part of Feynman's lecture here and more of it in the Internet supplement; see Volume II, Lecture 19, of the *Feynman Lectures on Physics* for the entire lecture.

The Principle of Least Action, by Richard Feynman

When I was in high school, my physics teacher—whose name was Mr. Bader—called me down one day after physics class and said, "You look bored; I want to tell you something interesting." Then he told me

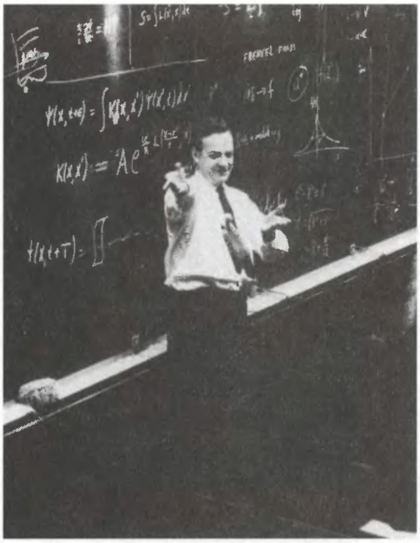

Figure 4.1.6 Feynman lecturing at Caltech.

something which I found absolutely fascinating, and have, since then, always found fascinating. Every time the subject comes up, I work on it. In fact, when I began to prepare this lecture I found myself making more analyses on the thing. Instead of worrying about the lecture, I got involved in a new problem. The subject is this—the principle of least action.

Mr. Bader told me the following: Suppose you have a particle (in a gravitational field, for instance) which starts somewhere and moves to some other point by free motion—you throw it, and it goes up and comes down [see Figure 4.1.7].

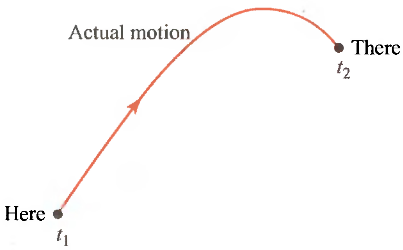

Figure 4.1.7

It goes from the original place to the final place in a certain amount of time. Now, you try a different motion. Suppose that to get from here to there, it went like this [see Figure 4.1.8], but got there in just the same amount of time.

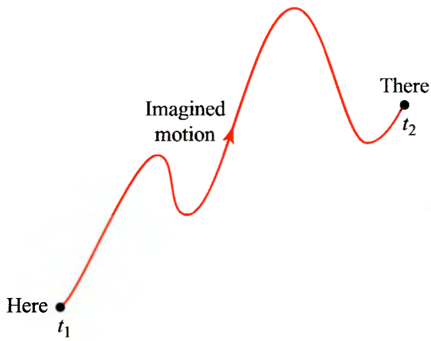

Figure 4.1.8

Then he said this: "If you calculate the kinetic energy at every moment on the path, take away the potential energy, and integrate it over the time during the whole path, you'll find that the number you'll get is bigger than that for the actual motion."

In other words, the laws of Newton could be stated not in the form $F = ma$ but in the form: The average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another.

Let me illustrate a little better what this means. If you take the case of the gravitational field, then if the particle has the path $x(t)$ (let's just take one dimension for a moment; we take a trajectory that goes up and down and not sideways), where x is the height above the ground, the kinetic energy is $\frac{1}{2}m(dx/dt)^2$, and the potential energy at any time is mgx. Now I take the kinetic energy minus the potential energy at every moment along the path and integrate that with respect to time from the initial time to the final time. Let's suppose that at the original time t_1 we started at some height and at the end of the time t_2 we are definitely ending at some other place [see Figure 4.1.9].

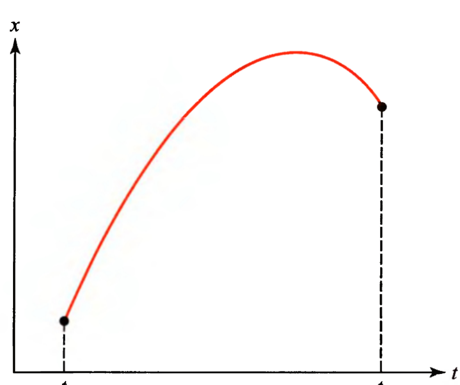

Figure 4.1.9

Then the integral is

$$\int_{t_1}^{t_2} \left[\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - mgx \right] dt.$$

The actual motion is some kind of curve—it's a parabola if we plot against the time—and gives a certain value for the integral. But we could *imagine* some other motion that went very high and came up and down in some peculiar way [see Figure 4.1.10].

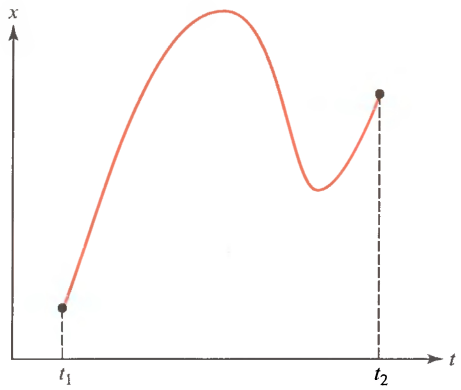

Figure 4.1.10

We can calculate the kinetic energy minus the potential energy and integrate for such a path . . . or for any other path we want. The miracle is that the true path is the one for which that *integral is least*.

Real-Life Trajectories

Interesting paths in $\mathbb{R}^3$ that obey Newton's second law occur in our own solar system and are used by NASA to plan space missions. One such mission, the *Genesis* Discovery Mission, launched from earth August 8, 2001 (and is due to return to earth in September 2004), has a particularly interesting trajectory, as shown in Figure 4.1.11. More information about this trajectory and the mission objectives can be found at http://genesismission.jpl.nasa.gov/.

The points denoted L_1 and L_2 in this figure denote places of balance (discovered by Euler) between the earth and the sun. A motionless spacecraft positioned there will remain there. There are periodic orbits about these points that we have (loosely)

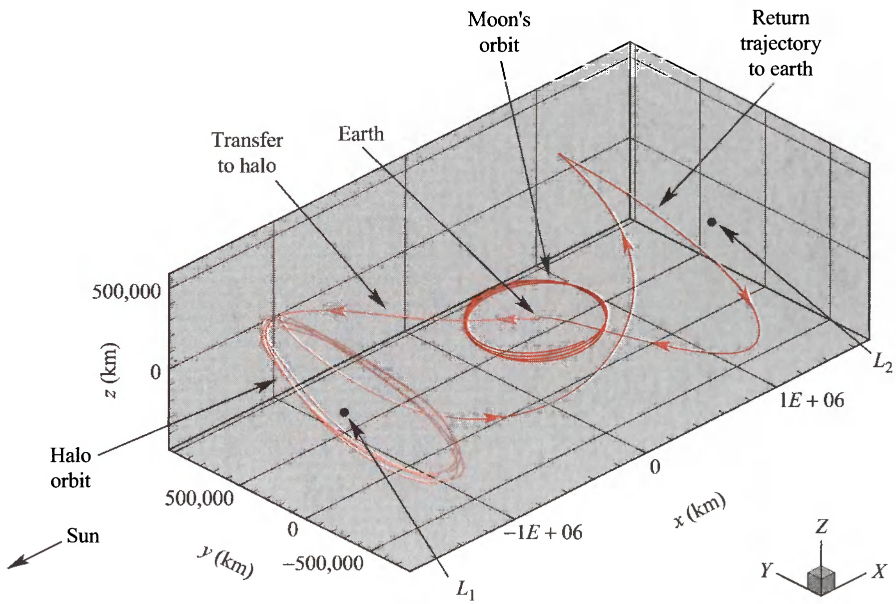

Figure 4.1.11 Trajectory of the *Genesis* spacecraft from the earth to a periodic orbit about a million and a half kilometers from earth and the interesting return trajectory to earth.

called *halo orbits*. The main dynamics of the spacecraft is governed by the pull of both the earth and the sun (and to a very small extent the moon) on the spacecraft. This is thus part of the famous *three-body problem* studied and made famous by Poincaré around 1890.[2]

Emmy Noether and Hamilton's Principle

Emmy Noether (1882–1935) (see Figure 4.1.12) is perhaps best known for her work in algebra, but she made a significant contribution to Hamilton's principle as well.[3] For planetary motion, the angular momentum vector $\mathbf{J} = \mathbf{r}(t) \times m\dot{\mathbf{r}}(t)$ is time-independent (so is a *conserved quantity*), as one can readily see by computing the time derivative of $\mathbf{J}$ and using $\mathbf{F} = m\mathbf{a}$ (see Exercise 20). What Noether discovered was a deep connection between such conserved quantities and symmetries in Hamilton's principle—in the case of angular momentum, this is rotational symmetry. Noether's discoveries have had a profound influence on the study of mechanical systems, from classical to quantum, ever since.

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[2]For more information about Poincaré, see F. Diacu and P. Holmes, *Celestial Encounters. The Origins of Chaos and Stability*, Princeton University Press: Princeton, NJ, 1996.

[3]"Invariante Variationsprobleme," *Göttingen Math. Phys.* **2** (1918): 235–257.

Figure 4.1.12 Emmy Noether (1882–1935).

EXERCISES

In Exercises 1 to 4, find the velocity and acceleration vectors and the equation of the tangent line for each of the following curves, at the given value of t.

1. $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin 2t)\mathbf{j}$, at $t = 0$
2. $\mathbf{c}(t) = (t\ \sin t, t\ \cos t, \sqrt{3}t)$, at $t = 0$
3. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, at $t = 0$
4. $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k}$, at $t = 9$

In Exercises 5 to 8, let $\mathbf{c}_1(t) = e^t\mathbf{i} + (\sin t)\mathbf{j} + t^3\mathbf{k}$ *and* $\mathbf{c}_2(t) = e^{-t}\mathbf{i} + (\cos t)\mathbf{j} - 2t^3\mathbf{k}$. *Find each of the stated derivatives in two different ways to verify the rules in the box preceding Example 1.*

5. $\dfrac{d}{dt}[\mathbf{c}_1(t) + \mathbf{c}_2(t)]$

6. $\dfrac{d}{dt}[\mathbf{c}_1(t) \cdot \mathbf{c}_2(t)]$

7. $\dfrac{d}{dt}[\mathbf{c}_1(t) \times \mathbf{c}_2(t)]$

8. $\dfrac{d}{dt}\{\mathbf{c}_1(t) \cdot [2\mathbf{c}_2(t) + \mathbf{c}_1(t)]\}$

9. If $\mathbf{r}(t) = 6t\mathbf{i} + 3t^2\mathbf{j} + t^3\mathbf{k}$, what force acts on a particle of mass m moving along $\mathbf{r}$ at $t = 0$?

10. Let a particle of mass 1 gram (g) follow the path in Exercise 1, with units in seconds and centimeters. What force acts on it at $t = 0$? (Give the units in your answer.)

11. A body of mass 2 kilograms moves on a circle of radius 3 meters, making one revolution every 5 seconds. Find the centripetal force acting on the body.

12. Find the centripetal force acting on a body of mass 4 kilograms, moving on a circle of radius 10 meters with a frequency of 2 revolutions per second.

13. Show that if the acceleration of an object is always perpendicular to the velocity, then the speed of the object is constant. (HINT: See Example 1.)

14. Show that, at a local maximum or minimum of $\|\mathbf{r}(t)\|$, the vector $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$.

15. A satellite is in a circular orbit 500 miles above the surface of the earth. What is the period of the orbit? (You may take the radius of the earth to be 4000 miles, or 6.436×10^6 meters).

16. What is the acceleration of the satellite in Exercise 15? The centripetal force?

17. Find the path $\mathbf{c}$ such that $\mathbf{c}(0) = (0, -5, 1)$ and $\mathbf{c}'(t) = (t, e^t, t^2)$.

18. Let $\mathbf{c}$ be a path in $\mathbb{R}^3$ with zero acceleration. Prove that $\mathbf{c}$ is a straight line or a point.

19. Find paths $\mathbf{c}(t)$ that represent the following curves or trajectories.

(a) $\{(x, y) \mid y = e^x\}$
(b) $\{(x, y) \mid 4x^2 + y^2 = 1\}$
(c) A straight line in $\mathbb{R}^3$ passing through the origin and the point (a, b, c)
(d) $\{(x, y) \mid 9x^2 + 16y^2 = 4\}$

20. Let $\mathbf{c}(t)$ be a path, $\mathbf{v}(t)$ its velocity, and $\mathbf{a}(t)$ the acceleration. Suppose $\mathbf{F}$ is a C^1 mapping of $\mathbb{R}^3$ to $\mathbb{R}^3$, $m > 0$, and $\mathbf{F}(\mathbf{c}(t)) = m\mathbf{a}(t)$ (Newton's second law). Prove that

$$\frac{d}{dt}[m\mathbf{c}(t) \times \mathbf{v}(t)] = \mathbf{c}(t) \times \mathbf{F}(\mathbf{c}(t))$$

(i.e., "rate of change of angular momentum = torque"). What can you conclude if $\mathbf{F}(\mathbf{c}(t))$ is parallel to $\mathbf{c}(t)$? Is this the case in planetary motion?

21. Continue the investigations in Exercise 20 to prove Kepler's law that a planet moving under the influence of gravity about the sun does so in a fixed plane.

4.2 Arc Length

Definition of Arc Length

What is the length of a path $\mathbf{c}(t)$? Because the speed $\|\mathbf{c}'(t)\|$ is the rate of change of distance traveled with respect to time, the distance traveled by a point moving along the curve should be the integral of speed with respect to the time over the interval

$[t_0, t_1]$ of travel time; that is, the length of the path, also called its ***arc length***, is

$$L(\mathbf{c}) = \int_{t_0}^{t_1} \|\mathbf{c}'(t)\| dt.$$

There is the question as to whether or not this formula actually corresponds to the true arc length. For example, suppose we take a curve in space and glue a string tightly to it, cutting the string so it exactly fits the curve. If we then remove the string, straighten it out and measure it with a straight edge, we surely should obtain the length of the curve. That our formula for arc length agrees with such a process is justified in the supplement at the end of this section.

EXAMPLE 1 The arc length of the path $\mathbf{c}(t) = (r\cos t, r\sin t)$, for t lying in the interval $[0, 2\pi]$; that is, for $0 \le t \le 2\pi$, is

$$L(\mathbf{c}) = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt = 2\pi r,$$

which is the circumference of a circle of radius r. If we had allowed $0 \le t \le 4\pi$, we would have obtained $4\pi r$, because the path traverses the same circle *twice* (Figure 4.2.1). ▲

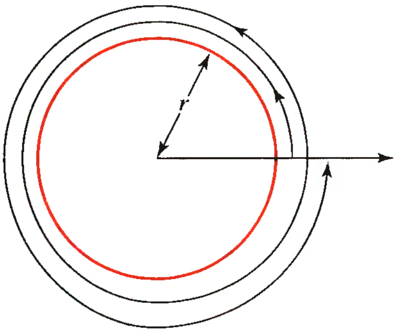

Figure 4.2.1 The arc length of a circle traversed twice is $4\pi r$.

> **Arc Length** The length of the path $\mathbf{c}(t) = (x(t), y(t), z(t))$ for $t_0 \le t \le t_1$, is
>
> $$L(\mathbf{c}) = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

For planar curves, one omits the $z'(t)$ term, as in Example 1. Here is an example in $\mathbb{R}^3$.

EXAMPLE 2 Find the arc length of $(\cos t, \sin t, t^2)$, $0 \le t \le \pi$.

SOLUTION The path $\mathbf{c}(t) = (\cos t, \sin t, t^2)$ has the velocity vector given by $\mathbf{v} = (-\sin t, \cos t, 2t)$. Because

$$\|\mathbf{v}\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2} = \sqrt{1 + 4t^2} = 2\sqrt{t^2 + \left(\frac{1}{2}\right)^2},$$

the arc length is

$$L(\mathbf{c}) = \int_0^{\pi} 2\sqrt{t^2 + \left(\frac{1}{2}\right)^2}\, dt.$$

This integral may be evaluated using the following formula from the table of integrals:

$$\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}[x\sqrt{x^2 + a^2} + a^2 \log(x + \sqrt{x^2 + a^2})] + C.$$

Thus,

$$L(\mathbf{c}) = 2 \cdot \frac{1}{2}\left[t\sqrt{t^2 + \left(\frac{1}{2}\right)^2} + \left(\frac{1}{2}\right)^2 \log\left(t + \sqrt{t^2 + \left(\frac{1}{2}\right)^2}\right)\right]\Bigg|_{t=0}^{\pi}$$

$$= \pi\sqrt{\pi^2 + \frac{1}{4}} + \frac{1}{4}\log\left(\pi + \sqrt{\pi^2 + \frac{1}{4}}\right) - \frac{1}{4}\log\left(\sqrt{\frac{1}{4}}\right)$$

$$= \frac{\pi}{2}\sqrt{1 + 4\pi^2} + \frac{1}{4}\log(2\pi + \sqrt{1 + 4\pi^2}) \approx 10.63.$$

As a check on our answer, we may note that the path $\mathbf{c}$ connects the points $(1, 0, 0)$ and $(-1, 0, \pi^2)$. The distance between these points is $\sqrt{4 + \pi^2} \approx 3.72$, which is less than 10.63, as it should be. ▲

If a curve is made up of a finite number of pieces each of which is C^1 (with bounded derivative), we compute the arc length by adding the lengths of the component pieces. Such curves are called ***piecewise*** C^1. Sometimes we just say "piecewise smooth."

EXAMPLE 3 A billiard ball on a pool table follows the path $\mathbf{c}\colon [-1, 1] \to \mathbb{R}^3$ defined by $\mathbf{c}(t) := (x(t), y(t), z(t)) = (|t|, |t - \frac{1}{2}|, 0)$. Find the distance traveled by the ball.

SOLUTION This path is not smooth, because $x(t) = |t|$ is not differentiable at 0, nor is $y(t) = |t - \frac{1}{2}|$ differentiable at $\frac{1}{2}$. However, if we divide the interval $[-1, 1]$ into the pieces $[-1, 0]$, $[0, \frac{1}{2}]$, and $[\frac{1}{2}, 1]$, we see that $x(t)$ and $y(t)$ have continuous derivatives on each of the intervals $[-1, 0]$, $[0, \frac{1}{2}]$, and $[\frac{1}{2}, 1]$. (See Figure 4.2.2.)

On $[-1, 0]$, $x(t) = -t$, $y(t) = -t + \frac{1}{2}$, and $z(t) = 0$, so $\|\mathbf{c}'(t)\| = \sqrt{2}$. Hence, the arc length of $\mathbf{c}$ between -1 and 0 is $\int_{-1}^{0} \sqrt{2}\, dt = \sqrt{2}$. Similarly, on $[0, \frac{1}{2}]$, $x(t) = t$, $y(t) = -t + \frac{1}{2}$, $z(t) = 0$, and again $\|\mathbf{c}'(t)\| = \sqrt{2}$, so that the arc length of $\mathbf{c}$ between 0

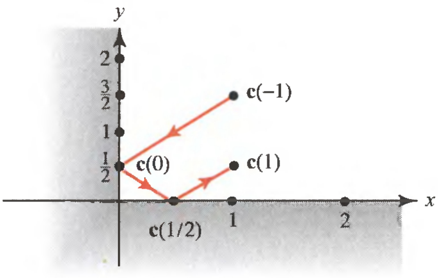

Figure 4.2.2 A piecewise smooth path.

and $\frac{1}{2}$ is $\frac{1}{2}\sqrt{2}$. Finally, on $[\frac{1}{2}, 1]$ we have $x(t) = t$, $y(t) = t - \frac{1}{2}$, $z(t) = 0$, and the arc length of $\mathbf{c}$ between $\frac{1}{2}$ and 1 is $\frac{1}{2}\sqrt{2}$. Thus, the total arc length of $\mathbf{c}$ is $2\sqrt{2}$. Of course, one can also compute the answer as the sum of the distances from $\mathbf{c}(-1)$ to $\mathbf{c}(0)$ to $\mathbf{c}(\frac{1}{2})$ to $\mathbf{c}(1)$. ▲

EXAMPLE 4 Consider the point with position function

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t),$$

which traces out the cycloid discussed in Section 2.4 (see Figure 2.4.6). Find the velocity, the speed, and the length of one arch.

SOLUTION The velocity vector is $\mathbf{c}'(t) = (1 - \cos t, \sin t)$, so the speed of the point $\mathbf{c}(t)$ is

$$\|\mathbf{c}'(t)\| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t}.$$

Hence, $\mathbf{c}(t)$ moves at variable speed although the circle rolls at constant speed. Furthermore, the speed of $\mathbf{c}(t)$ is zero when t is an integral multiple of 2π. At these values of t, the y coordinate of the point $\mathbf{c}(t)$ is zero and so the point lies on the x axis. The arc length of one cycle is

$$\begin{aligned} L(\mathbf{c}) &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = 2\int_0^{2\pi} \sqrt{\frac{1 - \cos t}{2}}\, dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\, dt \left(\text{because } 1 - \cos t = 2\sin^2\frac{t}{2} \text{ and } \sin\frac{t}{2} \geq 0 \text{ on } [0, 2\pi]\right) \\ &= 4\left(-\cos\frac{t}{2}\right)\Big|_0^{2\pi} = 8. \end{aligned}$$ ▲

The Differential of Arc Length

The arc-length formula suggests that one introduce the following notation, which will be useful in Chapter 7 in our discussion of line integrals.

Arc-Length Differential An ***infinitesimal displacement*** of a particle following a path $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right) dt,$$

and its length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

is the ***differential of arc length***. See Figure 4.2.3.

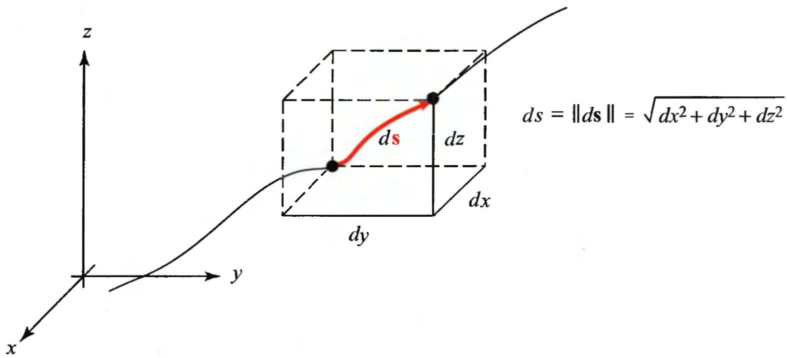

Figure 4.2.3 Differential of arc length.

These formulas help us remember the arc-length formula as

$$\text{arc length} = \int_{t_0}^{t_1} ds.$$

As we have done before with such geometric concepts as length and angle, we can extend the notion of arc length to paths in n-dimensional space.

Arc Length in $\mathbb{R}^n$ Let $\mathbf{c}: [t_0, t_1] \to \mathbb{R}^n$ be a piecewise C^1 path. Its ***length*** is defined to be

$$L(\mathbf{c}) = \int_{t_0}^{t_1} \|\mathbf{c}'(t)\|\, dt.$$

The integrand is the square root of the sum of the squares of the coordinate functions of $\mathbf{c}'(t)$: If

$$\mathbf{c}(t) = (x_1(t), x_2(t), \ldots, x_n(t)),$$

then

$$L(\mathbf{c}) = \int_{t_0}^{t_1} \sqrt{(x_1'(t))^2 + (x_2'(t))^2 + \cdots + (x_n'(t))^2}\, dt.$$

EXAMPLE 5 Find the length of the path $\mathbf{c}(t) = (\cos t, \sin t, \cos 2t, \sin 2t)$ in $\mathbb{R}^4$, defined on the interval from 0 to π.

SOLUTION We have $\mathbf{c}'(t) = (-\sin t, \cos t, -2 \sin 2t, 2 \cos 2t)$, and so

$$\|\mathbf{c}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4 \sin^2 2t + 4 \cos^2 2t} = \sqrt{1+4} = \sqrt{5},$$

a constant, so the length of the path is

$$\int_0^{\pi} \sqrt{5}\, dt = \sqrt{5}\pi. \quad \blacktriangle$$

It is common to introduce the ***arc-length function*** $s(t)$ associated to a path $\mathbf{c}(t)$ given by

$$s(t) = \int_a^t \|\mathbf{c}'(u)\|\, du,$$

so that (by the fundamental theorem of calculus)

$$s'(t) = \|\mathbf{c}'(t)\|$$

and

$$\int_a^b s'(t)\, dt = s(b) - s(a) = s(b).$$

EXAMPLE 6 Consider the graph of a function of one variable $y = f(x)$ for x in the interval $[a, b]$. We can consider it to be a curve parametrized by $t = x$, namely, $\mathbf{c}(x) = (x, f(x))$ for x ranging from a to b. The arc-length formula gives

$$L(\mathbf{c}) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx,$$

which agrees with the formula for the length of a graph from one-variable calculus. $\blacktriangle$

Justification for the Arc-Length Formula

The following discussion assumes an acquaintance with the definite integral defined in terms of Riemann sums. If your background in this topic needs reinforcement, the material may be postponed until after Chapter 5.

In $\mathbb{R}^3$ there is another way to justify the arc-length formula based on polygonal approximations. We partition the interval $[a, b]$ into N subintervals of equal length:

$$a = t_0 < t_1 < \cdots < t_N = b;$$

$$t_{i+1} - t_i = \frac{b-a}{N} \qquad \text{for} \qquad 0 \leq i \leq N-1.$$

We then consider the polygonal line obtained by joining the successive pairs of points $\mathbf{c}(t_i)$, $\mathbf{c}(t_{i+1})$ for $0 \leq i \leq N-1$. This yields a polygonal approximation to $\mathbf{c}$ as in Figure 4.2.4. By the formula for distance in $\mathbb{R}^3$, it follows that the line segment from $\mathbf{c}(t_i)$ to $\mathbf{c}(t_{i+1})$ has length

$$\|\mathbf{c}(t_{i+1}) - \mathbf{c}(t_i)\| = \sqrt{[x(t_{i+1}) - x(t_i)]^2 + [y(t_{i+1}) - y(t_i)]^2 + [z(t_{i+1}) - z(t_i)]^2},$$

where $\mathbf{c}(t) = (x(t), y(t), z(t))$. Applying the mean-value theorem to $x(t)$, $y(t)$, and $z(t)$ on $[t_i, t_{i+1}]$, we obtain three points t_i^*, t_i^{**}, and t_i^{***} such that

$$x(t_{i+1}) - x(t_i) = x'(t_i^*)(t_{i+1} - t_i),$$

$$y(t_{i+1}) - y(t_i) = y'(t_i^{**})(t_{i+1} - t_i),$$

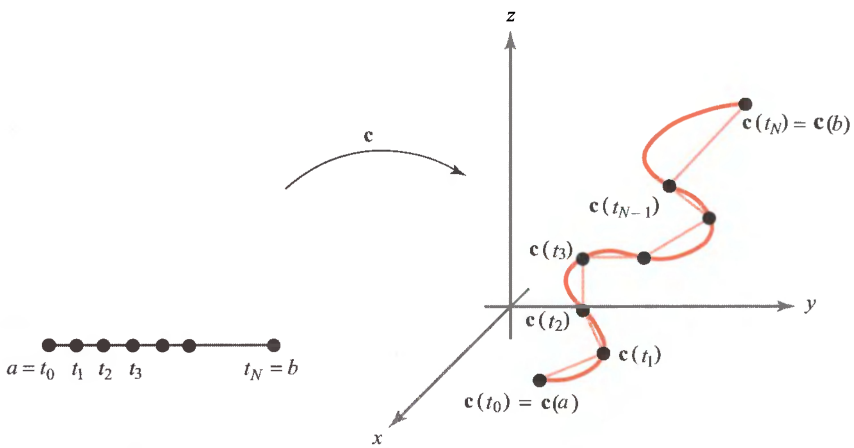

Figure 4.2.4 A path $\mathbf{c}$ may be approximated by a polygonal path obtained by joining each $\mathbf{c}(t_i)$ to $\mathbf{c}(t_{i+1})$ by a straight line.

and

$$z(t_{i+1}) - z(t_i) = z'(t_i^{***})(t_{i+1} - t_i).$$

Thus, the line segment from $\mathbf{c}(t_i)$ to $\mathbf{c}(t_{i+1})$ has length

$$\sqrt{[x'(t_i^*)]^2 + [y'(t_i^{**})]^2 + [z'(t_i^{***})]^2}(t_{i+1} - t_i).$$

Therefore, the length of our approximating polygonal line is

$$S_N = \sum_{i=0}^{N-1} \sqrt{[x'(t_i^*)]^2 + [y'(t_i^{**})]^2 + [z'(t_i^{***})]^2}(t_{i+1} - t_i).$$

As $N \to \infty$, this polygonal line approximates the image of $\mathbf{c}$ more closely. Therefore, we define the arc length of $\mathbf{c}$ as the limit, if it exists, of the sequence S_N as $N \to \infty$. Because the derivatives x', y', and z' are all assumed to be continuous on $[a, b]$, we can conclude that, in fact, the limit does exist and is given by

$$\operatorname*{limit}_{N\to\infty} S_N = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

(The theory of integration relates the integral to sums by the formula

$$\int_a^b f(t)\, dt = \operatorname*{limit}_{N\to\infty} \sum_{i=0}^{N-1} f(t_i^*)(t_{i+1} - t_i),$$

where $t_0, \ldots, t_N$ is a partition of $[a, b]$, $t_i^* \in [t_i, t_{i+1}]$ is arbitrary, and f is a continuous function. Here we have *possibly different points* t_i^*, t_i^{**}, and t_i^{***}, and so this formula must be extended slightly.)

EXERCISES

Find the arc length of the given curve on the specified interval in Exercises 1 to 6.[4]

1. $(2\cos t, 2\sin t, t)$, for $0 \le t \le 2\pi$

2. $(1, 3t^2, t^3)$, for $0 \le t \le 1$

3. $(\sin 3t, \cos 3t, 2t^{3/2})$, for $0 \le t \le 1$

---

[4] Several of these problems make use of the formula

$$\int \sqrt{x^2 + a^2}\, dx = \tfrac{1}{2}\left[x\sqrt{x^2 + a^2} + a^2 \log(x + \sqrt{x^2 + a^2})\right] + C$$

from the table of integrals in the back of the book.

4. $\left(t+1, \dfrac{2\sqrt{2}}{3}t^{3/2}+7, \dfrac{1}{2}t^2\right)$, for $1 \le t \le 2$ **5.** (t, t, t^2), for $1 \le t \le 2$

6. $(t, t \sin t, t \cos t)$, for $0 \le t \le \pi$

7. Find the length of the path $\mathbf{c}(t)$, defined by $\mathbf{c}(t) = (2 \cos t, 2 \sin t, t)$, if $0 \le t \le 2\pi$ and $\mathbf{c}(t) = (2, t - 2\pi, t)$, if $2\pi \le t \le 4\pi$.

8. Let $\mathbf{c}$ be the path $\mathbf{c}(t) = (t, t \sin t, t \cos t)$. Find the arc length of $\mathbf{c}$ between the two points $(0, 0, 0)$ and $(\pi, 0, -\pi)$.

9. Let $\mathbf{c}$ be the path $\mathbf{c}(t) = (2t, t^2, \log t)$, defined for $t > 0$. Find the arc length of $\mathbf{c}$ between the points $(2, 1, 0)$ and $(4, 4, \log 2)$.

10. The arc-length function $s(t)$ for a given path $\mathbf{c}(t)$, defined by $s(t) = \int_a^t \|\mathbf{c}'(\tau)\|\, d\tau$, represents the distance a particle traversing the trajectory of $\mathbf{c}$ will have traveled by time t if it starts out at time a; that is, it gives the length of $\mathbf{c}$ between $\mathbf{c}(a)$ and $\mathbf{c}(t)$. Find the arc-length functions for the curves $\alpha(t) = (\cosh t, \sinh t, t)$ and $\beta(t) = (\cos t, \sin t, t)$, with $a = 0$.

11. Let $\mathbf{c}(t)$ be a given path, $a \le t \le b$. Let $s = \alpha(t)$ be a new variable, where α is a strictly increasing C^1 function given on $[a, b]$. For each s in $[\alpha(a), \alpha(b)]$ there is a unique t with $\alpha(t) = s$. Define the function $\mathbf{d}\colon [\alpha(a), \alpha(b)] \to \mathbb{R}^3$ by $\mathbf{d}(s) = \mathbf{c}(t)$.

(a) Argue that the image curves of $\mathbf{c}$ and $\mathbf{d}$ are the same.
(b) Show that $\mathbf{c}$ and $\mathbf{d}$ have the same arc length.
(c) Let $s = \alpha(t) = \int_a^t \|\mathbf{c}'(\tau)\|\, d\tau$. Define $\mathbf{d}$ as above by $\mathbf{d}(s) = \mathbf{c}(t)$. Show that

$$\left\| \frac{d}{ds}\mathbf{d}(s) \right\| = 1.$$

The path $s \mapsto \mathbf{d}(s)$ is said to be an ***arc length reparametrization*** of $\mathbf{c}$ (see also Exercise 13).

Exercises 12 to 17 develop some of the classic differential geometry of curves.

12. Let $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ be an infinitely differentiable path (derivatives of all orders exist). Assume $\mathbf{c}'(t) \ne \mathbf{0}$ for any t. The vector $\mathbf{c}'(t)/\|\mathbf{c}'(t)\| = \mathbf{T}(t)$ is tangent to $\mathbf{c}$ at $\mathbf{c}(t)$, and, because $\|\mathbf{T}(t)\| = 1$, $\mathbf{T}$ is called the ***unit tangent*** to $\mathbf{c}$.

(a) Show that $\mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$. [Hint: Differentiate $\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$.]
(b) Write down a formula for $\mathbf{T}'(t)$ in terms of $\mathbf{c}$.

13. (a) A path $\mathbf{c}(s)$ is said to be ***parametrized by arc length*** or, what is the same thing, to have ***unit speed*** if $\|\mathbf{c}'(s)\| = 1$. For a path parametrized by arc length on $[a, b]$, show that $l(\mathbf{c}) = b - a$.

(b) The ***curvature*** at a point $\mathbf{c}(s)$ on a path is defined by $k = \|\mathbf{T}'(s)\|$ when the path is parametrized by arc length. Show that $k = \|\mathbf{c}''(s)\|$.

(c) If $\mathbf{c}$ is given in terms of some other parameter t and $\mathbf{c}'(t)$ is never $\mathbf{0}$, show that $k = \|\mathbf{c}'(t) \times \mathbf{c}''(t)\| / \|\mathbf{c}'(t)\|^3$.

(d) Calculate the curvature of the helix $\mathbf{c}(t) = (1/\sqrt{2})(\cos t, \sin t, t)$. (This $\mathbf{c}$ is a scalar multiple of the right-circular helix.)

14. If $\mathbf{T}'(t) \neq \mathbf{0}$, it follows from Exercise 12 that $\mathbf{N}(t) = \mathbf{T}'(t)/\|\mathbf{T}'(t)\|$ is normal (i.e., perpendicular) to $\mathbf{T}(t)$; $\mathbf{N}$ is called the ***principal normal vector***. Let a third unit vector that is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$ be defined by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$; $\mathbf{B}$ is called the ***binormal vector***. Together, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ form a right-handed system of mutually orthogonal vectors that may be thought of as moving along the path (Figure 4.2.5). Show that

(a) $\dfrac{d\mathbf{B}}{dt} \cdot \mathbf{B} = 0$.

(b) $\dfrac{d\mathbf{B}}{dt} \cdot \mathbf{T} = 0$.

(c) $d\mathbf{B}/dt$ is a scalar multiple of $\mathbf{N}$.

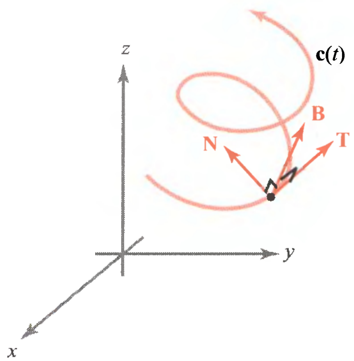

Figure 4.2.5 The tangent $\mathbf{T}$, principal normal $\mathbf{N}$, and binormal $\mathbf{B}$.

15. If $\mathbf{c}(s)$ is parametrized by arc length, we use the result of Exercise 14(c) to define a scalar-valued function τ, called the ***torsion***, by

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

(a) Show that $\tau = [\mathbf{c}'(s) \times \mathbf{c}''(s)] \cdot \mathbf{c}'''(s)/\|\mathbf{c}''(s)\|^2$.
(b) Show that if $\mathbf{c}$ is given in terms of some other parameter t,

$$\tau = \frac{[\mathbf{c}'(t) \times \mathbf{c}''(t)] \cdot \mathbf{c}'''(t)}{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|^2}.$$

Compare with Exercise 13(c).
(c) Compute the torsion of the helix $\mathbf{c}(t) = (1/\sqrt{2})(\cos t, \sin t, t)$.

16. Show that if a path lies in a plane, then the torsion is zero. Do this by demonstrating that $\mathbf{B}$ is constant and is a normal vector to the plane in which $\mathbf{c}$ lies. (If the torsion is not zero, it gives a measure of how fast the curve is twisting out of the plane of $\mathbf{T}$ and $\mathbf{N}$.)

17. (a) Use the results of Exercises 13, 14, and 15 to prove the following ***Frenet formulas*** for a unit-speed curve:

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N}; \qquad \frac{d\mathbf{N}}{ds} = -k\mathbf{T} + \tau\mathbf{B}; \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

(b) Reexpress the results of part (a) as

$$\frac{d}{ds}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix} = \boldsymbol{\omega} \times \begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}$$

for a suitable vector $\boldsymbol{\omega}$.

18. In special relativity, the ***proper time*** of a path $\mathbf{c}\colon [a, b] \to \mathbb{R}^4$ with components given by $\mathbf{c}(\lambda) = (x(\lambda), y(\lambda), z(\lambda), t(\lambda))$ is defined to be the quantity

$$\int_a^b \sqrt{-[x'(\lambda)]^2 - [y'(\lambda)]^2 - [z'(\lambda)]^2 + c^2[t'(\lambda)]^2}\, d\lambda,$$

where c is the velocity of light, a constant. In Figure 4.2.6, show that, using self-explanatory notation, the "twin paradox inequality" holds:

proper time (AB) + proper time (BC) < proper time (AC).

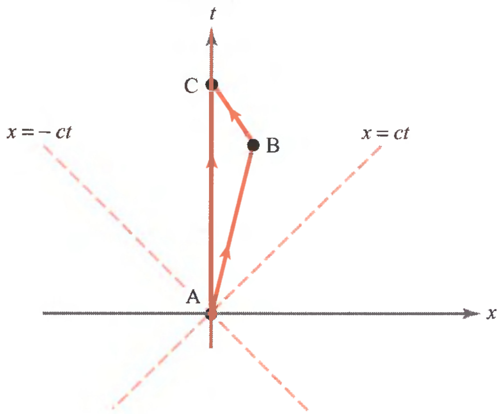

Figure 4.2.6 The relativistic triangle inequality.

19. The early Greeks knew that a straight line was the shortest possible path between two points. Euclid, in his book *Optics*, stated the "principle of the reflection of light"—that is, light traveling in a plane travels in a straight line, and when it is reflected across a mirror, the angle of incidence equals the angle of reflection.

The Greeks could not have had a proof that straight lines provided the shortest path between two points because they, in the first place, had no definition of the length of a path. They saw this property of straight lines as more or less "obvious."

Using the justification of arc length in this section and the triangle inequality of Section 1.5, argue that if $\mathbf{c}_0$ is the straight-line path $c_0(t) = t\mathrm{P} + (1-t)\mathrm{Q}$ between P and Q in $\mathbb{R}^3$, then

$$L(\mathbf{c}_0) \le L(\mathbf{c})$$

for any other path $\mathbf{c}$ joining P and Q.

4.3 Vector Fields

The Concept of a Vector Field

In Chapter 2, we introduced a particular kind of vector field, the gradient. In this section we study *general* vector fields, discussing their geometric and physical significance.

Vector Fields A ***vector field*** in $\mathbb{R}^n$ is a map $\mathbf{F}: A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns to each point $\mathbf{x}$ in its domain A a vector $\mathbf{F}(\mathbf{x})$. If $n = 2$, $\mathbf{F}$ is called a ***vector field in the plane***, and if $n = 3$, $\mathbf{F}$ is a ***vector field in space***.

Picture $\mathbf{F}$ as attaching an *arrow* to each point (Figure 4.3.1). By contrast, a map $f: A \subset \mathbb{R}^n \to \mathbb{R}$ that assigns a *number* to each point is a ***scalar field***. A vector field $\mathbf{F}(x, y, z)$ on $\mathbb{R}^3$ has three ***component scalar fields*** F_1, F_2, and F_3, so that

$$\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)).$$

Similarly, a vector field on $\mathbb{R}^n$ has n components $F_1, \ldots, F_n$. If each component is a C^k function, we say the vector field $\mathbf{F}$ is of ***class*** C^k. Vector fields will be assumed to be at least of class C^1 unless otherwise noted.

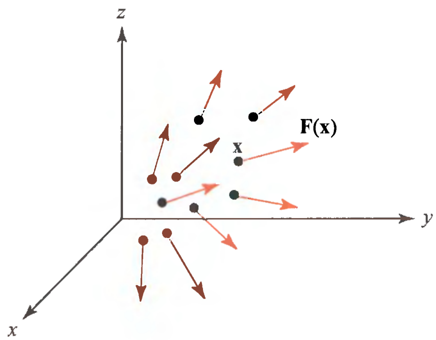

Figure 4.3.1 A vector field $\mathbf{F}$ assigns a vector $\mathbf{F}(\mathbf{x})$ to each point $\mathbf{x}$ of its domain.

In many applications, $\mathbf{F}(\mathbf{x})$ represents a physical vector quantity (force, velocity, etc.) associated with the position $\mathbf{x}$, as in the following examples.

EXAMPLE 1 The flow of water through a pipe is said to be ***steady*** if, at each point inside the pipe, the velocity of the fluid passing through that point does not change with time. (Note that this is quite different from saying that the water in the pipe is not moving.) Attaching to each point the fluid velocity at that point, we obtain the ***velocity field*** $\mathbf{V}$ of the fluid (see Figure 4.3.2). Notice that the length of the arrows (the speed), as well as the direction of flow, may change from point to point. ▲

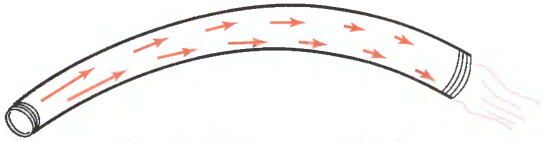

Figure 4.3.2 A vector field describing the velocity of flow in a pipe.

EXAMPLE 2 Some forms of rotary motion (such as the motion of particles on a turntable) can be described by the vector field

$$\mathbf{V}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

See Figure 4.3.3, in which we have shown instead of $\mathbf{V}$ the shorter vector field $\frac{1}{4}\mathbf{V}$ so that the arrows do not overlap. This is a common convention in drawing pictures of vector fields. ▲

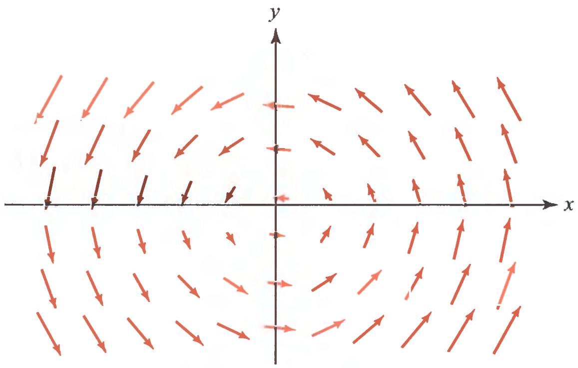

Figure 4.3.3 A rotary vector field.

EXAMPLE 3 In the plane, $\mathbb{R}^2$, let the vector field x be defined by

$$\mathbf{V}(x, y) = \frac{y\mathbf{i}}{x^2 + y^2} - \frac{x\mathbf{j}}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}\right)$$

(except at the origin, where $\mathbf{V}$ is not defined). This vector field is a good approximation to the planar part of the velocity of water flowing toward a hole in the bottom of a tub (Figure 4.3.4). Notice that the velocity becomes *larger* as you approach the hole. ▲

Gradient Vector Fields

In Section 2.6 we introduced the gradient of a function by

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k}.$$

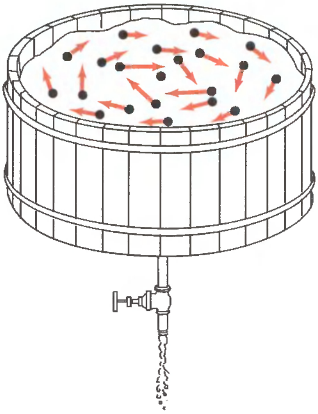

Figure 4.3.4 The vector field describing circular flow in a tub.

Now we want to think of this as an example of a vector field—it assigns a vector to each point (x, y, z). As such, we refer to ∇f as a ***gradient vector field***. Gradient fields come up in a variety of situations, as the next two examples show.

EXAMPLE 4 A piece of material is heated on one side and cooled on another. The temperature at each point within the body is described at a given moment by a scalar field $T(x, y, z)$. The flow of heat may be marked by a field of arrows indicating the direction and magnitude of the flow (Figure 4.3.5). This ***energy*** or ***heat flux vector field*** is given by $\mathbf{J} = -k\nabla T$, where $k > 0$ is a constant called the ***conductivity*** and ∇T is the gradient of the real-valued function T. Level sets of T are called ***isotherms***. Note that the heat flows from hot regions toward cold ones, since $-\nabla T$ points in the direction of decreasing T. ▲

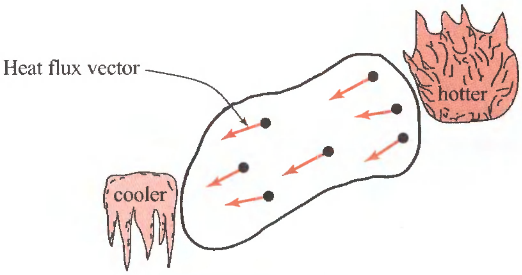

Figure 4.3.5 A vector field describing the direction and magnitude of heat flow.

EXAMPLE 5 The force of attraction of the earth on a mass m can be described by a vector field called the ***gravitational force field***. Place the origin of a coordinate system at the center of the earth (assumed spherical). According to Newton's law of gravity, this field is given by

$$\mathbf{F} = -\frac{mMG}{r^3}\mathbf{r},$$

where $\mathbf{r}(x, y, z) = (x, y, z)$, and $r = \|\mathbf{r}\|$ (see Figure 4.3.6). The domain of this vector field consists of those $\mathbf{r}$ for which $\|\mathbf{r}\|$ is greater than the radius of the earth. As we saw in Example 6, Section 2.6, $\mathbf{F}$ is a gradient field, $\mathbf{F} = -\nabla V$, where

$$V = -\frac{mMG}{r}$$

is the ***gravitational potential***. Note again that $\mathbf{F}$ points in the direction of *decreasing* V. Writing $\mathbf{F}$ in terms of components, we see that

$$\mathbf{F}(x, y, z) = \left(\frac{-mMG}{r^3}x, \frac{-mMG}{r^3}y, \frac{-mMG}{r^3}z\right). \quad \blacktriangle$$

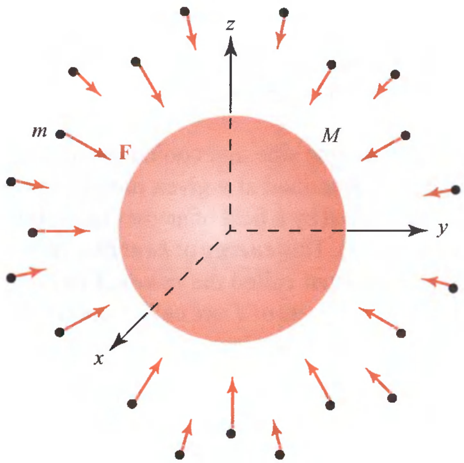

Figure 4.3.6 The vector field $\mathbf{F}$ given by Newton's law of gravitation.

EXAMPLE 6 According to ***Coulomb's law***, the force acting on a charge e at position $\mathbf{r}$ due to a charge Q at the origin is

$$\mathbf{F} = \frac{\varepsilon Qe}{r^3}\mathbf{r} = -\nabla V,$$

where $V = \varepsilon Qe/r$ and ε is a constant that depends on the units used. For $Qe > 0$ (like charges) the force is repulsive [Figure 4.3.7(a)], and for $Qe < 0$ (unlike charges) the force is attractive [Figure 4.3.7(b)]. Because the potential V is constant on the level surfaces of V, they are called ***equipotential surfaces***. Note that the force field is orthogonal to the equipotential surfaces (the force field is radial and the equipotential surfaces are concentric spheres). $\blacktriangle$

The next example shows that not every vector field is a gradient.

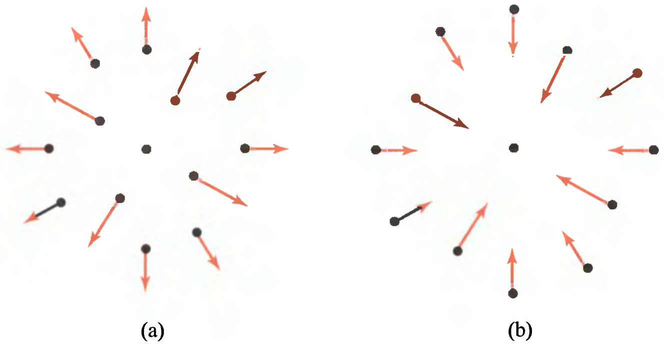

Figure 4.3.7 The vector fields associated with (a) like charges ($Qe > 0$), and (b) unlike charges ($Qe < 0$).

EXAMPLE 7 Show that the vector field $\mathbf{V}$ on $\mathbb{R}^2$ defined by $\mathbf{V}(x, y) = y\mathbf{i} - x\mathbf{j}$ is not a gradient vector field; that is, there is no C^1 function f such that

$$\mathbf{V}(x, y) = \nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

SOLUTION Suppose that such an f exists. Then $\partial f/\partial x = y$ and $\partial f/\partial y = -x$. Because these are C^1 functions, f itself must have continuous first- *and* second-order partial derivatives. But, $\partial^2 f/\partial x\,\partial y = -1$, and $\partial^2 f/\partial y\,\partial x = 1$, which violates the equality of mixed partials. Thus, $\mathbf{V}$ cannot be a gradient vector field. ▲

Conservation of Energy and Escaping the Earth's Gravitational Field

Consider a particle of mass m moving in a force field $\mathbf{F}$ that is a potential field. That is, assume $\mathbf{F} = -\nabla V$ for a real-valued function V, and that the particle moves according to $\mathbf{F} = m\mathbf{a}$. Thus, if the path is $\mathbf{r}(t)$, then

$$m\ddot{\mathbf{r}}(t) = -\nabla V(\mathbf{r}(t)). \tag{1}$$

A basic fact about such motion is the *conservation of energy*. The energy E of the particle is defined to be the sum of the kinetic and potential energies, defined as

$$E = \frac{1}{2}m\|\dot{\mathbf{r}}(t)\|^2 + V(\dot{\mathbf{r}}(t)). \tag{2}$$

The principle of *conservation of energy* states that if Newton's second law holds, then E is independent of time; that is, $dE/dt = 0$. The proof of this fact is a simple calculation; we use equation (2), the chain rule, and equation (1):

$$\begin{aligned}\frac{dE}{dt} &= m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + (\nabla V)\cdot\dot{\mathbf{r}}\\ &= \dot{\mathbf{r}}\cdot(-\nabla V + \nabla V) = 0.\end{aligned}$$

Escape Velocity

As an application of conservation of energy, we compute the velocity required for a rocket to escape the earth's gravitational influence. Assume the rocket has mass m and is at a distance R_0 from the center of the earth (or another planet) when its escape velocity v_e has been reached, and that it will coast thereafter. The energy at this time is

$$E_0 = \frac{1}{2}mv_e^2 - \frac{mMG}{R_0}. \qquad (3)$$

By conservation of energy, E_0 will equal the energy at a later time, which we write as

$$E_0 = E = \frac{1}{2}mv^2 - \frac{mMG}{R}, \qquad (4)$$

where v is the velocity and R is the distance from the center of the earth (or the other planet). What we mean by the term *escape velocity* is that v_e is chosen such that the rocket gets to great distances, at which time it is barely moving. That is, v is close to zero and R is very large. Thus, from equation (4), we see that $E = 0$ and hence $E_0 = 0$; solving $E_0 = 0$ for v_e using equation (3) gives:

$$v_e = \sqrt{\frac{2MG}{R_0}}.$$

Now GM/R_0 is exactly g, the acceleration due to gravity at the distance R_0 from the center of the planet. Thus, we can write:

$$v_e = \sqrt{2gR_0}.$$

For the earth, if the escape velocity were to be achieved at the surface of the earth (of course, this is a bit unrealistic), this would give

$$v_e = \sqrt{2 \cdot 9.8 \text{ m/s}^2 \cdot 6{,}371{,}000 \text{ m}} = 11{,}127 \text{ m/s}.$$

However, this is a good approximation to the velocity that a satellite in low earth orbit needs in order to escape the earth's gravitational field.

Flow Lines

An important concept related to general (not necessarily gradient) vector fields is that of a flow line, defined as follows.

Flow Lines If $\mathbf{F}$ is a vector field, a ***flow line*** for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

That is, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}(t)$.

In the context of Example 1, a flow line is the path followed by a small particle suspended in the fluid (Figure 4.3.8). Flow lines are also appropriately called ***streamlines*** or ***integral curves***.

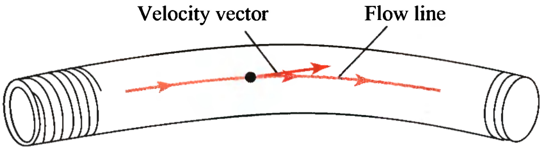

Figure 4.3.8 The velocity vector of a fluid is tangent to a flow line.

Geometrically, a flow line for a given vector filed **F** is a curve that threads its way through the domain of the vector field in such a way that the tangent vector of the curve coincides with the vector field, as in Figure 4.3.9.

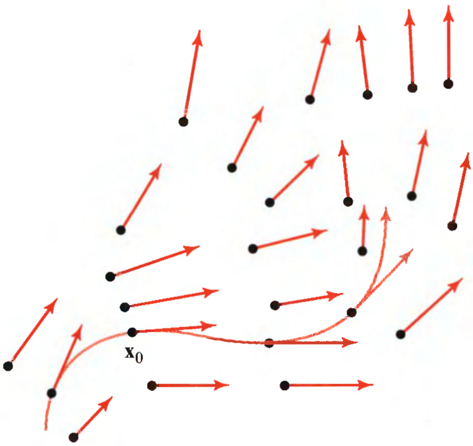

Figure 4.3.9 A flow line threading its way through a vector field in the plane.

A flow line may be viewed as a solution of a system of differential equations. Indeed, we can write the definition $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$ as

$$x'(t) = P(x(t), y(t), z(t)),$$
$$y'(t) = Q(x(t), y(t), z(t)),$$
$$z'(t) = R(x(t), y(t), z(t)),$$

where $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, and where

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

One learns about such systems in courses on differential equations, but we are not presuming such a course has been taken.

EXAMPLE 8 Show that the path $\mathbf{c}(t) = (\cos t, \sin t)$ is a flow line of the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Can you find others?

SOLUTION We must verify that $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t))$. The left side is $(-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, while the right side is $\mathbf{F}(\cos t, \sin t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, so we have a flow line. As suggested by Figure 4.3.3, the other flow lines are also circles. They have the form

$$\mathbf{c}(t) = (r\,\cos(t - t_0), r\,\sin(t - t_0))$$

for constants r and t_0. ▲

In many cases, explicit formulas for flow lines are not available, so one must resort to numerical methods. Figure 4.3.10 shows some output from a program that computes flow lines numerically and plots them on the screen.

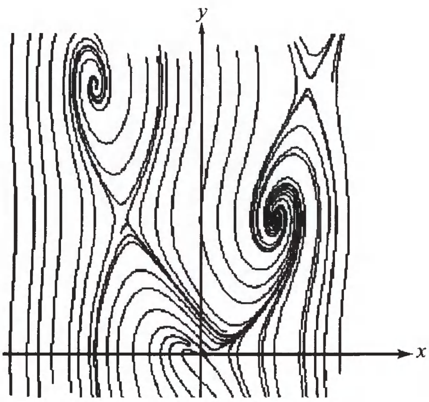

Figure 4.3.10 Computer-generated integral curves of the vector field $\mathbf{F}(x, y) = (\sin y)\mathbf{i} + (x^2 - y)\mathbf{j}$. This figure was created using *3D-XplorMath*, available from Richard Palais' Web site at rsp.math.brandeis.edu/3D-XplorMath.

Historical Note

The Field Concept

The concept of a "field," such as a vector field, has had an enormous impact on the development of conceptual frameworks for physics and engineering. It is truly one of the great breakthrough ideas in the history of human thought. It is the notion that allows one to describe, in a systematic way, influences on objects and between objects that are spatially separated.

The idea of a field began with Newton's concept of the gravitational field. In this case, the gravitational field describes the attractive influence of one body or group of bodies on one another. Similarly, the electric field produced by a charged object or group of objects creates, according to

Coulomb's law, a force on another charged object. Using vector fields to describe these sorts of forces has led to a deeper understanding of attractive and repulsive forces in nature.

However, it was the monumental discovery of the Maxwell field equations, which describe the propagation of electromagnetic energy, that cemented the concept of "field" in scientific thought. This example is particularly interesting because these fields can *propagate*. The contrast between the electromagnetic field that can propagate and the gravitational field that involves instantaneous *action at a distance* has caused great interest among philosophers of science.

Einstein's idea is that gravitation can be described in terms of the metric properties of space–time and that in this theory the associated field can also propagate, just like the electromagnetic field, thus providing profound philosophical evidence that Einstein's version of gravity may be correct. These ideas have also led to modern efforts to detect gravitational waves. For a further discussion of Einstein's work, see Section 7.7.

The idea of a field is also used in engineering to describe elastic systems and interesting microstructural properties of materials. In modern theoretical physics, the field concept is used to describe elementary particles and is central to attempts by modern theoretical physicists to unify gravity with the quantum mechanical physics of elementary particles. It is impossible to imagine a modern theoretical framework that does not incorporate some sort of field concept as a central ingredient.

EXERCISES

In Exercises 1 to 8, sketch the given vector field or a small multiple of it.

1. $\mathbf{F}(x, y) = (2, 2)$

2. $\mathbf{F}(x, y) = (4, 0)$

3. $\mathbf{F}(x, y) = (x, y)$

4. $\mathbf{F}(x, y) = (-x, y)$

5. $\mathbf{F}(x, y) = (2y, x)$

6. $\mathbf{F}(x, y) = (y, -2x)$

7. $\mathbf{F}(x, y) = \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}\right)$

8. $\mathbf{F}(x, y) = \left(\dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}}\right)$

In Exercises 9 to 12, sketch a few flow lines of the given vector field.

9. $\mathbf{F}(x, y) = (y, -x)$

10. $\mathbf{F}(x, y) = (x, -y)$

11. $\mathbf{F}(x, y) = (x, x^2)$

12. $\mathbf{F}(x, y, z) = (y, -x, 0)$

In Exercises 13 to 16, show that the given curve $\mathbf{c}(t)$ *is a flow line of the given velocity vector field* $\mathbf{F}(x, y, z)$.

13. $\mathbf{c}(t) = (e^{2t}, \log|t|, 1/t), t \neq 0; \mathbf{F}(x, y, z) = (2x, z, -z^2)$

14. $\mathbf{c}(t) = (t^2, 2t - 1, \sqrt{t}), t > 0; \mathbf{F}(x, y, z) = (y + 1, 2, 1/2z)$

15. $\mathbf{c}(t) = (\sin t,\ \cos t, e^t); \mathbf{F}(x, y, z) = (y, -x, z)$

16. $\mathbf{c}(t) = \left(\frac{1}{t^3}, e^t, \frac{1}{t}\right); \mathbf{F}(x, y, z) = (-3z^4, y, -z^2)$

17. Show that it takes half as much energy to launch a satellite into an orbit just above the earth as it does to escape the earth. (Ignore the rotation of the earth.)

18. Let $\mathbf{c}(t)$ be a flow line of a gradient field $\mathbf{F} = -\nabla V$. Prove that $V(\mathbf{c}(t))$ is a decreasing function of t.

19. Suppose that the isotherms in a region are all concentric spheres centered at the origin. Prove that the energy flux vector field points either toward or away from the origin.

20. Sketch the gradient field $-\nabla V$ for $V(x, y) = (x + y)/(x^2 + y^2)$ and the equipotential surface $V = 1$.

4.4 Divergence and Curl

For each of the divergence and curl operations, we will make use of the ***del operator***, defined by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

For functions of one variable, taking a derivative can be thought of as an operation or process; that is, given a function $y = f(x)$, its derivative is the result of *operating* on y by the derivative *operator* d/dx. Similarly, we can write the gradient as

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}\right) f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y}$$

for functions of two variables, and

$$\nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}$$

for three variables. In operational terms, the gradient of f is obtained by taking the ∇ operator and applying it to f.

Definition of Divergence

We define the divergence of a vector field $\mathbf{F}$ by taking the *dot product* of ∇ with $\mathbf{F}$.

Divergence If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the ***divergence*** of $\mathbf{F}$ is the scalar field

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Similarly, if $\mathbf{F} = (F_1, \ldots, F_n)$ is a vector field on $\mathbb{R}^n$, its divergence is

$$\operatorname{div} \mathbf{F} = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

EXAMPLE 1 Compute the divergence of

$$\mathbf{F} = x^2 y\mathbf{i} + z\mathbf{j} + xyz\mathbf{k}.$$

SOLUTION

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz) = 2xy + 0 + xy = 3xy. \quad \blacktriangle$$

Interpretation

The divergence has an important physical interpretation. If we imagine $\mathbf{F}$ to be the velocity field of a gas (or a fluid), then div $\mathbf{F}$ *represents the rate of expansion per unit volume under the flow of the gas (or fluid)*. If div $\mathbf{F} < 0$, the gas (or fluid) is *compressing*. For a vector field $\mathbf{F}(x, y) = F_1\mathbf{i} + F_2\mathbf{j}$ on the plane, the *divergence*

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

measures the rate of expansion of area.

This interpretation is explained graphically, as follows. Choose a small region W about a point $\mathbf{x}_0$. For each point $\mathbf{x}$ in W, let $\mathbf{x}(t)$ be the flow line emanating from $\mathbf{x}$. The set of points $\mathbf{x}(t)$ describe how the set W flows after time t (see Figure 4.4.1).

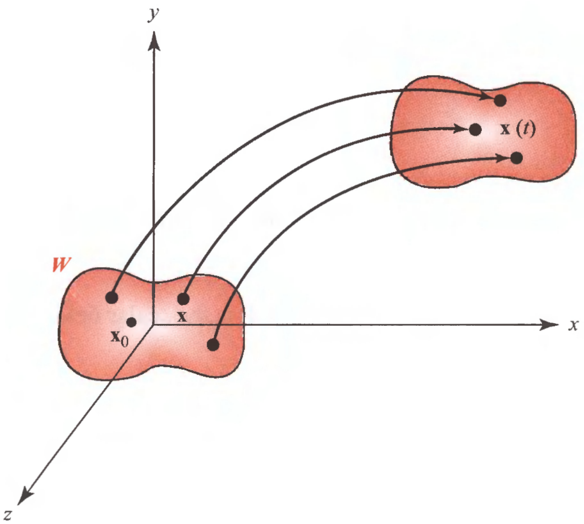

Figure 4.4.1 Flowing a region W along the flow lines of a vector field.

Call the region that results after time t has elapsed $W(t)$, and let $\mathcal{V}(t)$ be its volume (or area in two dimensions). Then the relative rate of change of volume is the divergence; more precisely,

$$\frac{1}{\mathcal{V}(0)}\frac{d}{dt}\mathcal{V}(t)\bigg|_{t=0} \approx \operatorname{div}\mathbf{F}(\mathbf{x}_0),$$

with the approximation being more exact as W shrinks to $\mathbf{x}_0$. A direct proof of this is given in the Internet supplement, but a more natural argument is given in Chapter 8, in the context of the integral theorems of vector calculus.

EXAMPLE 2 Consider the vector field in the plane given by $\mathbf{V}(x, y) = x\mathbf{i}$. Relate the sign of the divergence of $\mathbf{V}$ with the rate of change of areas under the flow.

SOLUTION We think of $\mathbf{V}$ as the velocity field of a fluid in the plane. The vector field $\mathbf{V}$ points to the right for $x > 0$ and to the left if $x < 0$, as we see in Figure 4.4.2. The length of $\mathbf{V}$ gets shorter toward the origin. As the fluid moves, it expands (the area of the shaded rectangle increases), so we expect $\operatorname{div}\mathbf{V} > 0$. Indeed, $\operatorname{div}\mathbf{V} = 1$. ▲

EXAMPLE 3 The flow lines of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ are straight lines directed away from the origin (Figure 4.4.3).

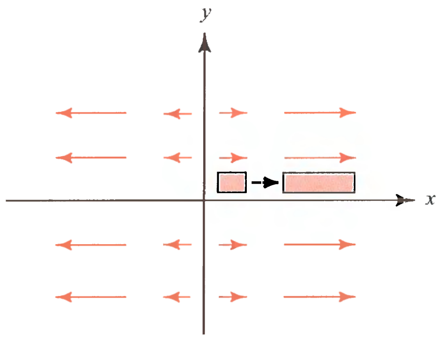

Figure 4.4.2 This fluid is expanding.

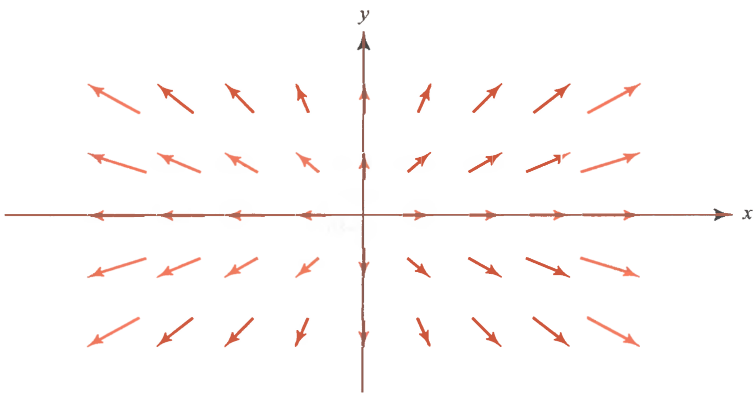

Figure 4.4.3 The vector field $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.

If these flow lines are those of a fluid, the fluid is expanding as it moves out from the origin, so div **F** should be positive. In fact,

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y = 2 > 0. \quad \blacktriangle$$

EXAMPLE 4 Consider the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$. Here the flow lines point toward the origin instead of away from it (see Figure 4.4.4). Therefore, the fluid is compressing, so we expect (div **F**) < 0. Calculating, we see that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -1 - 1 = -2 < 0. \quad \blacktriangle$$

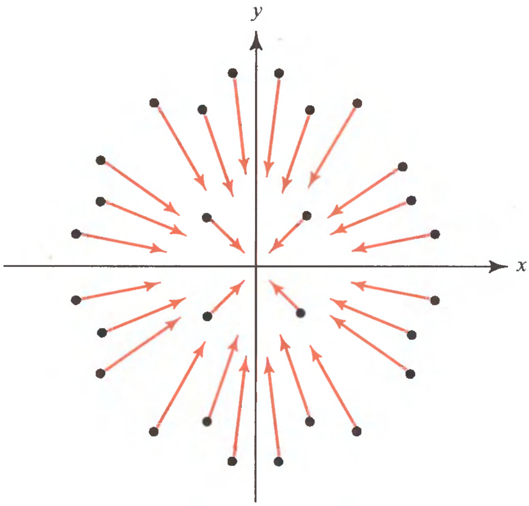

Figure 4.4.4 The vector field $\mathbf{F}(x, y) = -x\mathbf{i} - y\mathbf{j}$.

EXAMPLE 5 As we saw in the last section, the flow lines of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ are concentric circles about the origin, moving counterclockwise (see Figure 4.4.5). From this figure, it appears that the fluid is neither compressing nor expanding. This is confirmed by calculating

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0. \quad \blacktriangle$$

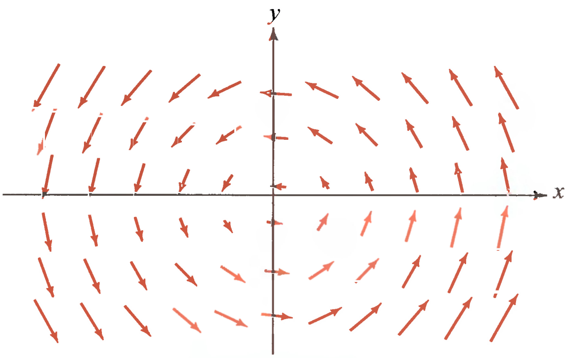

Figure 4.4.5 The vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ has zero divergence.

EXAMPLE 6 Some flow lines of $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ are shown in Figure 4.4.6. Here our intuition about expansion or compression is less clear. However, it is true that the shaded regions shown have the same area, and we calculate that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}(-y) = 1 + (-1) = 0. \quad \blacktriangle$$

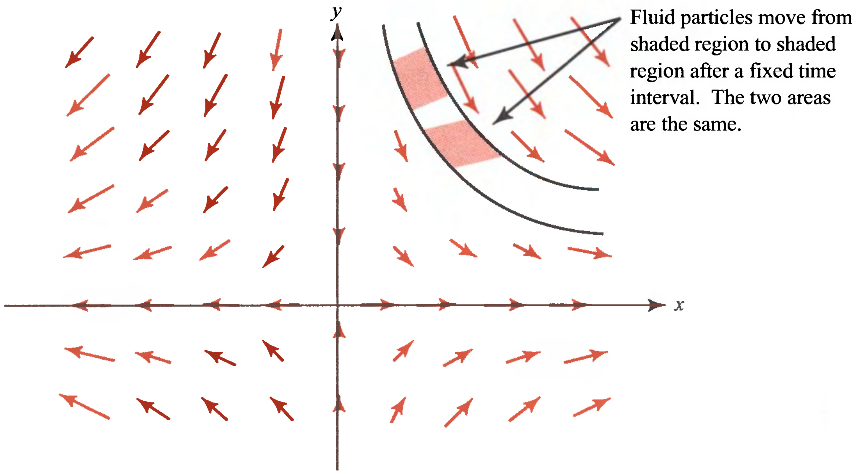

Figure 4.4.6 The vector field $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$.

Curl

To calculate the curl, the second basic operation performed on vector fields, we take the *cross product* of ∇ with $\mathbf{F}$.

Curl of a Vector Field If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the ***curl*** of $\mathbf{F}$ is the *vector field*

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

If we write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, which is alternative notation, the same formula for the curl reads

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

EXAMPLE 7 Let $\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

SOLUTION We use the preceding formula:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xy & 1 \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (y-0)\mathbf{k}.$$

Thus, $\nabla \times \mathbf{F} = y\mathbf{k}$. ▲

EXAMPLE 8 Find the curl of $xy\mathbf{i} - \sin z\mathbf{j} + \mathbf{k}$.

SOLUTION Letting $\mathbf{F} = xy\mathbf{i} - \sin z\mathbf{j} + \mathbf{k}$,

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} \\ &= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\sin z & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ xy & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ xy & -\sin z \end{vmatrix}\mathbf{k} \\ &= \cos z\mathbf{i} - x\mathbf{k}. \end{aligned}$$ ▲

Unlike the divergence, which can be defined in $\mathbb{R}^n$ for any n, we define the curl only in three-dimensional space (or for planar vector fields, regarding their third component as zero).

The Curl and Rotations

The physical significance of the curl will be discussed in Chapter 8, when we study Stokes' theorem. However, we can now consider a specific situation, in which the curl is associated with rotations.

EXAMPLE 9 Consider a solid rigid body B rotating about an axis L. The rotational motion of the body can be described by a vector $\boldsymbol{\omega}$ along the axis of rotation, the direction being chosen so that the body rotates about $\boldsymbol{\omega}$, as in Figure 4.4.7. We call $\boldsymbol{\omega}$ the ***angular velocity vector***. The length $\omega = \|\boldsymbol{\omega}\|$ is taken to be the angular speed of the body B, that is, the speed of any point in B divided by its distance from the axis L of rotation. The motion of points in the rotating body is described by the vector field $\mathbf{v}$ whose value at each point is the velocity at that point. To find $\mathbf{v}$, let Q be any point in B and let α be the distance from Q to L.

Figure 4.4.7 shows that $\alpha = \|\mathbf{r}\| \sin\theta$, where $\mathbf{r}$ is the vector whose initial point is the origin and whose terminal point is Q and θ is the angle between $\mathbf{r}$ and the axis L of rotation. The tangential velocity $\mathbf{v}$ of Q is directed counterclockwise along the

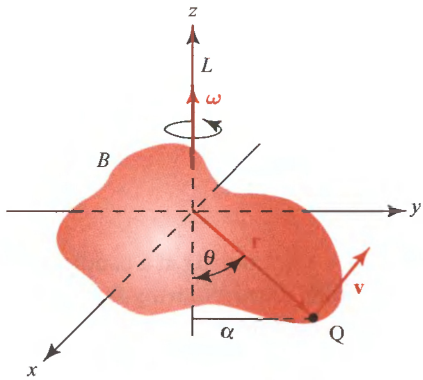

Figure 4.4.7 The velocity $\mathbf{v}$ and angular velocity $\boldsymbol{\omega}$ of a rotating body are related by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$.

tangent to a circle parallel to the xy plane with radius α and has magnitude

$$\|\mathbf{v}\| = \omega\alpha = \omega\|\mathbf{r}\| \sin\theta = \|\boldsymbol{\omega}\|\|\mathbf{r}\| \sin\theta.$$

The direction and magnitude of $\mathbf{v}$ imply that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Selecting a coordinate system in which L is the z axis, we can write $\boldsymbol{\omega} = \omega\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j},$$

and so

$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}.$$

Hence, for the rotation of a rigid body, the curl of the velocity vector field is a vector field whose value is the same at each point. It is directed along the axis of rotation with magnitude *twice* the angular speed. ▲

The Curl and Rotational Flow

If a vector field represents the flow of a *fluid*, then the value of $\nabla \times \mathbf{F}$ at a point is twice the angular velocity vector of a *rigid* body that rotates as the fluid does near that point. In particular, $\nabla \times \mathbf{F} = \mathbf{0}$ at a point P means that the fluid is free from rigid rotations at P; that is, it has no whirlpools. Another justification of this idea depends on Stokes' theorem from Chapter 8. However, we can say informally that curl $\mathbf{F} = \mathbf{0}$ means that if a *small* rigid paddle wheel is placed in the fluid, it will move with the fluid but will not rotate around its own axis. Such a vector field is called ***irrotational***. For example, it has been determined from experiments that fluid draining from a tub is usually irrotational except right at the center, even though the fluid is "rotating" around the drain (see Figure 4.4.8). In Example 10, *the flow lines of the vector field* $\mathbf{V}$ *are circles about the origin, yet we show that the flow is irrotational*. Thus, the reader should be warned of the possible confusion the word "irrotational" can cause.

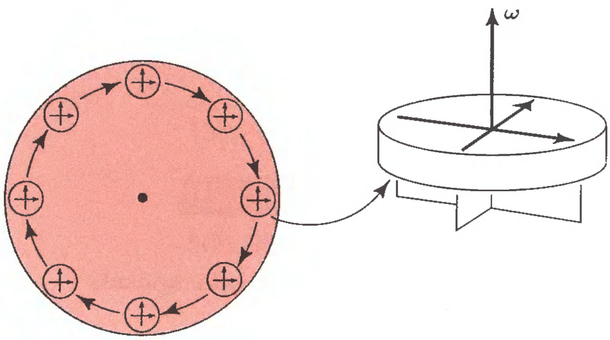

Figure 4.4.8 Looking at a paddle wheel from above a moving fluid. The velocity field $\mathbf{V}(x, y, z) = (y\mathbf{i} - x\mathbf{j})/(x^2 + y^2)$ is irrotational; the paddle wheel does not rotate around its axis $\boldsymbol{\omega}$.

EXAMPLE 10 Verify that the vector field

$$\mathbf{V}(x, y, z) = \frac{y\mathbf{i} - x\mathbf{j}}{x^2 + y^2}$$

is irrotational when $(x, y) \neq (0, 0)$ (i.e., except where $\mathbf{V}$ is not defined).

SOLUTION The curl is

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{y}{x^2 + y^2} & \dfrac{-x}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right)\right]\mathbf{k}$$

$$= \left[\frac{-(x^2 + y^2) + 2x^2}{(x^2 + y^2)^2} + \frac{-(x^2 + y^2) + 2y^2}{(x^2 + y^2)^2}\right]\mathbf{k} = \mathbf{0}. \quad \blacktriangle$$

Gradients are Curl Free

The following identity is a basic relation between the gradient and curl, which should be compared with the fact that for any vector $\mathbf{v}$, we have $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

THEOREM 1: Curl of a Gradient For any C^2 function f,

$$\nabla \times (\nabla f) = \mathbf{0}.$$

That is, the curl of any gradient is the zero vector.

PROOF Because $\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ we have, by definition,

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z\,\partial x} - \frac{\partial^2 f}{\partial x\,\partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\,\partial y} - \frac{\partial^2 f}{\partial y\,\partial x}\right)\mathbf{k}.$$

Each component is zero because of the equality of mixed partial derivatives. ■

The converse to this theorem (a vector field with zero curl is a gradient, under suitable hypotheses) will be discussed in Chapter 8.

EXAMPLE 11 Let $\mathbf{V}(x, y, z) = y\mathbf{i} - x\mathbf{j}$. Show that $\mathbf{V}$ is not a gradient field.

SOLUTION If $\mathbf{V}$ were a gradient field, then it would satisfy curl $\mathbf{V} = \mathbf{0}$ by Theorem 1. But

$$\text{curl}\,\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\mathbf{k} \neq \mathbf{0},$$

so $\mathbf{V}$ cannot be a gradient. ▲

Scalar Curl

There is an operation on vector fields in the plane that is closely related to the curl. If $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field in the plane, it can also be regarded as a vector field in space for which the $\mathbf{k}$ component is zero and the other two components are independent of z. The curl of $\mathbf{F}$ then reduces to

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

and always points in the **k** direction. The function

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

of x and y is called the ***scalar curl*** of **F**.

EXAMPLE 12 Find the scalar curl of $\mathbf{V}(x, y) = -y^2\mathbf{i} + x\mathbf{j}$.

SOLUTION The curl is

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & 0 \end{vmatrix} = (1 + 2y)\,\mathbf{k},$$

so the scalar curl, which is the coefficient of **k**, is $1 + 2y$. ▲

Curls are Divergence Free

A basic relation between the divergence and curl operations is given next.

THEOREM 2: Divergence of a Curl For any C^2 vector field **F**,

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

That is, the divergence of any curl is zero.

As with the curl of a gradient, the proof rests on the equality of the mixed partial derivatives. The student should write out the details. A converse will be discussed in Chapter 8.

EXAMPLE 13 Show that the vector field $\mathbf{V}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ cannot be the curl of some vector field **F**; that is, there is no **F** with $\mathbf{V} = \operatorname{curl} \mathbf{F}$.

SOLUTION If this were so, then div **V** would be zero by Theorem 2. But

$$\operatorname{div} \mathbf{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0,$$

so **V** cannot be curl **F** for any **F**. ▲

Laplacian

The ***Laplace operator*** ∇^2, which operates on functions f, is defined to be the divergence of the gradient:

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

This operator plays an important role in many physical laws, as we have mentioned in Section 3.1.

EXAMPLE 14 Show that $\nabla^2 f = 0$ for

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \quad \text{and} \quad (x, y, z) \neq (0, 0, 0),$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\|$.

SOLUTION The first derivatives are

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \quad \frac{\partial f}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}.$$

Computing the second derivatives, we find that

$$\frac{\partial^2 f}{\partial x^2} = \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}},$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}.$$

Thus,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3}{(x^2 + y^2 + z^2)^{3/2}} \\ &= \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3}{(x^2 + y^2 + z^2)^{3/2}} = 0. \quad \blacktriangle\end{aligned}$$

Vector Identities

We now have these basic operations on hand: gradient, divergence, curl, and the Laplace operator. The following box contains some basic general formulas that are useful when computing with vector fields.

Basic Identities of Vector Analysis

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$, for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at points $\mathbf{x}$ where $g(\mathbf{x}) \neq 0$
5. $\operatorname{div}(\mathbf{F}+\mathbf{G}) = \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$
6. $\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}\mathbf{F} + \operatorname{curl}\mathbf{G}$
7. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$
8. $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot\operatorname{curl}\mathbf{F} - \mathbf{F}\cdot\operatorname{curl}\mathbf{G}$
9. $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$
10. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \nabla f\times\mathbf{F}$
11. $\operatorname{curl}\nabla f = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g)$
13. $\operatorname{div}(\nabla f\times\nabla g) = 0$
14. $\operatorname{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

EXAMPLE 15 Prove identity 7 in the preceding box.

SOLUTION The vector field $f\mathbf{F}$ has components fF_i, for $i = 1, 2, 3$, and so

$$\operatorname{div}(f\mathbf{F}) = \frac{\partial}{\partial x}(fF_1) + \frac{\partial}{\partial y}(fF_2) + \frac{\partial}{\partial z}(fF_3).$$

However, $(\partial/\partial x)(fF_1) = f\,\partial F_1/\partial x + F_1\,\partial f/\partial x$ by the product rule, with similar expressions for the other terms. Therefore,

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= f\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) + F_1\frac{\partial f}{\partial x} + F_2\frac{\partial f}{\partial y} + F_3\frac{\partial f}{\partial z}\\ &= f(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla f. \quad\blacktriangle\end{aligned}$$

Let us use these identities to redo Example 14.

EXAMPLE 16 Show that for $\mathbf{r} \neq \mathbf{0}$, $\nabla^2(1/r) = 0$.

SOLUTION As in the case of the gravitational potential, $\nabla(1/r) = -\mathbf{r}/r^3$. In general, $\nabla(r^n) = nr^{n-2}\mathbf{r}$ (see Exercise 30). By the identity $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \nabla f \cdot \mathbf{F}$, we get

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) = \frac{1}{r^3}\nabla \cdot \mathbf{r} + \mathbf{r} \cdot \nabla\left(\frac{1}{r^3}\right)$$

$$= \frac{3}{r^3} + \mathbf{r} \cdot \left(\frac{-3\mathbf{r}}{r^5}\right) = \frac{3}{r^3} - \frac{3}{r^3} = 0. \quad \blacktriangle$$

Historical Note

Divergence and Curl

William Rowan Hamilton, in his investigation of quaternions (discussed in Section 1.3) introduced the *del operator*, defined formally as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Hamilton firmly believed in the significance of this operator. If $f(x, y, z)$ is a scalar function on $\mathbb{R}^3$, then "multiplication" by ∇ gives the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

which, of course, gives the direction of steepest ascent (see Section 2.6). If

$$\mathbf{V}(x,y,z) = V_1(x,y,z)\mathbf{i} + V_2(x,y,z)\mathbf{j} + V_3(x,y,z)\mathbf{k}$$

is a vector field, then the "quaternionic multiplication" of ∇ with $\mathbf{V}$ yields

$$\nabla\mathbf{V} = -\text{div}\,\mathbf{V} + \text{curl}\,\mathbf{V}.$$

Thus, what we now call the divergence of $\mathbf{V}$ is the negative of the scalar part of this product, and curl $\mathbf{V}$ is the vector part (c.f. the quaternion discussion in Section 1.3).

As far as we are aware, Hamilton never gave a physical interpretation of divergence and curl, but he surely believed that, as a consequence of his faith in them, they must have an important physical interpretation. His faith in his mathematical formalism was justified, but a physical explanation of divergence and curl had to wait for James Clerk Maxwell's *Treatise on Electricity and Magnetism*. Here, Maxwell used both the divergence and the curl in his equations for the interaction of electric and magnetic fields (the Maxwell equations are discussed in Chapter 8).

Curiously, Maxwell referred to divergence as *convergence* and to curl as *rotation*, a term still used in the literature. It was Josiah Gibbs (Figure 4.4.9) who renamed convergence and rotation as the more familiar terms we use today—divergence and curl.

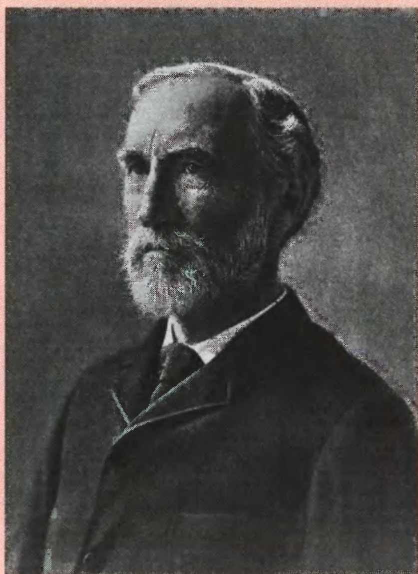

Figure 4.4.9 Josiah Willard Gibbs (1839–1903).

Maxwell gave a physical interpretation of the divergence using the Gauss divergence theorem, as we do in Section 8.4. His physical interpretation of the curl as a rotation was rather brief. Gibbs provided a more elementary interpretation of divergence, as we do in this section. In the spirit of Leibniz (who believed in infinitesimal quantities dx,dy,dz), Gibbs imagined placing a small cube of dimensions dx by dy by dz in a fluid. The faces of this cube have areas $dx\,dy$, $dy\,dz$, and $dx\,dz$.

At this point, students may be interested to hear Gibbs through the words of his student E. B. Wilson:

> Consider the amount of fluid which passes through those faces of the cube which are parallel to the YZ plane, i.e., perpendicular to the X axis [see Figure 4.4.10].
>
> The normal to the face whose x coordinate is the lesser, that is, the normal to the left-hand face of the cube is $-\mathbf{i}$. The flux of substance through this face is
>
> $$-\mathbf{i}\cdot\mathbf{V}(x, y, z)\,dy\,dz.$$

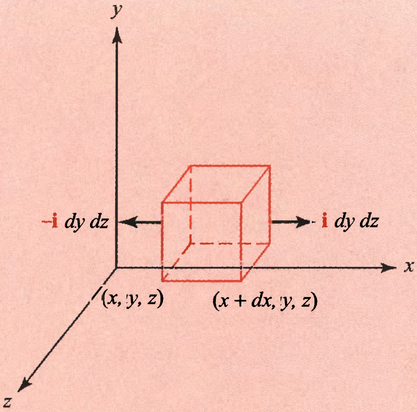

Figure 4.4.10 Cube with faces parallel to the *YZ* plane.

The normal to the opposite face, the face whose x coordinate is greater by the amount dx, is $+\mathbf{i}$, and the flux through it is therefore

$$\begin{aligned}\mathbf{i}\cdot\mathbf{V}(x+dx, y, z)\,dy\,dz &= \mathbf{i}\cdot\left[\mathbf{V}(x, y, z) + \frac{\partial \mathbf{V}}{\partial x}\,dx\right]dy\,dz\\ &= \mathbf{i}\cdot\mathbf{V}(x, y, z)\,dy\,dz + \mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}\,dx\,dy\,dz.\end{aligned}$$

The total flux outward from the cube through these two faces is therefore the algebraic sum of these quantities. This is simply

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}\,dx\,dy\,dz = \frac{\partial V_1}{\partial x}\,dx\,dy\,dz.$$

In like manner the fluxes through the other pairs of faces of the cube are

$$\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}\,dx\,dy\,dz \qquad \text{and} \qquad \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}\,dx\,dy\,dz.$$

The total flux out from the cube is therefore

$$\left(\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}\right)dx\,dy\,dz.$$

This is the net quantity of fluid that leaves the cube per unit time. The quotient of this by the volume $dx\,dy\,dz$ of the cube gives the rate of diminution of density. This is

$$\nabla \cdot \mathbf{V} = \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Because $\nabla \cdot \mathbf{V}$ thus represents the diminution of density or the rate at which matter is leaving a point per unit volume per unit time, it is called the *divergence*. Maxwell employed the term *convergence* to denote the rate at which fluid approaches a point per unit volume per unit time. This is the negative of the divergence. In the case that the fluid is *incompressible*, as much matter must leave the cube as enters it. The total change of contents must therefore be zero. For this reason, the characteristic differential equation that any incompressible fluid must satisfy is

$$\nabla \cdot \mathbf{V} = 0,$$

where $\mathbf{V}$ is the velocity of the fluid. This equation is often known as the *hydrodynamic equation*. It is satisfied by any flow of water, since water is practically incompressible. The great importance of the equation for work in electricity is due to the fact that according to Maxwell's hypothesis, electric displacement obeys the same laws as an incompressible fluid. If, then, $\mathbf{D}$ is the electric displacement,

$$\operatorname{div} \mathbf{D} = \nabla \cdot \mathbf{D} = 0.$$

Gibbs' interpretation of curl was much like the one we gave in Example 9 for the rotation of a rigid body. Wilson remarks that an analysis of the meaning of curl for fluid motion was "rather difficult." It remains a bit elusive, even today, as can be seen from our discussion following Example 9. We provide another interpretation in Chapter 8.

EXERCISES

Find the divergence of the vector fields in Exercises 1 to 4.

1. $\mathbf{V}(x, y, z) = e^{xy}\mathbf{i} - e^{xy}\mathbf{j} + e^{yz}\mathbf{k}$

2. $\mathbf{V}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

3. $\mathbf{V}(x, y, z) = x\mathbf{i} + (y + \cos x)\mathbf{j} + (z + e^{xy})\mathbf{k}$

4. $\mathbf{V}(x, y, z) = x^2\mathbf{i} + (x + y)^2\mathbf{j} + (x + y + z)^2\mathbf{k}$

5. Figure 4.4.11 shows some flow lines and moving regions for a fluid moving in the plane field velocity field $\mathbf{V}$. Where is div $\mathbf{V} > 0$, and also where is div $\mathbf{V} < 0$?

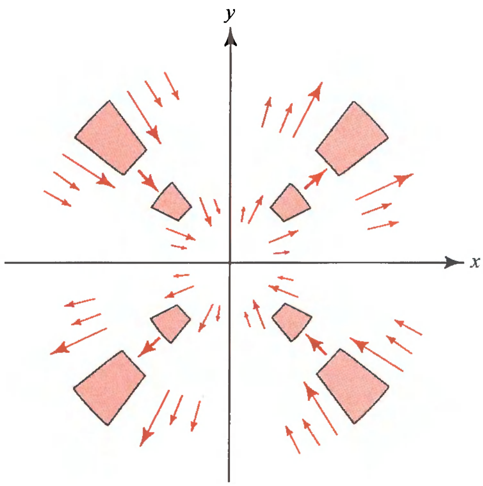

Figure 4.4.11 The flow lines of a fluid moving in the plane.

6. Let $V(x, y, z) = x\mathbf{i}$ be the velocity field of a fluid in space. Relate the sign of the divergence with the rate of change of volume under the flow.

7. Sketch a few flow lines for $\mathbf{F}(x, y) = y\mathbf{i}$. Calculate $\nabla \cdot \mathbf{F}$ and explain why your answer is consistent with your sketch.

8. Sketch a few flow lines for $\mathbf{F}(x, y) = -3x\mathbf{i} - y\mathbf{j}$. Calculate $\nabla \cdot \mathbf{F}$ and explain why your answer is consistent with your sketch.

Calculate the divergence of the vector fields in Exercises 9 to 12.

9. $\mathbf{F}(x, y) = x^3\mathbf{i} - x\sin(xy)\mathbf{j}$

10. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

11. $\mathbf{F}(x, y) = \sin(xy)\mathbf{i} - \cos(x^2y)\mathbf{j}$

12. $\mathbf{F}(x, y) = xe^y\mathbf{i} - [y/(x + y)]\mathbf{j}$

Compute the curl, $\nabla \times \mathbf{F}$, of the vector fields in Exercises 13 to 16.

13. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

14. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

15. $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$

16. $\mathbf{F}(x, y, z) = \dfrac{yz\mathbf{i} - xz\mathbf{j} + xy\mathbf{k}}{x^2 + y^2 + z^2}$

Calculate the scalar curl of each of the vector fields in Exercises 17 to 20.

17. $\mathbf{F}(x, y) = \sin x\mathbf{i} + \cos x\mathbf{j}$

18. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

19. $\mathbf{F}(x, y) = xy\mathbf{i} + (x^2 - y^2)\mathbf{j}$

20. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

Verify that $\nabla \times (\nabla f) = \mathbf{0}$ for the functions in Exercises 21 to 24.

21. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

22. $f(x, y, z) = xy + yz + xz$

23. $f(x, y, z) = 1/(x^2 + y^2 + z^2)$

24. $f(x, y, z) = x^2y^2 + y^2z^2$

25. Show that $\mathbf{F} = y(\cos x)\mathbf{i} + x(\sin y)\mathbf{j}$ is *not* a gradient vector field.

26. Show that $\mathbf{F} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j}$ is *not* a gradient field.

27. Prove identity 10 in the list of vector identities.

28. Suppose that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \cdot \mathbf{G} = 0$. Which of the following necessarily have zero divergence?

(a) $\mathbf{F} + \mathbf{G}$ (b) $\mathbf{F} \times \mathbf{G}$

29. Let $\mathbf{F} = 2xz^2\mathbf{i} + \mathbf{j} + y^3zx\mathbf{k}$ and $f = x^2y$. Compute the following quantities:

(a) ∇f (b) $\nabla \times \mathbf{F}$ (c) $\mathbf{F} \times \nabla f$ (d) $\mathbf{F} \cdot (\nabla f)$

30. Let $\mathbf{r}(x, y, z) = (x, y, z)$ and $r = \sqrt{x^2 + y^2 + z^2} = \|\mathbf{r}\|$. Prove the following identities.

(a) $\nabla(1/r) = -\mathbf{r}/r^3$, $r \neq 0$; and, in general, $\nabla(r^n) = nr^{n-2}\mathbf{r}$ and $\nabla(\log r) = \mathbf{r}/r^2$.
(b) $\nabla^2(1/r) = 0$, $r \neq 0$; and, in general, $\nabla^2 r^n = n(n+1)r^{n-2}$.
(c) $\nabla \cdot (\mathbf{r}/r^3) = 0$; and, in general, $\nabla \cdot (r^n\mathbf{r}) = (n+3)r^n$.
(d) $\nabla \times \mathbf{r} = \mathbf{0}$; and, in general, $\nabla \times (r^n\mathbf{r}) = \mathbf{0}$.

31. Does $\nabla \times \mathbf{F}$ have to be perpendicular to $\mathbf{F}$?

32. Let $\mathbf{F}(x, y, z) = 3x^2y\mathbf{i} + (x^3 + y^3)\mathbf{j}$.

(a) Verify that curl $\mathbf{F} = \mathbf{0}$.
(b) Find a function f such that $\mathbf{F} = \nabla f$. (Techniques for constructing f in general are given in Chapter 8. The one in this problem should be sought by trial and error.)

33. Show that the real and imaginary parts of each of the following complex functions form the components of an irrotational and incompressible vector field in the plane; here $i = \sqrt{-1}$.

(a) $(x - iy)^2$ (b) $(x - iy)^3$ (c) $e^{x-iy} = e^x(\cos y - i \sin y)$

REVIEW EXERCISES FOR CHAPTER 4

For Exercises 1 to 4, at the indicated point, compute the velocity vector, the acceleration vector, the speed, and the equation of the tangent line.

1. $\mathbf{c}(t) = (t^3 + 1, e^{-t}, \cos(\pi t/2))$, at $t = 1$

2. $\mathbf{c}(t) = (t^2 - 1, \cos(t^2), t^4)$, at $t = \sqrt{\pi}$

3. $\mathbf{c}(t) = (e^t, \sin t, \cos t)$, at $t = 0$

4. $\mathbf{c}(t) = \dfrac{t^2}{1+t^2}\mathbf{i} + t\mathbf{j} + \mathbf{k}$, at $t = 2$

5. Calculate the tangent and acceleration vectors for the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$ at $t = \pi/4$.

6. Calculate the tangent and acceleration vector for the cycloid $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ at $t = \pi/4$ and sketch.

7. Let a particle of mass m move on the path $\mathbf{c}(t) = (t^2, \sin t, \cos t)$. Compute the force acting on the particle at $t = 0$.

8. (a) Let $\mathbf{c}(t)$ be a path with $\|\mathbf{c}(t)\| =$ constant; that is, the curve lies on a sphere. Show that $\mathbf{c}'(t)$ is orthogonal to $\mathbf{c}(t)$.

(b) Let $\mathbf{c}$ be a path whose speed is never zero. Show that $\mathbf{c}$ has constant speed if and only if the acceleration vector $\mathbf{c}''$ is always perpendicular to the velocity vector $\mathbf{c}'$.

9. Express the arc length of the curve $x^2 = y^3 = z^5$ between $x = 1$ and $x = 4$ as an integral, using a suitable parametrization.

10. Find the arc length of $\mathbf{c}(t) = t\mathbf{i} + (\log t)\mathbf{j} + 2\sqrt{2t}\mathbf{k}$ for $1 \le t \le 2$.

11. A particle is constrained to move around the unit circle in the xy plane according to the formula $(x, y, z) = (\cos(t^2), \sin(t^2), 0)$, $t \ge 0$.

(a) What are the velocity vector and speed of the particle as functions of t?

(b) At what point on the circle should the particle be released to hit a target at $(2, 0, 0)$? (Be careful about which direction the particle is moving around the circle.)

(c) At what time t should the release take place? (Use the smallest $t > 0$ that will work.)

(d) What are the velocity and speed at the time of release?

(e) At what time is the target hit?

12. A particle of mass m moves under the influence of a force $\mathbf{F} = -k\mathbf{r}$, where k is a constant and $\mathbf{r}(t)$ is the position of the particle at time t.

(a) Write down differential equations for the components of $\mathbf{r}(t)$.

(b) Solve the equations in part (a) subject to the initial conditions $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}'(0) = 2\mathbf{j} + \mathbf{k}$.

13. Write the curve described by the equations $x - 1 = 2y + 1 = 3z + 2$ in parametric form.

14. Write the curve $x = y^3 = z^2 + 1$ in parametric form.

15. Show that $\mathbf{c}(t) = (1/(1-t), 0, e^t/(1-t))$ is a flow line of the vector field defined by $\mathbf{F}(x, y, z) = (x^2, 0, z(1+x))$.

16. Let $\mathbf{F}(x, y) = f(x^2 + y^2)[-y\mathbf{i} + x\mathbf{j}]$ for a function f of one variable. What equation must $g(t)$ satisfy for

$$\mathbf{c}(t) = [\cos g(t)]\mathbf{i} + [\sin g(t)]\mathbf{j}$$

to be a flow line for $\mathbf{F}$?

Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for the vector fields in Exercises 17 to 20.

17. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

18. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

19. $\mathbf{F} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + (z + x)\mathbf{k}$

20. $\mathbf{F} = x\mathbf{i} + 3xy\mathbf{j} + z\mathbf{k}$

Compute the divergence and curl of the vector fields in Exercises 21 and 22 at the points indicated.

21. $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, at the point $(1, 1, 1)$

22. $\mathbf{F}(x, y, z) = (x + y)^3\mathbf{i} + (\sin xy)\mathbf{j} + (\cos xyz)\mathbf{k}$, at the point $(2, 0, 1)$

Calculate the gradients of the functions in Exercises 23 to 26, and verify that $\nabla \times \nabla f = \mathbf{0}$.

23. $f(x, y) = e^{xy} + \cos(xy)$

24. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

25. $f(x, y) = e^{x^2} - \cos(xy^2)$

26. $f(x, y) = \tan^{-1}(x^2 + y^2)$

27. (a) Let $f(x, y, z) = xyz^2$; compute ∇f.

(b) Let $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zy\mathbf{k}$; compute $\nabla \times \mathbf{F}$.

(c) Compute $\nabla \times (f\mathbf{F})$ using identity 10 of the list of vector identities. Compare with a direct computation.

28. (a) Let $\mathbf{F} = 2xye^z\mathbf{i} + e^z x^2\mathbf{j} + (x^2ye^z + z^2)\mathbf{k}$. Compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

(b) Find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.

29. Let $\mathbf{F}(x, y) = f(x^2 + y^2)[-y\mathbf{i} + x\mathbf{j}]$, as in Exercise 16. Calculate div $\mathbf{F}$ and curl $\mathbf{F}$ and discuss your answers in view of the results of Exercise 16.

30. Let a particle of mass m move along the elliptical helix $\mathbf{c}(t) = (4\cos t, \sin t, t)$.

(a) Find the equation of the tangent line to the helix at $t = \pi/4$.

(b) Find the force acting on the particle at time $t = \pi/4$.

(c) Write an expression (in terms of an integral) for the arc length of the curve $\mathbf{c}(t)$ between $t = 0$ and $t = \pi/4$.

31. (a) Let $g(x, y, z) = x^3 + 5yz + z^2$ and let $h(u)$ be a function of one variable such that $h'(1) = 1/2$. Let $f = h \circ g$. Starting at $(1, 0, 0)$, in what directions is f changing at 50% of its maximum rate?

(b) For $g(x, y, z) = x^3 + 5yz + z^2$, calculate $\mathbf{F} = \nabla g$, the gradient of g, and verify directly that $\nabla \times \mathbf{F} = 0$ at each point (x, y, z).

32. (a) Write in parametric form the curve that is the intersection of the surfaces $x^2 + y^2 + z^2 = 3$ and $y = 1$.

(b) Find the equation of the line tangent to this curve at $(1, 1, 1)$.

(c) Write an integral expression for the arc length of this curve. What is the value of this integral?

33. In meteorology, the ***negative pressure gradient*** $\mathbf{G}$ is a vector quantity that points from regions of high pressure to regions of low pressure, normal to the lines of constant pressure (***isobars***).

(a) In an xy coordinate system,

$$\mathbf{G} = -\frac{\partial P}{\partial x}\mathbf{i} - \frac{\partial P}{\partial y}\mathbf{j}.$$

Write a formula for the magnitude of the negative pressure gradient.

(b) If the horizontal pressure gradient provided the only horizontal force acting on the air, the wind would blow directly across the isobars in the direction of $\mathbf{G}$, and for a given air mass, with acceleration proportional to the magnitude of $\mathbf{G}$. Explain, using Newton's second law.

(c) Because of the rotation of the earth, the wind does not blow in the direction that part (b) would suggest. Instead, it obeys ***Buys–Ballot's law***, which states: "If in the Northern Hemisphere, you stand with your back to the wind, the high pressure is on your right and the low pressure is on your left." Draw a figure and introduce xy coordinates so that $\mathbf{G}$ points in the proper direction.

(d) State and graphically illustrate Buys–Ballot's law for the Southern Hemisphere, in which the orientation of high and low pressure is reversed.

34. A sphere of mass m, radius a, and uniform density has potential u and gravitational force $\mathbf{F}$, at a distance r from the center (0, 0, 0), given by

$$u = \frac{3m}{2a} - \frac{mr^2}{2a^3}, \qquad \mathbf{F} = -\frac{m}{a^3}\mathbf{r} \qquad (r \leq a);$$

$$u = \frac{m}{r}, \qquad \mathbf{F} = -\frac{m}{r^3}\mathbf{r} \qquad (r > a).$$

Here, $r = \|\mathbf{r}\|$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Verify that $\mathbf{F} = \nabla u$ on the inside and outside of the sphere.

(b) Check that u satisfies Poisson's equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 =$ constant inside the sphere.

(c) Show that u satisfies Laplace's equation: $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0$ outside the sphere.

35. A circular helix that lies on the cylinder $x^2 + y^2 = R^2$ with pitch ρ may be described parametrically by

$$x = R\cos\theta, \qquad y = R\sin\theta, \qquad z = \rho\theta, \qquad \theta \geq 0.$$

A particle slides under the action of gravity (which acts parallel to the z axis) without friction along the helix. If the particle starts out at the height $z_0 > 0$, then when it reaches the height z along the helix, its speed is given by

$$\frac{ds}{dt} = \sqrt{(z_0 - z)2g},$$

where s is arc length along the helix, g is the constant of gravity, t is time, and $0 \leq z \leq z_0$.

(a) Find the length of the part of the helix between the planes $z = z_0$ and $z = z_1$, $0 \leq z_1 < z_0$.

(b) Compute the time T_0 it takes the particle to reach the plane $z = 0$.

36. A sphere of radius 10 centimeters (cm) with center at (0, 0, 0) rotates about the z axis with angular velocity 4 in such a direction that the rotation looks counterclockwise from the positive z axis.

(a) Find the rotation vector $\boldsymbol{\omega}$ (see Example 9, in Section 4.4).

(b) Find the velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ when $\mathbf{r} = 5\sqrt{2}(\mathbf{i} - \mathbf{j})$ is on the "equator."

(c) Find the velocity of the point $(0, 5\sqrt{3}, 5)$ on the sphere.

37. Find the speed of the students in a classroom located at a latitude 49°N due to the rotation of the earth. (Ignore the motion of the earth about the sun, the sun in the galaxy, etc.; the radius of the earth is 3960 miles.)