5

Double and Triple Integrals

It is to Archimedes himself (c. 225 B.C.) that we owe the nearest approach to actual integration to be found among the Greeks. His first noteworthy advance in this direction was concerned with his proof that the area of a parabolic segment is four thirds of the triangle with the same base and vertex, or two thirds of the circumscribed parallelogram.

D. E. Smith, *History of Mathematics*

In this chapter and the next we study the integration of real-valued functions of several variables; this chapter treats integrals of functions of two and three variables, or *double* and *triple integrals*. The double integral has a basic geometric interpretation as volume, and can be defined rigorously as a limit of approximating sums. We shall present several techniques for evaluating double and triple integrals and consider some applications.

5.1 Introduction

This section discusses some geometric aspects of the double integral, deferring a more rigorous discussion in terms of Riemann sums until Section 5.2.

Double Integrals as Volumes

Consider a continuous function of two variables $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ whose domain R is a rectangle with sides parallel to the coordinate axes. The rectangle R can be described in terms of the two closed intervals $[a, b]$ and $[c, d]$, representing the sides of R along the x and y axes, respectively, as in Figure 5.1.1. In this case, we say that R is the ***Cartesian product*** of $[a, b]$ and $[c, d]$ and write $R = [a, b] \times [c, d]$.

Assume that $f(x, y) \geq 0$ on R, so that the graph of $z = f(x, y)$ is a surface lying above the rectangle R. This surface, the rectangle R, and the four planes $x = a$, $x = b$, $y = c$, and $y = d$ form the boundary of a region V in space (see Figure 5.1.1).

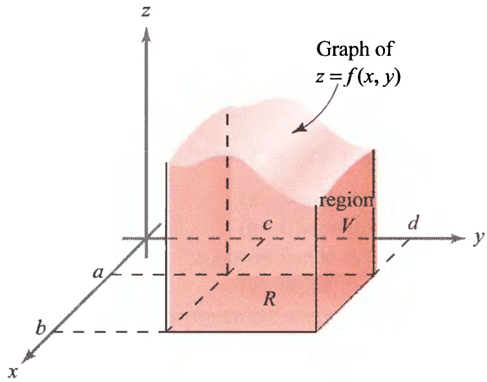

Figure 5.1.1 The region V in space is bounded by the graph of f, the rectangle R, and the four vertical sides indicated.

The problem of how to rigorously define the volume of V has to be faced, and we shall solve it in Section 5.2 by the classic method of exhaustion, or rather, in more modern terms, the method of Riemann sums. To gain an intuitive grasp of the double integral, we provisionally assume that the volume of a region has been defined.

Double Integrals The volume of the region above R and under the graph of a nonnegative function f is called the ***(double) integral*** of f over R and is denoted by

$$\iint_R f(x, y)\, dA, \qquad \text{or} \qquad \iint_R f(x, y)\, dx\, dy.$$

EXAMPLE 1 (a) If f is defined by $f(x, y) = k$, where k is a positive constant, then

$$\iint_R f(x, y)\, dA = k(b - a)(d - c),$$

because the integral is equal to the volume of a rectangular box with base R and height k.

(b) If $f(x, y) = 1 - x$ and $R = [0, 1] \times [0, 1]$, then

$$\iint_R f(x, y)\, dA = \frac{1}{2},$$

because the integral is equal to the volume of the triangular solid shown in Figure 5.1.2. ▲

EXAMPLE 2 Suppose $z = f(x, y) = x^2 + y^2$ and $R = [-1, 1] \times [0, 1]$. Then the integral $\iint_R (x^2 + y^2)\, dx\, dy$ is equal to the volume of the solid sketched in Figure 5.1.3. We shall compute this integral in Example 3. ▲

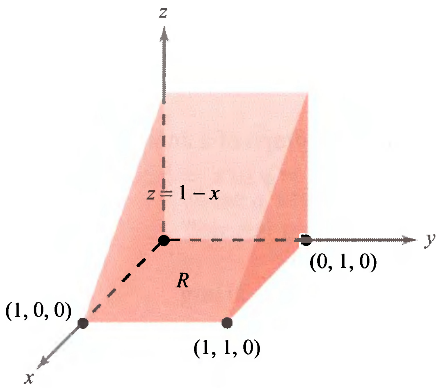

Figure 5.1.2 Volume under the graph $z = 1 - x$ and over $R = [0, 1] \times [0, 1]$.

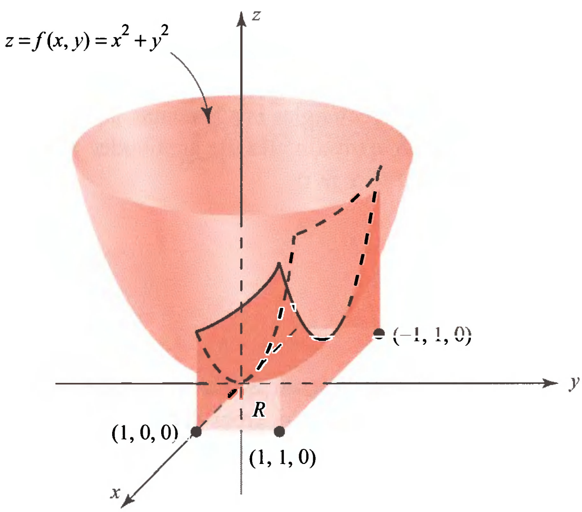

Figure 5.1.3 Volume under $z = x^2 + y^2$ and over $R = [-1, 1] \times [0, 1]$.

These ideas are similar to those for a single integral $\int_a^b f(x)\,dx$, which represents the area under the graph of f if $f \geq 0$; see Figure 5.1.4.[1]

Single integrals $\int_a^b f(x)\,dx$ can be rigorously defined, without recourse to the area concept, as a limit of Riemann sums. The idea is to approximate $\int_a^b f(x)\,dx$ by choosing a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$, selecting points $c_i \in [x_i, x_{i+1}]$, and forming the Riemann sum

$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) \approx \int_a^b f(x)\,dx$$

[1]Readers not already familiar with this idea should review the appropriate sections of their introductory calculus text.

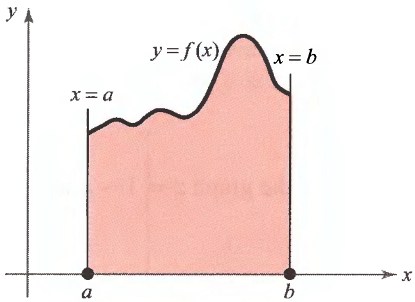

Figure 5.1.4 Area under the graph of a nonnegative continuous function f from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx$.

(see Figure 5.1.5). We examine the analogous process for double integrals in the next section.

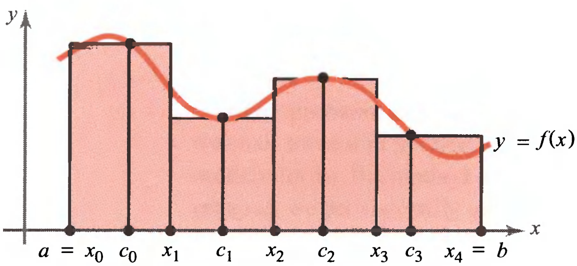

Figure 5.1.5 The sum of the areas of the shaded rectangles is a Riemann sum, which approximates the area under f from $x = a$ to $x = b$.

Cavalieri's Principle

There is a useful method for computing volumes, known as *Cavalieri's principle.* Suppose we have a solid body and we let $A(x)$ denote its cross-sectional area in a plane P_x measured at a distance x from a reference plane (Figure 5.1.6).

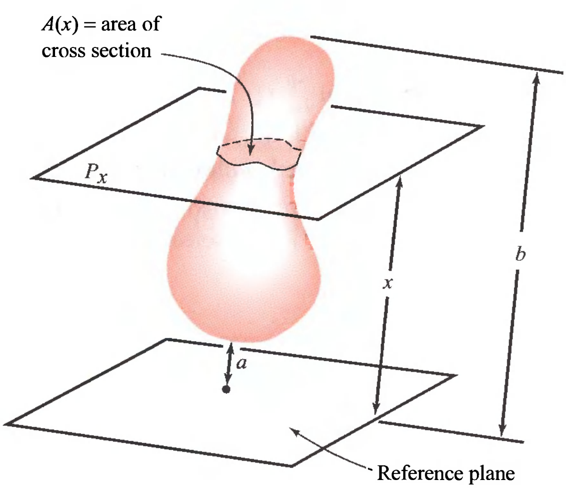

Figure 5.1.6 A solid body with cross-sectional area $A(x)$ at distance x from a reference plane.

According to Cavalieri's principle, the volume of the body is given by

$$\text{volume} = \int_a^b A(x)\,dx,$$

where a and b are the minimum and maximum distances from the reference plane. This can be made intuitively clear as follows. If we partition $[a, b]$ into $a = x_0 < x_1 < \cdots < x_n = b$, then an approximating Riemann sum for the preceding integral is

$$\sum_{i=0}^{n-1} A(c_i)(x_{i+1} - x_i).$$

But this sum also approximates the volume of the body, because $A(x)\,\Delta x$ is the volume of a slab with cross-sectional area $A(x)$ and thickness Δx (Figure 5.1.7). Therefore, it is reasonable to accept the preceding formula for the volume. A more careful justification of this method is given in the Internet supplement for Chapter 5.

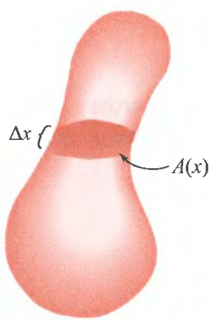

Figure 5.1.7 Volume of a slab with cross-sectional area $A(x)$ and thickness Δx equals $A(x)\Delta x$. The total volume of the body is $\int_a^b A(x)\,dx$.

The Slice Method—Cavalieri's Principle Let S be a solid and, for x satisfying $a \le x \le b$, let P_x be a family of parallel planes such that:

1. S lies between P_a and P_b;
2. The area of the slice of S cut by P_x is $A(x)$.

Then the volume of S is equal to

$$\int_a^b A(x)\,dx.$$

Historical Note

Bonaventura Cavalieri (1598–1647) was a pupil of Galileo and a professor in Bologna. His investigations into area and volume were important building blocks of the foundations of calculus. Although his methods were criticized by his contemporaries, similar ideas had been used by Archimedes in antiquity, and were later taken up by the "fathers" of calculus, Newton and Leibniz.

Reduction to Iterated Integrals

We now use Cavalieri's principle to evaluate double integrals. Consider the solid region under a graph $z = f(x, y)$ defined on the region $[a, b] \times [c, d]$, where f is continuous and greater than zero. There are two natural cross-sectional area functions: one obtained by using cutting planes perpendicular to the x axis, and the other obtained by using cutting planes perpendicular to the y axis. The cross section determined by a cutting plane $x = x_0$, of the first sort, is the plane region under the graph of $z = f(x_0, y)$ from $y = c$ to $y = d$ (Figure 5.1.8).

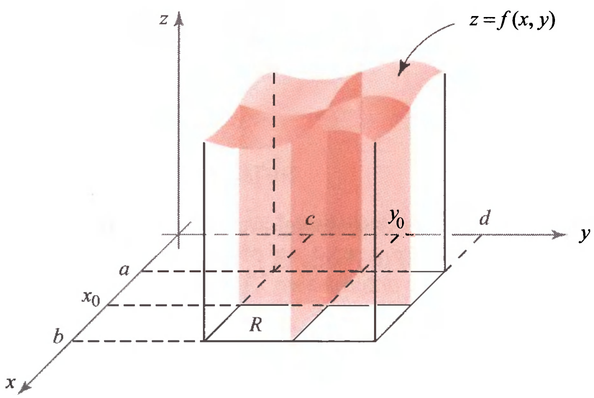

Figure 5.1.8 Two different cross sections sweeping out the volume under $z = f(x, y)$.

When we fix $x = x_0$, we obtain the function $y \mapsto f(x_0, y)$, which is continuous on $[c, d]$. The cross-sectional area $A(x_0)$ is, therefore, equal to the integral $\int_c^d f(x_0, y)\,dy$. Thus, the cross-sectional area function A has domain $[a, b]$, and is given by the rule $A\colon x \mapsto \int_c^d f(x, y)\,dy$. By Cavalieri's principle, the volume V of the region under $z = f(x, y)$ must be equal to

$$V = \int_a^b A(x)\,dx = \int_a^b \left[\int_a^d f(x, y)\,dy\right] dx.$$

The integral $\int_a^b \left[\int_c^d f(x, y)\,dy\right] dx$ is known as an ***iterated integral*** because it is obtained by integrating with respect to y and then integrating the result with respect to x. Because $\iint_R f(x, y)\,dA$ is equal to the volume V, we get the following result.

Double and Iterated Integrals

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx. \tag{1}$$

If we use cutting planes perpendicular to the y axis, we obtain

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy. \tag{2}$$

The expression on the right of formula (2) is the iterated integral obtained by integrating with respect to x and then integrating the result with respect to y.

Thus, if our intuition about volumes is correct, formulas (1) and (2) ought to be valid. This is in fact true when the concepts we are discussing are defined rigorously, and is known as *Fubini's theorem*. We give a proof of this theorem in the next section.

As the following examples illustrate, the notion of the iterated integral and equations (1) and (2) provide a powerful method for *computing* the double integral of a function of two variables.

EXAMPLE 3 Evaluate the integral

$$\iint_R (x^2+y^2)\,dx\,dy,$$

where $R=[-1,1]\times[0,1]$.

SOLUTION By equation (2),

$$\iint_R (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_{-1}^1 (x^2+y^2)\,dx\right] dy.$$

To find $\int_{-1}^1 (x^2+y^2)\,dx$, we treat y as a constant and integrate with respect to x. Because $x^3/3+y^2x$ is an antiderivative of x^2+y^2 with respect to x, we can integrate, using the fundamental theorem of calculus, to obtain

$$\int_{-1}^1 (x^2+y^2)\,dx = \left[\frac{x^3}{3}+y^2x\right]_{x=-1}^1 = \frac{2}{3}+2y^2.$$

Next, we integrate $\frac{2}{3}+2y^2$ with respect to y from 0 to 1, to obtain

$$\int_0^1 \left(\frac{2}{3}+2y^2\right) dy = \left[\frac{2}{3}y+\frac{2}{3}y^3\right]_{y=0}^1 = \frac{4}{3}.$$

Hence, the volume of the solid we saw in Figure 5.1.3 is $4/3$.

For completeness, let us evaluate $\iint_R (x^2+y^2)\,dx\,dy$ using equation (1)—that is, integrating with respect to y first and then with respect to x. We have

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left[\int_0^1 (x^2+y^2)\,dy\right]dx.$$

Treating x as a constant in the y integration, we obtain

$$\int_0^1 (x^2+y^2)\,dy = \left[x^2y+\frac{y^3}{3}\right]_{y=0}^{1} = x^2+\frac{1}{3}.$$

Next, we evaluate $\int_{-1}^{1}\left(x^2+\frac{1}{3}\right)dx$ to obtain

$$\int_{-1}^{1}\left(x^2+\frac{1}{3}\right)dx = \left[\frac{x^3}{3}+\frac{x}{3}\right]_{x=-1}^{1} = \frac{4}{3},$$

which agrees with our previous answer. ▲

EXAMPLE 4 Compute the double integral $\iint_S \cos x \sin y\,dx\,dy$, where S is the square $[0,\pi/2]\times[0,\pi/2]$ (see Figure 5.1.9).

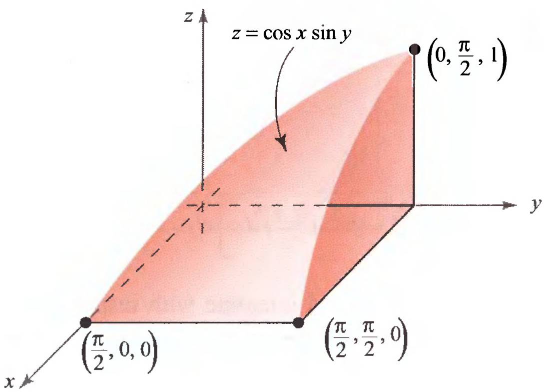

Figure 5.1.9 Volume under $z=\cos x\sin y$ and over the rectangle $[0,\pi/2]\times[0,\pi/2]$.

SOLUTION By equation (2),

$$\iint_S \cos x\sin y\,dx\,dy = \int_0^{\pi/2}\left[\int_0^{\pi/2}\cos x\sin y\,dx\right]dy$$

$$= \int_0^{\pi/2}\sin y\left[\int_0^{\pi/2}\cos x\,dx\right]dy = \int_0^{\pi/2}\sin y\,dy = 1. \quad ▲$$

In the next section, we shall use Riemann sums to rigorously define the double integral for a large class of functions of two variables without recourse to the notion of volume. Although we shall drop the requirement that $f(x, y) \geq 0$, equations (1) and (2) will remain valid. Therefore, the iterated integral will again provide the key to computing the double integral. In Section 5.3, we treat double integrals over regions more general than rectangles.

Finally, we remark that it is common to delete the brackets in iterated integrals such as equations (1) and (2) and write

$$\int_a^b \int_c^d f(x, y)\, dy\, dx \qquad \text{in place of} \qquad \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy \qquad \text{in place of} \qquad \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy.$$

EXERCISES

1. Evaluate the following iterated integrals:

(a) $\int_{-1}^1 \int_0^1 (x^4 y + y^2)\, dy\, dx$

(b) $\int_0^{\pi/2} \int_0^1 (y \cos x + 2)\, dy\, dx$

(c) $\int_0^1 \int_0^1 (xye^{x+y})\, dy\, dx$

(d) $\int_{-1}^0 \int_1^2 (-x \log y)\, dy\, dx$

2. Evaluate the integrals in Exercise 1 by integrating with respect to x and then with respect to y. [The solution to part (b) only is in the Study Guide to this text.]

3. Use Cavalieri's principle to show that the volumes of two cylinders with the same base and height are equal (see Figure 5.1.10).

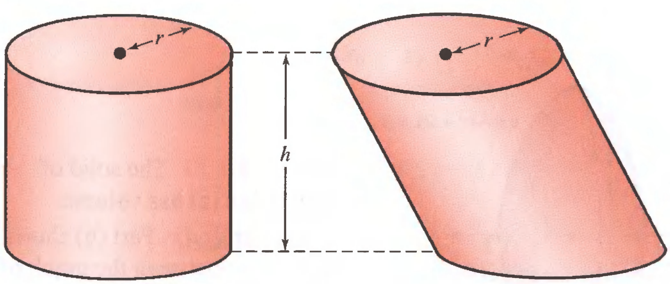

Figure 5.1.10 Two cylinders with the same base and height have the same volume.

4. Using Cavalieri's principle, compute the volume of the structure shown in Figure 5.1.11; each cross section is a rectangle of length 5 and width 3.

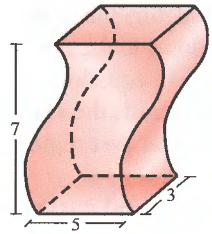

Figure 5.1.11 Compute this volume.

5. A lumberjack cuts out a wedge-shaped piece W of a cylindrical tree of radius r obtained by making two saw cuts to the tree's center, one horizontally and one at an angle θ. Compute the volume of the wedge W using Cavalieri's principle. (See Figure 5.1.12.)

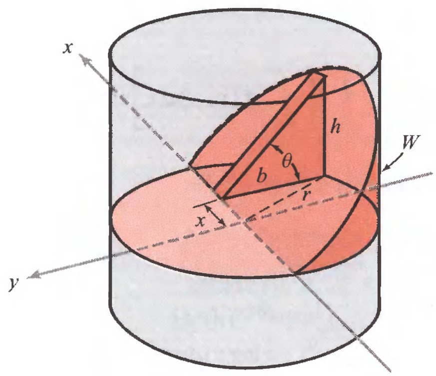

Figure 5.1.12 Find the volume of W.

6. (a) Show that the volume of the solid of revolution shown in Figure 5.1.13(a) is

$$\pi \int_a^b [f(x)]^2\, dx.$$

(b) Show that the volume of the region obtained by rotating the region under the graph of the parabola $y = -x^2 + 2x + 3$, $-1 \le x \le 3$, about the x axis is $512\pi/15$ [see Figure 5.1.13(b)].

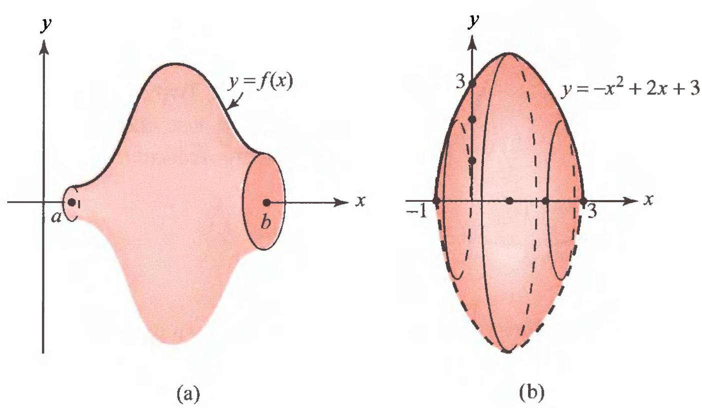

Figure 5.1.13 The solid of revolution (a) has volume $\pi \int_a^b [f(x)]^2\, dx$. Part (b) shows the region between the graph of $y = -x^2 + 2x + 3$ and the x axis rotated about the x axis.

Evaluate the double integrals in Exercises 7 to 9, where R is the rectangle $[0, 2] \times [-1, 0]$.

7. $\displaystyle\iint_R (x^2y^2 + x)\, dy\, dx$

8. $\displaystyle\iint_R \left(|y| \cos \frac{1}{4}\pi x\right) dy\, dx$

9. $\displaystyle\iint_R \left(-xe^x \sin \frac{1}{2}\pi y\right) dy\, dx$

10. Find the volume bounded by the graph of $f(x, y) = 1 + 2x + 3y$, the rectangle $[1, 2] \times [0, 1]$, and the four vertical sides of the rectangle R, as in Figure 5.1.1.

11. Repeat Exercise 10 for the function $f(x, y) = x^4 + y^2$ and the rectangle $[-1, 1] \times [-3, -2]$.

5.2 The Double Integral Over a Rectangle

We are ready to give a rigorous definition of the double integral as the limit of a sequence of sums. This will then be used to *define* the volume of the region under the graph of a function $f(x, y)$. We shall not require that $f(x, y) \geq 0$; but if $f(x, y)$ assumes negative values, we shall interpret the integral as a signed volume, just as for the area under the graph of a function of one variable. In addition, we shall discuss some of the fundamental algebraic properties of the double integral and prove Fubini's theorem, which states that the double integral can be calculated as an iterated integral. To begin, let us establish some notation for partitions and sums.

Definition of the Integral

Consider a closed rectangle $R \subset \mathbb{R}^2$; that is, R is a Cartesian product of two intervals: $R = [a, b] \times [c, d]$. By a ***regular partition*** of R of order n we mean the two ordered collections of $n + 1$ equally spaced points $\{x_j\}_{j=0}^n$ and $\{y_k\}_{k=0}^n$, that is, the points satisfying

$$a = x_0 < x_1 < \cdots < x_n = b, \qquad c = y_0 < y_1 < \cdots < y_n = d$$

and

$$x_{j+1} - x_j = \frac{b - a}{n}, \qquad y_{k+1} - y_k = \frac{d - c}{n}$$

(see Figure 5.2.1).

A function $f(x, y)$ is said to be ***bounded*** if there is a number $M > 0$ such that $-M \leq f(x, y) \leq M$ for all (x, y) in the domain of f. A continuous function on a *closed* rectangle is always bounded, but, for example, $f(x, y) = 1/x$ on $(0, 1] \times [0, 1]$ is continuous but is not bounded, because $1/x$ becomes arbitrarily large for x near 0. The rectangle $(0, 1] \times [0, 1]$ is not closed, because the endpoint 0 is missing in the first factor.

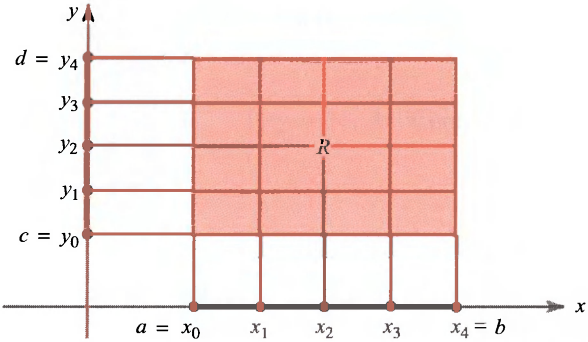

Figure 5.2.1 A regular partition of a rectangle R, with $n = 4$.

Let R_{jk} be the rectangle $[x_j, x_{j+1}] \times [y_k, y_{k+1}]$, and let $\mathbf{c}_{jk}$ be *any* point in R_{jk}. Suppose $f: R \to \mathbb{R}$ is a bounded real-valued function. Form the sum

$$S_n = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta x\, \Delta y = \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta A, \tag{1}$$

where

$$\Delta x = x_{j+1} - x_j = \frac{b-a}{n}, \qquad \Delta y = y_{k+1} - y_k = \frac{d-c}{n},$$

and

$$\Delta A = \Delta x\, \Delta y.$$

This sum is taken over all j's and k's from 0 to $n-1$, and so there are n^2 terms. A sum of this type is called a ***Riemann sum*** for f.

DEFINITION: Double Integral If the sequence $\{S_n\}$ converges to a limit S as $n \to \infty$ and if the limit S is the same for any choice of points $\mathbf{c}_{jk}$ in the rectangles R_{jk}, then we say that f is ***integrable*** over R and we write

$$\iint_R f(x, y)\, dA, \qquad \iint_R f(x, y)\, dx\, dy, \qquad \text{or} \qquad \iint_R f\, dx\, dy$$

for the limit S.

Thus, we can rewrite integrability in the following way:

$$\operatorname*{limit}_{n\to\infty} \sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta x\, \Delta y = \iint_R f\, dx\, dy$$

for any choice of $\mathbf{c}_{jk} \in R_{jk}$.

Properties of the Integral

The proof of the following basic theorem is presented in the Internet supplement for Chapter 5.

THEOREM 1 Any continuous function defined on a closed rectangle R is integrable.

If $f(x, y) \geq 0$, the existence of $\text{limit}_{n\to\infty} S_n$ has a straightforward geometric meaning. Consider the graph of $z = f(x, y)$ as the top of a solid whose base is the rectangle R. If we take each $\mathbf{c}_{jk}$ to be a point where $f(x, y)$ has its minimum value[2] on R_{jk}, then $f(\mathbf{c}_{jk})\,\Delta x\,\Delta y$ represents the volume of a rectangular box with base R_{jk}. The sum $\sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\,\Delta x\,\Delta y$ equals the volume of an inscribed solid, part of which is shown in Figure 5.2.2.

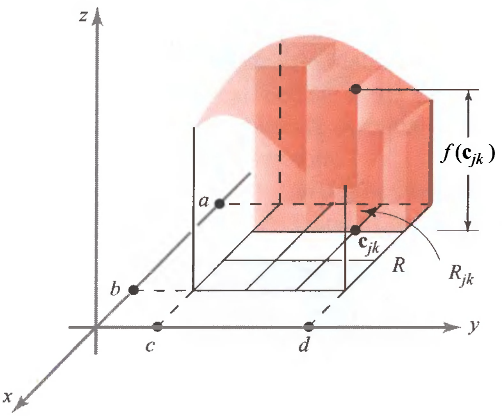

Figure 5.2.2 The sum of inscribed boxes approximates the volume under the graph of $z = f(x, y)$.

Similarly, if $\mathbf{c}_{jk}$ is a point where $f(x, y)$ has its maximum on R_{jk}, then the sum $\sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\,\Delta x\,\Delta y$ is equal to the volume of a circumscribed solid (see Figure 5.2.3).

Therefore, if $\text{limit}_{n\to\infty} S_n$ exists and is independent of $\mathbf{c}_{jk} \in R_{jk}$, it follows that the volumes of the inscribed and circumscribed solids approach the same limit as $n \to \infty$. It is therefore reasonable to call this limit the exact volume of the solid under the graph of f. Thus, the method of Riemann sums supports the concepts introduced on an intuitive basis in Section 5.1.

There is a theorem guaranteeing the existence of the integral of certain discontinuous functions as well. We shall need this result in the next section in order to discuss the integrals of functions over regions more general than rectangles. We shall be specifically interested in functions whose discontinuities lie on curves in the xy plane. Figure 5.2.4 shows two functions defined on a rectangle R whose discontinuities

[2]Such $\mathbf{c}_{jk}$ exist by virtue of the continuity of f on R; see Theorem 7 in Section 3.3.

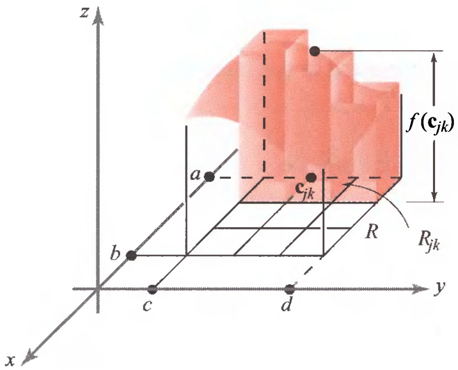

Figure 5.2.3 The volume of circumscribed boxes also approximates the volume under $z = f(x, y)$.

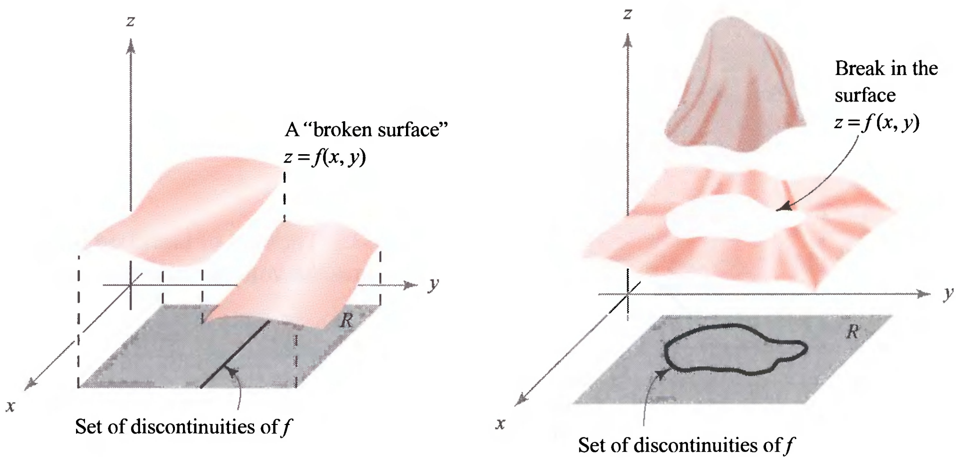

Figure 5.2.4 What the graphs of discontinuous functions of two variables might look like.

lie along curves. In other words, f is continuous at each point that is in R, but not necessarily on the curve.

Useful curves are graphs of functions such as $y = \phi(x)$, $a \le x \le b$, or $x = \psi(y)$, $c \le y \le d$, or finite unions of such graphs. Some examples are shown in Figure 5.2.5.

The next theorem provides an important criterion for determining whether a function is integrable. The proof is discussed in the Internet supplement.

THEOREM 2: Integrability of Bounded Functions Let $f : R \to \mathbb{R}$ be a bounded real-valued function on the rectangle R, and suppose that the set of points where f is discontinuous lies on a finite union of graphs of continuous functions. Then f is integrable over R.

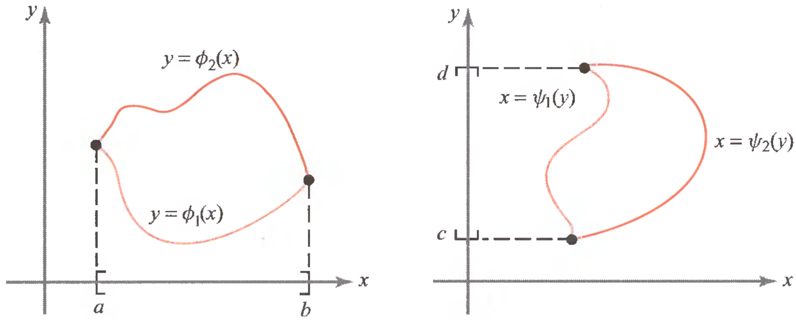

Figure 5.2.5 Curves in the plane represented as graphs.

Using Theorem 2 and the remarks preceding it, we see that the functions sketched in Figure 5.2.4 are integrable over R, because these functions are bounded and continuous except on graphs of continuous functions.

From the definition of the integral as a limit of sums and the limit theorems, we can deduce some fundamental properties of the integral $\iint_R f(x,y)\,dA$; these properties are essentially the same as for the integral of a real-valued function of a single variable.

Let f and g be integrable functions on the rectangle R, and let c be a constant. Then $f+g$ and cf are integrable, and

(i) Linearity

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

(ii) Homogeneity

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

(iii) Monotonicity If $f(x,y) \geq g(x,y)$, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

(iv) Additivity If R_i, $i=1,\dots,m$, are pairwise disjoint rectangles such that f is bounded and integrable over each R_i and if $Q = R_1 \cup R_2 \cup \cdots \cup R_m$ is a rectangle, then $f\colon Q \to \mathbb{R}$ is integrable over Q and

$$\iint_Q f(x,y)\,dA = \sum_{i=1}^{m} \iint_{R_i} f(x,y)\,dA.$$

Properties (i) and (ii) are a consequence of the definition of the integral as a limit of a sum and the following facts for convergent sequences $\{S_n\}$ and $\{T_n\}$, which are proved as with the limit theorems in Chapter 2:

$$\operatorname*{limit}_{n\to\infty} (T_n + S_n) = \operatorname*{limit}_{n\to\infty} T_n + \operatorname*{limit}_{n\to\infty} S_n$$

$$\operatorname*{limit}_{n\to\infty} (cS_n) = c \operatorname*{limit}_{n\to\infty} S_n.$$

To demonstrate monotonicity, we first observe that if $h(x, y) \geq 0$ and $\{S_n\}$ is a sequence of Riemann sums that converges to $\iint_R h(x, y)\, dA$, then $S_n \geq 0$ for all n, so that $\iint_R h(x, y)\, dA = \text{limit}_{n\to\infty} S_n \geq 0$. If $f(x, y) \geq g(x, y)$ for all $(x, y) \in R$, then $(f - g)(x, y) \geq 0$ for all (x, y), and, using properties (i) and (ii), we have

$$\iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA = \iint_R [f(x, y) - g(x, y)]\, dA \geq 0.$$

This proves property (iii). The proof of property (iv) is more technical and a special case is proved in the Internet supplement. It should be intuitively obvious.

Another important result is the inequality

$$\left| \iint_R f\, dA \right| \leq \iint_R |f|\, dA. \tag{2}$$

To see why formula (2) is true, note that, by the definition of absolute value,

$$-|f| \leq f \leq |f|;$$

therefore, from the monotonicity and homogeneity of integration (with $c = -1$),

$$-\iint_R |f|\, dA \leq \iint_R f\, dA \leq \iint_R |f|\, dA,$$

which is equivalent to formula (2).

Fubini's Theorem

Although we have noted the integrability of a variety of functions, we have not yet established rigorously a general method of computing integrals. In the case of one variable, we avoid computing $\int_a^b f(x)\, dx$ from its definition as a limit of a sum by using the *fundamental theorem of integral calculus*. This important theorem tells us that *if f is continuous, then*

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where F is an antiderivative of f; that is, $F' = f$.

This technique will not work as stated for functions $f(x, y)$ of two variables. However, as we indicate in Section 5.1, we can often reduce a double integral over a rectangle to iterated single integrals; the fundamental theorem then applies to each of these single integrals. Fubini's theorem, which was mentioned in the last section, establishes this reduction to iterated integrals rigorously, by using Riemann sums. As we saw in Section 5.1, the reduction,

$$\iint_R f(x, y)\, dA = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy,$$

is a consequence of Cavalieri's principle, at least if $f(x, y) \geq 0$. In terms of Riemann sums, it corresponds to the following equality:

$$\sum_{j,k=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta x\, \Delta y = \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta y\right) \Delta x = \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} f(\mathbf{c}_{jk})\, \Delta x\right) \Delta y,$$

which may be proved more generally as follows: *Let* $[a_{jk}]$ *be an* $n \times n$ *matrix, where* $0 \leq j \leq n-1$ *and* $0 \leq k \leq n-1$. *Let* $\sum_{j,k=0}^{n-1} a_{jk}$ *be the sum of the* n^2 *matrix entries. Then*

$$\sum_{j,k=0}^{n-1} a_{jk} = \sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} a_{jk}\right) = \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} a_{jk}\right). \tag{3}$$

In the first equality, the right-hand side represents summing the matrix entries first by rows and then adding the results:

$$\begin{array}{cc}
\begin{bmatrix}
a_{00} & a_{01} & a_{02} & \cdots & \overrightarrow{a_{0k} \quad \cdots \quad a_{0(n-1)}} \\
\vdots & & & & \vdots \\
a_{j0} & a_{j1} & & \cdots & \overrightarrow{a_{jk} \quad \cdots \quad a_{j(n-1)}} \\
\vdots & & & & \vdots \\
a_{(n-1)0} & a_{(n-1)1} & & \cdots & \overrightarrow{a_{(n-1)k} \quad \cdots \quad a_{(n-1)(n-1)}}
\end{bmatrix}
&
\begin{array}{l}
\displaystyle\sum_{k=0}^{n-1} a_{0k} \\
\vdots \\
\displaystyle\sum_{k=0}^{n-1} a_{jk} \\
\vdots \\
\displaystyle\sum_{k=0}^{n-1} a_{(n-1)k} \\
\hline
\displaystyle\sum_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} a_{jk}\right).
\end{array}
\end{array}$$

Clearly, this is equal to $\sum_{j,k=0}^{n-1} a_{jk}$, that is, the sum of all the a_{jk}. Similarly, the sum $\sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} a_{jk}\right)$ represents a summing of the matrix entries by columns. This establishes equation (3) and makes the reduction to iterated integrals quite plausible if we remember that integrals can be approximated by the corresponding Riemann sums. The actual proof of Fubini's theorem exploits this idea.

THEOREM 3: Fubini's Theorem Let f be a continuous function with a rectangular domain $R = [a, b] \times [c, d]$. Then

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy = \iint_R f(x, y)\, dA. \tag{4}$$

PROOF We shall first show that

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \iint_R f(x, y)\, dA.$$

Let $c = y_0 < y_1 < \cdots < y_n = d$ be a partition of $[c, d]$ into n equal parts. Define

$$F(x) = \int_c^d f(x, y)\, dy.$$

Then

$$F(x) = \sum_{k=0}^{n-1} \int_{y_k}^{y_{k+1}} f(x, y)\, dy.$$

Using the integral version of the mean-value theorem,[3] for each fixed x and for each k we have

$$\int_{y_k}^{y_{k+1}} f(x, y)\, dy = f(x, Y_k(x))(y_{k+1} - y_k)$$

(see Figure 5.2.6), where the point $Y_k(x)$ belongs to $[y_k, y_{k+1}]$ and may depend on x, k, and n.

We have thus shown that

$$F(x) = \sum_{k=0}^{n-1} f(x, Y_k(x))(y_{k+1} - y_k). \tag{5}$$

---

[3]This states that if $g(x)$ is continuous on $[a, b]$, then $\int_a^b g(x)\, dx = g(c)(b - a)$ for some point $c \in [a, b]$. The more general second mean-value theorem was proved in Section 3.2.

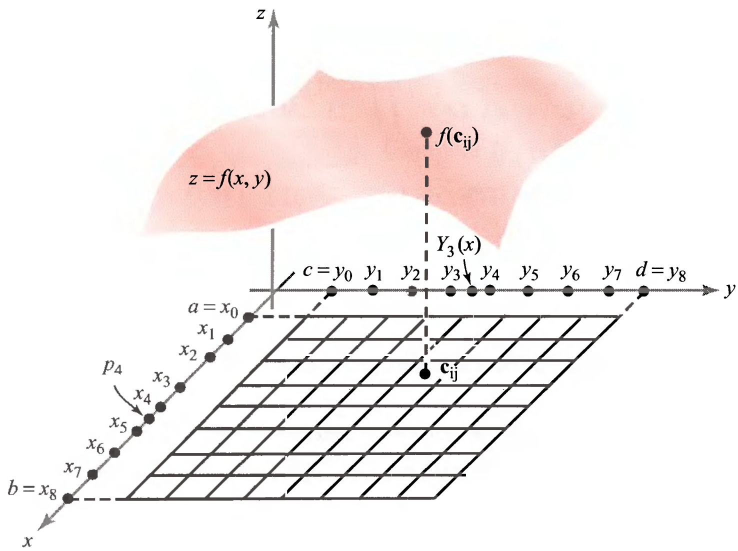

Figure 5.2.6 The notation needed in the proof of Fubini's theorem; $n = 8$.

By the definition of the integral in one variable as a limit of Riemann sums,

$$\int_a^b F(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \operatorname*{limit}_{n\to\infty} \sum_{j=0}^{n-1} F(p_j)(x_{j+1} - x_j),$$

where $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the interval $[a, b]$ into n equal parts and p_j is any point in $[x_j, x_{j+1}]$. Setting $\mathbf{c}_{jk} = (p_j, Y_k(p_j)) \in R_{jk}$, we have [substituting p_j for x in equation (5)]

$$F(p_j) = \sum_{k=0}^{n-1} f(\mathbf{c}_{jk})(y_{k+1} - y_k).$$

Therefore,

$$\begin{aligned}\int_a^b \int_c^d f(x,y)\,dy\,dx &= \int_a^b F(x)\,dx = \operatorname*{limit}_{n\to\infty} \sum_{j=0}^{n-1} F(p_j)(x_{j+1} - x_j)\\ &= \operatorname*{limit}_{n\to\infty} \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(\mathbf{c}_{jk})(y_{k+1} - y_k)(x_{j+1} - x_j)\\ &= \iint_R f(x,y)\,dA.\end{aligned}$$

Thus, we have proved that

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \iint_R f(x, y)\, dA.$$

By the same reasoning we can show that

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \iint_R f(x, y)\, dA.$$

These two conclusions are exactly what we wanted to prove. ■

Fubini's theorem can be generalized to the case where f is not necessarily continuous. Although we shall not present a proof, we state here this more general version.

THEOREM 3′: Fubini's Theorem Let f be a bounded function with domain a rectangle $R = [a, b] \times [c, d]$, and suppose the discontinuities of f lie on a finite union of graphs of continuous functions. If the integral $\int_c^d f(x, y)\, dy$ exists for each $x \in [a, b]$, then

$$\int_a^b \left[\int_a^d f(x, y)\, dy\right] dx$$

exists and

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \iint_R f(x, y)\, dA.$$

Similarly, if $\int_a^b f(x, y)\, dx$ exists for each $y \in [c, d]$, then

$$\int_c^d \left[\int_a^b f(x, y)\, dx\right] dy$$

exists and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \iint_R f(x, y)\, dA.$$

Thus, if all these conditions hold simultaneously,

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy = \iint_R f(x, y)\, dA.$$

The assumptions made for this version of Fubini's theorem are more complicated than those we made in Theorem 3. They are necessary because if f is not continuous everywhere, for example, there is no guarantee that $\int_c^d f(x, y)\, dy$ will exist for each x.

EXAMPLE 1 Compute $\iint_R (x^2 + y)\, dA$, where R is the square $[0, 1] \times [0, 1]$.

SOLUTION By Fubini's theorem,

$$\iint_R (x^2 + y)\, dA = \int_0^1 \int_0^1 (x^2 + y)\, dx\, dy = \int_0^1 \left[\int_0^1 (x^2 + y)\, dx \right] dy.$$

By the fundamental theorem of calculus, the x integration may be performed:

$$\int_0^1 (x^2 + y)\, dx = \left[\frac{x^3}{3} + yx \right]_{x=0}^{1} = \frac{1}{3} + y.$$

Thus,

$$\iint_R (x^2 + y)\, dA = \int_0^1 \left[\frac{1}{3} + y \right] dy = \left[\frac{1}{3} y + \frac{y^2}{2} \right]_0^1 = \frac{5}{6}.$$

What we have done is hold y fixed, integrate with respect to x, and then evaluate the result between the given limits for the x variable. Next, we integrated the remaining function (of y alone) with respect to y to obtain the final answer. ▲

EXAMPLE 2 A consequence of Fubini's theorem is that interchanging the order of integration in the iterated integrals does not change the answer. Verify this for Example 1.

SOLUTION We carry out the integration in the other order:

$$\int_0^1 \int_0^1 (x^2 + y)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{y=0}^{1} dx = \int_0^1 \left[x^2 + \frac{1}{2} \right] dx$$

$$= \left[\frac{x^3}{3} + \frac{x}{2} \right]_0^1 = \frac{5}{6}. \quad ▲$$

We have seen that when $f(x, y) \geq 0$ on $R = [a, b] \times [c, d]$, the integral $\iint_R f(x, y)\, dA$ can be interpreted as a volume. If the function also takes on negative values, then the double integral can be thought of as the sum of all volumes lying between the surface $z = f(x, y)$ and the plane $z = 0$, bounded by the planes $x = a$, $x = b$, $y = c$, and $y = d$; here the volumes above $z = 0$ are counted as positive and those below as negative. However, Fubini's theorem as stated remains valid in the case where $f(x, y)$ is negative or changes sign on R; that is, there is no restriction on the sign of f in the hypotheses of the theorem.

EXAMPLE 3 Let R be the rectangle $[-2, 1] \times [0, 1]$ and let f be defined by $f(x, y) = y(x^3 - 12x)$; $f(x, y)$ takes on both positive and negative values on R. Evaluate the integral $\iint_R f(x, y)\,dx\,dy = \iint_R y(x^3 - 12x)\,dx\,dy$.

SOLUTION By Fubini's theorem, we may write

$$\iint_R y(x^3 - 12x)\,dx\,dy = \int_0^1 \left[\int_{-2}^1 y(x^3 - 12x)\,dx\right] dy = \frac{57}{4}\int_0^1 y\,dy = \frac{57}{8}.$$

Alternatively, integrating first with respect to y, we find

$$\begin{aligned}\iint_R y(x^3 - 12x)\,dy\,dx &= \int_{-2}^1 \left[\int_0^1 (x^3 - 12x)y\,dy\right] dx \\ &= \frac{1}{2}\int_{-2}^1 (x^3 - 12x)\,dx = \frac{1}{2}\left[\frac{x^4}{4} - 6x^2\right]_{-2}^1 = \frac{57}{8}.\end{aligned}$$ ▲

Historical Note

The Riemann Integral

The first time most mathematics students encounter the name of Bernhard Riemann is in their calculus courses, where they read about the Riemann integral. Leibniz had thought of the integral of a function of one variable as an infinite sum (the $\int$ standing for a sum) of infinitesimal areas $f(x)\,dx$, where dx is an "infinitesimal width" and $f(x)$ is the height of the corresponding "infinitesimally thin" rectangle. This intuitive approach sufficed for most purposes because the fundamental theorem

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

showed how to evaluate this (nebulously defined) integral when one knows the antiderivative F of f.

However, Riemann was interested in applying integration to functions of one variable where the antiderivative was not known, and to functions in number theory or in general to those functions that "one need not find in nature."

Cauchy had already known that all continuous functions could be integrated and that the fundamental theorem was valid—that is, every continuous function had an antiderivative. However, his proofs were not entirely rigorous. For applications to number theory and to certain series (called *Fourier series*), Riemann needed a clear, precise definition of the integral, which he presented in a paper in 1854. In this paper he defines his integral and gives necessary and sufficient conditions for a bounded function f to be integrable over an interval $[a, b]$.

In 1876, the German mathematician Karl J. Thomae generalized Riemann's integral to apply to functions of several variables, as we do in this chapter. We further develop this approach in the Internet supplement.

In the first half of the nineteenth century, Cauchy had observed that for continuous function of two variables, Fubini's theorem was valid. But Cauchy also gave an example of an unbounded function of two variables for which the iterated integrals were not equal. In 1878, Thomae gave the first example of a bounded function of two variables where one iterated integral exists and the other does not. In these examples, the functions were not "Riemann integrable" in the sense described in this section. Cauchy and Thomae's examples demonstrated that one must apply caution and not necessarily assume that iterated integrals are always equal.

In 1902, the French mathematician Henri Lebesgue developed a truly sweeping generalization of the Riemann integral. Lebesgue's theory allowed integration of vastly more functions than did Riemann's approach. Perhaps, unforeseen by Lebesgue, his theory was to have a profound impact on the development of many areas of mathematics in the twentieth century—in particular the theory of partial differential equations. Mathematics students go into more depth about the Lebesgue integral in their first year of graduate study.

In 1907, the Italian mathematician Guido Fubini used the Lebesgue integral to state the most general form of the theorem on the equality of iterated integrals, the form that is studied today and used by working mathematicians and scientists in their research.

EXERCISES

1. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\iint_R (x^3 + y^2)\, dA$

(b) $\iint_R ye^{xy}\, dA$

(c) $\iint_R (xy)^2 \cos x^3\, dA$

(d) $\iint_R \ln[(x + 1)(y + 1)]\, dA$

2. Evaluate each of the following integrals if $R = [0, 1] \times [0, 1]$.

(a) $\iint_R (x^m y^n)\, dx\, dy$, where $m, n > 0$

(b) $\iint_R (ax + by + c)\, dx\, dy$

(c) $\iint_R \sin(x + y)\, dx\, dy$

(d) $\iint_R (x^2 + 2xy + y\sqrt{x})\, dx\, dy$

3. Compute the volume of the region over the rectangle $[0, 1] \times [0, 1]$ and under the graph of $z = xy$.

4. Compute the volume of the solid bounded by the xz plane, the yz plane, the xy plane, the planes $x = 1$ and $y = 1$, and the surface $z = x^2 + y^4$.

5. Let f be continuous on $[a, b]$ and g continuous on $[c, d]$. Show that

$$\iint_R [f(x)g(y)]\, dx\, dy = \left[\int_a^b f(x)\, dx\right]\left[\int_c^d g(y)\, dy\right],$$

where $R = [a, b] \times [c, d]$.

6. Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the xy plane.

7. Compute the volume of the solid bounded by the graph $z = x^2 + y$, the rectangle $R = [0, 1] \times [1, 2]$, and the "vertical sides" of R.

8. Let f be continuous on $R = [a, b] \times [c, d]$; for $a < x < b, c < y < d$, define

$$F(x, y) = \int_a^x \int_c^y f(u, v)\, dv\, du.$$

Show that $\partial^2 F/\partial x\, \partial y = \partial^2 F/\partial y\, \partial x = f(x, y)$. Use this example to discuss the relationship between Fubini's theorem and the equality of mixed partial derivatives.

9. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational.} \end{cases}$$

Show that the iterated integral $\int_0^1 \left[\int_0^1 f(x, y)\, dy\right] dx$ exists but that f is not integrable.

10. Express $\iint_R \cosh xy\, dx\, dy$ as a convergent sequence, where $R = [0, 1] \times [0, 1]$.

11. Although Fubini's theorem holds for most functions met in practice, one must still exercise some caution. This exercise gives a function for which it fails. By using a substitution involving the tangent function, show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx = \frac{\pi}{4}, \qquad \text{yet} \qquad \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy = -\frac{\pi}{4}.$$

Why does this not contradict Theorem 3 or 3′?

12. Let f be continuous, $f \geq 0$, on the rectangle R. If $\iint_R f\, dA = 0$, prove that $f = 0$ on R.

5.3 The Double Integral Over More General Regions

Our goal in this section is twofold: First, we wish to define the double integral of a function $f(x, y)$ over regions D more general than rectangles; second, we want to develop a technique for evaluating this type of integral. To accomplish this, we shall define three special types of subsets of the xy plane, and then extend the notion of the double integral to them.

Elementary Regions

Suppose we are given two continuous real-valued functions $\phi_1: [a, b] \to \mathbb{R}$ and $\phi_2: [a, b] \to \mathbb{R}$ that satisfy $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a, b]$. Let D be the set of all points (x, y) such that $x \in [a, b]$ and $\phi_1(x) \leq y \leq \phi_2(x)$. This region D is said to be ***y-simple***. Figure 5.3.1 shows various examples of y-simple regions. The curves and straight-line segments that bound the region together constitute the ***boundary*** of D, denoted ∂D. We use the phrase y-simple because the region is described in a relatively simple way, using y as a function of x.

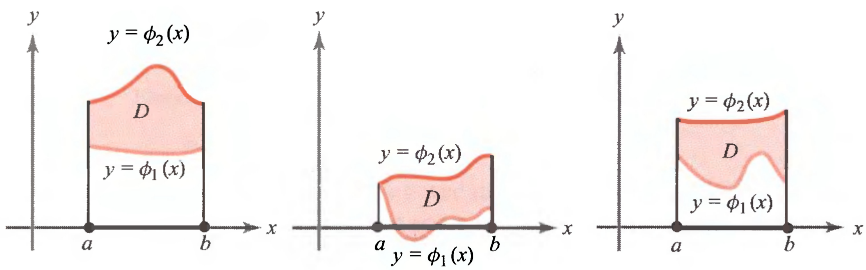

Figure 5.3.1 Some y-simple regions.

We say that a region D is ***x-simple*** if there are continuous functions ψ_1 and ψ_2 defined on $[c, d]$ such that D is the set of points (x, y) satisfying

$$y \in [c, d] \qquad \text{and} \qquad \psi_1(y) \leq x \leq \psi_2(y)$$

where $\psi_1(y) \leq \psi_2(y)$ for all $y \in [c, d]$. Again, the curves that bound the region D constitute its boundary ∂D. Some examples of x-simple regions are shown in Figure 5.3.2. In this situation, x is the distinguished variable, given as a function of y. Thus, the phrase x-simple is appropriate.

Finally, a ***simple*** region is one that is both x- and y-simple; that is, a simple region can be described as both an x-simple region and a y-simple region. An example of a simple region is a unit disk (see Figure 5.3.3).

Sometimes we will refer to any of the regions as ***elementary regions***. Note that the boundary ∂D of an elementary region is the type of set of discontinuities of a function allowed in Theorem 2.

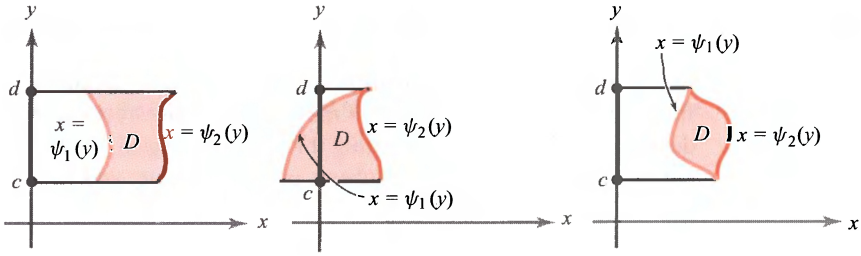

Figure 5.3.2 Some x-simple regions.

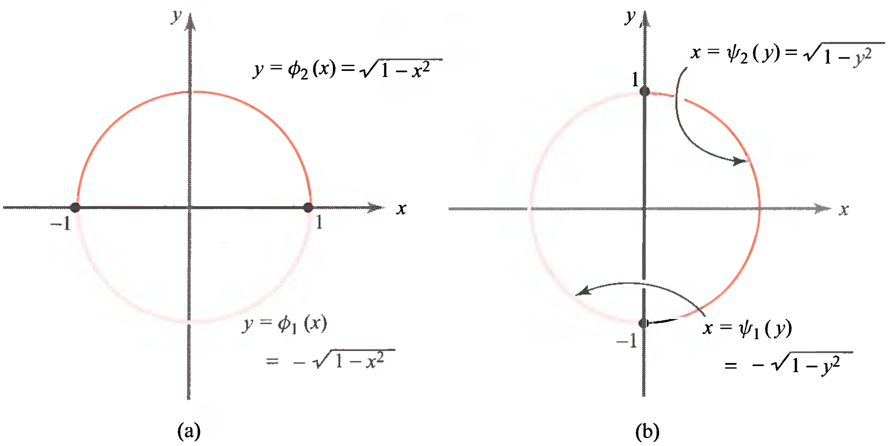

Figure 5.3.3 The unit disk, a simple region: (a) as a y-simple region, and (b) as an x-simple region.

The Integral over an Elementary Region

We can now use an interesting "trick" to extend the definition of the integral from rectangles to elementary regions.

DEFINITION: Integral over an Elementary Region If D is an elementary region in the plane, choose a rectangle R that contains D. Given $f\colon D \to \mathbb{R}$, where f is continuous (and hence bounded), define $\iint_D f(x, y)\, dA$, the ***integral of f over the set*** D as follows: Extend f to a function f^* defined on all of R by

$$f^*(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \text{ and } (x, y) \in R. \end{cases}$$

Note that f^* is bounded (because f is) and continuous except possibly on the boundary of D (see Figure 5.3.4). The boundary of D consists of graphs of continuous functions, and so f^* is integrable over R by Theorem 2, Section 5.2. Therefore, we can define

$$\iint_D f(x, y)\, dA = \iint_R f^*(x, y)\, dA.$$

When $f(x, y) \geq 0$ on D, we can interpret the integral $\iint_D f(x, y)\, dA$ as the volume of the three-dimensional region between the graph of f and D, as is evident from Figure 5.3.4.

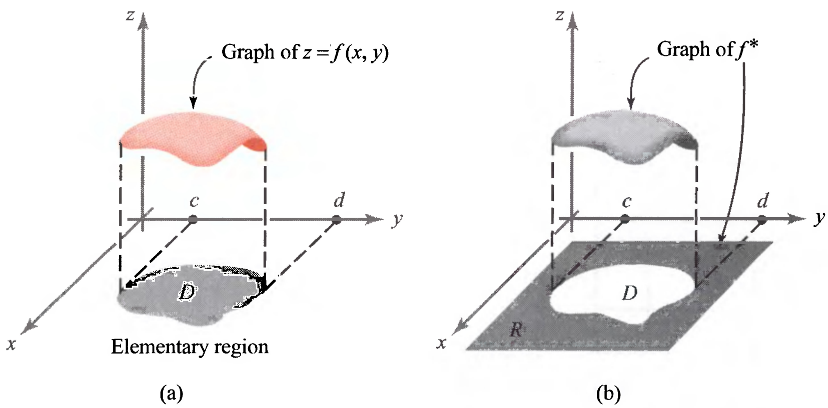

Figure 5.3.4 (a) Graph of $z = f(x, y)$ over an elementary region D. (b) Shaded region shows graph of $z = f^*(x, y)$ on some rectangle R containing D. From this picture we see that boundary points of D may be points of discontinuity of f^*, because the graph of $z = f^*(x, y)$ can be broken at these points.

We have defined $\iint_D f(x, y)\, dx\, dy$ by choosing a rectangle R that encloses D. It should be intuitively clear that the value of $\iint_D f(x, y)\, dx\, dy$ does not depend on the particular R we select; we shall demonstrate this fact at the end of this section.

Reduction to Iterated Integrals

If $R = [a, b] \times [c, d]$ is a rectangle containing D, we can use the results on iterated integrals in Section 5.2 to obtain

$$\begin{aligned}\iint_D f(x, y)\, dA = \iint_R f^*(x, y)\, dA &= \int_a^b \int_c^d f^*(x, y)\, dy\, dx\\ &= \int_c^d \int_a^b f^*(x, y)\, dx\, dy,\end{aligned}$$

where f^* equals f in D and zero outside D, as before. Assume that D is a y-simple region determined by functions $\phi_1: [a, b] \to \mathbb{R}$ and $\phi_2: [a, b] \to \mathbb{R}$.

Consider the iterated integral

$$\int_a^b \int_c^d f^*(x, y)\, dy\, dx$$

and, in particular, the inner integral $\int_c^d f^*(x, y)\, dy$ for some fixed x (Figure 5.3.5). By definition, $f^*(x, y) = 0$ if $y < \phi_1(x)$ or $y > \phi_2(x)$, so we obtain

$$\int_c^d f^*(x, y)\, dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x, y)\, dy = \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy.$$

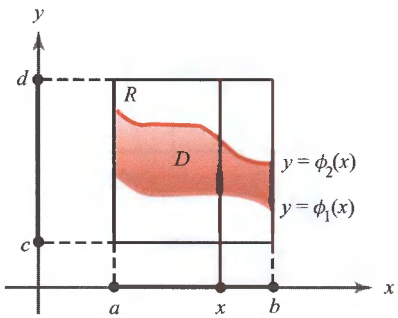

Figure 5.3.5 The region between two graphs—a y-simple region.

We summarize what we have obtained in the following.

THEOREM 4: Reduction to Iterated Integrals If D is a y-simple region, as shown in Figure 5.3.5, then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx. \tag{1}$$

In the case $f(x, y) = 1$ for all $(x, y) \in D$, $\iint_D f(x, y)\, dA$ is the area of D. On the other hand, in this case, the right-hand side of formula (1) becomes:

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx = \int_a^b [\phi_2(x) - \phi_1(x)]\, dx = A(D),$$

which is the formula for the area of D learned in one-variable calculus. Thus, formula (1) checks in this case.

EXAMPLE 1 Find $\iint_T (x^3y + \cos x)\, dA$, where T is the triangle consisting of all points (x, y) such that $0 \le x \le \pi/2$, $0 \le y \le x$.

SOLUTION Referring to Figure 5.3.6 and formula (1), we have

$$\iint_T (x^3y + \cos x)\, dA = \int_0^{\pi/2} \int_0^x (x^3y + \cos x)\, dy\, dx$$

$$= \int_0^{\pi/2} \left[\frac{x^3y^2}{2} + y\cos x \right]_{y=0}^{x} dx = \int_0^{\pi/2} \left(\frac{x^5}{2} + x\cos x \right) dx$$

$$= \left[\frac{x^6}{12} \right]_0^{\pi/2} + \int_0^{\pi/2} (x\cos x)\, dx = \frac{\pi^6}{(12)(64)} + [x\sin x + \cos x]_0^{\pi/2}$$

$$= \frac{\pi^6}{768} + \frac{\pi}{2} - 1. \quad \blacktriangle$$

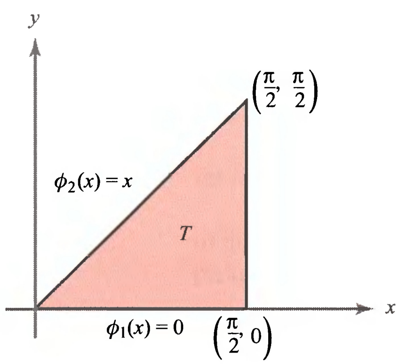

Figure 5.3.6 A triangle T represented as a y-simple region.

In the next example, we use formula (1) to find the volume of a solid whose base is a nonrectangular region D.

EXAMPLE 2 Find the volume of the tetrahedron bounded by the planes $y = 0$, $z = 0$, $x = 0$, and $y - x + z = 1$ (Figure 5.3.7).

SOLUTION We first note that the given tetrahedron has a triangular base D whose points (x, y) satisfy $-1 \le x \le 0$ and $0 \le y \le 1 + x$; hence, D is a y-simple region. In fact, D is a simple region; see Figure 5.3.8.

For any point (x, y) in D, the height of the surface z above (x, y) is $1 - y + x$. Thus, the volume we seek is given by the integral

$$\iint_D (1 - y + x)\, dA.$$

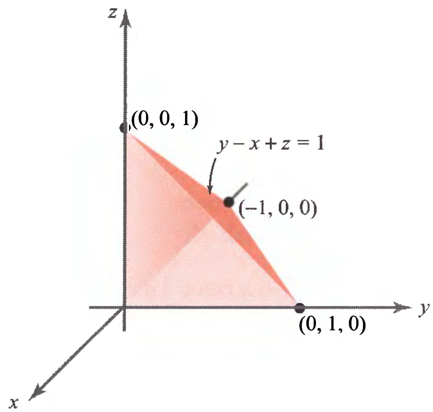

Figure 5.3.7 A tetrahedron bounded by the planes $y = 0, z = 0, x = 0$, and $y - x + z = 1$.

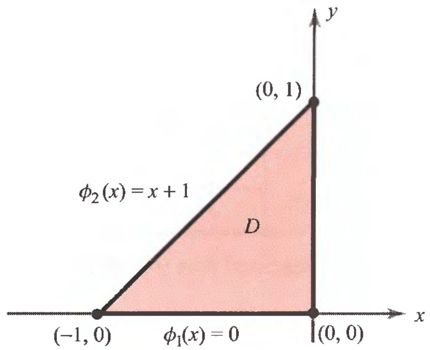

Figure 5.3.8 The base of the tetrahedron in Figure 5.3.7 represented as a y-simple region.

Using formula (1) with $\phi_1(x) = 0$ and $\phi_2(x) = x + 1$, we have

$$\iint_D (1 - y + x)\, dA = \int_{-1}^{0} \int_{0}^{1+x} (1 - y + x)\, dy\, dx = \int_{-1}^{0} \left[(1 + x)y - \frac{y^2}{2}\right]_{y=0}^{1+x} dx$$

$$= \int_{-1}^{0} \left[\frac{(1+x)^2}{2}\right] dx = \left[\frac{(1+x)^3}{6}\right]_{-1}^{0} = \frac{1}{6}. \quad \blacktriangle$$

EXAMPLE 3 Let D be a y-simple region. Describe its area $A(D)$ as a limit of Riemann sums.

SOLUTION If we recall the definition, $A(D) = \iint_D dx\, dy$ is the integral over a containing rectangle R of the function $f = 1$. A Riemann sum S_n for this integral is obtained by dividing R into subrectangles and forming the sum $S_n = \sum_{j,k=0}^{n-1} f^*(\mathbf{c}_{jk})\Delta x\, \Delta y$, as in formula (1) of Section 5.2. Now $f^*(\mathbf{c}_{jk})$ is 1 or 0, depending on whether or not $\mathbf{c}_{jk}$ is in D. Consider those subrectangles R_{jk} that have nonvoid intersection with D, and choose $\mathbf{c}_{jk}$ in $D \cap R_{jk}$. Thus, S_n is the sum of the areas of the subrectangles that meet D and $A(D)$ is the limit of these as $n \to \infty$.

Thus, $A(D)$ is the limit of the areas of the rectangles "circumscribing" D. The reader should draw a figure to accompany this discussion. ▲

The methods for treating x-simple regions are entirely analogous. Specifically, we have the following.

THEOREM 4′: Iterated Integrals for x-Simple Regions Suppose that D is the set of points (x, y) such that $y \in [c, d]$ and $\psi_1(y) \le x \le \psi_2(y)$. If f is continuous on D, then

$$\iint_D f(x, y)\, dA = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\right] dy. \tag{2}$$

To find the area of D, we substitute $f = 1$ in formula (2); this yields

$$\iint_D dA = \int_c^d (\psi_2(y) - \psi_1(y))\, dy.$$

Again, this result for area agrees with the results of single-variable calculus for the area of a region between two curves.

Either the method for y-simple or the method for x-simple regions can be used for integrals over simple regions.

It follows from formulas (1) and (2) that $\iint_D f\, dA$ is independent of the choice of the rectangle R enclosing D used in the definition of $\iint_D f\, dA$, because, if we had picked another rectangle enclosing D, we would have arrived at the same formula (1).

EXERCISES

1. Evaluate the following iterated integrals and draw the regions D determined by the limits. State whether the regions are x-simple, y-simple, or simple.

(a) $\displaystyle\int_0^1 \int_0^{x^2} dy\, dx$

(b) $\displaystyle\int_1^2 \int_{2x}^{3x+1} dy\, dx$

(c) $\displaystyle\int_0^1 \int_1^{e^x} (x + y)\, dy\, dx$

(d) $\displaystyle\int_0^1 \int_{x^3}^{x^2} y\, dy\, dx$

2. Evaluate the following integrals and sketch the corresponding regions.

(a) $\displaystyle\int_{-3}^2 \int_0^{y^2} (x^2 + y)\, dx\, dy$

(b) $\displaystyle\int_{-1}^1 \int_{-2|x|}^{|x|} e^{x+y}\, dy\, dx$

(c) $\displaystyle\int_0^1 \int_0^{(1-x^2)^{1/2}} dy\, dx$

(d) $\displaystyle\int_0^{\pi/2} \int_0^{\cos x} y \sin x\, dy\, dx$

(e) $\displaystyle\int_0^1 \int_{y^2}^{y} (x^n + y^m)\, dx\, dy, \qquad m, n > 0$

(f) $\displaystyle\int_{-1}^0 \int_0^{2(1-x^2)^{1/2}} x\, dy\, dx$

3. Use double integrals to compute the area of a circle of radius r.

4. Using double integrals, determine the area of an ellipse with semiaxes of length a and b.

5. What is the volume of a barn that has a rectangular base 20 ft by 40 ft, vertical walls 30 ft high at the front (which we assume is on the 20-ft side of the barn), and 40 ft high at the rear? The barn has a flat roof. Use double integrals to compute the volume.

6. Let D be the region bounded by the positive x and y axes and the line $3x + 4y = 10$. Compute

$$\iint_D (x^2 + y^2)\, dA.$$

7. Let D be the region bounded by the y axis and the parabola $x = -4y^2 + 3$. Compute

$$\iint_D x^3 y\, dx\, dy.$$

8. Evaluate $\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$. Describe this iterated integral as an integral over a certain region D in the xy plane.

9. Let D be the region given as the set of (x, y) where $1 \leq x^2 + y^2 \leq 2$ and $y \geq 0$. Is D an elementary region? Evaluate $\iint_D f(x, y)\, dA$ where $f(x, y) = 1 + xy$.

10. Use the formula $A(D) = \iint_D dx\, dy$ to find the area enclosed by one period of the sine function $\sin x$, for $0 \leq x \leq 2\pi$, and the x axis.

11. Find the volume of the region inside the surface $z = x^2 + y^2$ and between $z = 0$ and $z = 10$.

12. Set up the integral required to calculate the volume of a cone of base radius r and height h.

13. Evaluate $\iint_D y\, dA$ where D is the set of points (x, y) such that $0 \leq 2x/\pi \leq y, y \leq \sin x$.

14. From Exercise 5, Section 5.2, $\int_a^b \int_a^d f(x)g(y)\, dy\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^d g(y)\, dy\right)$. Is it true that $\iint_D f(x)g(y)\, dx\, dy = \left(\int_a^b f(x)\, dx\right)\left(\int_{\phi_1(a)}^{\phi_2(b)} g(y)\, dy\right)$ for y-simple regions?

15. Let D be a region given as the set of (x, y) with $-\phi(x) \leq y \leq \phi(x)$ and $a \leq x \leq b$, where ϕ is a nonnegative continuous function on the interval $[a, b]$. Let $f(x, y)$ be a function on D such that $f(x, y) = -f(x, -y)$ for all $(x, y) \in D$. Argue that $\iint_D f(x, y)\, dA = 0$.

16. Use the methods of this section to show that the area of the parallelogram D determined by two planar vectors $\mathbf{a}$ and $\mathbf{b}$ is $|a_1 b_2 - a_2 b_1|$, where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$.

17. Describe the area $A(D)$ of a region as a limit of areas of inscribed rectangles, as in Example 3.

5.4 Changing the Order of Integration

Suppose that D is a simple region—that is, it is both x-simple and y-simple. Thus, it can be given as the set of points (x, y) such that

$$a \leq x \leq b, \qquad \phi_1(x) \leq y \leq \phi_2(x),$$

and also as the set of points (x, y) such that

$$c \leq y \leq d, \qquad \psi_1(y) \leq x \leq \psi_2(y).$$

Hence, we have the formulas

$$\iint_D f(x, y)\, dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy.$$

If we are required to compute one of the preceding iterated integrals, we may do so by evaluating the other iterated integral; this technique is called *changing the order of integration*. It can be useful to make such a change when evaluating iterated integrals, because one of the iterated integrals may be more difficult to compute than the other.

EXAMPLE 1 By changing the order of integration, evaluate

$$\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2}\, dy\, dx.$$

SOLUTION Note that x varies between 0 and a, and for each such fixed x, we have $0 \leq y \leq (a^2 - x^2)^{1/2}$. Thus, the iterated integral is equivalent to the double integral

$$\iint_D (a^2 - y^2)^{1/2}\, dy\, dx,$$

where D is the set of points (x, y) such that $0 \leq x \leq a$ and $0 \leq y \leq (a^2 - x^2)^{1/2}$. But this is the representation of one quarter (the positive quadrant portion) of the disk of radius a; hence, D can also be described as the set of points (x, y) satisfying

$$0 \leq y \leq a, \qquad 0 \leq x \leq (a^2 - y^2)^{1/2}$$

(see Figure 5.4.1). Thus,

$$\begin{aligned}
\int_0^a \int_0^{(a^2-x^2)^{1/2}} (a^2 - y^2)^{1/2}\, dy\, dx &= \int_0^a \left[\int_0^{(a^2-y^2)^{1/2}} (a^2 - y^2)^{1/2}\, dx\right] dy \\
&= \int_0^a \left[x(a^2 - y^2)^{1/2}\right]_{x=0}^{(a^2-y^2)^{1/2}} dy \\
&= \int_0^a (a^2 - y^2)\, dy = \left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{2a^3}{3}. \quad \blacktriangle
\end{aligned}$$

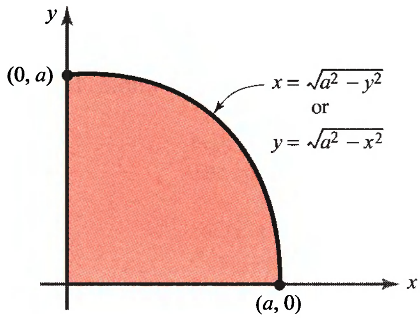

Figure 5.4.1 The positive-quadrant portion of a disk of radius a.

We could have evaluated the initial iterated integral directly, but, as the reader can easily verify, changing the order of integration makes the problem simpler. The next example shows that it may not be obvious how to evaluate an iterated integral, and yet it may be relatively simple to evaluate the iterated integral obtained by changing the order of integration.

EXAMPLE 2 Evaluate

$$\int_1^2 \int_0^{\log x} (x-1)\sqrt{1+e^{2y}}\, dy\, dx.$$

SOLUTION It will simplify matters if we first interchange the order of integration. First notice that the integral is equal to $\iint_D (x-1)\sqrt{1+e^{2y}}\, dA$, where D is the set of (x, y) such that

$$1 \le x \le 2 \qquad \text{and} \qquad 0 \le y \le \log x.$$

The region D is simple (see Figure 5.4.2) and can also be described by

$$0 \le y \le \log 2 \qquad \text{and} \qquad e^y \le x \le 2.$$

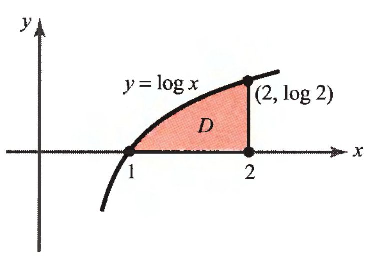

Figure 5.4.2 D is the region of integration for Example 2.

Thus, the given iterated integral is equal to

$$\int_0^{\log 2}\int_{e^y}^2 (x-1)\sqrt{1+e^{2y}}\,dx\,dy = \int_0^{\log 2}\sqrt{1+e^{2y}}\left[\int_{e^y}^2 (x-1)\,dx\right]dy$$

$$= \int_0^{\log 2}\sqrt{1+e^{2y}}\left[\frac{x^2}{2}-x\right]_{e^y}^2 dy$$

$$= -\int_0^{\log 2}\left(\frac{e^{2y}}{2}-e^y\right)\sqrt{1+e^{2y}}\,dy$$

$$= -\frac{1}{2}\int_0^{\log 2} e^{2y}\sqrt{1+e^{2y}}\,dy + \int_0^{\log 2} e^y\sqrt{1+e^{2y}}\,dy. \tag{1}$$

In the first integral in expression (1), we substitute $u = e^{2y}$, and in the second, $v = e^y$. Hence, we obtain

$$-\frac{1}{4}\int_1^4 \sqrt{1+u}\,du + \int_1^2 \sqrt{1+v^2}\,dv. \tag{2}$$

Both integrals in expression (2) are easily found with techniques of one-variable calculus (or by consulting the table of integrals at the back of the book). For the first integral, we get

$$\frac{1}{4}\int_1^4 \sqrt{1+u}\,du = \left[\frac{1}{6}(1+u)^{3/2}\right]_1^4 = \frac{1}{6}[(1+4)^{3/2}-2^{3/2}] = \frac{1}{6}[5^{3/2}-2^{3/2}]. \tag{3}$$

The second integral is

$$\int_1^2 \sqrt{1+v^2}\,dv = \frac{1}{2}\left[v\sqrt{1+v^2}+\log\left(\sqrt{1+v^2}+v\right)\right]_1^2$$

$$= \frac{1}{2}\left[2\sqrt{5}+\log(\sqrt{5}+2)\right] - \frac{1}{2}\left[\sqrt{2}+\log(\sqrt{2}+1)\right] \tag{4}$$

(see formula 43 in the table of integrals at the back of the book). Finally, we subtract equation (3) from equation (4) to obtain the answer

$$\frac{1}{2}\left(2\sqrt{5}-\sqrt{2}+\log\frac{\sqrt{5}+2}{\sqrt{2}+1}\right) - \frac{1}{6}[5^{3/2}-2^{3/2}]. \quad \blacktriangle$$

Mean Value Inequality

We conclude with an inequality that helps us estimate integrals. Suppose there are numbers m and M such that for all $(x, y) \in D$, and $m \leq f(x, y) \leq M$, then integrating over D, we get

$$m \cdot A(D) \leq \iint_D f(x, y)\, dA \leq M \cdot A(D), \tag{5}$$

where $A(D)$ is the area of the region D. Even though this inequality is obvious, it can help us *estimate* integrals that we cannot easily evaluate *exactly*.

EXAMPLE 3 Consider the integral

$$\iint_D \frac{1}{\sqrt{1+x^6+y^8}}\, dx\, dy,$$

where D is the unit square $[0, 1] \times [0, 1]$. Because the integrand satisfies, for x and y between 0 and 1,

$$\frac{1}{\sqrt{3}} \leq \frac{1}{\sqrt{1+x^6+y^8}} \leq 1,$$

and because the square has area 1, we get:

$$\frac{1}{\sqrt{3}} \leq \iint_D \frac{1}{\sqrt{1+x^6+y^8}}\, dx\, dy \leq 1. \quad \blacktriangle$$

Mean Value Equality

The mean value inequality can be turned into an equality when f is continuous. Here is the formal statement.

THEOREM 5: Mean Value Theorem: Double Integrals Suppose $f\colon D \to \mathbb{R}$ is continuous and D is an elementary region. Then for some point (x_0, y_0) in D we have

$$\iint_D f(x, y)\, dA = f(x_0, y_0) A(D),$$

where $A(D)$ denotes the area of D.

PROOF We cannot prove this theorem with complete rigor, because it requires some concepts about continuous functions not proved in this course; but we can sketch the main ideas that underlie the proof.

Because f is continuous on D, it has a maximum value M and a minimum value m. Thus, $m \leq f(x, y) \leq M$ for all $(x, y) \in D$. Furthermore, $f(x_1, y_1) = m$ and $f(x_2, y_2) = M$ for some pairs (x_1, y_1) and (x_2, y_2) in D.

Dividing through inequality (5) by $A(D)$, we get

$$m \leq \frac{1}{A(D)} \iint_D f(x, y)\, dA \leq M. \tag{6}$$

Because a continuous function on D takes on every value between its maximum and minimum values (this is the two-variable *intermediate value theorem* proved in advanced calculus; see also Review Exercise 32), and because the number $[1/A(D)] \iint_D f(x, y)\, dA$ is, by inequality (6), between these values, there must be a point $(x_0, y_0) \in D$ with

$$f(x_0, y_0) = \frac{1}{A(D)} \iint_D f(x, y)\, dA,$$

which is precisely the conclusion of Theorem 5. ■

EXERCISES

1. In the following integrals, change the order of integration, sketch the corresponding regions, and evaluate the integral both ways.

(a) $\displaystyle\int_0^1 \int_x^1 xy\, dy\, dx$

(b) $\displaystyle\int_0^{\pi/2} \int_0^{\cos\theta} \cos\theta\, dr\, d\theta$

(c) $\displaystyle\int_0^1 \int_y^{2-y} (x+y)^2\, dx\, dy$

(d) $\displaystyle\int_a^b \int_a^y f(x, y)\, dx\, dy$ (express your answer in terms of antiderivatives).

2. Find

(a) $\displaystyle\int_{-1}^1 \int_{|y|}^1 (x+y)^2\, dx\, dy$

(b) $\displaystyle\int_{-3}^1 \int_{-\sqrt{(9-y^2)}}^{\sqrt{(9-y^2)}} x^2\, dx\, dy$

(c) $\displaystyle\int_0^4 \int_{y/2}^2 e^{x^2}\, dx\, dy$

(d) $\displaystyle\int_0^1 \int_{\tan^{-1} y}^{\pi/4} (\sec^5 x)\, dx\, dy$

3. If $f(x, y) = e^{\sin(x+y)}$ and $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x, y)\, dA \leq e.$$

4. Show that

$$\frac{1}{2}(1 - \cos 1) \leq \iint_{[0,1]\times[0,1]} \frac{\sin x}{1 + (xy)^4}\, dx\, dy \leq 1.$$

5. If $D = [-1, 1] \times [-1, 2]$, show that

$$1 \leq \iint_D \frac{dx\, dy}{x^2 + y^2 + 1} \leq 6.$$

6. Using the mean value inequality, show that

$$\frac{1}{6} \leq \iint_D \frac{dA}{y - x + 3} \leq \frac{1}{4},$$

where D is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$.

7. Compute the volume of an ellipsoid with semiaxes a, b, and c. (Hint: Use symmetry and first find the volume of one half of the ellipsoid.)

8. Compute $\iint_D f(x, y)\,dA$, where $f(x, y) = y^2\sqrt{x}$ and D is the set of (x, y) where $x > 0$, $y > x^2$, and $y < 10 - x^2$.

9. Find the volume of the region determined by $x^2 + y^2 + z^2 \leq 10$, $z \geq 2$. Use the disk method from one-variable calculus and state how the method is related to Cavalieri's principle.

10. Evaluate $\iint_D e^{x-y}\,dx\,dy$, where D is the interior of the triangle with vertices $(0, 0)$, $(1, 3)$, and $(2, 2)$.

11. Evaluate $\iint_D y^3(x^2 + y^2)^{-3/2}\,dx\,dy$, where D is the region determined by the conditions $\frac{1}{2} \leq y \leq 1$ and $x^2 + y^2 \leq 1$.

12. Given that the double integral $\iint_D f(x, y)\,dx\,dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_{x^2}^{x} f(x, y)\,dy\right] dx$, sketch the region D and interchange the order of integration.

13. Given that the double integral $\iint_D f(x, y)\,dx\,dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \left[\int_y^{\sqrt{2-y^2}} f(x, y)\,dx\right] dy$, sketch the region D and interchange the order of integration.

14. Prove that $2\int_a^b \int_x^b f(x)f(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)^2$. [Hint: Notice that $\left(\int_a^b f(x)\,dx\right)^2 = \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy$.]

15. Show that (see Exercise 27, Section 2.5)

$$\frac{d}{dx}\int_a^x \int_c^d f(x, y, z)\,dz\,dy = \int_c^d f(x, y, z)\,dz + \int_a^x \int_c^d f_x(x, y, z)\,dz\,dy.$$

5.5 The Triple Integral

Triple integrals are needed for many physical problems. For example, if the temperature inside an oven is not uniform, determining the average temperature involves "summing" the values of the temperature function at all points in the solid region enclosed by the oven walls and then dividing the answer by the total volume of the oven. Such a sum is expressed mathematically as a triple integral.

Definition of the Triple Integral

Our objective now is to define the triple integral of a function $f(x, y, z)$ over a box (rectangular parallelepiped) $B = [a, b] \times [c, d] \times [p, q]$. Proceeding as in double integrals, we partition the three sides of B into n equal parts and form the sum

$$S_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(\mathbf{c}_{ijk})\, \Delta V,$$

where $\mathbf{c}_{ijk}$ is a point in B_{ijk}, the ijkth rectangular parallelepiped (or box) in the partition of B, and ΔV is the volume of B_{ijk} (see Figure 5.5.1).

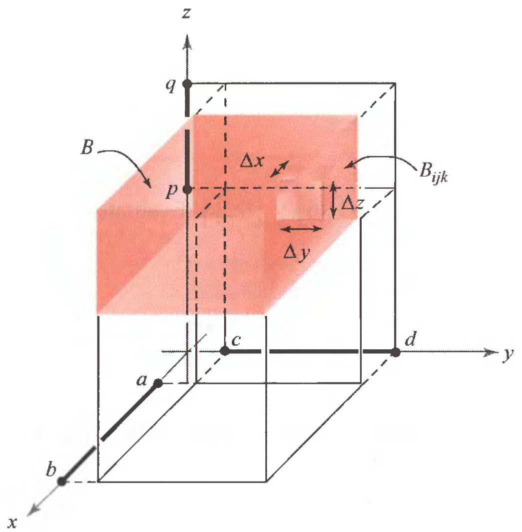

Figure 5.5.1 A partition of a box B into n^3 subboxes B_{ijk}.

DEFINITION: Triple Integrals Let f be a bounded function of three variables defined on B. If $\text{limit}_{n\to\infty} S_n = S$ exists and is independent of any choice of $\mathbf{c}_{ijk}$, we call f ***integrable*** and call S the ***triple integral*** (or simply the integral) of f over B and denote it by

$$\iiint_B f\, dV, \qquad \iiint_B f(x, y, z)\, dV \qquad \text{or} \qquad \iiint_B f(x, y, z)\, dx\, dy\, dz.$$

Properties of Triple Integrals

As before, one can prove that continuous functions defined on B are integrable. Moreover, bounded functions whose discontinuities are confined to graphs of continuous functions [such as $x = \alpha(y, z)$, $y = \beta(x, z)$, or $z = \gamma(x, y)$] are integrable. The other basic properties (such as the fact that the integral of a sum is the sum of the integrals) for double integrals also hold for triple integrals. Especially important is the reduction to iterated integrals:

Reduction to Iterated Integrals Let $f(x, y, z)$ be integrable on the box $B = [a, b] \times [c, d] \times [p, q]$. Then any iterated integral that exists is equal to the triple integral; that is,

$$\begin{aligned}\iiint_B f(x, y, z)\,dx\,dy\,dz &= \int_p^q \int_c^d \int_a^b f(x, y, z)\,dx\,dy\,dz \\ &= \int_p^q \int_a^b \int_c^d f(x, y, z)\,dy\,dx\,dz \\ &= \int_a^b \int_p^q \int_c^d f(x, y, z)\,dy\,dz\,dx,\end{aligned}$$

and so on. (There are six possible orders altogether.)

EXAMPLE 1 (a) Let B be the box $[0, 1] \times \left[-\frac{1}{2}, 0\right] \times \left[0, \frac{1}{3}\right]$. Evaluate

$$\iiint_B (x + 2y + 3z)^2\,dx\,dy\,dz.$$

(b) Verify that we get the same answer if the integration is done in the order y first, then z, and then x.

SOLUTION (a) According to the principle of reduction to iterated integrals, this integral may be evaluated as

$$\begin{aligned}&\int_0^{1/3} \int_{-1/2}^0 \int_0^1 (x + 2y + 3z)^2\,dx\,dy\,dz \\ &= \int_0^{1/3} \int_{-1/2}^0 \left[\left.\frac{(x + 2y + 3z)^3}{3}\right|_{x=0}^1\right] dy\,dz \\ &= \int_0^{1/3} \int_{-1/2}^0 \frac{1}{3}\left[(1 + 2y + 3z)^3 - (2y + 3z)^3\right] dy\,dz \\ &= \int_0^{1/3} \frac{1}{24}\left.\left[(1 + 2y + 3z)^4 - (2y + 3z)^4\right]\right|_{y=-1/2}^0 dz \\ &= \int_0^{1/3} \frac{1}{24}\left[(3z + 1)^4 - 2(3z)^4 + (3z - 1)^4\right] dz \\ &= \frac{1}{24 \cdot 15}\left.\left[(3z + 1)^5 - 2(3z)^5 + (3z - 1)^5\right]\right|_{z=0}^{1/3} \\ &= \frac{1}{24 \cdot 15}(2^5 - 2) = \frac{1}{12}.\end{aligned}$$

(b)

$$
\begin{aligned}
\iiint_B &(x + 2y + 3z)^2 \, dy\, dz\, dx \\
&= \int_0^1 \int_0^{1/3} \int_{-1/2}^{0} (x + 2y + 3z)^2 \, dy\, dz\, dx \\
&= \int_0^1 \int_0^{1/3} \left[\frac{(x + 2y + 3z)^3}{6} \bigg|_{y=-1/2}^{0} \right] dz\, dx \\
&= \int_0^1 \int_0^{1/3} \frac{1}{6} \left[(x + 3z)^3 - (x + 3z - 1)^3 \right] dz\, dx \\
&= \int_0^1 \frac{1}{6} \left\{ \left[\frac{(x + 3z)^4}{12} - \frac{(x + 3z - 1)^4}{12} \right] \bigg|_{z=0}^{1/3} \right\} dx \\
&= \int_0^1 \frac{1}{72} \left[(x + 1)^4 + (x - 1)^4 - 2x^4 \right] dx \\
&= \frac{1}{72} \frac{1}{5} \left[(x + 1)^5 + (x - 1)^5 - 2x^5 \right]_{x=0}^{1} = \frac{1}{12}. \quad \blacktriangle
\end{aligned}
$$

EXAMPLE 2 Integrate e^{x+y+z} over the box $[0, 1] \times [0, 1] \times [0, 1]$.

SOLUTION We perform the integrations in the standard order:

$$
\begin{aligned}
\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} \, dx\, dy\, dz &= \int_0^1 \int_0^1 \left(e^{x+y+z} \big|_{x=0}^{1} \right) dy\, dz \\
&= \int_0^1 \int_0^1 (e^{1+y+z} - e^{y+z}) \, dy\, dz = \int_0^1 \left[e^{1+y+z} - e^{y+z} \right]_{y=0}^{1} dz \\
&= \int_0^1 \left[e^{2+z} - 2e^{1+z} + e^z \right] dz = \left[e^{2+z} - 2e^{1+z} + e^z \right]_0^1 \\
&= e^3 - 3e^2 + 3e - 1 = (e - 1)^3. \quad \blacktriangle
\end{aligned}
$$

As in the two-variable case, we define the integral of a function f over a bounded region W by defining a new function f^*, equal to f on W and zero outside W, and then setting

$$
\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_B f^*(x, y, z)\, dx\, dy\, dz,
$$

where B is any box containing the region W.

Elementary Regions

As before, we restrict our attention to particularly simple regions. An ***elementary region*** in three-dimensional space is one defined by restricting one of the variables to be between two functions of the remaining variables, the domains of these functions being an elementary (i.e., an x-simple or a y-simple) region in the plane. For example, if D is an elementary region in the xy plane and if $\gamma_1(x, y)$ and $\gamma_2(x, y)$ are two functions with $\gamma_2(x, y) \geq \gamma_1(x, y)$, an elementary region consists of all (x, y, z) such that (x, y) lies in D and $\gamma_1(x, y) \leq z \leq \gamma_2(x, y)$. Figure 5.5.2 shows two elementary regions.

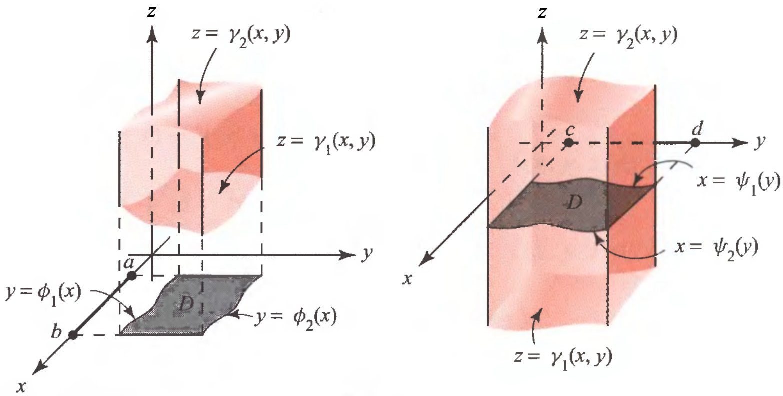

Figure 5.5.2 Two elementary regions in space. The domain D in the figure on the left is y-simple, while on the right it is x-simple.

EXAMPLE 3 Describe the unit ball $x^2 + y^2 + z^2 \leq 1$ as an elementary region.

SOLUTION This can be done in several ways. One, in which D is y-simple, is:

$$\begin{gathered} -1 \leq x \leq 1, \\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, \\ -\sqrt{1-x^2-y^2} \leq z \leq \sqrt{1-x^2-y^2}. \end{gathered}$$

In doing this, we first write the top and bottom hemispheres as $z = \sqrt{1-x^2-y^2}$ and $z = -\sqrt{1-x^2-y^2}$, respectively, where x and y vary over the unit disk (that is, $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and x varies between -1 and 1). (See Figure 5.5.3.) We can describe the region in other ways by interchanging the roles of x, y, and z in the defining inequalities. ▲

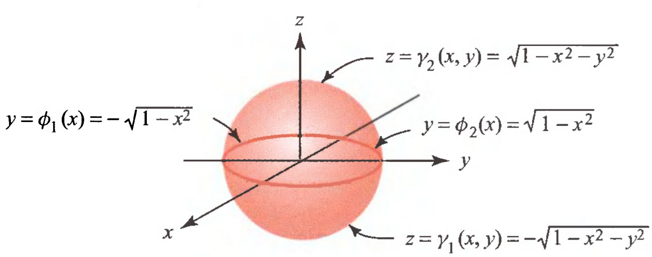

Figure 5.5.3 The unit ball as an elementary region in space.

Integrals over Elementary Regions

As with integrals in the plane, any function of three variables that is continuous over an elementary region is integrable on that region. An argument like that for double integrals shows that a triple integral over an elementary region can be rewritten as an iterated integral in which the limits of integration are functions. The formulas for such iterated integrals are given in the following box.

Triple Integrals by Iterated Integration Suppose that W is an elementary region described by bounding z between two functions of x and y. Then either

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

[see Figure 5.5.2 (left)] or

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\, dz\, dx\, dy$$

[see Figure 5.5.2 (right)].

If $f = 1$, we get the integral $\iiint_W dx\, dy\, dz$, which is the ***volume*** of the region W.

EXAMPLE 4 Verify the volume formula for the ball of radius 1:

$$\iiint_W dx\, dy\, dz = \frac{4}{3}\pi,$$

where W is the set of (x, y, z) with $x^2 + y^2 + z^2 \leq 1$.

SOLUTION We use the description of the unit ball from Example 3. From the first formula in the preceding box, the integral is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

Holding y and x fixed and integrating with respect to z yields

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[z \Big|_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \right] dy\, dx = 2 \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\, dy \right] dx.$$

Because x is fixed in the y-integral, it can be expressed as $\int_{-a}^{a} (a^2 - y^2)^{1/2}\, dy$, where $a = (1 - x^2)^{1/2}$. This integral is the area of a semicircular region of radius

a, so that

$$\int_{-a}^{a} (a^2 - y^2)^{1/2}\, dy = \frac{a^2}{2}\pi.$$

(We could also have used a trigonometric substitution or a table of integrals.) Thus,

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\, dy = \frac{1-x^2}{2}\pi,$$

and so

$$2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2}\, dy\, dx = 2\int_{-1}^{1} \pi \frac{1-x^2}{2}\, dx$$

$$= \pi \int_{-1}^{1} (1 - x^2)\, dx = \pi\left(x - \frac{x^3}{3}\right)\bigg|_{x=-1}^{1} = \frac{4}{3}\pi. \quad \blacktriangle$$

Other types of elementary regions are shown in Figure 5.5.4. For instance, in the second region, (y, z) lies in an elementary region in the yz plane and x lies between two graphs:

$$\rho_1(y, z) \le x \le \rho_2(y, z).$$

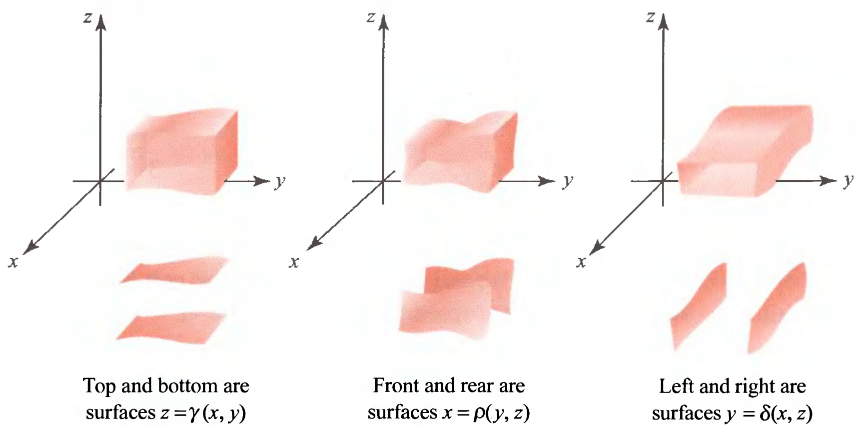

Figure 5.5.4 Types of elementary regions in space.

As shown in Figure 5.5.5, some elementary regions can be simultaneously described in all three ways. We shall call these regions ***symmetric elementary regions***.

Corresponding to each description of a region as an elementary region is an integration formula. For instance, if W is expressed as the set of all (x, y, z) such that

$$c \le y \le d, \qquad \psi_1(y) \le z \le \psi_2(y), \qquad \rho_1(y, z) \le x \le \rho_2(y, z),$$

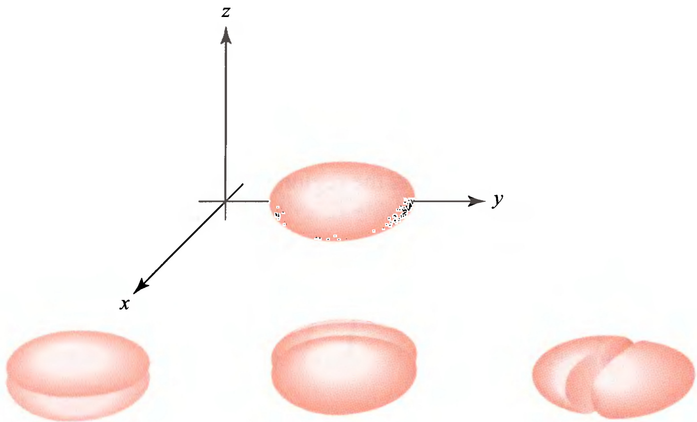

Figure 5.5.5 A symmetric elementary region can be described in three overall ways.

then

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \int_{\rho_1(y,z)}^{\rho_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

EXAMPLE 5 Let W be the region bounded by the planes $x = 0$, $y = 0$, and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0$, $y \geq 0$. Compute $\iiint_W x\,dx\,dy\,dz$ and sketch the region.

SOLUTION *Method 1.* The region W is sketched in Figure 5.5.6. As indicated in the figure, we may describe this region by the inequalities

$$0 \leq x \leq \sqrt{2}, \qquad 0 \leq y \leq \sqrt{2 - x^2}, \qquad x^2 + y^2 \leq z \leq 2.$$

Therefore,

$$\iiint_W x\,dx\,dy\,dz = \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^2 x\,dz \right) dy \right] dx$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^{\sqrt{2}} x \left[(2 - x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3} \right] dx$$

$$= \int_0^{\sqrt{2}} \frac{2x}{3}(2 - x^2)^{3/2}\,dx = \left. \frac{-2(2-x^2)^{5/2}}{15} \right|_0^{\sqrt{2}}$$

$$= 2 \cdot \frac{2^{5/2}}{15} = \frac{8\sqrt{2}}{15}.$$

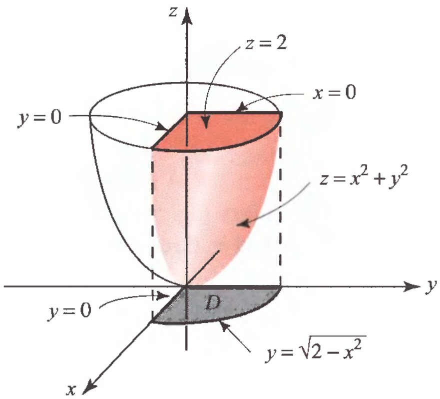

Figure 5.5.6 W is the region below the plane $z = 2$, above the paraboloid $z = x^2 + y^2$, and on the positive sides of the planes $x = 0$, $y = 0$.

Method 2. We can also place limits on x first and describe W by $0 \le x \le (z - y^2)^{1/2}$ and (y, z) in D, where D is the subset of the yz plane with $0 \le z \le 2$ and $0 \le y \le z^{1/2}$ (see Figure 5.5.7).

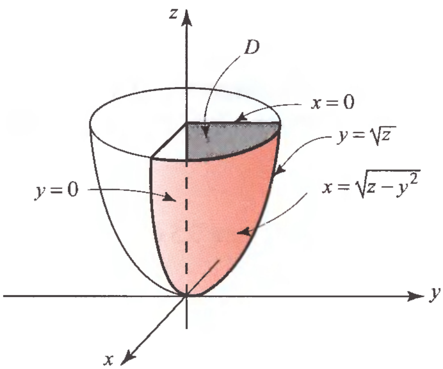

Figure 5.5.7 A different description of the region in Example 5.

Therefore,

$$\begin{aligned}\iiint_W x\,dx\,dy\,dz &= \iint_D \left(\int_0^{(z-y^2)^{1/2}} x\,dx\right) dy\,dz\\ &= \int_0^2 \left[\int_0^{z^{1/2}} \left(\int_0^{(z-y^2)^{1/2}} x\,dx\right) dy\right] dz\\ &= \int_0^2 \int_0^{z^{1/2}} \left(\frac{z-y^2}{2}\right) dy\,dz\\ &= \frac{1}{2}\int_0^2 \left(z^{3/2} - \frac{z^{3/2}}{3}\right) dz = \frac{1}{2}\int_0^2 \frac{2}{3} z^{3/2}\,dz\\ &= \left[\frac{2}{15} z^{5/2}\right]_0^2 = \frac{2}{15} 2^{5/2} = \frac{8\sqrt{2}}{15},\end{aligned}$$

which agrees with our previous answer. ▲

EXAMPLE 6 Evaluate

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\, dy\, dx.$$

Sketch the region W of integration and interpret.

SOLUTION

$$\int_0^1 \int_0^x \int_{x^2+y^2}^2 dz\, dy\, dx = \int_0^1 \int_0^x (2 - x^2 - y^2)\, dy\, dx$$
$$= \int_0^1 \left(2x - x^3 - \frac{x^3}{3}\right) dx = 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3}.$$

This integral is the volume of the region sketched in Figure 5.5.8. ▲

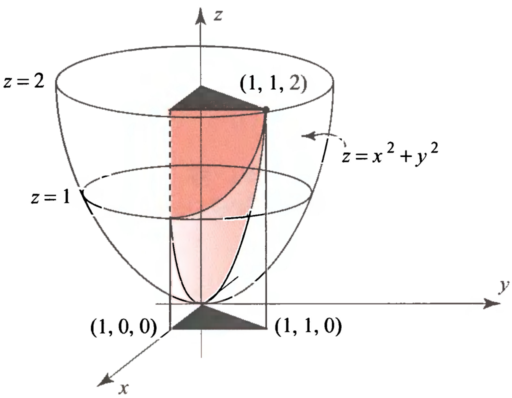

Figure 5.5.8 The region W lies between the paraboloid $z = x^2 + y^2$ and the plane $z = 2$, and above the region D.

EXERCISES

In Exercises 1 to 4, perform the indicated integration over the given box.

1. $\iiint_B x^2\, dx\, dy\, dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

2. $\iiint_B e^{-xy} y\, dx\, dy\, dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

3. $\iiint_B (2x + 3y + z)\, dx\, dy\, dz$, $B = [0, 2] \times [-1, 1] \times [0, 1]$

4. $\iiint_B ze^{x+y}\, dx\, dy\, dz$, $B = [0, 1] \times [0, 1] \times [0, 1]$

In Exercises 5 to 8, describe the given region as an elementary region.

5. The region between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$.

6. The region cut out of the ball $x^2 + y^2 + z^2 \leq 4$ by the elliptic cylinder $2x^2 + z^2 = 1$, that is, the region inside the cylinder and the ball.

7. The region inside the sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.

8. The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$, and $x = z - y - 1$.

Find the volume of the region in Exercises 9 to 12.

9. The region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.

10. The solid bounded by $x^2 + 2y^2 = 2$, $z = 0$, and $x + y + 2z = 2$.

11. The solid bounded by $x = y$, $z = 0$, $y = 0$, $x = 1$, and $x + y + z = 0$.

12. The region common to the intersecting cylinders $x^2 + y^2 \leq a^2$ and $x^2 + z^2 \leq a^2$.

Evaluate the integrals in Exercises 13 to 21.

13. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos\,[\pi(x + y + z)]\, dx\, dy\, dz$

14. $\displaystyle\int_0^1 \int_0^x \int_0^y (y + xz)\, dz\, dy\, dx$

15. $\displaystyle\iiint_y (x^2 + y^2 + z^2)\, dx\, dy\, dz$; W is the region bounded by $x + y + z = a$ (where $a > 0$), $x = 0$, $y = 0$, and $z = 0$.

16. $\displaystyle\iiint_W z\, dx\, dy\, dz$; W is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, and the cylinder $x^2 + y^2 = 1$, with $x \geq 0$, $y \geq 0$.

17. $\displaystyle\iiint_W x^2 \cos z\, dx\, dy\, dz$; W is the region bounded by $z = 0$, $z = \pi$, $y = 0$, $y = 1$, $x = 0$, and $x + y = 1$.

18. $\displaystyle\int_0^2 \int_0^x \int_0^{x+y} dz\, dy\, dx$

19. $\displaystyle\iiint_y (1 - z^2)\, dx\, dy\, dz$; W is the pyramid with top vertex at $(0, 0, 1)$ and base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(1, 1, 0)$.

20. $\displaystyle\iiint_W (x^2 + y^2)\, dx\, dy\, dz$; W is the same pyramid as in Exercise 19.

21. $\displaystyle\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz\, dy\, dx.$

22. (a) Sketch the region for the integral $\displaystyle\int_0^1 \int_0^x \int_0^y f(x, y, z)\, dz\, dy\, dx.$

(b) Write the integral with the integration order $dx\, dy\, dz$.

For the regions in Exercises 23 to 26, find the appropriate limits $\phi_1(x)$, $\phi_2(x)$, $\gamma_1(x, y)$, and $\gamma_2(x, y)$, and write the triple integral over the region W as an iterated integral in the form

$$\iiint_W f\,dV = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)\,dz \right] dy \right\} dx.$$

23. $W = \{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\}$

24. $W = \{(x, y, z) \mid \frac{1}{2} \le z \le 1 \text{ and } x^2 + y^2 + z^2 \le 1\}$

25. $W = \{(x, y, z) \mid x^2 + y^2 \le 1, z \ge 0 \text{ and } x^2 + y^2 + z^2 \le 4\}$

26. $W = \{(x, y, z) \mid |x| \le 1, |y| \le 1, z \ge 0 \text{ and } x^2 + y^2 + z^2 \le 1\}$

27. Show that the formula using triple integrals for the volume under the graph of a positive function $f(x, y)$, on an elementary region D in the plane, reduces to the double integral of f over D.

28. Let W be the region bounded by the planes $x = 0, y = 0, z = 0, x + y = 1$, and $z = x + y$.

(a) Find the volume of W.
(b) Evalute $\iiint_W x\,dx\,dy\,dz$.
(c) Evalute $\iiint_W y\,dx\,dy\,dz$.

29. Let f be continuous and let B_ε be the ball of radius ε centered at the point (x_0, y_0, z_0). Let vol (B_ε) be the volume of B_ε. Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\operatorname{vol}(B_\varepsilon)} \iiint_{B_\varepsilon} f(x, y, z)\,dV = f(x_0, y_0, z_0).$$

REVIEW EXERCISES FOR CHAPTER 5

Evaluate the integrals in Exercises 1 to 4.

1. $\displaystyle\int_0^3 \int_{-x^2+1}^{x^2+1} xy\,dy\,dx$

2. $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 (x + y)^2\,dy\,dx$

3. $\displaystyle\int_0^1 \int_{e^x}^{e^{2x}} x \ln y\,dy\,dx$

4. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x + y + z)]\,dx\,dy\,dz.$

Reverse the order of integration of the integrals in Exercises 5 to 8 and evaluate.

5. The integral in Exercise 1.

6. The integral in Exercise 2.

7. The integral in Exercise 3.

8. The integral in Exercise 4.

9. Evaluate the integral $\int_0^1 \int_0^x \int_0^y (y + xz)\, dz\, dy\, dx$.

10. Evaluate $\int_0^1 \int_y^{y^2} e^{x/y}\, dx\, dy$.

11. Evaluate $\int_0^1 \int_0^{(\arcsin y)/y} y \cos xy\, dx\, dy$.

12. Change the order of integration and evaluate

$$\int_0^2 \int_{y/2}^1 (x + y)^2\, dx\, dy.$$

13. Show that evaluating $\iint_D dx\, dy$, where D is a y-simple region, reproduces the formula from one-variable calculus for the area between two curves.

14. Change the order of integration and evaluate

$$\int_0^1 \int_{y^{1/2}}^1 (x^2 + y^3 x)\, dx\, dy.$$

15. Let D be the region in the xy plane inside the unit circle $x^2 + y^2 = 1$. Evaluate $\iint_D f(x, y)\, dx\, dy$ in each of the following cases:

(a) $f(x, y) = xy$ (b) $f(x, y) = x^2 y^2$ (c) $f(x, y) = x^3 y^3$

16. Find $\iint_D y[1 - \cos(\pi x/4)]\, dx\, dy$, where D is the region in Figure 5.R.1.

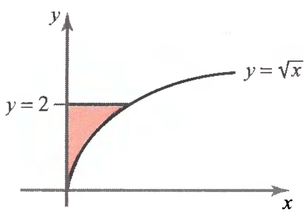

Figure 5.R.1 The region of integration for Exercise 16.

Evaluate the integrals in Exercises 17 to 24. Sketch and identify the type of the region (corresponding to the way the integral is written).

17. $\displaystyle\int_0^{\pi} \int_{\sin x}^{3 \sin x} x(1 + y)\, dy\, dx$

18. $\displaystyle\int_0^1 \int_{x-1}^{x \cos(\pi x/2)} (x^2 + xy + 1)\, dy\, dx$

19. $\displaystyle\int_{-1}^1 \int_{y^{2/3}}^{(2-y)^2} \left(\frac{3}{2}\sqrt{x} - 2y\right) dx\, dy$

20. $\displaystyle\int_0^2 \int_{-3(\sqrt{4-x^2})/2}^{3(\sqrt{4-x^2})/2} \left(\frac{5}{\sqrt{2 + x}} + y^3\right) dy\, dx$

21. $\displaystyle\int_0^1 \int_0^{x^2} (x^2 + xy - y^2)\, dy\, dx$

22. $\displaystyle\int_2^4 \int_{y^2-1}^{y^3} 3\, dx\, dy$

23. $\int_0^1 \int_{x^2}^{x} (x+y)^2\, dy\, dx$

24. $\int_0^1 \int_0^{3y} e^{x+y}\, dx\, dy$

In Exercises 25 to 27, integrate the given function f over the given region D.

25. $f(x, y) = x - y$; D is the triangle with vertices (0, 0), (1, 0), and (2, 1).

26. $f(x, y) = x^3 y + \cos x$; D is the triangle defined by $0 \leq x \leq \pi/2, 0 \leq y \leq x$.

27. $f(x, y) = x^2 + 2xy^2 + 2$; D is the region bounded by the graph of $y = -x^2 + x$, the x axis, and the lines $x = 0$ and $x = 2$.

In Exercises 28 and 29, sketch the region of integration, interchange the order, and evaluate.

28. $\int_1^4 \int_1^{\sqrt{x}} (x^2 + y^2)\, dy\, dx$

29. $\int_0^1 \int_{1-y}^{1} (x + y^2)\, dx\, dy$

30. Show that

$$4e^5 \leq \iint_{[1,3]\times[2,4]} e^{x^2+y^2}\, dA \leq 4e^{25}.$$

31. Show that

$$4\pi \leq \iint_D (x^2 + y^2 + 1)\, dx\, dy \leq 20\pi,$$

where D is the disk of radius 2 centered at the origin.

32. Suppose W is a ***path-connected region***, that is, given any two points of W there is a continuous path joining them. If f is a continuous function on W, use the intermediate-value theorem to show that there is at least one point in W at which the value of f is equal to the average of f over W, that is, the integral of f over W divided by the volume of W. (Compare this with the mean-value theorem for double integrals.) What happens if W is not connected?

33. Prove: $\int_0^x [\int_0^t F(u)\, du]\, dt = \int_0^x (x - u) F(u)\, du$.

Evaluate the integrals in Exercises 34 to 36.

34. $\int_0^1 \int_0^z \int_0^y xy^2 z^3\, dx\, dy\, dz$

35. $\int_0^1 \int_0^y \int_0^{x/\sqrt{3}} \frac{x}{x^2 + z^2}\, dz\, dx\, dy$

36. $\int_1^2 \int_1^z \int_{1/y}^{2} yz^2\, dx\, dy\, dz$

37. Write the iterated integral $\int_0^1 \int_{1-x}^1 \int_x^1 f(x, y, z)\, dz\, dy\, dx$ as an integral over a region in $\mathbb{R}^3$ and then rewrite it in five other possible orders of integration.