6

The Change of Variables Formula and Applications of Integration

If you are stuck in a calculus problem and don't know what else to do, try integrating by parts or changing variables.

Jerry Kazdan

If that fails, go away, have a cup of coffee, and think!

Ute Müller

The change of variables formula is one of the most powerful integration methods in single-variable calculus; it enables us to evaluate integrals such as

$$\int_0^1 x e^{x^2}\,dx$$

by using the substitution, or *change of variables* $u = x^2$, which reduces the problem to the easy task of integrating e^u with respect to u. In this chapter, we develop the *multidimensional change of variables formula*, which is especially important and useful in evaluating multiple integrals in polar, cylindrical, and spherical coordinates.

One of the key ingredients in the change of variables formula is how to change variables in multidimensions. This involves the notion of mapping, which occurs in various interesting situations. For example, consider a deforming object, such as a swimming fish. As it changes its shape, one can imagine the instantaneous correspondence between points on the fish in its rest state and in its current shape. This type of correspondence is, in fact, the main idea behind a change of variables, in this case, of one three-dimensional region (the fish in its rest state) to another (the fish in its current shape).

The first section in this chapter describes the key concepts for mappings between regions of the plane. It goes on to develop the change of variables technique for double and then triple integrals. The chapter also includes some of the important physical applications of the integral.

6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

In this section, we shall be interested in maps from subsets of $\mathbb{R}^2$ to $\mathbb{R}^2$. The resulting geometric understanding will be useful in the next section, when we discuss the change of variables formula for multiple integrals.

Maps of One Region to Another

Let D^* be a subset of $\mathbb{R}^2$; suppose we consider a continuously differentiable map $T: D^* \to \mathbb{R}^2$, so T takes points in D^* to points in $\mathbb{R}^2$. We denote the set of image points by D or by $T(D^*)$; hence, $D = T(D^*)$ is the set of all points $(x, y) \in \mathbb{R}^2$ such that

$$(x, y) = T(x^*, y^*) \qquad \text{for some} \qquad (x^*, y^*) \in D^*.$$

One way to understand the geometry of a map T is to see how it *deforms* or changes D^*. For example, Figure 6.1.1 illustrates a map T that takes a slightly twisted region into a disk.

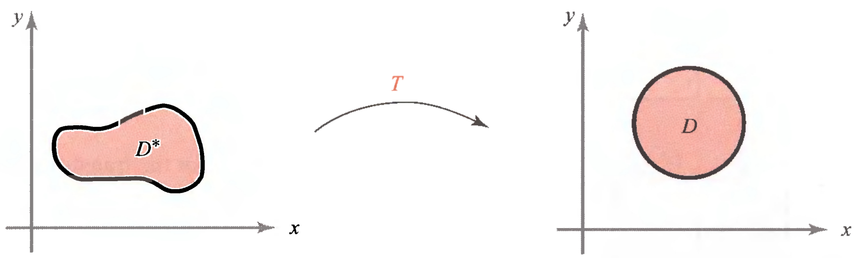


Figure 6.1.1 A function T from a region D^* to a disk D.

EXAMPLE 1 Let $D^* \subset \mathbb{R}^2$ be the rectangle $D^* = [0, 1] \times [0, 2\pi]$. Then all points in D^* are of the form (r, θ), where $0 \le r \le 1, 0 \le \theta \le 2\pi$. Let T be the polar coordinate "change of variables" defined by $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Find the image set D.

SOLUTION Let $(x, y) = (r\cos\theta, r\sin\theta)$. Because of the identity $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 1$, we see that the set of points $(x, y) \in \mathbb{R}^2$ such that $(x, y) \in D$ has the property that $x^2 + y^2 \le 1$, and so D is contained in the unit disk. In addition, any point (x, y) in the unit disk can be written as $(r\cos\theta, r\sin\theta)$ for some $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Thus, D is the unit disk (see Figure 6.1.2). ▲

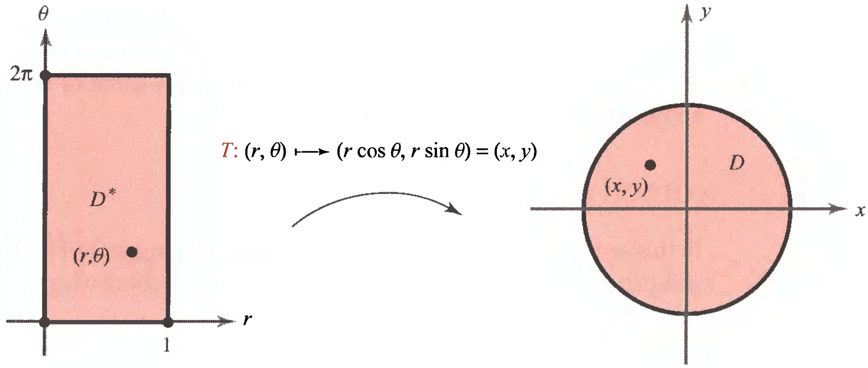


Figure 6.1.2 T gives a change of variables between Euclidean and polar coordinates. The unit circle is the image of a rectangle.

EXAMPLE 2 Let T be defined by $T(x, y) = ((x + y)/2, (x - y)/2)$ and let $D^* = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ be a square with side of length 2 centered at the origin. Determine the image D obtained by applying T to D^*.

SOLUTION Let us first determine the effect of T on the line $\mathbf{c}_1(t) = (t, 1)$, where $-1 \leq t \leq 1$ (see Figure 6.1.3). We have $T(\mathbf{c}_1(t)) = ((t + 1)/2, (t - 1)/2)$. The map $t \mapsto T(\mathbf{c}_1(t))$ is a parametrization of the line $y = x - 1, 0 \leq x \leq 1$, because $(t - 1)/2 = (t + 1)/2 - 1$. This is the straight line segment joining $(1, 0)$ and $(0, -1)$.

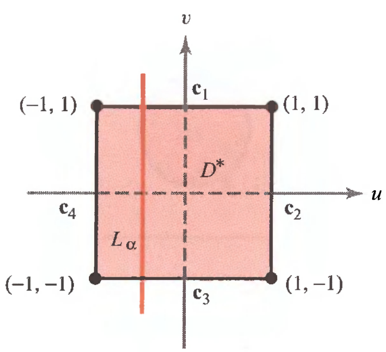


Figure 6.1.3 Domain for the transformation T of Example 2.

Let

$$
\begin{aligned}
\mathbf{c}_2(t) &= (1, t), \qquad & -1 \leq t \leq 1 \\
\mathbf{c}_3(t) &= (t, -1), \qquad & -1 \leq t \leq 1 \\
\mathbf{c}_4(t) &= (-1, t), \qquad & -1 \leq t \leq 1
\end{aligned}
$$

be parametrizations of the other edges of the square D^*. Using the same argument as before, we see that $T \circ \mathbf{c}_2$ is a parametrization of the line $y = 1 - x, 0 \leq x \leq 1$

[the straight line segment joining (0, 1) and (1, 0)]; $T \circ \mathbf{c}_3$ is the line $y = x + 1$, $-1 \leq x \leq 0$ joining (0, 1) and (−1, 0); and $T \circ \mathbf{c}_4$ is the line $y = -x - 1$, $-1 \leq x \leq 0$ joining (−1, 0) and (0, −1). By this time it seems reasonable to guess that T "flips" the square D^* over and takes it to the square D whose vertices are (1, 0), (0, 1), (−1, 0), (0, −1) (Figure 6.1.4).

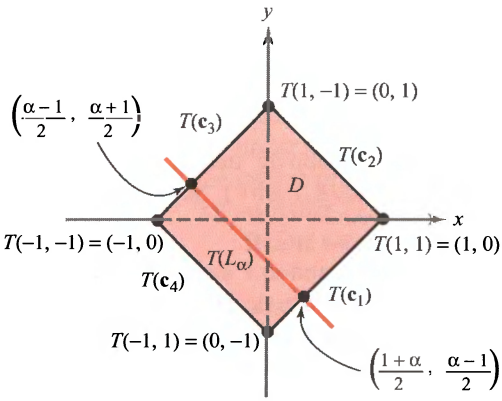


Figure 6.1.4 The effect of T on the region D^*.

To prove that this is indeed the case, let $-1 \leq \alpha \leq 1$ and let L_α (Figure 6.1.3) be a fixed line parametrized by $\mathbf{c}(t) = (\alpha, t)$, $-1 \leq t \leq 1$; then $T(\mathbf{c}(t)) = ((\alpha + t)/2, (\alpha - t)/2)$ is a parametrization of the line $y = -x + \alpha$, $(\alpha - 1)/2 \leq x \leq (\alpha + 1)/2$. This line begins, for $t = -1$, at the point $((\alpha - 1)/2, (1 + \alpha)/2)$ and ends at the point $((1 + \alpha)/2, (\alpha - 1)/2)$; as is easily checked, these points lie on the lines $T \circ \mathbf{c}_3$ and $T \circ \mathbf{c}_1$, respectively. Thus, as α varies between −1 and 1, L_α sweeps out the square D^* while $T(L_\alpha)$ sweeps out the square D determined by the vertices (−1, 0), (0, 1), (1, 0), and (0, −1). ▲

Images of Maps

The following theorem is a useful way to describe the image $T(D^*)$.

THEOREM 1 Let A be a 2×2 matrix with $\det A \neq 0$ and let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ (matrix multiplication). Then T transforms parallelograms into parallelograms and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

The proof of Theorem 1 is left as Exercise 10 at the end of this section. This theorem simplifies the result of Example 2, because we need only find the vertices of $T(D^*)$ and then connect them by straight lines.

One-to-One Maps

Although we cannot visualize the graph of a function $T: \mathbb{R}^2 \to \mathbb{R}^2$, it does help to consider how the function deforms subsets. However, simply looking at these deformations does not give us a complete picture of the behavior of T. We may characterize T further by using the notion of a one-to-one correspondence.

DEFINITION A mapping T is ***one-to-one*** on D^* if for (u, v) and $(u', v') \in D^*$, $T(u, v) = T(u', v')$ implies that $u = u'$ and $v = v'$.

This statement means that *two different points of D^* are not sent into the same point of D by T*. For example, the function $T(x, y) = (x^2 + y^2, y^4)$ is not one-to-one, because $T(1, -1) = (2, 1) = T(1, 1)$ and yet $(1, -1) \neq (1, 1)$.

EXAMPLE 3 Consider the polar coordinate mapping function $T: \mathbb{R}^2 \to \mathbb{R}^2$ described in Example 1, defined by $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Show that T is not one-to-one if its domain is all of $\mathbb{R}^2$.

SOLUTION If $\theta_1 \neq \theta_2$, then $T(0, \theta_1) = T(0, \theta_2)$, and so T cannot be one-to-one. This observation implies that if L is the side of the rectangle $D^* = [0, 1] \times [0, 2\pi]$ where $0 \leq \theta \leq 2\pi$ and $r = 0$ (Figure 6.1.5), then T maps all of L into a single point, the center of the unit disk D. However, if we consider the set $S^* = (0, 1] \times [0, 2\pi)$, then $T: S^* \to S$ is one-to-one (see Exercise 1). Evidently, in determining whether a function is one-to-one, the domain chosen must be carefully considered. ▲

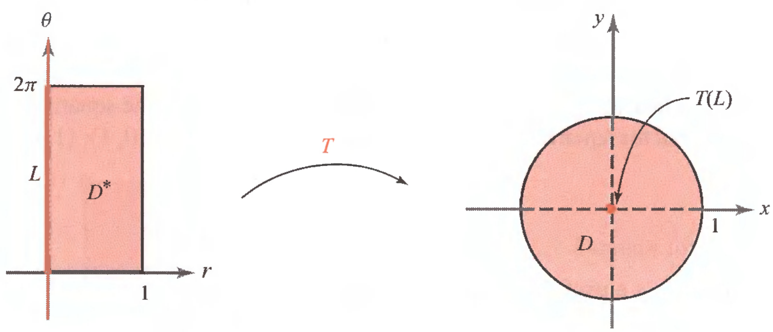


Figure 6.1.5 The polar-coordinate transformation T takes the line L to the point $(0, 0)$.

EXAMPLE 4 Show that the function $T: \mathbb{R}^2 \to \mathbb{R}^2$ of Example 2 is one-to-one.

SOLUTION Suppose $T(x, y) = T(x', y')$; then

$$\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \left(\frac{x'+y'}{2}, \frac{x'-y'}{2}\right)$$

and we have

$$x + y = x' + y',$$
$$x - y = x' - y'.$$

Adding, we have

$$2x = 2x'.$$

Thus, $x = x'$ and, similarly, subtracting gives $y = y'$, which shows that T is one-to-one (with domain all of $\mathbb{R}^2$). Actually, because T is linear and $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix, it would also suffice to show that $\det A \neq 0$ (see Exercise 8). ▲

Onto Maps

In Examples 1 and 2, we have been determining the image $D = T(D^*)$ of a region D^* under a mapping T. What will be of interest to us in the next section is, in part, the inverse problem: Namely, given D and a one-to-one mapping T of $\mathbb{R}^2$ to $\mathbb{R}^2$, find D^* such that $T(D^*) = D$.

Before we examine this question in more detail, we introduce the notion of "onto."

DEFINITION The mapping T is ***onto*** D if for every point $(x, y) \in D$ there exists at least one point (u, v) in the domain of T such that $T(u, v) = (x, y)$.

Thus, if T is onto, we *can solve* the equation $T(u, v) = (x, y)$ for (u, v), given $(x, y) \in D$. If T is, in addition, one-to-one, this *solution is unique*.

For *linear* mappings T of $\mathbb{R}^2$ to $\mathbb{R}^2$ (or $\mathbb{R}^n$ to $\mathbb{R}^n$) it turns out that one-to-one and onto are equivalent notions (see Exercises 8 and 9).

If we are given a region D and a mapping T, the determination of a region D^* such that $T(D^*) = D$ will be possible only when for every $(x, y) \in D$ there is a (u, v) in the domain of T such that $T(u, v) = (x, y)$ (that is, T must be onto D). The next example shows that this cannot always be done.

EXAMPLE 5 Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(u, v) = (u, 0)$. Let D be the square, $D = [0, 1] \times [0, 1]$. Because T takes all of $\mathbb{R}^2$ to one axis, it is impossible to find a D^* such that $T(D^*) = D$. ▲

Let us revisit Example 2 using these methods.

EXAMPLE 6 Let T be defined as in Example 2 and let D be the square whose vertices are $(1, 0), (0, 1), (-1, 0), (0, -1)$. Find a D^* with $T(D^*) = D$.

SOLUTION Because T is linear and $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix satisfying $\det A \neq 0$, we know that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is onto (see Exercises 8 and 9), and thus D^* can be found. By Theorem 1, D^* must be a parallelogram. In order to find D^*, it suffices to find the four points that are mapped onto vertices of D; then, by connecting these points, we will have found D^*. For the vertex $(1, 0)$ of D, we must solve $T(x, y) = (1, 0) = ((x + y)/2, (x - y)/2)$, so that $(x + y)/2 = 1$, $(x - y)/2 = 0$. Thus, $(x, y) = (1, 1)$ is a vertex of D^*. Solving for the other vertices, we find that $D^* = [-1, 1] \times [-1, 1]$. This is in agreement with what we found more laboriously in Example 2. ▲

EXAMPLE 7 Let D be the region in the first quadrant lying between the arcs of the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$, $0<a<b$ (see Figure 6.1.6). These circles have equations $r=a$ and $r=b$ in polar coordinates. Let T be the polar-coordinate transformation given by $T(r,\theta)=(r\cos\theta, r\sin\theta)=(x,y)$. Find D^* such that $T(D^*)=D$.

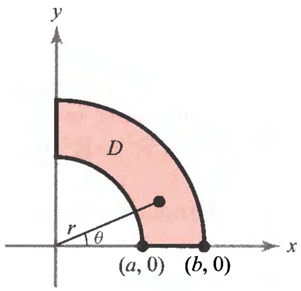


Figure 6.1.6 We seek a region D^* in the θr plane whose image under the polar-coordinate mapping is D.

SOLUTION In the region D, $a^2 \le x^2+y^2 \le b^2$; and because $r^2=x^2+y^2$, we see that $a \le r \le b$. Clearly, for this region θ varies between $0 \le \theta \le \pi/2$. Thus, if $D^*=[a,b]\times[0,\pi/2]$, we have $T(D^*)=D$ and T is one-to-one. ▲

REMARK The inverse function theorem discussed in Section 3.5 is relevant to the material here. It states that if the determinant of $\mathbf{D}T(u_0,v_0)$ [which is the matrix of partial derivatives of T evaluated at (u_0,v_0)] is not zero, then for (u,v) near (u_0,v_0) and (x,y) near $(x_0,y_0)=T(u_0,v_0)$, the equation $T(u,v)=(x,y)$ can be *uniquely solved* for (u,v) as functions of (x,y). In particular, by uniqueness, T is one-to-one near (u_0,v_0); also, T is onto a neighborhood of (x_0,y_0), because $T(u,v)=(x,y)$ is solvable for (u,v) if (x,y) is near (x_0,y_0).

However, even if T is one-to-one near every point, and also onto, T need not be *globally* one-to-one. Thus, one must exercise caution (see Exercise 12).

Surprisingly, if D^* and D are elementary regions and $T\colon D^*\to D$ has the property that the determinant of $\mathbf{D}T(u,v)$ is not zero for any (u,v) in D^* and if T maps the boundary of D^* in a one-to-one and onto manner to the boundary of D, then T is one-to-one and onto from D^* to D. (This proof is beyond the scope of this text.)

In summary, we have:

One-to-One and Onto Mappings A mapping $T\colon D^*\to D$ is ***one-to-one*** when it maps distinct points to distinct points. It is ***onto*** when the image of D^* under T is all of D.

A *linear* transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by multiplication by a matrix A is one-to-one and onto when and only when $\det A \ne 0$.

EXERCISES

1. Let $S^* = (0, 1] \times [0, 2\pi)$ and define $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Determine the image set S. Show that T is one-to-one on S^*.

2. Define

$$T(x^*, y^*) = \left(\frac{x^* - y^*}{\sqrt{2}}, \frac{x^* + y^*}{\sqrt{2}}\right).$$

Show that T rotates the unit square, $D^* = [0, 1] \times [0, 1]$.

3. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(u, v) = (-u^2 + 4u, v)$. Find the image D. Is T one-to-one?

4. Let D^* be the parallelogram bounded by the lines $y = 3x - 4$, $y = 3x$, $y = \frac{1}{2}x$, and $y = \frac{1}{2}(x + 4)$. Let $D = [0, 1] \times [0, 1]$. Find a T such that D is the image of D^* under T.

5. Let $D^* = [0, 1] \times [0, 1]$ and define T on D^* by $T(x^*, y^*) = (x^*y^*, x^*)$. Determine the image set D. Is T one-to-one? If not, can we eliminate some subset of D^* so that on the remainder T is one-to-one?

6. Let D^* be the parallelogram with vertices at $(-1, 3)$, $(0, 0)$, $(2, -1)$, and $(1, 2)$, and D be the rectangle $D = [0, 1] \times [0, 1]$. Find a T such that D is the image set of D^* under T.

7. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate mapping defined by $(\rho, \phi, \theta) \mapsto (x, y, z)$, where

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

Let D^* be the set of points (ρ, ϕ, θ) such that $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$, $\rho \in [0, 1]$. Find $D = T(D^*)$. Is T one-to-one? If not, can we eliminate some subset of D^* so that, on the remainder, T will be one-to-one?

In Exercises 8 and 9, let $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix.

8. Show that T is one-to-one if and only if the determinant of A is not zero.

9. Show that $\det A \neq 0$ if and only if T is onto.

10. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear and is given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix. Show that if $\det A \neq 0$, then T takes parallelograms onto parallelograms. [Hint: The general parallelogram in $\mathbb{R}^2$ can be described by the set of points $\mathbf{q} = \mathbf{p} + \lambda\mathbf{v} + \mu\mathbf{w}$ for $\lambda, \mu \in (0, 1)$ where $\mathbf{p}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^2$ with $\mathbf{v}$ not a scalar multiple of $\mathbf{w}$.]

11. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is as in Exercise 10 and that $T(P^*) = P$ is a parallelogram. Show that P^* is a parallelogram.

12. Consider the map $T\colon D \to D$, where D is the unit disk in the plane, given by

$$T(r\cos\theta, r\sin\theta) = (r^2\cos 2\theta, r^2\sin 2\theta).$$

Using complex notation, $z = x + iy$, the map T can be written as $T(z) = z^2$. Show that the Jacobian determinant of T vanishes only at the origin. Thus, away from the origin, T is locally one-to-one. However, show that T is not globally one-to-one on $\mathbb{R}^2$ minus the origin.

6.2 The Change of Variables Theorem

Given two regions D and D^* in $\mathbb{R}^2$, a differentiable map T on D^* with image D, that is, $T(D^*) = D$, and any real-valued integrable function $f\colon D \to \mathbb{R}$, we would like to express $\iint_D f(x, y)\, dA$ as an integral over D^* of the composite function $f \circ T$. In this section we shall see how to do this.

Assume that D^* is a region in the uv plane and that D is a region in the xy plane. The map T is given by two coordinate functions:

$$T(u, v) = (x(u, v), y(u, v)) \qquad \text{for} \qquad (u, v) \in D^*.$$

At first, one might conjecture that

$$\iint_D f(x, y)\, dx\, dy \stackrel{?}{=} \iint_{D^*} f(x(u, v), y(u, v))\, du\, dv, \tag{1}$$

where $f \circ T(u, v) = f(x(u, v), y(u, v))$ is the composite function defined on D^*. However, if we consider the function $f\colon D \to \mathbb{R}^2$ where $f(x, y) = 1$, then equation (1) would imply

$$A(D) = \iint_D dx\, dy \stackrel{?}{=} \iint_{D^*} du\, dv = A(D^*). \tag{2}$$

But equation (2) will hold for only a few special cases and not for a general map T. For example, define T by $T(u, v) = (-u^2 + 4u, v)$. Restrict T to the unit square; that is, to the region $D^* = [0, 1] \times [0, 1]$ in the uv plane (see Figure 6.2.1). Then, as in Exercise 3, Section 6.1, T takes D^* onto $D = [0, 3] \times [0, 1]$. Clearly, $A(D) \neq A(D^*)$, and so formula (2) is *not valid.*

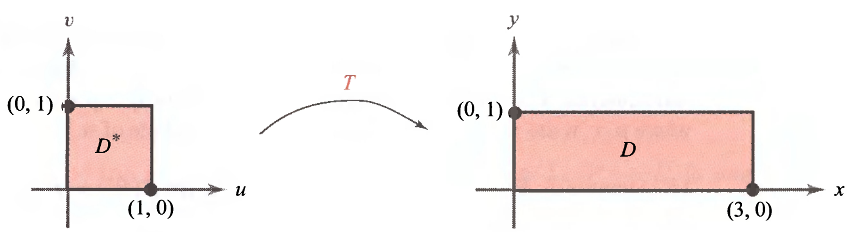


Figure 6.2.1 The map $T\colon (u, v) \mapsto (-u^2 + 4u, v)$ takes the square D^* onto the rectangle D.

Jacobian Determinants

To rectify the incorrect formula (1), we need a measure of how a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ distorts the area of a region. This is given by the *Jacobian determinant*, which is defined as follows.

DEFINITION: Jacobian Determinant Let $T\colon D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u, v)$ and $y = y(u, v)$. The ***Jacobian determinant*** of T, written $\partial(x, y)/\partial(u, v)$, is the determinant of the derivative matrix $\mathbf{D}T(u, v)$ of T:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

EXAMPLE 1 The function from $\mathbb{R}^2$ to $\mathbb{R}^2$ that transforms polar coordinates into Cartesian coordinates is given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

and its Jacobian determinant is

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r. \quad \blacktriangle$$

Under suitable restrictions on the function T, we will argue below that the area of $D = T(D^*)$ is obtained by integrating the absolute value of the Jacobian $\partial(x, y)/\partial(u, v)$ over D^*; that is, we have the equation

$$A(D) = \iint_D dx\, dy = \iint_{D^*} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \tag{3}$$

To illustrate: From Example 1 in Section 6.1, take $T\colon D^* \to D$, where $D = T(D^*)$ is the set of (x, y) with $x^2 + y^2 \le 1$ and $D^* = [0, 1] \times [0, 2\pi]$, and $T(r, \theta) = (r\cos\theta, r\sin\theta)$. By formula (3),

$$A(D) = \iint_{D^*} \left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| dr\, d\theta = \iint_{D^*} r\, dr\, d\theta \tag{4}$$

(here r and θ play the role of u and v). From the preceding computation it follows that

$$\iint_{D^*} r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi$$

is the area of the unit disk D, confirming formula (3) in this case. In fact, we may recall from first-year calculus that equation (4) is the correct formula for the area of a region in polar coordinates.

It is not so easy to rigorously prove assertion (3). However, looked at in the proper way, it becomes quite plausible. Recall that $A(D) = \iint_D dx\,dy$ was obtained by dividing up D into little rectangles, summing their areas, and then taking the limit of this sum as the size of the subrectangles tended to zero. The problem is that T may map rectangles into regions whose area is not easy to compute. The solution is to approximate these images by simpler regions whose area we can compute. A useful tool for doing this is the derivative of T, which we know (from Chapter 2) gives the best linear approximation to T.

Consider a small rectangle D^* in the uv plane as shown in Figure 6.2.2. Let T' denote the derivative of T evaluated at (u_0, v_0), so T' is a 2×2 matrix. From our work in Chapter 2, we know that a good approximation to $T(u, v)$ is given by

$$T(u_0, v_0) + T'\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix},$$

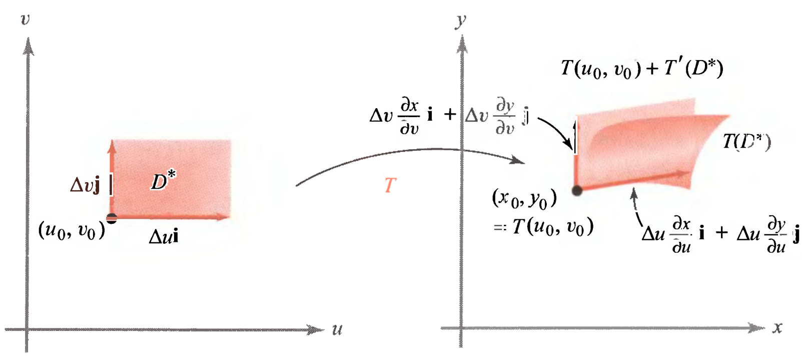


Figure 6.2.2 The effect of the transformation T on a small rectangle D^*.

where $\Delta u = u - u_0$ and $\Delta v = v - v_0$. This mapping T' takes D^* into a parallelogram with vertex at $T(u_0, v_0)$ and with adjacent sides given by the vectors

$$T'(\Delta u\mathbf{i}) = \begin{bmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{bmatrix}\begin{bmatrix}\Delta u\\ 0\end{bmatrix} = \Delta u\begin{bmatrix}\dfrac{\partial x}{\partial u}\\[2mm] \dfrac{\partial y}{\partial u}\end{bmatrix} = \Delta u\mathbf{T}_u$$

and

$$T'(\Delta v\mathbf{j}) = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} = \Delta v \begin{bmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \end{bmatrix} = \Delta v \mathbf{T}_v,$$

where

$$\mathbf{T}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} \qquad \text{and} \qquad \mathbf{T}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

are evaluated at (u_0, v_0).

Recall from Section 1.3 that the area of the parallelogram with sides equal to the vectors $a\mathbf{i} + b\mathbf{j}$ and $c\mathbf{i} + d\mathbf{j}$ is equal to the absolute value of the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Thus, the area of $T(D^*)$ is approximately equal to the *absolute value* of

$$\begin{vmatrix} \dfrac{\partial x}{\partial u}\Delta u & \dfrac{\partial x}{\partial v}\Delta v \\ \dfrac{\partial y}{\partial u}\Delta u & \dfrac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \Delta u\, \Delta v = \frac{\partial(x, y)}{\partial(u, v)} \Delta u\, \Delta v$$

evaluated at (u_0, v_0).

This fact and a partitioning argument should make formula (3) plausible. Indeed, if we partition D^* into small rectangles with sides of length Δu and Δv, the images of these rectangles are approximated by parallelograms with sides $\mathbf{T}_u\, \Delta u$ and $\mathbf{T}_v\, \Delta v$, and hence with area $|\partial(x, y)/\partial(u, v)|\, \Delta u\, \Delta v$. Thus, the area of D^* is approximately $\sum \Delta u\, \Delta v$, where the sum is taken over all the rectangles R inside D^* (see Figure 6.2.3). Hence, the area of $T(D^*)$ is approximately the sum

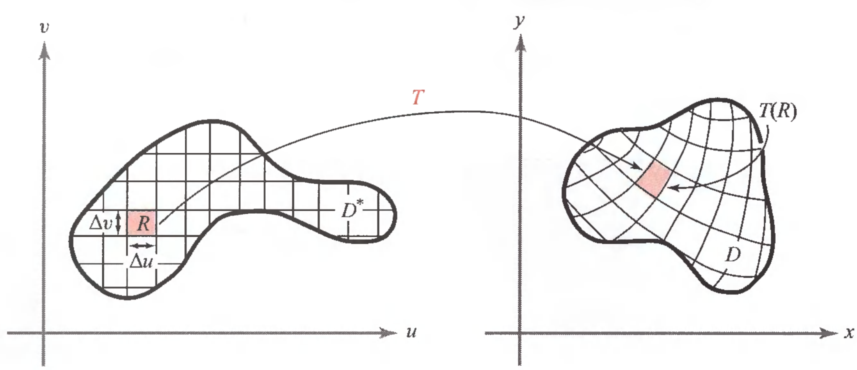


Figure 6.2.3 The area of the little rectangle R is $\Delta u\, \Delta v$. The area of $T(R)$ is approximately $|\partial(x, y)/\partial(u, v)|\Delta u \Delta v$.

$\sum |\partial(x, y)/\partial(u, v)| \Delta u \, \Delta v$. In the limit, this sum becomes

$$\iint_{D^*} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

Let us give another informal argument for the special case (4) of formula (3), that is, the case of polar coordinates. Consider a region D in the xy plane and a grid corresponding to a partition of the r and θ variables (Figure 6.2.4). The area of the shaded region shown is approximately $(\Delta r)(r_{jk}\, \Delta\theta)$, because the arc length of a segment of a circle of radius r subtending an angle ϕ is $r\phi$. The total area is then the limit of $\sum r_{jk}\, \Delta r\, \Delta\theta$; that is, $\iint_{D^*} r\, dr\, d\theta$. The key idea is thus that the jkth "polar rectangle" in the grid has area approximately equal to $r_{jk}\, \Delta r\, \Delta\theta$. (For n large, the jkth polar rectangle will look like a rectangle with sides of lengths $r_{jk}\, \Delta\theta$ and Δr). This should provide some insight into why we say *the "area element $dx\, dy$" is transformed into the "area element $r\, dr\, d\theta$."*

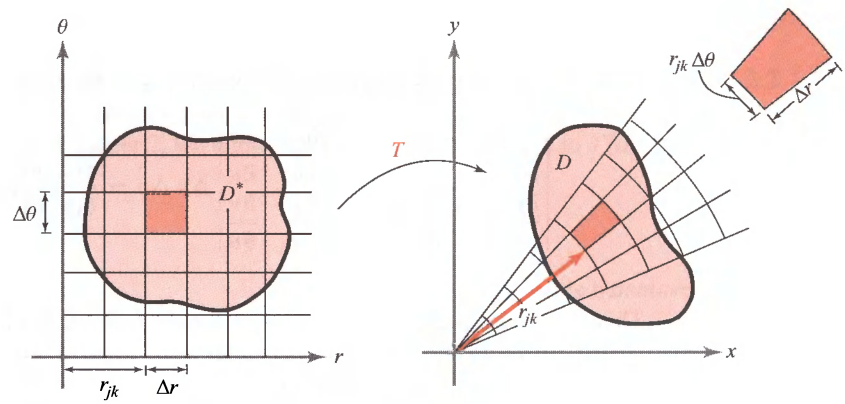


Figure 6.2.4 D^* is mapped to D under the polar-coordinate mapping T.

EXAMPLE 2 Let the elementary region D in the xy plane be bounded by the graph of a polar equation $r = f(\theta)$, where $\theta_0 \le \theta \le \theta_1$ and $f(\theta) \ge 0$ (see Figure 6.2.5). In the $r\theta$ plane we consider the r-simple region D^* where $\theta_0 \le \theta \le \theta_1$

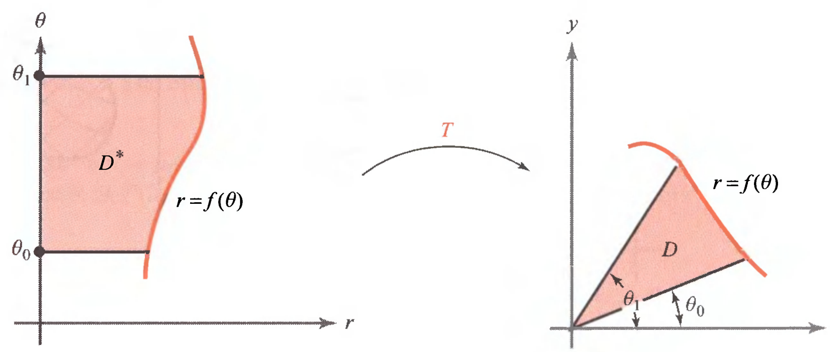


Figure 6.2.5 The effect on the region D^* of the polar-coordinate mapping.

and $0 \leq r \leq f(\theta)$. Under the transformation $x = r\cos\theta, y = r\sin\theta$, the region D^* is carried onto the region D. Use equation (4) to calculate the area of D.

SOLUTION

$$A(D) = \iint_D dx\, dy = \iint_{D^*} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\, d\theta$$

$$= \iint_{D^*} r\, dr\, d\theta = \int_{\theta_0}^{\theta_1}\left[\int_0^{f(\theta)} r\, dr\right] d\theta$$

$$= \int_{\theta_0}^{\theta_1}\left[\frac{r^2}{2}\right]_0^{f(\theta)} d\theta = \int_{\theta_0}^{\theta_1} \frac{[f(\theta)]^2}{2} d\theta$$

This formula for $A(D)$ should be familiar from one-variable calculus. ▲

Change of Variables Formula

Before stating the two-variable change of variables formula, which is the culmination of this discussion, let us recall the corresponding theorem from one-variable calculus that goes under the name the *method of substitution*:

$$\int_a^b f(x(u))\frac{dx}{du}\, du = \int_{x(a)}^{x(b)} f(x)\, dx, \tag{5}$$

where f is continuous and $u \mapsto x(u)$ is continuously differentiable on $[a, b]$.

PROOF Let F be an antiderivative of f; that is, $F' = f$, whose existence is guaranteed by the fundamental theorem of calculus. The right-hand side of equation (5) becomes

$$\int_{x(a)}^{x(b)} f(x)\, dx = F(x(b)) - F(x(a)).$$

To evaluate the left-hand side of equation (5), let $G(u) = F(x(u))$. By the chain rule, $G'(u) = F'(x(u))x'(u) = f(x(u))x'(u)$. Hence, again by the fundamental theorem,

$$\int_a^b f(x(u))x'(u)\, du = \int_a^b G'(u)\, du = G(b) - G(a) = F(x(b)) - F(x(a)),$$

as required. ■

Suppose now that we have a C^1 *function* $u \mapsto x(u)$ that is one-to-one on $[a, b]$. Thus, we must have either $dx/du \geq 0$ on $[a, b]$ or $dx/du \leq 0$ on $[a, b]$.[1] Let I^* denote the interval $[a, b]$, and let I denote the closed interval with endpoints $x(a)$

[1]If dx/du is positive and then negative, the function $x = x(u)$ rises and then falls, and thus is not one-to-one; a similar statement applies if dx/du is negative and then positive.

and $x(b)$. (Thus, $I = [x(a), x(b)]$ if $u \mapsto x(u)$ is increasing and $I = [x(b), x(a)]$ if $u \mapsto x(u)$ is decreasing.) With these conventions we can rewrite formula (5) as

$$\int_{I^*} f(x(u)) \left| \frac{dx}{du} \right| du = \int_I f(x)\, dx.$$

This formula generalizes to double integrals, as was already given informally in formula (3): I^* becomes D^*, I becomes D, and $|dx/du|$ is replaced by $|\partial(x, y)/\partial(u, v)|$. Let us state the result formally (the technical proof is omitted).

THEOREM 2: Change of Variables: Double Integrals Let D and D^* be elementary regions in the plane and let $T\colon D^* \to D$ be of class C^1; suppose that T is one-to-one on D^*. Furthermore, suppose that $D = T(D^*)$. Then for any integrable function $f\colon D \to \mathbb{R}$, we have

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \qquad (6)$$

One of the purposes of the change of variables theorem is to supply a method by which some double integrals can be simplified. One might encounter an integral $\iint_D f\, dA$ for which either the integrand f or the region D is complicated and for which direct evaluation is difficult. Therefore, a mapping T is chosen so that the integral is easier to evaluate with the new integrand $f \circ T$ and with the new region D^* [defined by $T(D^*) = D$]. Unfortunately, the problem may actually become more complicated if T is not selected carefully.

EXAMPLE 3 Let P be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$ (see Figure 6.2.6). Evaluate $\iint_P xy\, dx\, dy$ by making the

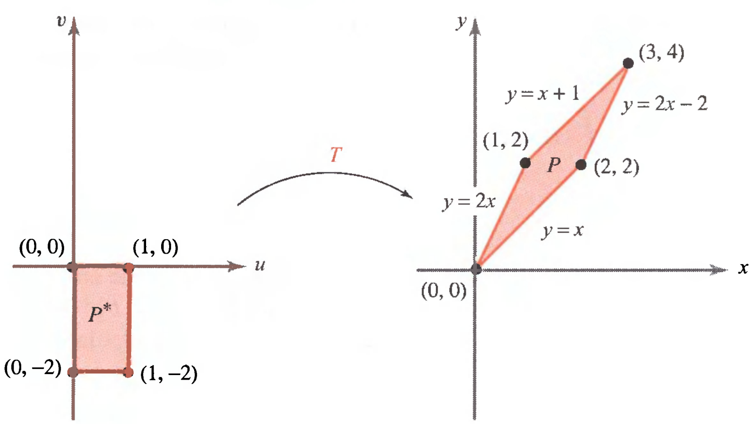


Figure 6.2.6 The effect of $T(u, v) = (u - v, 2u - v)$ on the rectangle P^*.

change of variables

$$x = u - v, \qquad y = 2u - v,$$

that is, $T(u, v) = (u - v, 2u - v)$.

SOLUTION The transformation T has nonzero determinant and so is one-to-one (see Exercise 8, Section 6.1). It is designed so that it takes the *rectangle* P^* bounded by $v = 0$, $v = -2$, $u = 0$, $u = 1$ onto P. The use of T simplifies the region of integration from P to P^*. Moreover,

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\det\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\right| = 1.$$

Therefore, by the change of variables formula,

$$\begin{aligned}
\iint_P xy\,dx\,dy &= \iint_{P^*}(u - v)(2u - v)\,du\,dv = \int_{-2}^{0}\int_{0}^{1}(2u^2 - 3vu + v^2)\,du\,dv \\
&= \int_{-2}^{0}\left[\frac{2}{3}u^3 - \frac{3u^2v}{2} + v^2u\right]_0^1 dv = \int_{-2}^{0}\left[\frac{2}{3} - \frac{3}{2}v + v^2\right] dv \\
&= \left[\frac{2}{3}v - \frac{3}{4}v^2 + \frac{v^3}{3}\right]_{-2}^{0} = -\left[\frac{2}{3}(-2) - 3 - \frac{8}{3}\right] \\
&= -\left[-\frac{12}{3} - 3\right] = 7. \quad \blacktriangle
\end{aligned}$$

Integrals in Polar Coordinates

Suppose we consider the rectangle D^* defined by $0 \le \theta \le 2\pi$, $0 \le r \le a$ in the $r\theta$ plane. The transformation T given by $T(r, \theta) = (r\cos\theta, r\sin\theta)$ takes D^* onto the disk D with equation $x^2 + y^2 \le a^2$ in the xy plane. This transformation represents the change from Cartesian coordinates to polar coordinates. However, T does not satisfy the requirements of the change of variables theorem, because it is not one-to-one on D^*: In particular, T sends all points with $r = 0$ to $(0, 0)$ (see Figure 6.2.7 and Example 3 of Section 6.1). Nevertheless, the change of variables theorem is valid in this case. Basically, the reason for this is that the set of points where T is not one-to-one lies on an edge of D^*, which is the graph of a smooth curve and therefore, for the purpose of integration, can be neglected. In summary, the formula

Change of Variables—Polar Coordinates

$$\iint_D f(x, y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{7}$$

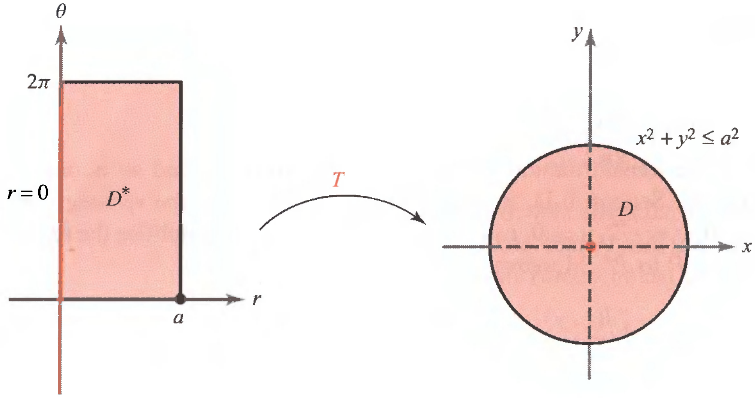


Figure 6.2.7 The image of the rectangle D^* under the polar-coordinate transformation is the disk D.

is valid when T sends D^* onto D in a one-to-one fashion except possibly for points on the boundary of D^*.

EXAMPLE 4 Evaluate $\iint_D \log(x^2 + y^2)\, dx\, dy$, where D is the region in the first quadrant lying between the arcs of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $0 < a < b$ (Figure 6.2.8).

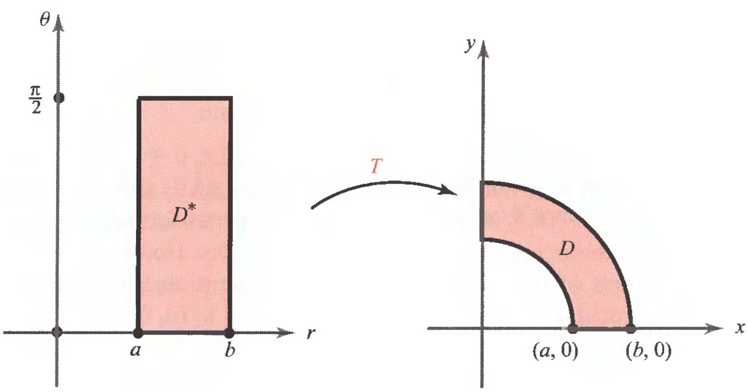


Figure 6.2.8 The polar-coordinate mapping takes a rectangle D^* onto part of an annulus D.

SOLUTION These circles have the simple equations $r = a$ and $r = b$ in polar coordinates. Moreover, $r^2 = x^2 + y^2$ appears in the integrand. Thus, a change to polar coordinates will simplify both the integrand and the region of integration. From Example 7, Section 6.1, the polar-coordinate transformation

$$x = r\cos\theta, \qquad y = r\sin\theta$$

sends the rectangle D^* given by $a \leq r \leq b, 0 \leq \theta \leq \pi/2$ onto the region D. This transformation is one-to-one on D^* and so, by formula (7), we have

$$\iint_D \log(x^2+y^2)\,dx\,dy = \int_a^b \int_0^{\pi/2} r \log r^2 \,d\theta\,dr$$

$$= \frac{\pi}{2}\int_a^b r\log r^2\,dr = \frac{\pi}{2}\int_a^b 2r\log r\,dr.$$

Applying integration by parts, or using the formula

$$\int x\log x\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$$

from the table of integrals at the back of the book, we obtain the result

$$\frac{\pi}{2}\int_a^b 2r\log r\,dr = \frac{\pi}{2}\left[b^2\log b - a^2\log a - \frac{1}{2}(b^2-a^2)\right]. \quad ▲$$

EXAMPLE 5 **The Gaussian Integral** One of the most beautiful applications of the change of variables formula, polar coordinates, and the reduction to iterated integrals is their application to the following formula, known as the ***Gaussian integral***:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Not only is this formula very attractive in its own right, but it is also useful in areas such as statistics. It also illustrates the unity of the transcendental numbers e and π nearly as well as does the classic formula $e^{i\pi} = -1$.

To carry out the integration of the Gaussian integral,[2] we first evaluate the double integral

$$\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy,$$

[2]The method that follows is admittedly not straightforward but requires a trick. The trick is to start with the desired formula and square both sides. You will then observe that the left-hand side resembles an iterated integral. There are several other ways to evaluate the Gaussian integral, but all of them require some nonobvious method. For the use of complex variables to evaluate it, see, for example, J. Marsden and M. Hoffman, *Basic Complex Analysis*, 3rd ed., W. H. Freeman, New York, 1998.

where D_a is the disk $x^2 + y^2 \leq a^2$. Because $r^2 = x^2 + y^2$, and $dx\,dy = r\,dr\,d\theta$, the change of variables formula gives

$$\iint_{D_a} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi} \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a d\theta$$
$$= -\frac{1}{2}\int_0^{2\pi} (e^{-a^2} - 1)\,d\theta = \pi(1 - e^{-a^2}).$$

If we let $a \to \infty$ in this expression, we give meaning to the improper integral and get

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

Assuming (as shown in the Internet supplement) that we can also evaluate this improper integral as the limit of the integrals over the rectangles $R_a = [-a, a] \times [-a, a]$ as $a \to \infty$, we get

$$\lim_{a\to\infty} \iint_{R_a} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

By reduction to iterated integrals, we can write this as

$$\lim_{a\to\infty}\left[\int_{-a}^{a} e^{-x^2}\,dx \int_{-a}^{a} e^{-y^2}\,dy\right] = \left[\lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx\right]^2 = \pi.$$

That is,

$$\left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]^2 = \pi.$$

Thus, taking square roots, we arrive at the desired result.

Here is a variant of the Gaussian integral. Evaluate

$$\int_{-\infty}^{\infty} e^{-2x^2}\,dx.$$

To do this, use the change of variables formula $y = \sqrt{2}x$ to reduce the problem to the Gaussian integral just computed:

$$\int_{-\infty}^{\infty} e^{-2x^2}\,dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-2x^2}\,dx = \lim_{a\to\infty}\int_{-\sqrt{2}a}^{\sqrt{2}a} e^{-y^2}\,\frac{dy}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{2}}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}. \quad \blacktriangle$$

Change of Variables Formula for Triple Integrals

To state this formula, we first define the Jacobian of a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$—it is a simple extension of the two-variable case.

DEFINITION Let $T\colon W \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. Then the ***Jacobian*** of T, which is denoted $\partial(x, y, z)/\partial(u, v, w)$, is the determinant

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

The absolute value of this determinant is equal to the volume of the parallelepiped determined by the three vectors

$$\mathbf{T}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k},$$

$$\mathbf{T}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k},$$

$$\mathbf{T}_w = \frac{\partial x}{\partial w}\mathbf{i} + \frac{\partial y}{\partial w}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k}.$$

Just as in the two-variable case, the Jacobian measures how the transformation T distorts the volume of its domain. Hence, for volume (triple) integrals, the change of variables formula takes the following form:

Change of Variables Formula: Triple Integrals

$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw, \tag{8}$$

where W^* is an elementary region in uvw space corresponding to W in xyz space, under a mapping $T\colon (u, v, w) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w))$, provided T is of class C^1 and is one-to-one, except possibly on a set that is the union of graphs of functions of two variables.

Cylindrical Coordinates

Let us apply formula (8) to cylindrical and then to spherical coordinates. First, we compute the Jacobian for the map defining the change to cylindrical coordinates. Because

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Thus, we obtain the formula

Change of Variables—Cylindrical Coordinates

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz. \qquad (9)$$

Spherical Coordinates

Next we consider the spherical coordinate system. Recall that it is given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$$

Therefore, we have

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}.$$

Expanding along the last row, we get

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \cos\phi \begin{vmatrix} -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix} \\ &\quad -\rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\ &= -\rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta \\ &\quad -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta \\ &= -\rho^2\cos^2\phi\sin\phi - \rho^2\sin^3\phi = -\rho^2\sin\phi. \end{aligned}$$

Thus, we arrive at the formula:

Change of Variables—Spherical Coordinates

$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_{W^*} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi. \tag{10}$$

To prove formula (10), one must show that the transformation S on the set W^* is one-to-one except on a set that is the union of finitely many graphs of continuous functions. We shall leave this verification as Exercise 34.

EXAMPLE 6 Evaluate

$$\iiint_W \exp(x^2 + y^2 + z^2)^{3/2}\, dV,$$

where W is the unit ball in $\mathbb{R}^3$.

SOLUTION First note that we cannot *easily* integrate this function using iterated integrals (try it!). Hence (employing the strategy in the quote that opened this chapter), let us try a change of variables. The transformation into spherical coordinates seems appropriate, because then the entire quantity $x^2 + y^2 + z^2$ can be replaced by one variable, namely, ρ^2. If W^* is the region such that

$$0 \le \rho \le 1, \qquad 0 \le \theta \le 2\pi, \qquad 0 \le \phi \le \pi,$$

we may apply formula (10) and write

$$\iiint_W \exp(x^2 + y^2 + z^2)^{3/2}\, dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi\, d\rho\, d\theta\, d\phi.$$

This integral equals the iterated integral

$$\begin{aligned}
\int_0^1 \int_0^\pi \int_0^{2\pi} e^{\rho^3} \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho &= 2\pi \int_0^1 \int_0^\pi e^{\rho^3} \rho^2 \sin\phi\, d\phi\, d\rho \\
&= -2\pi \int_0^1 \rho^2 e^{\rho^3} [\cos\phi]_0^\pi\, d\rho \\
&= 4\pi \int_0^1 e^{\rho^3} \rho^2\, d\rho = \frac{4}{3}\pi \int_0^1 e^{\rho^3}(3\rho^2)\, d\rho \\
&= \left[\frac{4}{3}\pi e^{\rho^3}\right]_0^1 = \frac{4}{3}\pi(e - 1). \quad \blacktriangle
\end{aligned}$$

EXAMPLE 7 Let W be the ball of radius R and center $(0, 0, 0)$ in $\mathbb{R}^3$. Find the volume of W.

SOLUTION The volume of W is $\iiint_W dx\,dy\,dz$. This integral may be evaluated by reducing it to iterated integrals or by regarding W as a volume of revolution, but let us evaluate it here by using spherical coordinates. We get

$$\iiint_W dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^R \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{R^3}{3}\int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi$$
$$= \frac{2\pi R^3}{3}\int_0^{\pi} \sin\phi\, d\phi = \frac{2\pi R^3}{3}\{-[\cos(\pi) - \cos(0)]\} = \frac{4\pi R^3}{3},$$

which is the standard formula for the volume of a solid sphere. ▲

EXERCISES

1. Let D be the unit disk: $x^2 + y^2 \le 1$. Evaluate

$$\iint_D \exp(x^2 + y^2)\, dx\, dy$$

by making a change of variables to polar coordinates.

2. Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate

$$\iint_D (x + y)\, dx\, dy$$

by making the change of variables $x = u + v$, $y = u - v$. Check your answer by evaluating the integral directly by using an iterated integral.

3. Let $T(u, v) = (x(u, v), y(u, v))$ be the mapping defined by $T(u, v) = (4u, 2u + 3v)$. Let D^* be the rectangle $[0, 1] \times [1, 2]$. Find $D = T(D^*)$ and evaluate

(a) $\displaystyle\iint_D xy\, dx\, dy$ (b) $\displaystyle\iint_D (x - y)\, dx\, dy$

by making a change of variables to evaluate them as integrals over D^*.

4. Repeat Exercise 3 for $T(u, v) = (u, v(1 + u))$.

5. Evaluate

$$\iint_D \frac{dx\, dy}{\sqrt{1 + x + 2y}},$$

where $D = [0, 1] \times [0, 1]$, by setting $T(u, v) = (u, v/2)$ and evaluating an integral over D^*, where $T(D^*) = D$.

6. Define $T(u, v) = (u^2 - v^2, 2uv)$. Let D^* be the set of (u, v) with $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Find $T(D^*) = D$. Evaluate $\iint_D dx\, dy$.

7. Let $T(u, v)$ be as in Exercise 6. By making a change of variables, "formally" evaluate the "improper" integral

$$\iint_D \frac{dx\,dy}{\sqrt{x^2+y^2}}.$$

[NOTE: This integral (and the one in the next exercise) is *improper*, because the integrand $1/\sqrt{x^2+y^2}$ is neither continuous nor bounded on the domain of integration. (The theory of improper integrals is discussed in Section 6.4.)]

8. Calculate $\displaystyle\iint_R \frac{1}{x+y}\,dy\,dx$, where R is the region bounded by $x=0$, $y=0$, $x+y=1$, $x+y=4$, by using the mapping $T(u,v)=(u-uv, uv)$.

9. Evaluate $\displaystyle\iint_D (x^2+y^2)^{3/2}\,dx\,dy$ where D is the disk $x^2+y^2\le 4$.

10. Let D^* be a v-simple region in the uv plane bounded by $v=g(u)$ and $v=h(u)\le g(u)$ for $a\le u\le b$. Let $T\colon \mathbb{R}^2\to\mathbb{R}^2$ be the transformation given by $x=u$ and $y=\psi(u,v)$, where ψ is of class C^1 and $\partial\psi/\partial v$ is never zero. Assume that $T(D^*)=D$ is a y-simple region; show that if $f\colon D\to\mathbb{R}$ is continuous, then

$$\iint_D f(x,y)\,dx\,dy=\iint_{D^*} f(u,\psi(u,v))\left|\frac{\partial\psi}{\partial v}\right|du\,dv.$$

11. Use double integrals to find the area inside the curve $r=1+\sin\theta$.

12. (a) Express $\int_0^1\int_0^{x^2} xy\,dy\,dx$ as an integral over the triangle D^*, which is the set of (u,v) where $0\le u\le 1, 0\le v\le u$. (HINT: Find a one-to-one mapping T of D^* onto the given region of integration.)

(b) Evaluate this integral directly and as an integral over D^*.

13. Integrate $ze^{x^2+y^2}$ over the cylinder $x^2+y^2\le 4, 2\le z\le 3$.

14. Let D be the unit disk. Express $\displaystyle\iint_D (1+x^2+y^2)^{3/2}\,dx\,dy$ as an integral over $[0,1]\times[0,2\pi]$ and evaluate.

15. Using polar coordinates, find the area bounded by the *lemniscate* $(x^2+y^2)^2 = 2a^2\,(x^2-y^2)$.

16. Redo Exercise 11 of Section 5.3 using a change of variables and compare the effort involved in each method.

17. Calculate $\displaystyle\iint_R (x+y)^2 e^{x-y}\,dx\,dy$ where R is the region bounded by $x+y=1$, $x+y=4$, $x-y=-1$, and $x-y=1$.

18. Let $T\colon\mathbb{R}^3\to\mathbb{R}^3$ be defined by

$$T(u,v,w)=(u\cos v\cos w, u\sin v\cos w, u\sin w).$$

(a) Show that T is onto the unit sphere; that is, every (x, y, z) with $x^2 + y^2 + z^2 = 1$ can be written as $(x, y, z) = T(u, v, w)$ for some (u, v, w).

(b) Show that T is not one-to-one.

19. Integrate $x^2 + y^2 + z^2$ over the cylinder $x^2 + y^2 \le 2$, $-2 \le z \le 3$.

20. Evaluate $\int_0^\infty e^{-4x^2}\,dx$.

21. Let B be the unit ball. Evaluate

$$\iiint_B \frac{dx\,dy\,dz}{\sqrt{2 + x^2 + y^2 + z^2}}$$

by making the appropriate change of variables.

22. Evaluate $\iint_A [1/(x^2 + y^2)^2]\,dx\,dy$ where A is determined by the conditions $x^2 + y^2 \le 1$ and $x + y \ge 1$.

23. Evaluate $\iiint_W \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{3/2}}$, where W is the solid bounded by the two spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $0 < b < a$.

24. Evaluate $\iint_D x^2\,dx\,dy$ where D is determined by the two conditions $0 \le x \le y$ and $x^2 + y^2 \le 1$.

25. Integrate $\sqrt{x^2 + y^2 + z^2}\,e^{-(x^2+y^2+z^2)}$ over the region described in Exercise 23.

26. Evaluate the following by using cylindrical coordinates.

(a) $\iiint_B z\,dx\,dy\,dz$ where B is the region within the cylinder $x^2 + y^2 = 1$ above the xy plane and below the cone $z = (x^2 + y^2)^{1/2}$.

(b) $\iiint_W (x^2 + y^2 + z^2)^{-1/2}\,dx\,dy\,dz$ where W is the region determined by the conditions $\frac{1}{2} \le z \le 1$ and $x^2 + y^2 + z^2 \le 1$.

27. Evaluate $\iint_B (x + y)\,dx\,dy$ where B is the rectangle in the xy plane with vertices at $(0, 1)$, $(1, 0)$, $(3, 4)$, and $(4, 3)$.

28. Evaluate $\iint_D (x + y)\,dx\,dy$ where D is the square with vertices at $(0, 0)$, $(1, 2)$, $(3, 1)$, and $(2, -1)$.

29. Let E be the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1$, where a, b, and c are positive.

(a) Find the volume of E.

(b) Evaluate $\iiint_E [(x^2/a^2) + (y^2/b^2) + (z^2/c^2)]\,dx\,dy\,dz$. (HINT: Change variables and then use spherical coordinates.)

30. Using spherical coordinates, compute the integral of $f(\rho, \phi, \theta) = 1/\rho$ over the region in the first octant of $\mathbb{R}^3$, which is bounded by the cones $\phi = \pi/4$, $\phi = \arctan 2$ and the sphere $\rho = \sqrt{6}$.

31. The mapping $T(u, v) = (u^2 - v^2, 2uv)$ transforms the rectangle $1 \le u \le 2$, $1 \le v \le 3$ of the uv plane into a region R of the xy plane.

(a) Show that T is one-to-one.
(b) Find the area of R using the change of variables formula.

32. Let R denote the region inside $x^2 + y^2 = 1$, but outside $x^2 + y^2 = 2y$ with $x \ge 0, y \ge 0$.

(a) Sketch this region.
(b) Let $u = x^2 + y^2$, $v = x^2 + y^2 - 2y$. Sketch the region D in the uv plane, which corresponds to R under this change of coordinates.
(c) Compute $\iint_R xe^y \, dx\, dy$ using this change of coordinates.

33. Let D be the region bounded by $x^{3/2} + y^{3/2} = a^{3/2}$, for $x \ge 0, y \ge 0$, and the coordinate axes $x = 0, y = 0$. Express $\iint_D f(x, y)\, dx\, dy$ as an integral over the triangle D^*, which is the set of points $0 \le u \le a$, $0 \le v \le a - u$. (Do not attempt to evaluate.)

34. Show that $S(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$, the spherical change-of-coordinate mapping, is one-to-one except on a set that is a union of finitely many graphs of continuous functions.

6.3 Applications

In this section, we shall discuss average values, centers of mass, moments of inertia, and the gravitational potential as applications.

Averages

If $x_1, \ldots, x_n$ are n numbers, their ***average*** is defined by

$$[x_i]_{\mathrm{av}} = \frac{x_1 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

Notice that if all the x_i happen to have a common value c, then their average, of course, also equals c.

This concept leads one to define the average values of functions as follows.

Average Values The ***average value*** of a function of one variable on the interval $[a, b]$ is defined by

$$[f]_{\mathrm{av}} = \frac{\int_a^b f(x)\, dx}{b - a}.$$

Likewise, for functions of two variables, the ratio of the integral to the area of D,

$$[f]_{\mathrm{av}} = \frac{\iint_D f(x, y)\, dx\, dy}{\iint_D dx\, dy}, \tag{1}$$

is called the ***average value*** of f over D. Similarly, the ***average value*** of a function f on a region W in three space is defined by

$$[f]_{\mathrm{av}} = \frac{\iiint_W f(x,y,z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}.$$

Again, notice that the denominator is chosen so that if f is a constant, say c, then $[f]_{\mathrm{av}} = c$.

EXAMPLE 1 Find the average value of $f(x,y) = x\sin^2(xy)$ on the region $D = [0,\pi]\times[0,\pi]$.

SOLUTION First, we compute

$$\begin{aligned}
\iint_D f(x,y)\,dx\,dy &= \int_0^\pi \int_0^\pi x\sin^2(xy)\,dx\,dy \\
&= \int_0^\pi \left[\int_0^\pi \frac{1-\cos(2xy)}{2}x\,dy\right] dx \\
&= \int_0^\pi \left[\frac{y}{2} - \frac{\sin(2xy)}{4x}\right] x\bigg|_{y=0}^{\pi} dx \\
&= \int_0^\pi \left[\frac{\pi x}{2} - \frac{\sin(2\pi x)}{4}\right] dx = \left[\frac{\pi x^2}{4} + \frac{\cos(2\pi x)}{8\pi}\right]\bigg|_0^\pi \\
&= \frac{\pi^3}{4} + \frac{\cos(2\pi^2)-1}{8\pi}.
\end{aligned}$$

Thus, the average value of f, by formula (1), is

$$\frac{\pi^3/4 + [\cos(2\pi^2)-1]/8\pi}{\pi^2} = \frac{\pi}{4} + \frac{\cos(2\pi^2)-1}{8\pi^3} \approx 0.7839. \quad \blacktriangle$$

EXAMPLE 2 The temperature at points in the cube $W = [-1,1]\times[-1,1]\times[-1,1]$ is proportional to the square of the distance from the origin.

(a) What is the average temperature?

(b) At which points of the cube is the temperature equal to the average temperature?

SOLUTION (a) Let c be the constant of proportionality so $T = c(x^2+y^2+z^2)$ and the average temperature is $[T]_{\mathrm{av}} = \frac{1}{8}\iiint_W T\,dx\,dy\,dz$,because the volume of the cube is 8. Thus,

$$[T]_{\text{av}} = \frac{c}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz.$$

The triple integral is the sum of the integrals of x^2, y^2, and z^2. Because x, y, and z enter symmetrically into the description of the cube, the three integrals will be equal, so that

$$[T]_{\text{av}} = \frac{3c}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} z^2\,dx\,dy\,dz = \frac{3c}{8}\int_{-1}^{1} z^2\left(\int_{-1}^{1}\int_{-1}^{1} dx\,dy\right) dz.$$

The inner integral is equal to the area of the square $[-1,1]\times[-1,1]$. The area of that square is 4, and so

$$[T]_{\text{av}} = \frac{3c}{8}\int_{-1}^{1} 4z^2\,dz = \frac{3c}{2}\left(\frac{z^3}{3}\right)\Bigg|_{-1}^{1} = c.$$

(b) The temperature is equal to the average temperature at all points satisfying $c(x^2+y^2+z^2)=c$, that is, which lie on the sphere $x^2+y^2+z^2=1$, which is inscribed in the cube W. ▲

Centers of Mass

If masses $m_1,\dots,m_n$ are placed at points $x_1,\dots,x_n$ on the x axis, their ***center of mass*** is defined to be

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}. \tag{2}$$

This definition arises from the following observation: If one is balancing masses on a lever (Figure 6.3.1), the balance point $\bar{x}$ occurs where the total moment (mass times distance from the balance point) is zero, that is, where $\sum m_i(x_i-\bar{x})=0$. A physical principle, going back first to Archimedes and then in this generality to Newton, states that this condition means there is no tendency for the lever to rotate.

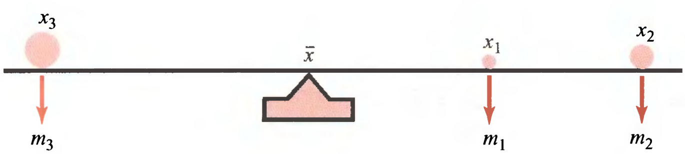


Figure 6.3.1 The lever is balanced if $\Sigma(x_i-\bar{x})m_i=0$.

For a continuous mass density $\delta(x)$ along the lever (measured in, say, grams/cm), the analog of formula (2) is

$$\bar{x} = \frac{\int x\delta(x)\,dx}{\int \delta(x)\,dx}. \tag{3}$$

For two-dimensional plates, this generalizes to:

The Center of Mass of Two-Dimensional Plates

$$\bar{x} = \frac{\iint_D x\delta(x, y)\, dx\, dy}{\iint_D \delta(x, y)\, dx\, dy} \quad \text{and} \quad \bar{y} = \frac{\iint_D y\delta(x, y)\, dx\, dy}{\iint_D \delta(x, y)\, dx\, dy}, \tag{4}$$

where again $\delta(x, y)$ is the mass density (see Figure 6.3.2).

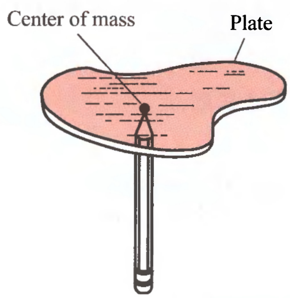


Figure 6.3.2 The plate balances when supported at its center of mass.

EXAMPLE 3 Find the center of mass of the rectangle $[0, 1] \times [0, 1]$ if the mass density is e^{x+y}.

SOLUTION First we compute the total mass:

$$\iint_D e^{x+y}\, dx\, dy = \int_0^1 \int_0^1 e^{x+y}\, dx\, dy = \int_0^1 (e^{x+y}|_{x=0}^1)\, dy = \int_0^1 (e^{1+y} - e^y)\, dy$$
$$= (e^{1+y} - e^y)|_{y=0}^1 = e^2 - e - (e - 1) = e^2 - 2e + 1.$$

The numerator in formula (4) for $\bar{x}$ is

$$\int_0^1 \int_0^1 xe^{x+y}\, dx\, dy = \int_0^1 (xe^{x+y} - e^{x+y})|_{x=0}^1\, dy = \int_0^1 [e^{1+y} - e^{1+y} - (0e^y - e^y)]\, dy$$
$$= \int_0^1 e^y\, dy = e^y|_{y=0}^1 = e - 1,$$

so that

$$\bar{x} = \frac{e - 1}{e^2 - 2e + 1} = \frac{e - 1}{(e - 1)^2} = \frac{1}{e - 1} \approx 0.582.$$

The roles of x and y may be interchanged in all these calculations, so that $\bar{y} = 1/(e - 1) \approx 0.582$ as well. ▲

For a region W in space with mass density $\delta(x, y, z)$, we know that

$$\text{volume} = \iiint_W dx\,dy\,dz, \tag{5}$$

$$\text{mass} = \iiint_W \delta(x, y, z)\,dx\,dy\,dz. \tag{6}$$

If one denotes the coordinates of the center of mass by $(\bar{x}, \bar{y}, \bar{z})$, then the generalization of the formulas in the preceding box are as follows.

Coordinates for the Center of Mass of Three-Dimensional Regions

$$\begin{aligned}\bar{x} &= \frac{\iiint_W x\delta(x, y, z)\,dx\,dy\,dz}{\text{mass}},\\ \bar{y} &= \frac{\iiint_W y\delta(x, y, z)\,dx\,dy\,dz}{\text{mass}},\\ \bar{z} &= \frac{\iiint_W z\delta(x, y, z)\,dx\,dy\,dz}{\text{mass}}.\end{aligned} \tag{7}$$

EXAMPLE 4 The cube $[1, 2] \times [1, 2] \times [1, 2]$ has mass density given by $\delta(x, y, z) = (1 + x)e^z y$. Find the total mass of the box.

SOLUTION The mass of the box is, by formula (6),

$$\int_1^2 \int_1^2 \int_1^2 (1 + x)e^z y\,dx\,dy\,dz = \int_1^2 \int_1^2 \left[\left(x + \frac{x^2}{2}\right) e^z y\right]_{x=1}^{x=2} dy\,dz$$

$$= \int_1^2 \int_1^2 \frac{5}{2} e^z y\,dy\,dz = \int_1^2 \frac{15}{4} e^z\,dz = \left[\frac{15}{4} e^z\right]_{z=1}^{z=2} = \frac{15}{4}(e^2 - e). \quad \blacktriangle$$

If a region and its mass density are reflection-symmetric across a plane, then the center of mass lies on that plane. For example, in formula (7) for $\bar{x}$, if the region and mass density are symmetric in the yz plane, then the integrand is odd in x, and so $\bar{x} = 0$. This kind of use of symmetry is illustrated in the next example.

EXAMPLE 5 Find the center of mass of the hemispherical region W defined by the inequalities $x^2 + y^2 + z^2 \leq 1, z \geq 0$. (Assume that the density is unity.)

SOLUTION By symmetry, the center of mass must lie on the z axis, and so $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we must compute, by formula (7), the numerator $I = \iiint_W z\,dx\,dy\,dz$. The hemisphere is an elementary region, and thus the integral becomes

$$I = \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} z\,dx\,dy\,dz.$$

Because z is a constant for the x and y integrations, we can remove it from the first two integral signs, to obtain

$$I = \int_0^1 z\left(\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx\,dy\right) dz.$$

Instead of calculating the inner two integrals explicitly, we observe that they equal the double integral $\iint_D dx\,dy$ over the disk $x^2 + y^2 \leq 1 - z^2$, considered as an x-simple region in the plane. The area of this disk is $\pi(1 - z^2)$, and so

$$I = \pi \int_0^1 z(1 - z^2)\,dz = \pi \int_0^1 (z - z^3)\,dz = \pi \left[\frac{z^2}{2} - \frac{z^4}{4}\right]_0^1 = \frac{\pi}{4}.$$

The volume of the hemisphere is $\frac{2}{3}\pi$, and so $\bar{z} = (\pi/4)/(\frac{2}{3}\pi) = \frac{3}{8}$. ▲

Historical Note

It is common knowledge that Archimedes observed the principle of the lever. Perhaps less known is that he was also responsible for discovering the concepts of center of mass and center of gravity. Only two of his works on mechanics have been handed down to us: *On Floating Bodies* and *On the Equilibrium and Centers of Mass of Plane Figures*. Both were translated into Latin by Niccolo Tartaglia, circa 1543.

In *Equilibrium*..., Archimedes began the field of applied mathematics, doing for mechanics what Euclid had accomplished for geometry. In this work he describes the principles behind all the machines of antiquity, including the lever, inclined plane, and pulley system.

Surprisingly, Archimedes never carefully defined the center of mass; the first proper definition was given by Pappus of Alexandria in 340 C.E. The concept of equilibrium was to have a profound effect on the development of mechanical engineering (through the introduction of gears), architecture, and in art, permitting the construction of complex machines, large-scale buildings, and sculptures. Figure 6.3.3 shows sketches by Leonardo DaVinci, illustrating equilibrium positions of the human body.

Figure 6.3.3 Equilibrium positions of the human body, to be observed by the painter. The center of mass should be supported to maintain equilibrium.

Moments of Inertia

Another important concept in mechanics, one that is needed in studying the dynamics of a rotating rigid body, is that of *moment of inertia*. If the solid W has uniform density δ, the ***moments of inertia*** I_x, I_y, and I_z about the x, y, and z axes, respectively, are defined by:

Moments of Inertia About the Coordinate Axes

$$I_x = \iiint_W (y^2 + z^2)\,\delta\, dx\, dy\, dz, \qquad I_y = \iiint_W (x^2 + z^2)\,\delta\, dx\, dy\, dz,$$
$$I_z = \iiint_W (x^2 + y^2)\,\delta\, dx\, dy\, dz. \tag{8}$$

The moment of inertia measures a body's response to efforts to rotate it; for example, as when one tries to rotate a merry-go-round. The moment of inertia is analogous to the mass of a body, which measures its response to efforts to translate it. In contrast to translational motion, however, the moments of inertia *depend on the shape and not just the total mass*. It is harder to spin up a large plate than a compact ball of the same mass.

For example, I_x measures the body's response to forces attempting to rotate it about the x axis. The factor $y^2 + z^2$, which is the square of the distance to the x axis, weights masses farther away from the rotation axis more heavily. This is in agreement with the intuition just explained.

EXAMPLE 6 Compute the moment of inertia I_z for the solid above the xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$, assuming a and the mass density to be constants.

SOLUTION The paraboloid and cylinder intersect at the plane $z = a^2$. Using cylindrical coordinates, we find from equation (8),

$$I_z = \int_0^a \int_0^{2\pi} \int_0^{r^2} \delta r^2 \cdot r\, dz\, d\theta\, dr = \delta \int_0^a \int_0^{2\pi} \int_0^{r^2} r^3\, dz\, d\theta\, dr = \frac{\pi \delta a^6}{3}. \quad ▲$$

Gravitational Fields of Solid Objects

Another interesting physical application of triple integration is the determination of the gravitational fields of solid objects. Example 6, Section 2.6, showed that the gravitational force field $\mathbf{F}(x, y, z)$ of a particle is the negative of the gradient of a function $V(x, y, z)$ called the ***gravitational potential***. If there is a point mass M at (x, y, z), then the gravitational potential acting on a mass m at (x_1, y_1, z_1) due to this mass is $-GmM[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{-1/2}$, where G is the universal gravitational constant.

If our attracting object occupies a domain W with mass density $\delta(x, y, z)$, we may think of it as made of infinitesimal box-shaped regions with masses $dM = \delta(x, y, z)\, dx\, dy\, dz$ located at points (x, y, z). The total gravitational potential V for W is then obtained by "summing" the potentials from the infinitesimal masses. Thus, we arrive at the triple integral (see Figure 6.3.4):

$$V(x_1, y_1, z_1) = -Gm \iiint_W \frac{\delta(x, y, z)\, dx\, dy\, dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}. \tag{9}$$

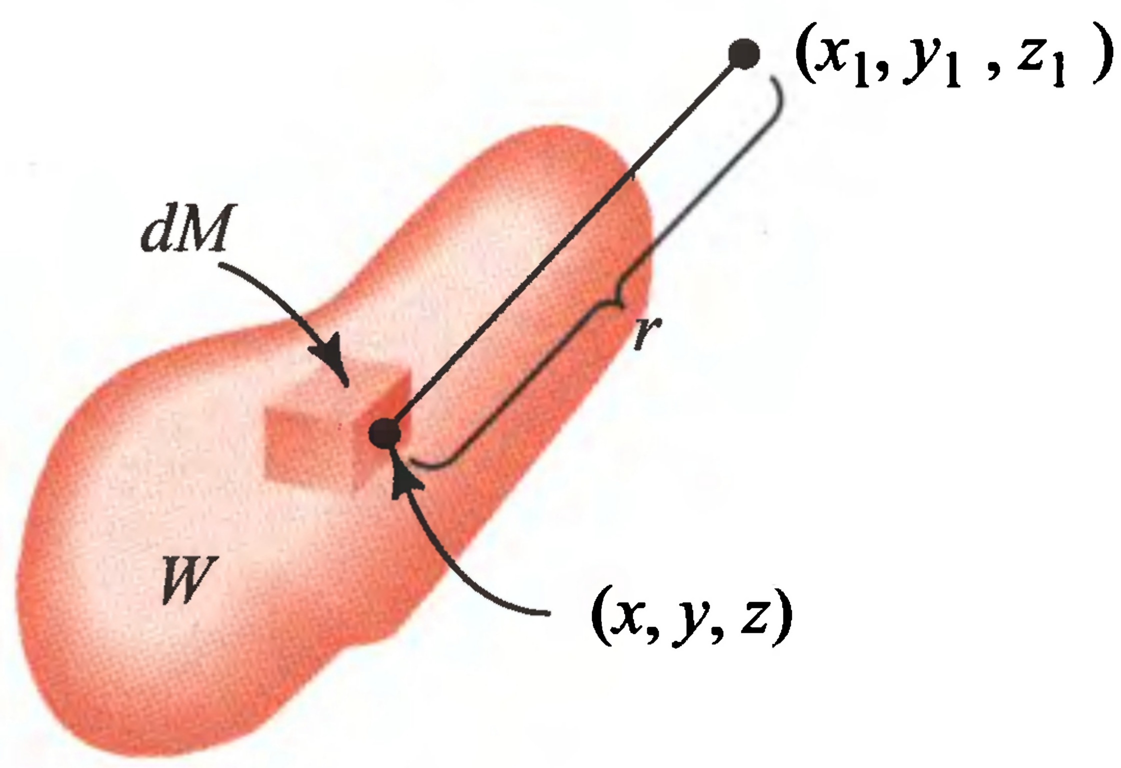


Figure 6.3.4 The gravitational potential that produces a force acting on a mass m at (x_1, y_1, z_1) arising from the mass $dM = \delta(x, y, z)\, dx\, dy\, dz$ at (x, y, z) is $-[Gm\delta(x, y, z)\, dx\, dy\, dz]/r$.

Historical Note

The theory of gravitational force fields and gravitational potentials was developed by Sir Isaac Newton (1642–1727). Newton withheld publication of his gravitational theories for quite some time. The result that a spherical planet has the same gravitational field as it would have if its mass were all concentrated at the planet's center first appeared in his famous *Philosophiae Naturalis Principia Mathematica*, the first edition of which appeared in 1687. Using multiple integrals and spherical coordinates, we shall solve Newton's problem here; remarkably, Newton's published solution used only Euclidean geometry.

EXAMPLE 7 Let W be a region of constant density and total mass M. Show that the gravitational potential is given by

$$V(x_1, y_1, z_1) = \left[\frac{1}{r}\right]_{\text{av}} GMm,$$

where $[1/r]_{\text{av}}$ is the average over W of

$$f(x, y, z) = \frac{1}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}.$$

SOLUTION According to formula (9),

$$
\begin{aligned}
-V(x_1, y_1, z_1) &= Gm \iiint_W \frac{\delta\, dx\, dy\, dz}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \\
&= Gm\delta \iiint_W \frac{dx\, dy\, dz}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} \\
&= Gm[\delta \text{ volume}\,(W)] \frac{\displaystyle\iiint_W \frac{dx\, dy\, dz}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}}}{\text{volume}\,(W)} \\
&= GmM\left[\frac{1}{r}\right]_{\text{av}}
\end{aligned}
$$

are required. ▲

Let us now use formula (9) and spherical coordinates to find the gravitational potential $V(x_1, y_1, z_1)$ for a region W with constant density between the concentric spheres $\rho = \rho_1$ and $\rho = \rho_2$, assuming the density is constant. Before evaluating the integral in formula (9), we make some observations that will simplify the computation. Because G, m, and the density are constants, we may ignore them at first. Because the attracting body, W, is symmetric with respect to rotations about the origin, the potential $V(x_1, y_1, z_1)$ should itself be symmetric—thus, $V(x_1, y_1, z_1)$ depends only on the distance $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$ from the origin. Our computation will be simplest if we look at the point $(0, 0, R)$ on the z axis (see Figure 6.3.5). Thus, we need to evaluate the integral

$$V(0, 0, R) = -\iiint_W \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + (z-R)^2}}.$$

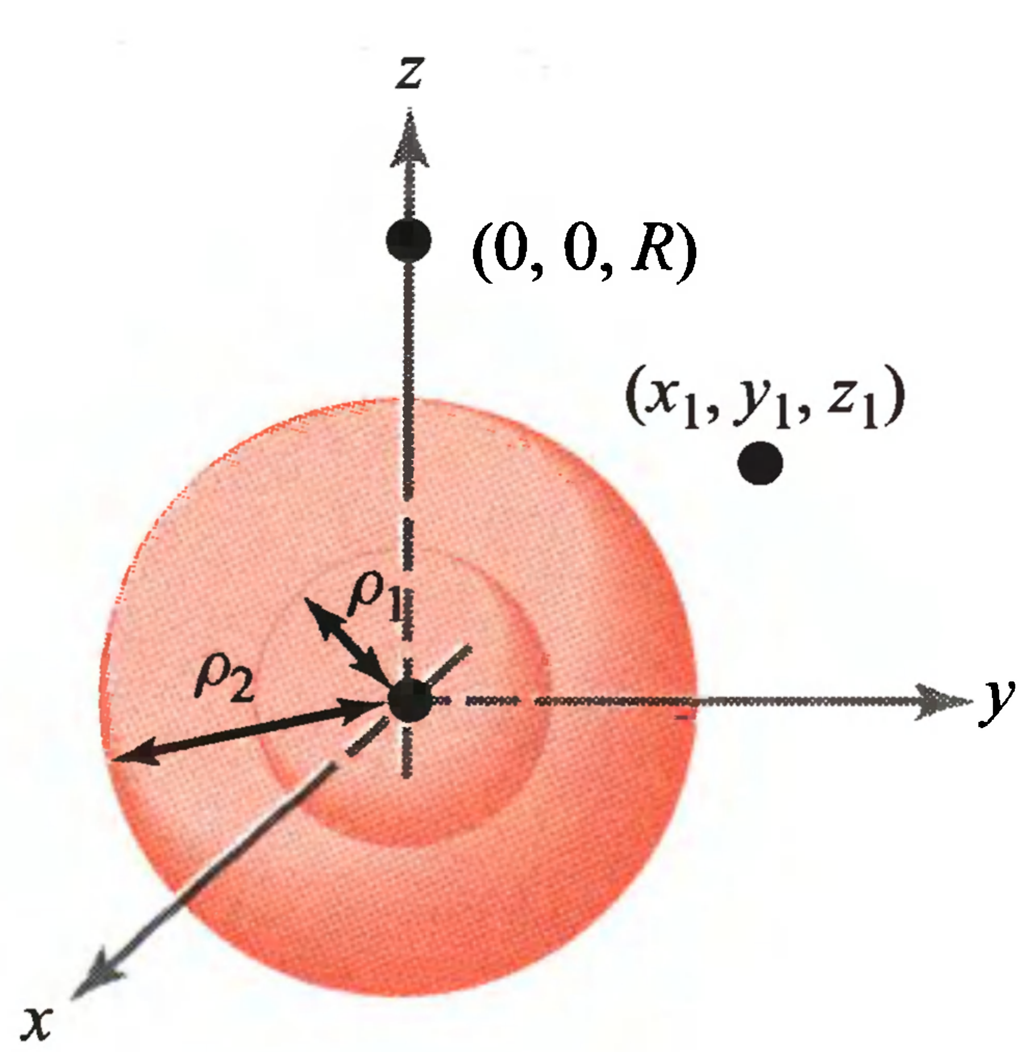


Figure 6.3.5 The gravitational potential at (x_1, y_1, z_1) is the same as at $(0, 0, R)$, where $R = \sqrt{x_1^2 + y_1^2 + z_1^2}$.

In spherical coordinates, W is described by the inequalities $\rho_1 \le \rho \le \rho_2$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$, and so

$$-V(0,0,R) = \int_{\rho_1}^{\rho_2}\int_0^{\pi}\int_0^{2\pi} \frac{\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho}{\sqrt{\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + (\rho\cos\phi - R)^2}}.$$

Replacing $\cos^2\theta + \sin^2\theta$ by 1, so that the integrand no longer involves θ, we may integrate over θ to get

$$\begin{aligned} -V(0,0,R) &= 2\pi \int_{\rho_1}^{\rho_2}\int_0^{\pi} \frac{\rho^2 \sin\phi\, d\phi\, d\rho}{\sqrt{\rho^2\sin^2\phi + (\rho\cos\phi - R)^2}} \\ &= 2\pi \int_{\rho_1}^{\rho_2} \rho^2 \left(\int_0^{\pi} \frac{\sin\phi\, d\phi}{\sqrt{\rho^2 - 2R\rho\cos\phi + R^2}}\right) d\rho. \end{aligned}$$

The inner integral over ϕ may be evaluated using the substitution $u = -2R\rho\cos\phi$. We get

$$\begin{aligned} \frac{1}{2R\rho}\int_{-2R\rho}^{2R\rho} (\rho^2 + u + R^2)^{-1/2}\, du &= \frac{2}{2R\rho}(\rho^2 + u + R^2)^{1/2}\Big|_{-2R\rho}^{2R\rho} \\ &= \frac{1}{R\rho}[(\rho^2 + 2R\rho + R^2)^{1/2} - (\rho^2 - 2R\rho + R^2)^{1/2} \\ &= \frac{1}{R\rho}\{[(\rho + R)^2]^{1/2} - [(\rho - R)^2]^{1/2}\} \\ &= \frac{1}{R\rho}(\rho + R - |\rho - R|). \end{aligned}$$

The expression $\rho + R$ is always positive, but $\rho - R$ may not be, so we must keep the absolute value sign. Substituting into the formula for V, we get

$$-V(0,0,R) = 2\pi \int_{\rho_1}^{\rho_2} \frac{\rho^2}{R\rho}(\rho + R - |\rho - R|)\, d\rho = \frac{2\pi}{R}\int_{\rho_1}^{\rho_2} \rho(\rho + R - |\rho - R|)\, d\rho.$$

We consider two possibilities for R, corresponding to the gravitational potential for objects *outside* and *inside* the hollow ball W.

Case 1. If $R \ge \rho_2$ [that is, if (x_1, y_1, z_1) is outside W], then $|\rho - R| = R - \rho$ for all ρ in the interval $[\rho_1, \rho_2]$, so that

$$-V(0,0,R) = \frac{2\pi}{R}\int_{\rho_1}^{\rho_2} \rho[\rho + R - (R - \rho)]\, d\rho = \frac{4\pi}{R}\int_{\rho_1}^{\rho_2} \rho^2\, d\rho = \frac{1}{R}\frac{4\pi}{3}(\rho_2^3 - \rho_1^3).$$

The factor $(4\pi/3)(\rho_2^3 - \rho_1^3)$ equals the volume of W. Putting back the constants G, m, and the mass density, we find that *the gravitational potential* is $-GmM/R$, *where M is the mass of W. Thus, V is just as it would be if all the mass of W were concentrated at the central point.*

Case 2. If $R \le \rho_1$ [that is, if (x_1, y_1, z_1) is inside the hole], then $|\rho - R| = \rho - R$ for ρ in $[\rho_1, \rho_2]$, and so

$$-V(0,0,R) = (Gm)\frac{2\pi}{R}\int_{\rho_1}^{\rho_2} \rho[\rho + R - (\rho - R)]\,d\rho = (Gm)4\pi \int_{\rho_1}^{\rho_2} \rho\,d\rho$$
$$= (Gm)2\pi(\rho_2^2 - \rho_1^2).$$

The result is independent of R, and so the potential V is *constant* inside the hole. Because the gravitational force is minus the gradient of V, we conclude that *there is no gravitational force inside a uniform hollow planet*!

We leave it to the reader to compute $V(0,0,R)$ for the case $\rho_1 < R < \rho_2$.

A similar argument shows that the gravitational potential outside any *spherically symmetric* body of mass M (even if the density is variable) is $V = GMm/R$, where R is the distance to its center (which is its center of mass).

EXAMPLE 8 Find the gravitational potential acting on a unit mass of a spherical star with a mass $M = 3.02 \times 10^{30}$ kg at a distance of 2.25×10^{11} m from its center ($G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$).

SOLUTION The negative potential is

$$-V = \frac{GM}{R} = \frac{6.67 \times 10^{-11} \times 3.02 \times 10^{30}}{2.25 \times 10^{11}} = 8.95 \times 10^{8}\ \ \mathrm{m^2/s^2}. \quad \blacktriangle$$

EXERCISES

1. Find the average of $f(x,y) = y \sin xy$ over $D = [0,\pi] \times [0,\pi]$.

2. Find the average of $f(x,y) = e^{x+y}$ over the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$.

3. Find the center of mass of the region between $y = x^2$ and $y = x$ if the density is $x + y$.

4. Find the center of mass of the region between $y = 0$ and $y = x^2$, where $0 \le x \le \frac{1}{2}$.

5. A sculptured gold plate D is defined by $0 \le x \le 2\pi$ and $0 \le y \le \pi$ (centimeters) and has mass density $\delta(x,y) = y^2 \sin^2 4x + 2$ (grams per square centimeter). If gold sells for \$7 per gram, how much is the gold in the plate worth?

6. In Exercise 5, what is the average mass density in grams per square centimeter?

7. (a) Find the mass of the box $[0, \frac{1}{2}] \times [0, 1] \times [0, 2]$, assuming the density to be uniform.
(b) Same as part (a), but with a mass density $\delta(x, y, z) = x^2 + 3y^2 + z + 1$.

8. Find the mass of the solid bounded by the cylinder $x^2 + y^2 = 2x$ and the cone $z^2 = x^2 + y^2$ if the density is $\delta = \sqrt{x^2 + y^2}$.

9. Find the center of mass of the region bounded by $x + y + z = 2$, $x = 0$, $y = 0$, and $z = 0$, assuming the density to be uniform.

10. Find the center of mass of the cylinder $x^2 + y^2 \leq 1$, $1 \leq z \leq 2$ if the density is $\delta = (x^2 + y^2)z^2$.

11. Find the average value of $\sin^2 \pi z \cos^2 \pi x$ over the cube $[0, 2] \times [0, 4] \times [0, 6]$.

12. Find the average value of e^{-z} over the ball $x^2 + y^2 + z^2 \leq 1$.

13. A solid with constant density is bounded above by the plane $z = a$ and below by the cone described in spherical coordinates by $\phi = k$, where k is a constant $0 < k < \pi/2$. Set up an integral for its moment of inertia about the z axis.

14. Find the moment of inertia around the y axis for the ball $x^2 + y^2 + z^2 \leq R^2$ if the mass density is a constant δ.

15. Find the gravitational potential on a mass m of a spherical planet with mass $M = 3 \times 10^{26}$ kg, at a distance of 2×10^8 m from its center.

16. Find the gravitational force exerted on a 70-kg object at the position in Exercise 15.

17. A body W in xyz coordinates is called *symmetric with respect to a given plane* if for every particle on one side of the plane there is a particle of equal mass located at its mirror image through the plane.

(a) Discuss the planes of symmetry for an automobile shell.

(b) Let the plane of symmetry be the xy plane, and denote by W^+ and W^- the portions of W above and below the plane, respectively. By our assumption, the mass density $\delta(x, y, z)$ satisfies $\delta(x, y, -z) = \delta(x, y, z)$. Justify the following steps:

$$\begin{aligned}
\iiint_W \delta(x, y, z)\, dx\, dy\, dz &= \iiint_W z\delta(x, y, z)\, dx\, dy\, dz \\
&= \iiint_{W^+} z\delta(x, y, z)\, dx\, dy\, dz + \iiint_{W^-} z\delta(x, y, z)\, dx\, dy\, dz \\
&= \iiint_{W^+} z\delta(x, y, z)\, dx\, dy\, dz + \iiint_{W^+} -w\delta(u, v, -w)\, du\, dv\, dw \\
&= 0.
\end{aligned}$$

(c) Explain why part (b) proves that if a body is symmetrical with respect to a plane, then its center of mass lies in that plane.

(d) Derive this law of mechanics: *If a body is symmetric with respect to two planes, then its center of mass lies on their line of intersection.*

18. A uniform rectangular steel plate of sides a and b rotates about its center of mass with constant angular velocity ω.

(a) The kinetic energy equals $\frac{1}{2}$(mass)(velocity)^2. Argue that the kinetic energy of any element of mass $\delta\, dx\, dy$ ($\delta =$ constant) is given by $\delta(\omega^2/2)(x^2+y^2)\, dx\, dy$, provided the origin $(0, 0)$ is placed at the center of mass of the plate.

(b) Justify the formula for kinetic energy:

$$\text{K.E.} = \iint_{\text{plate}} \delta \frac{\omega^2}{2}(x^2+y^2)\, dx\, dy.$$

(c) Evaluate the integral, assuming that the plate is described by the inequalities $-a/2 \le x \le a/2$, $-b/2 \le y \le b/2$.

19. As is well known, the density of a typical planet is not constant throughout the planet. Assume that planet C.M.W. has a radius of 5×10^8 cm and a mass density (in grams per cubic centimeter)

$$\rho(x, y, z) = \begin{cases} \dfrac{3 \times 10^4}{r}, & r \ge 10^4 \text{ cm}, \\ 3, & r \le 10^4 \text{ cm}, \end{cases}$$

where $r = \sqrt{x^2+y^2+z^2}$. Find a formula for the gravitational potential outside C.M.W.

6.4 Improper Integrals

In this section, we study improper integrals—that is, integrals in which the function may be unbounded or the region of integration is unbounded. We shall first recall the situation for functions of one variable.

One-Variable Improper Integrals

In the study of integrals of functions of one variable, one encounters various types of "improper" integrals; that is, integrals of unbounded functions defined on intervals or integrals of functions over unbounded intervals. For example,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx \qquad \text{and} \qquad \int_1^\infty \frac{dx}{x^2}$$

are improper integrals. They are evaluated using a limiting process; for instance,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a\to 0} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a\to 0} \left(2\sqrt{x}\Big|_a^1\right) = \lim_{a\to 0} (2 - 2\sqrt{a}) = 2$$

and

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2} = \lim_{b\to\infty} \left(-\frac{1}{x}\bigg|_1^b \right) = \lim_{b\to\infty} \left(1 - \frac{1}{b} \right) = 1.$$

If, in such a limiting process, the limit does not exist (or is infinite), we say that the integral does not exist (or that the integral diverges).

Improper Integrals in the Plane

Next, we describe three types of improper integrals of two variables over a region D. The first two types are described in the text below, and the third type (integrals over unbounded regions) is left to the exercises. We will evaluate all integrals using a limiting process, as in the one-variable case.

For simplicity of exposition, we first restrict ourselves to nonnegative functions f—that is, $f(x, y) \geq 0$ for all points $(x, y) \in D$—and to y-simple regions described as the set of (x, y) such that

$$a \leq x \leq b, \qquad \phi_1(x) \leq y \leq \phi_2(x),$$

as in Figure 6.4.1.

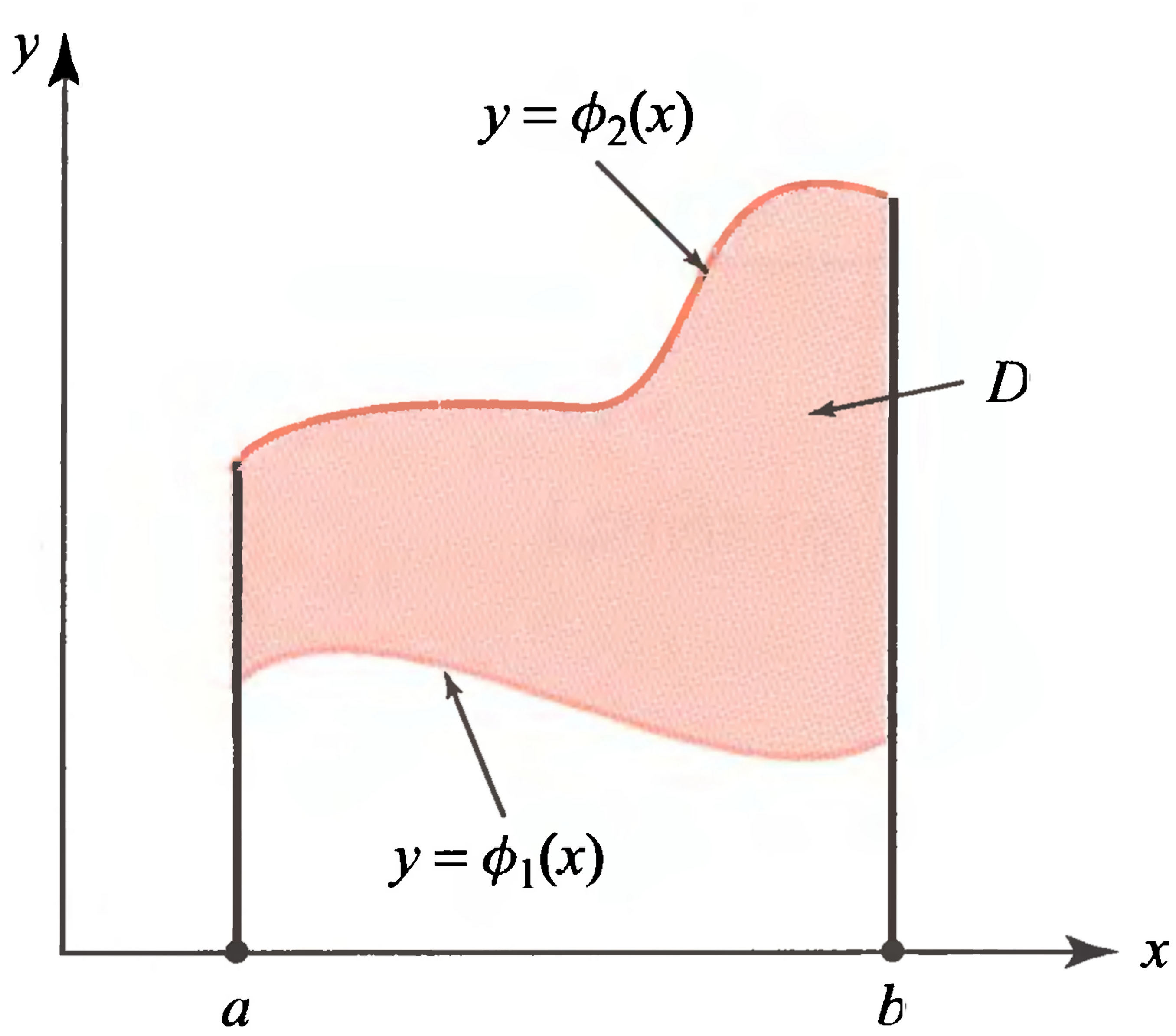


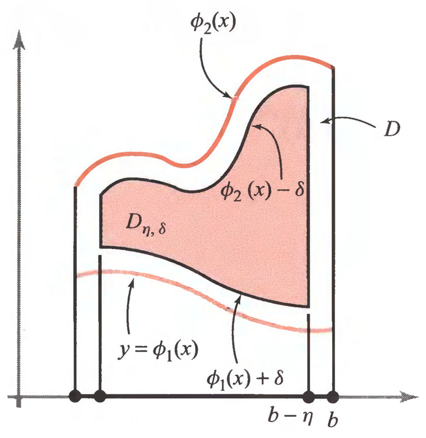
Figure 6.4.1 A y-simple domain.

In the first case we wish to consider, let's assume that $f: D \to \mathbb{R}$ is continuous except for points on the boundary of D. Consider, for example,

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}},$$

where D is the unit disk $D = \{(x, y) | x^2 + y^2 \leq 1\}$. Clearly, f is not defined on the boundary of D, where $x^2 + y^2 = 1$; yet it will be of practical interest to be able to evaluate $\iint_D f(x, y)\, dA$, because this integral represents the area of the upper hemisphere of the unit sphere in three space.

Exhausting Regions



Our basic idea will be to integrate such an f over a smaller region D', where we know the integral exists, and then let D' "tend" to D; that is, "exhaust" D and see if $\iint_D f\, dA$ tends to some limit. With this in mind, we pick a special kind of D', as follows.

Let η 0 be small enough so that $a+\eta < b-\eta$. Let $\delta > 0$ be small enough so that $\phi_1(x)+\delta < \phi_2(x)-\delta$ for all x, a x b (see Figure 6.4.2). If $\phi_2(x)$ $\phi_1(x)$ for some x, no such δ will exist, but we shall worry about this minor issue when it arises in our later examples. Then the region

$$D_{\eta,\delta} \quad \{(x,y) | a+\eta < x < b-\eta \quad \text{and} \quad \phi_1(x)+\delta < y \quad \phi_2(x) \quad \delta\}$$

is a subset of D, and as $(\eta, \delta) \to 0$, $D_{\eta,\delta}$ tends to D.

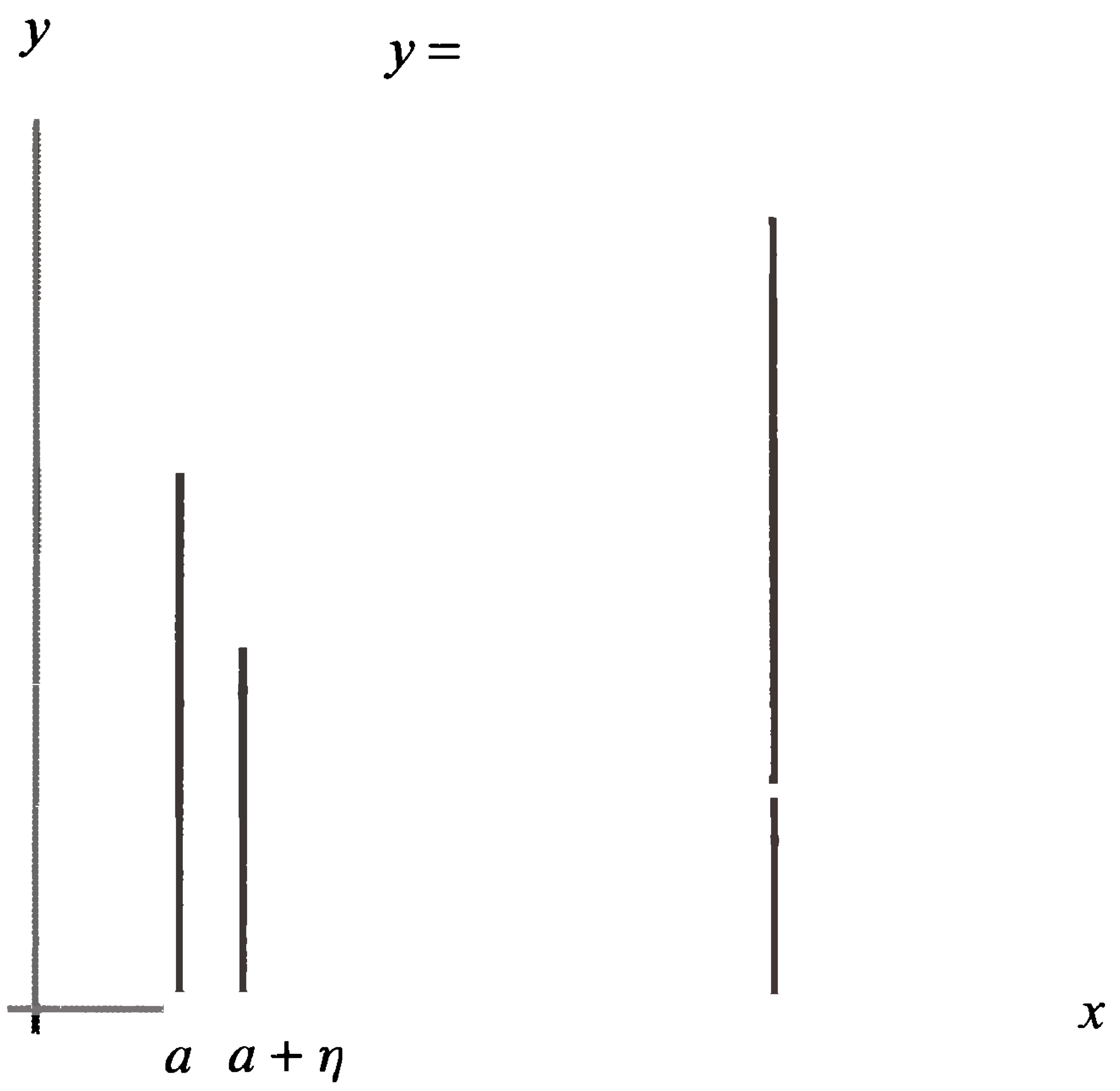


Figure 6.4.2 A shrunken domain $D_{\eta,\delta}$ for improper integrals.

Improper Integrals as Limits

Because f is continuous and bounded on $D_{\eta,\delta}$, the integral $\iint D_{\eta,\delta} f\, dA$ exists. We can now ask what happens as the region $D_{\eta,\delta}$ expands to fill the region, D—that is, as $(\eta, \delta) \to (0, 0)$. Provided that

$$\lim_{(0,0)} \iint_D \quad f\, dA$$

exists, we say that the integral of f over D is ***convergent*** or that f is ***integrable*** over D, and we define $\iint_D f\, dx\, dy$ to be equal to this limit.

Evaluate

$$_D \frac{}{\sqrt[3]{xy}}\, dA,$$

where D is the unit square $[0, 1] \times [0, 1]$.

SOLUTION D is clearly a y-simple region. Choose $\eta > 0$ and $\delta > 0$ so that $D_{\eta,\delta} \subset D$, as in Figure 6.4.3. Then, by Fubini's theorem:

$$\begin{aligned}\iint_{D_{\eta,\delta}} \frac{1}{\sqrt[3]{xy}}\, dA &= \int_{\eta}^{1-\eta} \int_{\delta}^{1-\delta} \frac{1}{\sqrt[3]{xy}}\, dy\, dx \\ &= \int_{\eta}^{1-\eta} \frac{1}{\sqrt[3]{x}}\, dx \int_{\delta}^{1-\delta} \frac{1}{\sqrt[3]{y}}\, dy \\ &= \frac{3}{2}\left((1-\eta)^{2/3} - \eta^{2/3}\right) \cdot \frac{3}{2}\left((1-\delta)^{2/3} - \delta^{2/3}\right).\end{aligned}$$

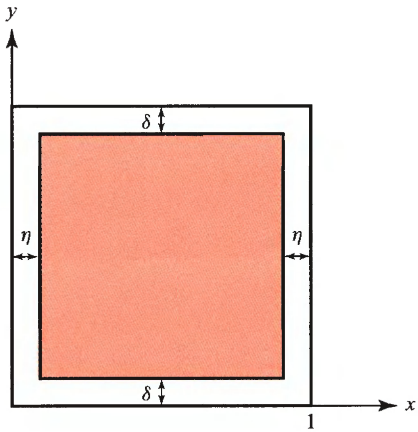


Figure 6.4.3 The slightly shrunken unit square.

Letting $(\eta, \delta) \to (0, 0)$, we see that

$$\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} \frac{1}{\sqrt[3]{xy}}\, dy\, dx = \frac{3}{2}\frac{3}{2} = \frac{9}{4}. \quad \blacktriangle$$

Unfortunately, it may not always be possible to evaluate such limits so directly and simply. This is often the case in the most interesting examples, as with the surface area of the hemisphere, mentioned earlier. It's as if the "real world" always presents the greatest challenges to the mathematician! So let us expand a bit on our theoretical discussion.

Improper Integrals as Limits of Iterated Integrals

Suppose f is integrable over $D_{\eta,\delta}$. We can then apply Fubini's theorem to obtain

$$\iint_{D_{\eta,\delta}} f\, dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dy\, dx.$$

Hence, if f is integrable over D,

$$\iint_D f\, dA = \lim_{(\eta,\delta)\to(0,0)} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dy\, dx. \tag{1}$$

Now $F(\eta, \delta) = \iint_{D_{\eta,\delta}} f\, dA$ is a function of two variables, η and δ, because as we change η and δ, we get another number. Now if f is integrable, then

$$\lim_{(\eta,\delta)\to 0} F(\eta, \delta) = L$$

exists. It follows that the iterated limits

$$\lim_{\eta\to 0} \lim_{\delta\to 0} F(\eta, \delta) \qquad \text{and} \qquad \lim_{\delta\to 0} \lim_{\eta\to 0} F(\eta, \delta)$$

also exist and are both equal to L, which in our case is $\iint_D f dA$. Thus, the iterated limit

$$\lim_{\eta\to 0} \lim_{\delta\to 0} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_1(x)-\delta} f(x, y)\, dy\, dx$$

also exists. Conversely, if the iterated limits exist, it does not generally follow that the limit $\lim_{(\eta,\delta)\to(0,0)} F(\eta, \delta)$ exists.

For example, if it were to turn out in some way that $F(\eta, \delta) = \eta\delta/(\eta^2 + \delta^2)$, then $\lim_{\eta\to 0} \lim_{\delta\to 0} F(\eta, \delta) = \lim_{\delta\to 0} \lim_{\eta\to 0} F(\eta, \delta) = 0$; yet $\lim_{(\eta,\delta)\to 0} F(\eta, \delta)$ does not exist, because $F(\eta, \eta) = 1/2$ (see Section 2.2).

In view of this, consider expression (1) again. If f is integrable, then

$$\begin{aligned} \iint_D f(x, y)\, dA &= \lim_{(\eta,\delta)\to(0,0)} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dy\, dx \\ &= \lim_{\eta\to 0} \lim_{\delta\to 0} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dy\, dx. \end{aligned}$$

Now suppose that for each x,

$$\lim_{\delta\to 0} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y)\, dy$$

exists. Denote this by $\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$. Suppose further that

$$\lim_{\eta\to 0} \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy$$

also exists. We denote this limit by $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx$. Then if all limits exist, *all limits must be equal*. Thus, f is integrable and the iterated improper integral exists, then necessarily

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx.$$

However, is it possible that the existence of *just* the iterated integrals *implies* the integrability of f? We turn to this important question next.

Fubini's Theorem for Improper Integrals

For *integrals*, something truly *remarkable* happens. Unlike the case for iterated limits (as in the counterexample considered earlier), the existence of the iterated limits *does imply* the integrability of f as long as $f \geq 0$. Thus, if $f \geq 0$ and if $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$ exists as an iterated limit, then f is integrable and

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx.$$

If D is an x-simple region with the x coordinate lying between two functions ψ_1 and ψ_2, and if

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

exists as an improper integral, it again follows that f is integrable and

$$\iint_D f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy.$$

All these results, which are the *improper* analogues of Theorems 4 and 4′ in Section 5.3, are known as *Fubini's theorem* for improper integrals, which we formally state.

THEOREM 3: Fubini's Theorem Let D be an elementary region in the plane and $f \geq 0$ a function continuous except for points possibly on the boundary of D. If either of the integrals

$$\iint_D f(x,y)\,dA,$$

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx, \qquad \text{for } y\text{-simple regions}$$

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy \qquad \text{for } x\text{-simple regions}$$

exist as improper integrals, f is integrable and they are all equal.

The proof of this involves advanced concepts of analysis, so we omit it here. This result can be quite useful in calculation, as the next example shows.

EXAMPLE 2 Let $f(x, y) = 1/\sqrt{1 - x^2 - y^2}$. Show that f is integrable and that $\iint_D f(x, y)\, dA = 2\pi$, half the surface area of the unit sphere.

SOLUTION For $-1 < x < 1$, we have

$$\begin{aligned}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} &= \lim_{\delta\to 0} \int_{-\sqrt{1-x^2}+\delta}^{\sqrt{1+x^2}-\delta} \frac{dy}{\sqrt{1-x^2-y^2}} \\ &= \lim_{\delta\to 0} \sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\Bigg|_{-\sqrt{1-x^2}+\delta}^{\sqrt{1-x^2}-\delta} \\ &= \lim_{\delta\to 0}\left\{\sin^{-1}\left(1-\frac{\delta}{\sqrt{1-x^2}}\right) - \sin^{-1}\left(-1+\frac{\delta}{\sqrt{1-x^2}}\right)\right\} \\ &= \sin^{-1}(1) - \sin^{-1}(-1) = \frac{\pi}{2} - \frac{(-\pi)}{2} = \pi.\end{aligned}$$

Clearly,

$$\lim_{\eta\to 0}\int_{-1+\eta}^{1-\eta}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy\, dx}{\sqrt{1-x^2-y^2}} = \lim_{\eta\to 0}\int_{-1+\eta}^{1-\eta} \pi\, dx = \lim_{\eta\to 0} \pi(2-2\eta) = 2\pi.$$

Thus, f is integrable. To see why this theorem is so useful, try to show directly from the definition that f is integrable. It is not easy to do so! ▲

EXAMPLE 3 Let $f(x, y) = 1/(x - y)$ and let D be the set of (x, y) satisfying $0 \le x \le 1$ and $0 \le y \le x$. Show that f is *not* integrable over D.

SOLUTION Because the denominator of f is zero on the line $y = x$, f is unbounded on part of the boundary of D. Let $0 < \eta < 1$ and $0 < \delta < \eta$, and let $D_{\eta,\delta}$ be the set of (x, y) with $\eta \le x \le 1 - \eta$ and $\delta \le y \le x - \delta$ (Figure 6.4.4).

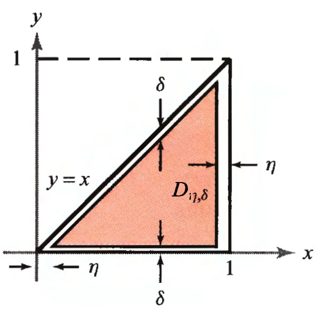


Figure 6.4.4 The shrunken domain $D_{\eta,\delta}$ for a triangular domain D.

Here the region D is y-simple with $\phi_1(x) = 0$, $\phi_2(x) = x$, and $\phi_1(0) = \phi_2(0)$. To ensure that $D_{\eta,\delta} \subset D$ and is depicted in the figure, we must choose δ a bit more carefully. A little geometry shows that we should choose $2\delta \leq \eta$. Consider

$$\begin{aligned}\iint_{D_{\eta,\delta}} f\,dA &= \int_{\eta}^{1-\eta}\int_{\delta}^{x-\delta} \frac{1}{x-y}\,dy\,dx \\ &= \int_{\eta}^{1-\eta} [-\log(x-y)]\Big|_{y=\delta}^{x-\delta}\,dx \\ &= \int_{\eta}^{1-\eta} [-\log(\delta)+\log(x-\delta)]\,dx \\ &= [-\log\delta]\int_{\eta}^{1-\eta} dx + \int_{\eta}^{1-\eta} \log(x-\delta)\,dx \\ &= -(1-2\eta)\log\delta + [(x-\delta)\log(x-\delta)-(x-\delta)]\Big|_{\eta}^{1-\eta}.\end{aligned}$$

In the last step, we used the fact that $\int \log u\,du = u\log u - u$. Continuing the preceding set of qualities, we have

$$\begin{aligned}\iint_{D_{\eta,\delta}} f\,dA = {}& -(1-2\eta)\log\delta + (1-\eta-\delta)\log(1-\eta-\delta) \\ & -1(1-\eta-\delta) - (\eta-\delta)\log(\eta-\delta) + (\eta-\delta).\end{aligned}$$

As $(\eta,\delta) \to (0,0)$, the second term converges to $1\log 1 = 0$, and the third and fifth terms converge to -1 and 0, respectively. Let $v = \eta - \delta$. Because $v\log v \to 0$ as $v \to 0$ (a limit established by using L'Hôpital's rule from calculus[3]), we see that the fourth term goes to zero as $(\eta,\delta)\to(0,0)$. It is the first term that will give us trouble. Now:

$$-(1-2\eta)\log\delta = -\log\delta + 2\eta\log\delta, \tag{2}$$

and it is not hard to see that this does not converge as $(\eta,\delta)\to(0,0)$. For example, let $\eta = 2\delta$; then expression (2) becomes $-\log\delta + 4\delta\log\delta$. As before, $4\delta\log\delta \to 0$ as $\delta\to 0$, but $-\log\delta \to +\infty$ as $\delta \to 0$, which shows that expression (2) does not converge. Hence, $\lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\,dA$ does not exist and so f is not integrable. ▲

Functions Unbounded at Isolated Points

We now consider nonnegative functions f that become "infinite" or are undefined at isolated points in an x-simple or y-simple region D. For example, consider the function $f(x,y) = 1/\sqrt{x^2+y^2}$ on the unit disk $D = \{(x,y)|x^2+y^2 \leq 1\}$. Again, $f \geq 0$, but f is unbounded and is not defined at the origin.

[3] L' Hôpital's rule was discovered by Bernoulli and was reported in L'Hôpital's textbook.

Let (x_0, y_0) be a point of a general region D where a nonnegative function f is undefined. Further, let $D_\delta = D_\delta(x_0, y_0)$ be the disk of radius δ centered at (x_0, y_0) and let $D \backslash D_\delta$ denote the region D with D_δ removed. Assume that f is continuous at every point of D except (x_0, y_0). Then $\iint_{D\backslash D_\delta} f\, dA$ is defined. We say that $\iint_D f\, dA$ is ***convergent***, or that f is ***integrable*** over D if

$$\lim_{\delta\to 0} \iint_{D\backslash D_\delta} f\, dA$$

exists.

EXAMPLE 4 Show that $f(x, y) = 1/\sqrt{x^2 + y^2}$ is integrable over the unit disk D and evaluate $\iint_D f dA$.

SOLUTION Let D_δ be the disk of radius δ centered at the origin. Then f is continuous everywhere on D except at (0, 0). Thus, $\iint_{D\backslash D_\delta} f\, dA$ exists. To evaluate this integral, we change variables to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. Then $f(r\cos\theta, r\sin\theta) = 1/r$, and

$$\iint_{D\backslash D_\delta} f\, dA = \int_\delta^1 \int_0^{2\pi} \frac{1}{r} f\, d\theta\, dr = \int_\delta^1 \int_0^{2\pi} d\theta\, dr = 2\pi(1-\delta).$$

Thus,

$$\iint_D f\, dA = \lim_{\delta\to 0} \iint_{D\backslash D_\delta} f\, dA = 2\pi. \quad \blacktriangle$$

More generally, one can, in an analogous manner, define the integral of nonnegative functions f that are continuous except at a finite number of points in D. One can also combine both types of improper integrals; that is, one may consider functions that are continuous except at a finite number of points on D or at points on the boundary of D, and define $\iint_D f\, dA$ appropriately.

If f takes both positive and negative values, one can use a more advanced integration theory, called the *Lebesgue integral*, to generalize the notion of convergent integral $\iint_D f dA$. Using this theory, it is possible to show that if $\iint_D f\, dA$ exists, it can then be evaluated as an iterated integral. This latter fact is also known as Fubini's theorem.

Unbounded Regions

As was mentioned previously, we will leave consideration of unbounded regions to the exercise section. However, we must point out that we have already addressed the main idea in Example 5 of Section 6.2 on the Gaussian integral. In that example, we integrated $\exp(-x^2 - y^2)$ over all of $\mathbb{R}^2$ by integrating first over a disk of radius a and then letting $a \to \infty$.

EXERCISES

In Exercises 1 to 4, evaluate the following integrals if they exist (discuss how you define the integral if it was not given in the text).

1. $\displaystyle\iint_D \frac{1}{\sqrt{xy}}\, dA$, where $D = [0, 1] \times [0, 1]$

2. $\displaystyle\iint_D \frac{1}{\sqrt{|x-y|}}\, dx\, dy$, where $D = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1, y \le x\}$

3. $\displaystyle\iint_D (y/x)\, dx\, dy$, where D is bounded by $x = 1$, $x = y$, and $x = 2y$

4. $\displaystyle\int_0^1 \int_0^{e^y} \log x\, dx\, dy$

5. (a) Evaluate

$$\iint_D \frac{dA}{(x^2 + y^2)^{2/3}},$$

where D is the unit disk in $\mathbb{R}^2$

(b) Determine the real numbers λ for which the integral

$$\iint_D \frac{dA}{(x^2 + y^2)^{\lambda}}$$

is convergent, where again D is the unit disk.

6. (a) Discuss how you would define $\iint_D f\, dA$ if D is an unbounded region, for example, the set of (x, y) such that $a \le x < \infty$ and $\phi_1(x) \le y \le \phi_2(x)$, where $\phi_1 \le \phi_2$ are given (Figure 6.4.5).

(b) Evaluate $\iint_D xye^{-(x^2+y^2)}\, dx\, dy$ if $x \ge 0$, $0 \le y \le 1$.

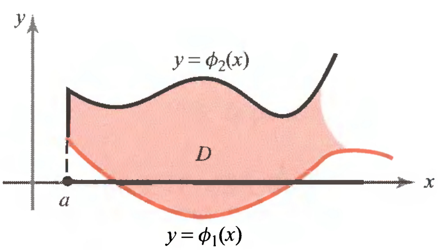


Figure 6.4.5 An unbounded region D.

7. Using Exercise 6, integrate e^{-xy} for $x \ge 0$, $1 \le y \le 2$ in two ways. Assuming Fubini's theorem can be used, show that

$$\int_0^{\infty} \frac{e^{-x} - e^{-2x}}{x}\, dx = \log 2.$$

8. Show that the integral

$$\int_0^1 \int_0^a (x/\sqrt{a^2 - y^2})\, dy\, dx$$

exists, and compute its value.

9. Discuss whether the integral

$$\iint_D \frac{x+y}{x^2+2xy+y^2}\, dx\, dy$$

exists where $D = [0, 1] \times [0, 1]$. If it exists, compute its value.

10. One can also consider improper integrals of functions that fail to be continuous on entire curves lying in some region D. For example, by breaking $D = [0, 1] \times [0, 1]$ into two regions, define and then discuss the convergence of the integral

$$\iint_D \frac{1}{\sqrt{|x-y|}}\, dx\, dy.$$

11. Let W be the first octant of the ball $x^2 + y^2 + z^2 \leq a^2$, where $x \geq 0, y \geq 0, z \geq 0$. Evaluate the improper integral

$$\iiint_W \frac{(x^2+y^2+z^2)^{1/4}}{\sqrt{z+(x^2+y^2+z^2)^2}}\, dx\, dy\, dz$$

by changing variables.

12. Let f be a nonnegative function that may be unbounded and discontinuous on the boundary of an elementary region D. Let g be a similar function such that $f(x, y) \leq g(x, y)$ whenever both are defined. Suppose $\iint_D g(x, y)\, dA$ exists. Argue informally that this implies the existence of $\iint_D f(x, y)\, dA$.

13. Use Exercise 12 to show that

$$\iint_D \frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}}\, dy\, dx$$

exists where D is the unit disk $x^2 + y^2 \leq 1$.

14. Let f be as in Exercise 12 and let g be a function such that $0 \leq g(x, y) \leq f(x, y)$ whenever both are defined. Suppose that $\iint_D g(x, y)\, dA$ does not exist. Argue informally that $\iint_D f(x, y)\, dA$ cannot exist.

15. Use Exercise 14 to show that

$$\iint_D \frac{e^{x^2+y^2}}{x-y}\, dy\, dx$$

does not exist, where D is the set of (x, y) with $0 \leq x \leq 1$ and $0 \leq y \leq x$.

16. Let D be the unbounded region defined as the set of (x, y, z) with $x^2 + y^2 + z^2 \geq 1$. By making a change of variables, evaluate the improper integral

$$\iiint_D \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^2}.$$

17. Evaluate

$$\int_0^1 \int_0^y \frac{x}{y}\, dx\, dy \qquad \text{and} \qquad \int_0^1 \int_x^1 \frac{x}{y}\, dy\, dx$$

Does Fubini's theorem apply?

18. In Exercise 11 of Section 5.2 we showed that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\, dx \neq \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\, dy.$$

Thus, Fubini's theorem does not hold here, even thought the iterated improper integrals both exist. What went wrong?

REVIEW EXERCISES FOR CHAPTER 6

1. (a) Find a linear transformation taking the square $S = [0, 1] \times [0, 1]$ to the parallelogram P with vertices $(0, 0)$, $(2, 0)$, $(1, 2)$, $(3, 2)$.

(b) Write down a change of variables formula appropriate to the transformation you found in part (a).

2. (a) Find the image of the square $[0, 1] \times [0, 1]$ under the transformation $T(x, y) = (2x, x + 3y)$.

(b) Write down a change of variables formula appropriate to the transformation and the region you found in part (a).

3. Let B be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Evaluate $\iint_B (x^2 + y^2)\, dx\, dy$ using the change of variables $u = x^2 - y^2$, $v = xy$.

4. In parts (a) to (d), make the indicated change of variables. (Do not evaluate.)

(a) $\displaystyle\int_0^1 \int_{-1}^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} (x^2 + y^2)^{1/2}\, dx\, dy\, dz$, cylindrical coordinates

(b) $\displaystyle\int_{-1}^1 \int_{-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} \int_{-\sqrt{(4-x^2-y^2)}}^{\sqrt{(4-x^2-y^2)}} xyz\, dz\, dx\, dy$, cylindrical coordinates

(c) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{(2-y^2)}}^{\sqrt{(2-y^2)}} \int_{\sqrt{(x^2+y^2)}}^{\sqrt{(4-x^2-y^2)}} z^2\, dz\, dx\, dy$, spherical coordinates

(d) $\displaystyle\int_0^1 \int_0^{\pi/4} \int_0^{2\pi} \rho^3 \sin 2\phi\, d\theta\, d\phi\, d\rho$, rectangular coordinates

5. Find the volume inside the surfaces $x^2 + y^2 = z$ and $x^2 + y^2 + z^2 = 2$.

6. Find the volume enclosed by the cone $x^2 + y^2 = z^2$ and the plane $2z - y - 2 = 0$.

7. A cylindrical hole of diameter 1 is bored through a sphere of radius 2. Assuming that the axis of the cylinder passes through the center of the sphere, find the volume of the solid that remains.

8. Let C_1 and C_2 be two cylinders of infinite extent, of diameter 2, and with axes on the x and y axes, respectively. Find the volume of their intersection, $C_1 \cap C_2$.

9. Find the volume bounded by $x/a + y/b + z/c = 1$ and the coordinate planes.

10. Find the volume determined by $z \leq 6 - x^2 - y^2$ and $z \geq \sqrt{x^2 + y^2}$.

11. The *tetrahedron* defined by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$ is to be sliced into n segments of equal volume by planes parallel to the plane $x + y + z = 1$. Where should the slices be made?

12. Let E be the solid ellipsoid $E = \{(x, y, z) \mid (x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1\}$ where $a > 0$, $b > 0$, and $c > 0$. Evaluate

$$\iiint xyz\, dx\, dy\, dz$$

(a) over the whole ellipsoid; and
(b) over that part of it in the first quadrant:

$$x \geq 0, \qquad y \geq 0, \qquad \text{and} \qquad z \geq 0, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

13. Find the volume of the "ice cream cone" defined by the inequalities $x^2 + y^2 \leq \frac{1}{5}z^2$, and $0 \leq z \leq 5 + \sqrt{5 - x^2 - y^2}$.

14. Let ρ, θ, ϕ be spherical coordinates in $\mathbb{R}^3$ and suppose that a surface surrounding the origin is described by a continuous positive function $\rho = f(\theta, \phi)$. Show that the volume enclosed by the surface is

$$V = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} [f(\theta, \phi)]^3 \sin\phi\, d\phi\, d\theta.$$

15. Using an appropriate change of variables, evaluate

$$\iint_B \exp\,[(y - x)/(y + x)]\, dx\, dy$$

where B is the interior of the triangle with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$.

16. Suppose the density of a solid of radius R is given by $(1 + d^3)^{-1}$ where d is the distance to the center of the sphere. Find the total mass of the sphere.

17. The density of the material of a spherical shell whose inner radius is 1 m and whose outer radius is 2 m is $0.4d^2$ g/cm^3, where d is the distance to the center of the sphere in meters. Find the total mass of the shell.

18. If the shell in Exercise 17 were dropped into a large tank of pure water, would it float? What if the shell leaked? (Assume that the density of water is exactly 1 g/cm^3.)

19. The temperature at points in the cube $C = \{(x, y, z) \mid -1 \le x \le 1, -1 \le y \le 1$, and $-1 \le z \le 1\}$ is $32d^2$, where d is the distance to the origin.

(a) What is the average temperature?
(b) At what points of the cube is the temperature equal to the average temperature?

20. Use cylindrical coordinates to find the center of mass of the region defined by

$$y^2 + z^2 \le \frac{1}{4}, \qquad (x-1)^2 + y^2 + z^2 \le 1, \qquad x \ge 1.$$

21. Find the center of mass of the solid hemisphere

$$V = \{(x, y, z) \mid x^2 + y^2 + z^2 \le a^2 \text{ and } z \ge 0\}$$

if the density is constant.

22. Evaluate $\iint_B e^{-x^2-y^2}\, dx\, dy$ where B consists of those (x, y) satisfying $x^2 + y^2 \le 1$ and $y \le 0$.

23. Evaluate

$$\iiint_S \frac{dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}},$$

where S is the solid bounded by the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, where $a > b > 0$.

24. Evaluate $\displaystyle\iiint_D (x^2 + y^2 + z^2)xyz\, dx\, dy\, dz$ over each of the following regions.

(a) The sphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2\}$
(b) The hemisphere $D = \{(x, y, z) \mid x^2 + y^2 + z^2 \le R^2 \text{ and } z \ge 0\}$
(c) The octant $D = \{(x, y, z) \mid x \ge 0, y \ge 0, z \ge 0, \text{ and } z^2 + y^2 + z^2 \le R^2\}$

25. Let C be the cone-shaped region $\{(x, y, z) \mid \sqrt{x^2 + y^2} \le z \le 1\}$ in $\mathbb{R}^3$ and evaluate the integral $\displaystyle\iiint_C (1 + \sqrt{x^2 + y^2})\, dx\, dy\, dz$.

26. Find $\displaystyle\iiint_{\mathbb{R}^3} f(x, y, z)\, dx\, dy\, dz$ where $f(x, y, z) = \exp\left[-(x^2 + y^2 + z^2)^{3/2}\right]$.

27. The *flexural rigidity EI* of a uniform beam is the product of its Young's modulus of elasticity E and the moment of inertia I of the cross section of the beam with respect to a horizontal line l passing through the center of gravity of this cross section. Here

$$I = \iint_R [d(x, y)]^2 \, dx \, dy,$$

where $d(x, y) =$ the distance from (x, y) to l and $R =$ the cross section of the beam being considered.

(a) Assume that the cross section R is the rectangle $-1 \le x \le 1, -1 \le y \le 2$, and l is the line $y = 1/2$. Find I.

(b) Assume the cross section R is a circle of radius 4 and l is the x axis. Find I, using polar coordinates.

28. Find, $\iiint_{\mathbb{R}^3} f(x, y, z) \, dx \, dy \, dz$ where

$$f(x, y, z) = \frac{1}{[1 + (x^2 + y^2 + z^2)^{3/2}]^{3/2}}.$$

29. Suppose D is the unbounded region of $\mathbb{R}^2$ given by the set of (x, y) with $0 \le x < \infty$, $0 \le y \le x$. Let $f(x, y) = x^{-3/2} e^{y-x}$. Does the improper integral $\iint_D f(x, y) \, dx \, dy$ exist?

30. If the world were two-dimensional, the laws of physics would predict that the gravitational potential of a mass point is proportional to the logarithm of the distance from the point. Using polar coordinates, write an integral giving the gravitational potential of a disk of constant density.

31. (a) Evaluate the improper integral

$$\int_0^\infty \int_0^y x e^{-y^3} \, dx \, dy.$$

(b) Evaluate

$$\iint_B (x^4 + 2x^2 y^2 + y^4) \, dx \, dy,$$

where B is the portion of the disk of radius 2 [centered at (0, 0) in the first quadrant].

32. Let f be a nonnegative function on an x-simple or a y-simple region $D \subset \mathbb{R}^2$, and that is continuous except for points on the boundary of D and at most finitely many points interior to D. Give a suitable definition of $\iint_D f \, dA$.

33. Evaluate $\iint_{\mathbb{R}^2} f(x, y) \, dx \, dy$ where $f(x, y) = 1/(1 + x^2 + y^2)^{3/2}$. (HINT: You may assume that changing variables and Fubini's theorem are valid for improper integrals.)