Answers to Odd-Numbered Exercises

Some solutions requiring proofs may be incomplete or be omitted.

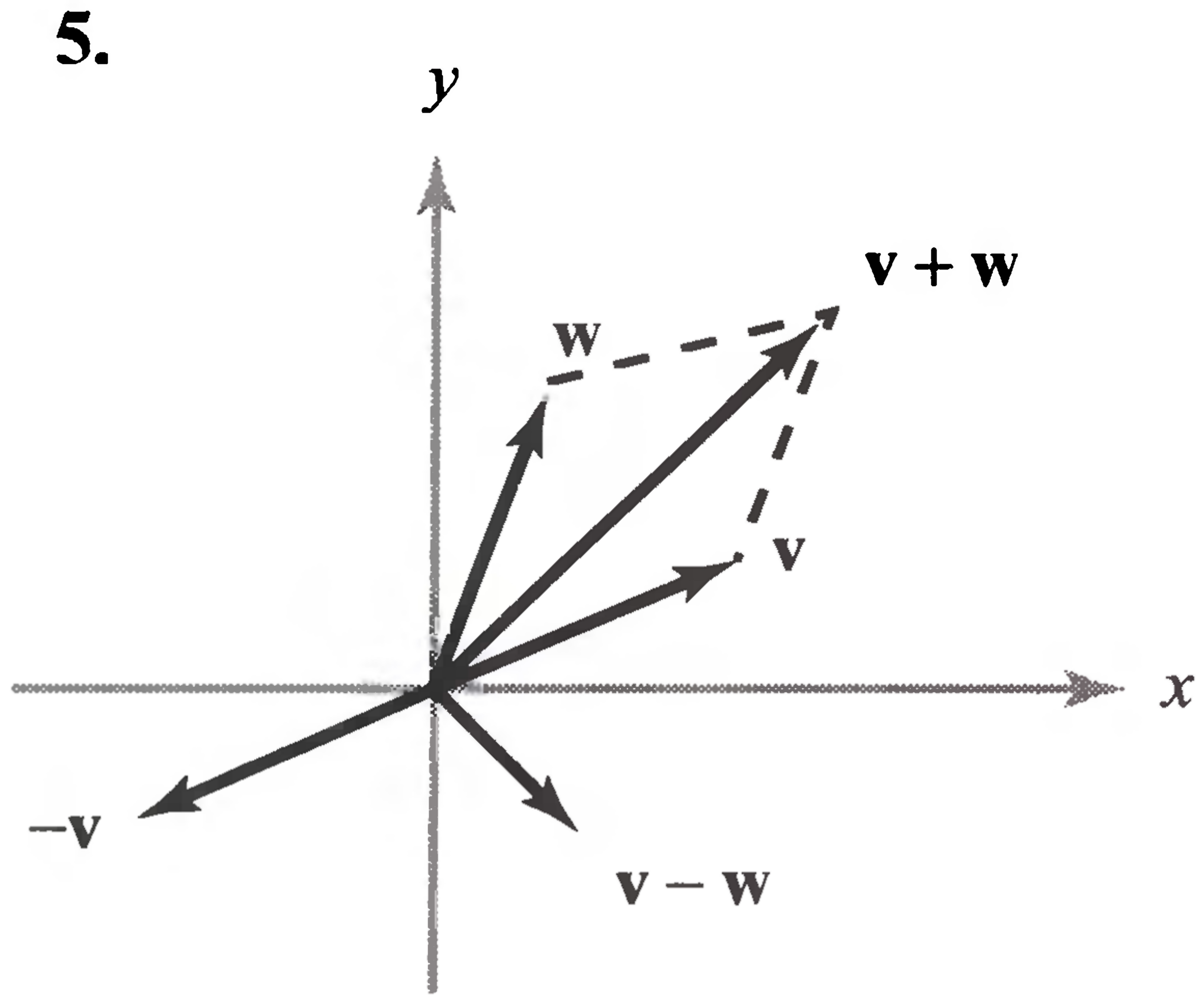
Chapter 1

Section 1.1

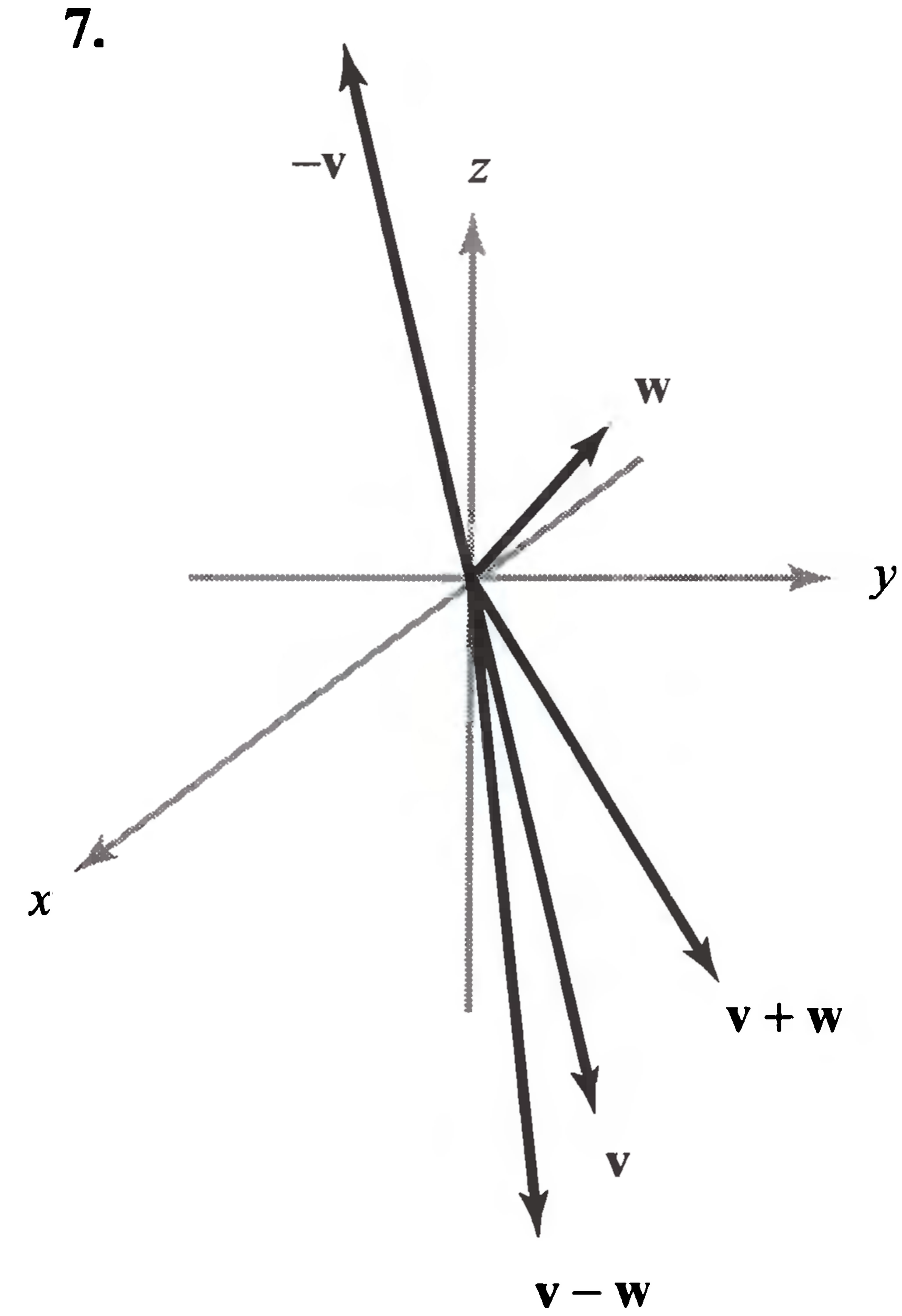
1. 4; 17

3. $(-104+16a, -24-4b, -22+26c)$

5.



7.



9. $x=0, z=0, y \in \mathbb{R}; x=0, y=0, z \in \mathbb{R}; y=0, x, z \in \mathbb{R}; x=0, y, z \in \mathbb{R}$

11. $\{(2s, 7s+2t, 7t) \mid s \in \mathbb{R}, t \in \mathbb{R}\}$

13. $\mathbf{l}(t) = -\mathbf{i} + (t-1)\mathbf{j} - \mathbf{k}$

15. $\mathbf{l}(t) = (2t-1)\mathbf{i} - \mathbf{j} + (3t-1)\mathbf{k}$

17. $\{s\mathbf{i} + 3s\mathbf{k} - 2t\mathbf{j} \mid 0 \le s \le 1, 0 \le t \le 1\}$

19. If (x, y, z) lies on the line, then $x = 2+t$, $y = -2+t$, and $z = -1+t$. Therefore, $2x - 3y + z - 2 = 4 + 2t + 6 - 3t - 1 + t - 2 = 7$, which is not zero. Hence, no (x, y, z) satisfies both conditions.

21. Yes.

23. The set of vectors of the form

$$\mathbf{v} = p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$$

where $0 \le p \le 1, 0 \le q \le 1$, and $0 \le r \le 1$.

25. All points of the form

$$(x_0 + t(x_1 - x_0) + s(x_2 - x_0), y_0 + t(y_1 - y_0) + s(y_2 - y_0), z_0 + t(z_1 - z_0) + s(z_2 - z_0))$$

for real numbers t and s.

27. If one vertex is placed at the origin and the two adjacent sides are $\mathbf{u}$ and $\mathbf{v}$, the new triangle has sides $b\mathbf{u}$, $b\mathbf{v}$, and $b(\mathbf{u} - \mathbf{v})$.

29. $(1, 0, 1) + (0, 2, 1) = (0, 2, 0) + (1, 0, 2)$

31. Two such lines (there are many others) are $x = 1, y = t, z = t$ and $x = 1, y = t, z = -t$.

Section 1.2

1. 6

3. 99°

5. No, it is 75.7; it would be zero only if the vectors were parallel.

7. $\|\mathbf{u}\| = \sqrt{5}$, $\|\mathbf{v}\| = \sqrt{2}$, $\mathbf{u} \cdot \mathbf{v} = -3$

9. $\|\mathbf{u}\| = \sqrt{11}$, $\|\mathbf{v}\| = \sqrt{62}$, $\mathbf{u} \cdot \mathbf{v} = -14$

11. $\|\mathbf{u}\| = \sqrt{14}$, $\|\mathbf{v}\| = \sqrt{26}$, $\mathbf{u} \cdot \mathbf{v} = -17$

13. In Exercise 9, $\cos^{-1}(-14/\sqrt{11}\sqrt{62})$; in Exercise 10, $\pi/2$; and in 11, $\cos^{-1}(-17/\sqrt{14}\sqrt{26})$.

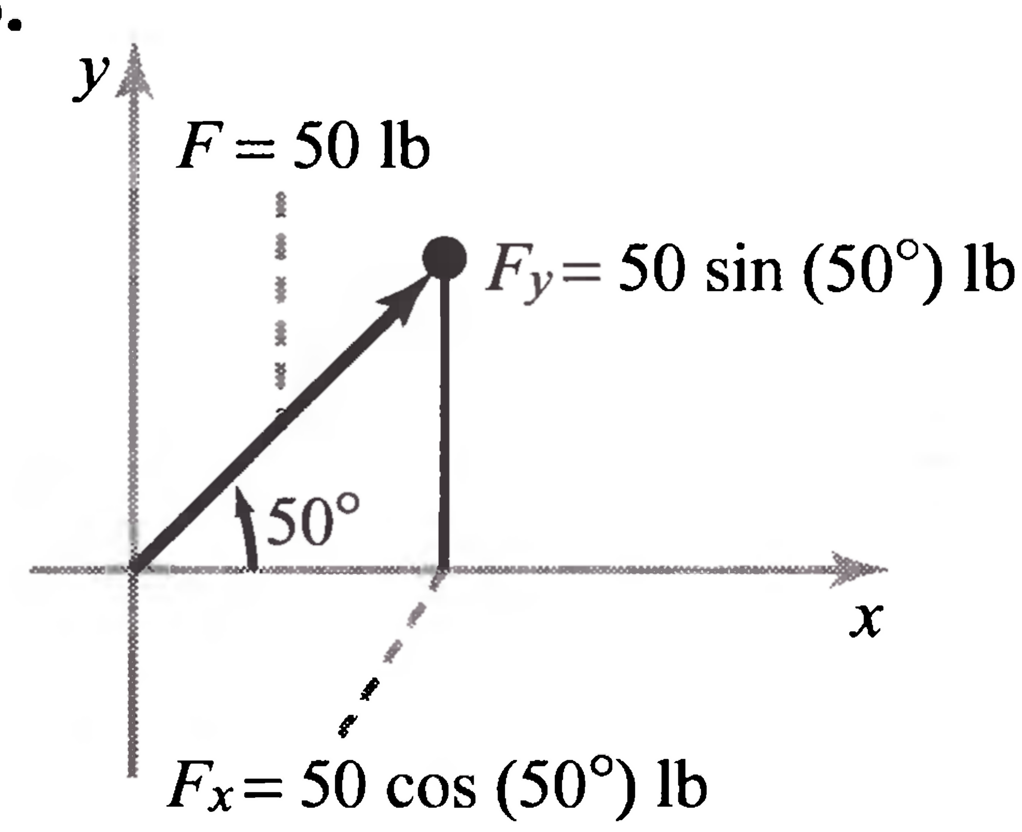
15. $-4(-\mathbf{i} + \mathbf{j} + \mathbf{k})/3$

17. Any (x, y, z) with $x + y + z = 0$; for example, $(1, -1, 0)$ and $(0, 1, -1)$.

19. $\mathbf{i} + 4\mathbf{j}$, $\theta \approx 0.24$ radian east of north

21. (a) 12:03 P.M. (b) 4.95 km

23.



25. $(4.9, 4.9, 4.9)$ and $(-4.9, -4.9, 4.9)$ N.

27. (a) $\mathbf{F} = (3\sqrt{2}\mathbf{i} + 3\sqrt{2}\mathbf{j})$ (b) ≈ 0.322 radian or 18.4° (c) $18\sqrt{2}$

Section 1.3

1. $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{vmatrix} = -8, \quad \begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 8.$

3. $-3\mathbf{i}+\mathbf{j}+5\mathbf{k}$

5. $\sqrt{35}$

7. 10

9. $\pm\mathbf{k}$

11. $\pm(113\mathbf{i}+17\mathbf{j}-103\mathbf{k})/\sqrt{23,667}$

13. $\mathbf{u}+\mathbf{v}=3\mathbf{i}-3\mathbf{j}+3\mathbf{k}$; $\mathbf{u}\cdot\mathbf{v}=6$; $\|\mathbf{u}\|=\sqrt{6}$; $\|\mathbf{v}\|=3$; $\mathbf{u}\times\mathbf{v}=-3\mathbf{i}+3\mathbf{k}$

15. (a) $x+y+z-1=0$
(b) $x+2y+3z-6=0$
(c) $5x+2z=25$
(d) $x+2y-3z=13$

17. (a) The parallel planes $Ax+By+Cz+D=0$ and $\sigma Ax+\sigma By+\sigma Cz+D'=0$ are identical when $D'=\sigma D$ and otherwise never intersect.
(b) In a line.

19. The line $x=t$, $y=2t$, $z=-5t$.

21. (a) Do the first by working out each side in coordinates, and then use that and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=-(\mathbf{b}\times\mathbf{c})\times\mathbf{a}$ to get the second.
(b) Use the identities in part (a) to write the quantity in terms of inner products.
(c) Use the identities in part (a) and collect terms.

23. Compute the results of Cramer's rule and check that they satisfy the equation.

25. $x-2y+3z+12=0$

27. $4x-6y-10z=14$

29. $10x-17y+z+25=0$

31. For Exercise 19, note that $(2,-3,1)\cdot(1,1,1)=0$, and so the line and plane are parallel and $(2,-2,-1)$ does not lie in the plane. For Exercise 20, the line and plane are parallel and $(1,-1,2)$ *does* lie in the plane.

33. $\sqrt{2}/13$

35. (a) Show that $\mathbf{M}$ satisfies the geometric properties of $\mathbf{R}\times\mathbf{F}$. (b) $2\sqrt{3}$

37. Show that $n_1(\mathbf{N}\times\mathbf{a})$ and $n_2(\mathbf{N}\times\mathbf{b})$ have the same magnitude and direction.

39. One method is to write out all terms in the left-hand side and see that the terms involving λ all cancel. Another method is to first observe that the determinant is linear in each row or column and that if any row or column is repeated, the answer is zero. Then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2+\lambda a_1 & b_2+\lambda b_1 & c_2+\lambda c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \lambda \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Section 1.4

1. (a)

Cylindrical r	θ	z	Rectangular x	y	z	Spherical ρ	θ	ϕ
1	45°	1	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	45°	45°
2	$\pi/2$	-4	0	2	-4	$2\sqrt{5}$	$\pi/2$	$\pi - \arccos(2\sqrt{5}/5)$
0	45°	10	0	0	10	10	45°	0
3	$\pi/6$	4	$3\sqrt{3}/2$	3/2	4	5	$\pi/6$	$\arccos \frac{4}{5}$
1	$\pi/6$	0	$\sqrt{3}/2$	$\frac{1}{2}$	0	1	$\pi/6$	$\pi/2$
2	$3\pi/4$	-2	$-\sqrt{2}$	$\sqrt{2}$	-2	$2\sqrt{2}$	$3\pi/4$	$3\pi/4$

(b)

Rectangular x	y	z	Spherical ρ	θ	ϕ	Cylindrical r	θ	z
2	1	-2	3	$\arctan \frac{1}{2}$	$\pi - \arccos(2/3)$	$\sqrt{5}$	$\arctan \frac{1}{2}$	-2
0	3	4	5	$\pi/2$	$\arccos(4/5)$	3	$\pi/2$	4
$\sqrt{2}$	1	1	2	$\arctan(\sqrt{2}/2)$	$\pi/3$	$\sqrt{3}$	$\arctan(\sqrt{2}/2)$	1
$-2\sqrt{3}$	-2	3	5	$7\pi/6$	$\arccos \frac{3}{5}$	4	$7\pi/6$	3

3. (a) Rotation by π around the z axis
(b) Reflection across the xy plane
(c) Rotation by $\pi/2$ about the z axis together with a radial expansion by a factor of 2

5. No; (ρ, θ, ϕ) and $(-\rho, \theta + \pi, \pi - \phi)$ represent the same point.

7. (a) $\mathbf{e}_\rho = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2+y^2+z^2} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho$
$\mathbf{e}_\theta = (-y\mathbf{i} + x\mathbf{j})/\sqrt{x^2+y^2} = (-y\mathbf{i} + x\mathbf{j})/r$
$\mathbf{e}_\phi = (xz\mathbf{i} + yz\mathbf{j} - (x^2+y^2)\mathbf{k})/r\rho$
(b) $\mathbf{e}_\theta \times \mathbf{j} = -y\mathbf{k}/\sqrt{x^2+y^2}$, $\mathbf{e}_\phi \times \mathbf{j} = (xz/r\rho)\mathbf{k} + (r/\rho)\mathbf{i}$

9. (a) The length of $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is $(x^2+y^2+z^2)^{1/2} = \rho$
(b) $\cos\phi = z/(x^2+y^2+z^2)^{1/2}$ (c) $\cos\theta = x/(x^2+y^2)^{1/2}$

11. $0 \le r \le a$, $0 \le \theta \le 2\pi$ means that (r, θ, z) is inside the cylinder with radius a centered on the z axis, and $|z| \le b$ means that it is no more than a distance b from the xy plane.

13. $-d/(6\cos\phi) \le \rho \le d/2, 0 \le \theta \le 2\pi$, and $\pi - \cos^{-1}(\frac{1}{3}) \le \phi \le \pi$

15. This is a surface whose cross section with each surface $z = c$ is four-petaled rose. The petals shrink to zero as $|c|$ changes from 0 to 1.

Section 1.5

1. 7

3. $|\mathbf{x}\cdot\mathbf{y}| = 10 = \sqrt{5}\sqrt{20} = \|\mathbf{x}\|\,\|\mathbf{y}\|$, so $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ is true.
$\|\mathbf{x}+\mathbf{y}\| = 3\sqrt{5} = \|\mathbf{x}\| + \|\mathbf{y}\|$, so $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ is true.

5. $|\mathbf{x}\cdot\mathbf{y}| = 5 < \sqrt{65} = \|\mathbf{x}\|\,\|\mathbf{y}\|$, so $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$ is true.
$\|\mathbf{x}+\mathbf{y}\| = \sqrt{28} < \sqrt{5} + \sqrt{13} = \|\mathbf{x}\| + \|\mathbf{y}\|$, so $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$ is true.

7. $AB = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 11 & 3 \\ -6 & 5 & 8 \end{bmatrix}$, $\det A = -5$, $\det B = -24$,
$\det AB = 120(=\det A \det B)$, $\det(A+B) = -61(\neq \det A + \det B)$

9. Hint: For $k = 2$ use the triangle inequality to show that $\|\mathbf{x}_1 + \mathbf{x}_2\| \le \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$; then for $k = i+1$ note that $\|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{i+1}\| \le \|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_i\| + \|\mathbf{x}_{i+1}\|$.

11. (a) Check $n = 1$ and $n = 2$ directly. Then reduce an $n \times n$ determinant to a sum of $(n-1)\times(n-1)$ determinants and use induction.
(b) The argument is similar to that for part (a). Suppose the first row is multiplied by λ. The first term of the sum will be λa_{11} times an $(n-1)\times(n-1)$ determinant with no factors of λ. The other terms obtained (by expanding across the first row) are similar.

13. Not necessarily. Try $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

15. (a) The sum of two continuous functions and a scalar multiple of a continuous function are continuous.
(b) (i)
$$\begin{aligned}(\alpha f + \beta g)\cdot h &= \int_0^1 (\alpha f + \beta g)(x) h(x)\,dx \\ &= \int_0^1 f(x)h(x)\,dx + \beta\int_0^1 g(x)h(x)\,dx \\ &= \alpha f\cdot h + \beta g\cdot h.\end{aligned}$$
(ii) $f\cdot g = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = g\cdot f.$
In conditions (iii) and (iv), the integrand is a perfect square. Therefore, the integral in nonnegative and can be 0 only if the integrand is 0 everywhere. If $f(x) \neq 0$ for some x, then it would be positive in a neighborhood of x by continuity, and the integral would be positive.

17. Compute the matrix product in both orders and check that you get the identity.

19. $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det(I) = 1$

Review Exercises for Chapter 1

1. $\mathbf{v}+\mathbf{w} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$; $3\mathbf{v} = 9\mathbf{i} + 12\mathbf{j} + 15\mathbf{k}$; $6\mathbf{v} + 8\mathbf{w} = 26\mathbf{i} + 16\mathbf{j} + 38\mathbf{k}$; $-2\mathbf{v} = -6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k}$; $\mathbf{v}\cdot\mathbf{w} = 4$; $\mathbf{v}\times\mathbf{w} = 9\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$. Your sketch should display $\mathbf{v}$, $\mathbf{w}$, $3\mathbf{v}$, $6\mathbf{v}$, $8\mathbf{w}$, $6\mathbf{v}+ 8\mathbf{w}$, $\mathbf{v}\cdot\mathbf{w}$ as the projection of $\mathbf{v}$ along $\mathbf{w}$ and $\mathbf{v}\times\mathbf{w}$ as a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

3. (a) $\mathbf{l}(t) = -\mathbf{i} + (2+t)\mathbf{j} - \mathbf{k}$ (c) $-2x + y + 2z = 9$
(b) $\mathbf{l}(t) = (3t-3)\mathbf{i} + (t+1)\mathbf{j} - t\mathbf{k}$

5. (a) 0 (b) 5 (c) -10

7. (a) $\pi/2$ (b) $5/(2\sqrt{15})$ (c) $-10/(\sqrt{6}\sqrt{17})$

9. $\{st\mathbf{a} + s(1-t)\mathbf{b} \mid 0 \le t \le 1 \text{ and } 0 \le s \le 1\}$

11. Let $\mathbf{v} = (a_1, a_2, a_3)$, $\mathbf{w} = (b_1, b_2, b_3)$, and apply the CBS inequality.

13. The area is the absolute value of

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 + \lambda a_1 & b_2 + \lambda a_2 \end{vmatrix}.$$

(A multiple of one row of a determinant can be added to another row without changing its value.) Your sketch should show two parallelograms with the same base and height.

15. The cosines of the two parts of the angle are equal, because
$\mathbf{a}\cdot\mathbf{v}/\|\mathbf{a}\|\,\|\mathbf{v}\| = (\mathbf{a}\cdot\mathbf{b} + \|\mathbf{a}\|\,\|\mathbf{b}\|)/\|\mathbf{v}\| = \mathbf{b}\cdot\mathbf{v}/\|\mathbf{b}\|\,\|\mathbf{v}\|$.

17. $\mathbf{i}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$; etc.

19. (a) HINT: The length of the projection of the vector connecting any pair of points, one on each line, onto $(\mathbf{a}_1 \times \mathbf{a}_2)/\|\mathbf{a}_1 \times \mathbf{a}_2\|$ is d.
(b) $\sqrt{2}$

21. (a) Note that

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ x_1 & x_2 - x_1 & x_3 - x_1 \\ y_1 & y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}.$$

(b) $\frac{1}{2}$

23.

Rectangular	Spherical	(plot omitted)
(a) $(\sqrt{2}/2, \sqrt{2}/2, 1)$	(a) $(\sqrt{2}, \pi/4, \pi/4)$	
(b) $(3\sqrt{3}/2, 3/2, -4)$	(b) $(5, \pi/6, \arccos(-4/5))$	
(c) $(0, 0, 1)$	(c) $(1, \pi/4, 0)$	
(d) $(0, -2, 1)$	(d) $(\sqrt{5}, 3\pi/2, \arccos(\sqrt{5}/5))$	
(e) $(0, 2, 1)$	(e) $(\sqrt{5}, \pi/2, \text{arceos}(\sqrt{5}/5))$	

25. $z = r^2\cos 2\theta$; $\cos\phi = \rho\sin^2\phi\cos 2\theta$

27. $|\mathbf{x}\cdot\mathbf{y}| = 6 < \sqrt{98} = \|\mathbf{x}\|\,\|\mathbf{y}\|$; $\|\mathbf{x}+\mathbf{y}\| = \sqrt{33} < \sqrt{14}+\sqrt{7} = \|\mathbf{x}\| + \|\mathbf{y}\|$

29. (a) The associative law for matrix multiplication may be checked as follows:

$$\begin{aligned}[(AB)C]_{ij} &= \sum_{k=1}^{n}(AB)_{ik}C_{kj} = \sum_{k=1}^{n}\sum_{l=1}^{n} A_{il}B_{lk}C_{kj}\\ &= \sum_{l=1}^{n} A_i(BC)_{lj} = [A(BC)]_{ij}.\end{aligned}$$

Use this with C taken to be a column vector.

(b) The matrix for the composition is the product matrix.

31. $\mathbb{R}^n$ is spanned by the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. If $\mathbf{v} \in \mathbb{R}^n$, then

$$T\mathbf{v} = T\left[\sum_{i=1}^{n}(\mathbf{v}\cdot\mathbf{e}_i)\mathbf{e}_i\right] = \sum_{i=1}^{n}(\mathbf{v}\cdot\mathbf{e}_i)T\mathbf{e}_i.$$

Let $a_{ij} = (T\mathbf{e}_j\cdot\mathbf{e}_i)$, so that

$$T\mathbf{e}_j = \sum_{j=1}^{n} a_{ij}\mathbf{e}_i.$$

Then

$$T\mathbf{v}\cdot\mathbf{e}_k = \sum_{i=1}^{n}(\mathbf{v}\cdot\mathbf{e}_i)a_{ki}.$$

That is, if

$$\mathbf{v} = \begin{bmatrix} v_1\\ \vdots\\ v_n\end{bmatrix}, \quad \text{then} \quad T\mathbf{v} = \begin{bmatrix} a_{11} & \cdots & a_{1n}\\ \vdots & & \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix} v_1\\ \vdots\\ v_n\end{bmatrix},$$

as desired.

33. (a) $70\cos\theta + 20\sin\theta$ (b) $(21\sqrt{3}+6)$ ft · lb

35. Each side equals $2xy - 7yz + 5z^2 - 48x + 54y - 5z - 96$. (Or switch the first two columns and then subtract the first row from the second.)

37. Add the last row to the first and subtract it from the second.

39. (a) $\dfrac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$ (b) $1/3$

41. Use the fact that $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$, expand both sides, and use the definition of $\mathbf{c}$.

43. $(1/\sqrt{38})\mathbf{i} - (6/\sqrt{38})\mathbf{j} + (1/\sqrt{38})\mathbf{k}$

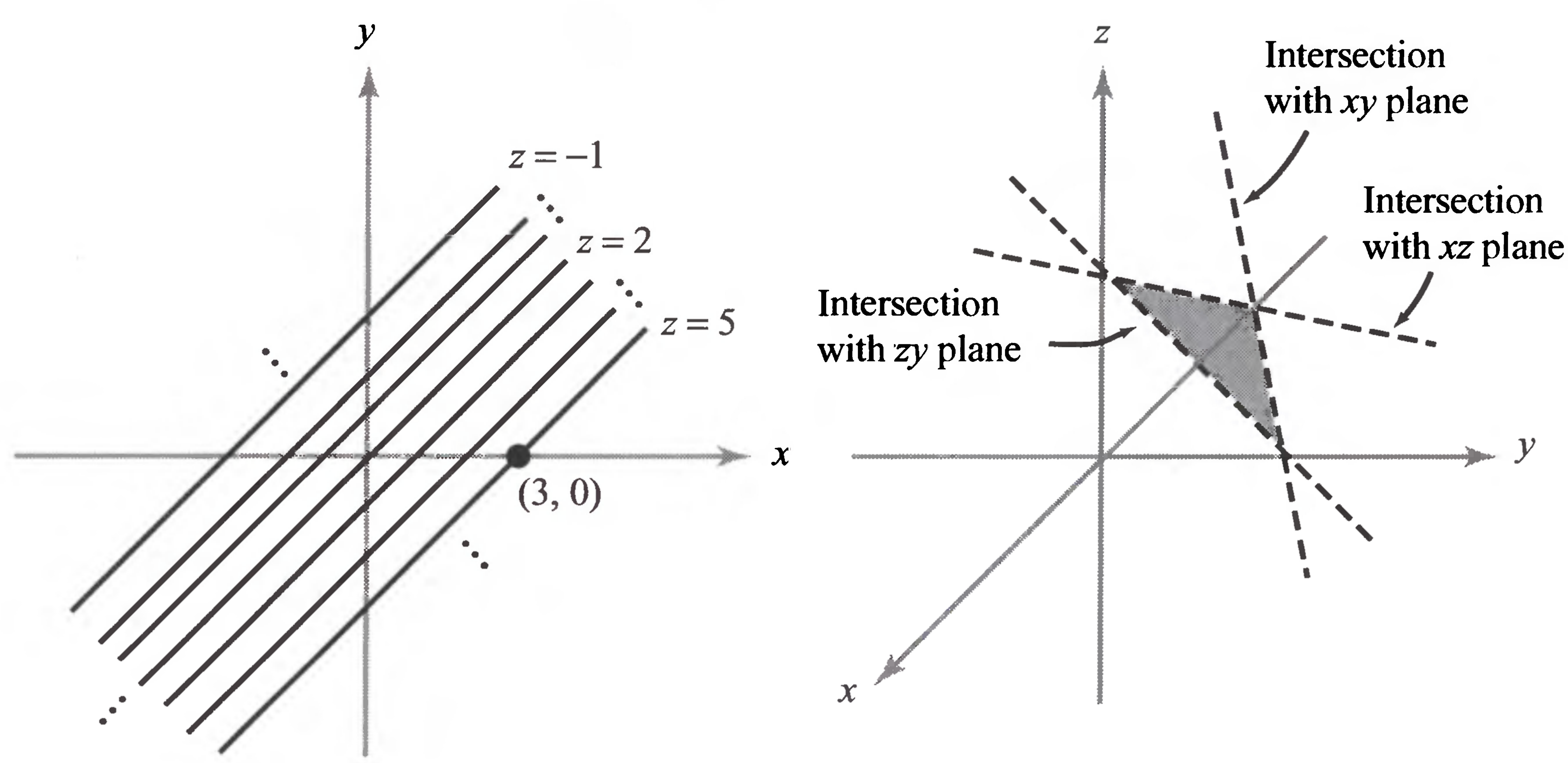
45. $(2/\sqrt{5})\mathbf{i} - (1/\sqrt{5})\mathbf{j}$

47. $(\sqrt{3}/2)\mathbf{i} + (1/2\sqrt{2})\mathbf{j} + (1/2\sqrt{2})\mathbf{k}$

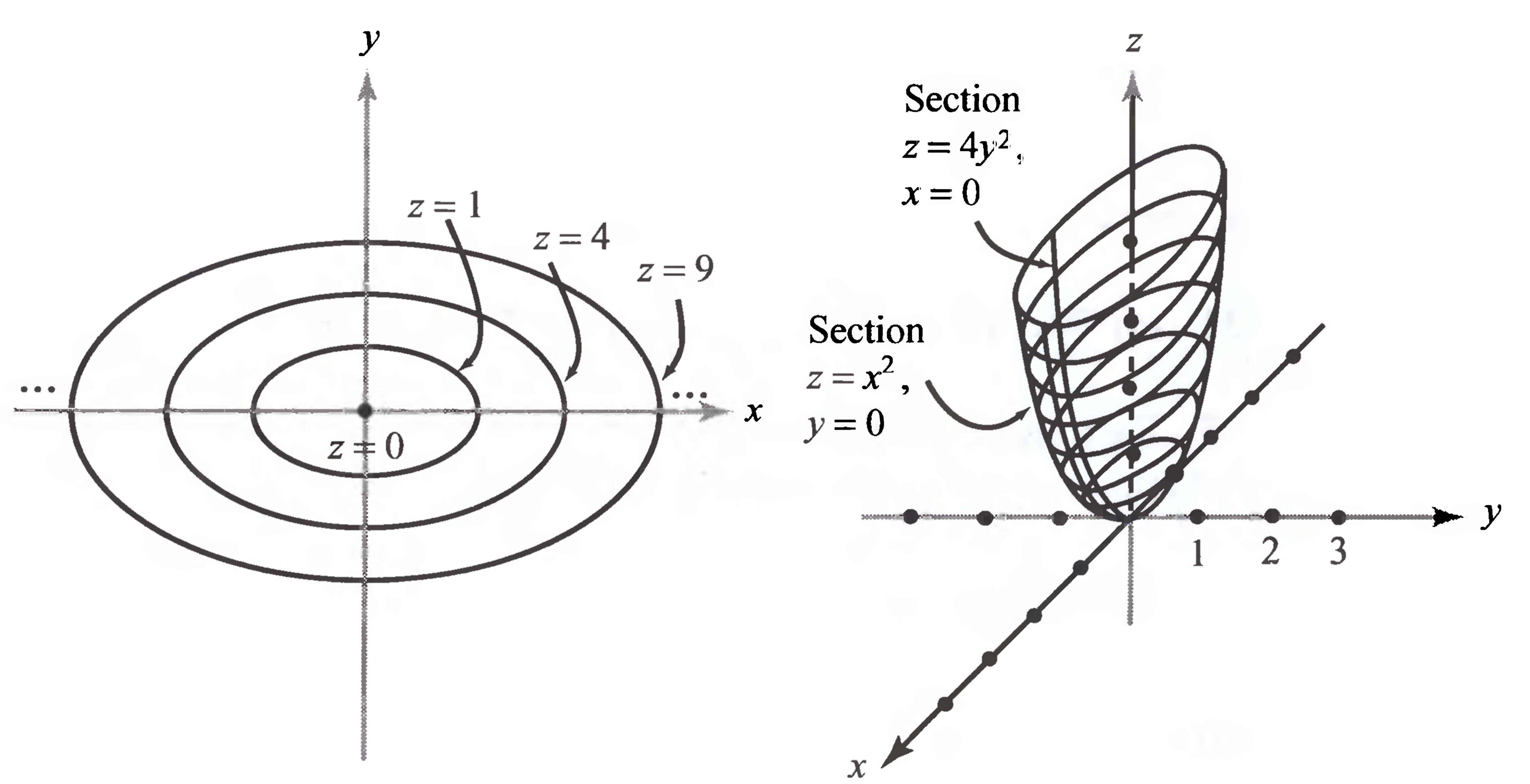
Chapter 2

Section 2.1

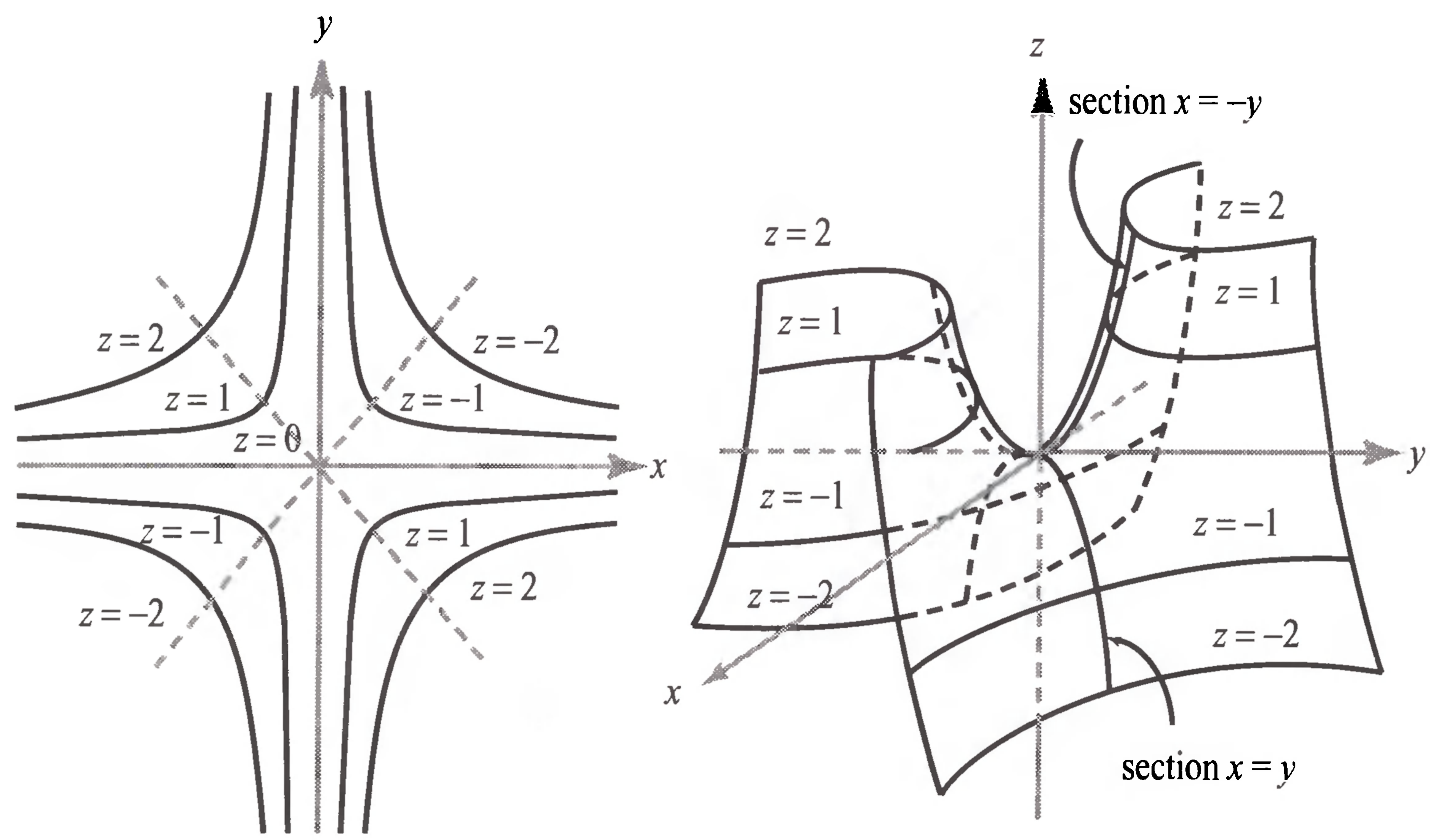
1. The level curves and graphs are sketched below. The graph in part (c) is a hyperbolic paraboloid like that of Example 4, but rotated 45° and vertically compressed by a factor of 1/4. To see this, use the variables $u = x + y$ and $v = x - y$. Then $z = (v^2 - u^2)/4$.



(a)



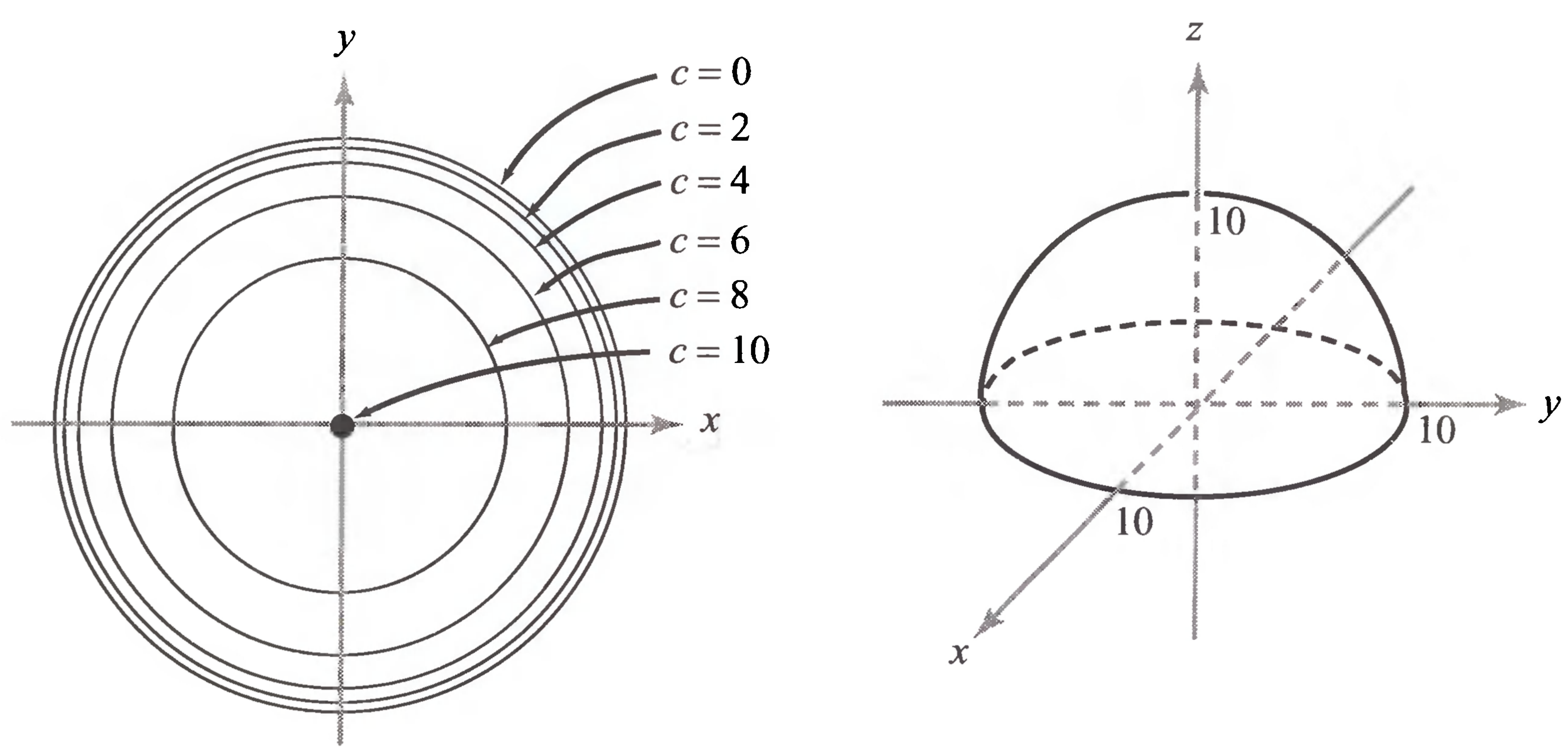
(b)



(c) $z = -xy$

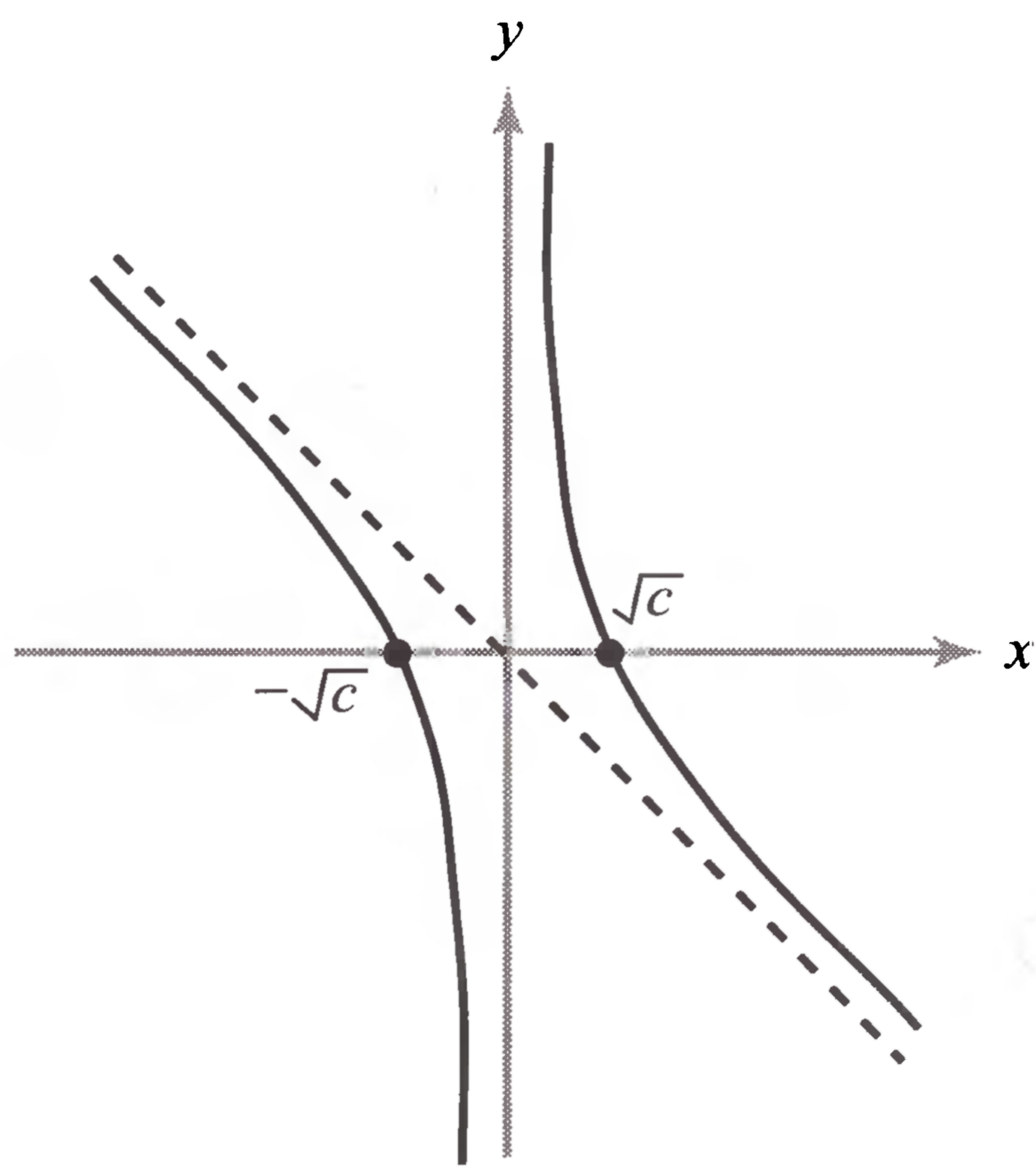
3. For Example 2, $z = r(\cos\theta + \sin\theta) + 2$, shape depends on θ; for Example 3, $z = r^2$, shape is independent of θ; for Example 4, $z = r^2(\cos^2\theta - \sin^2\theta)$, shape depends on θ.

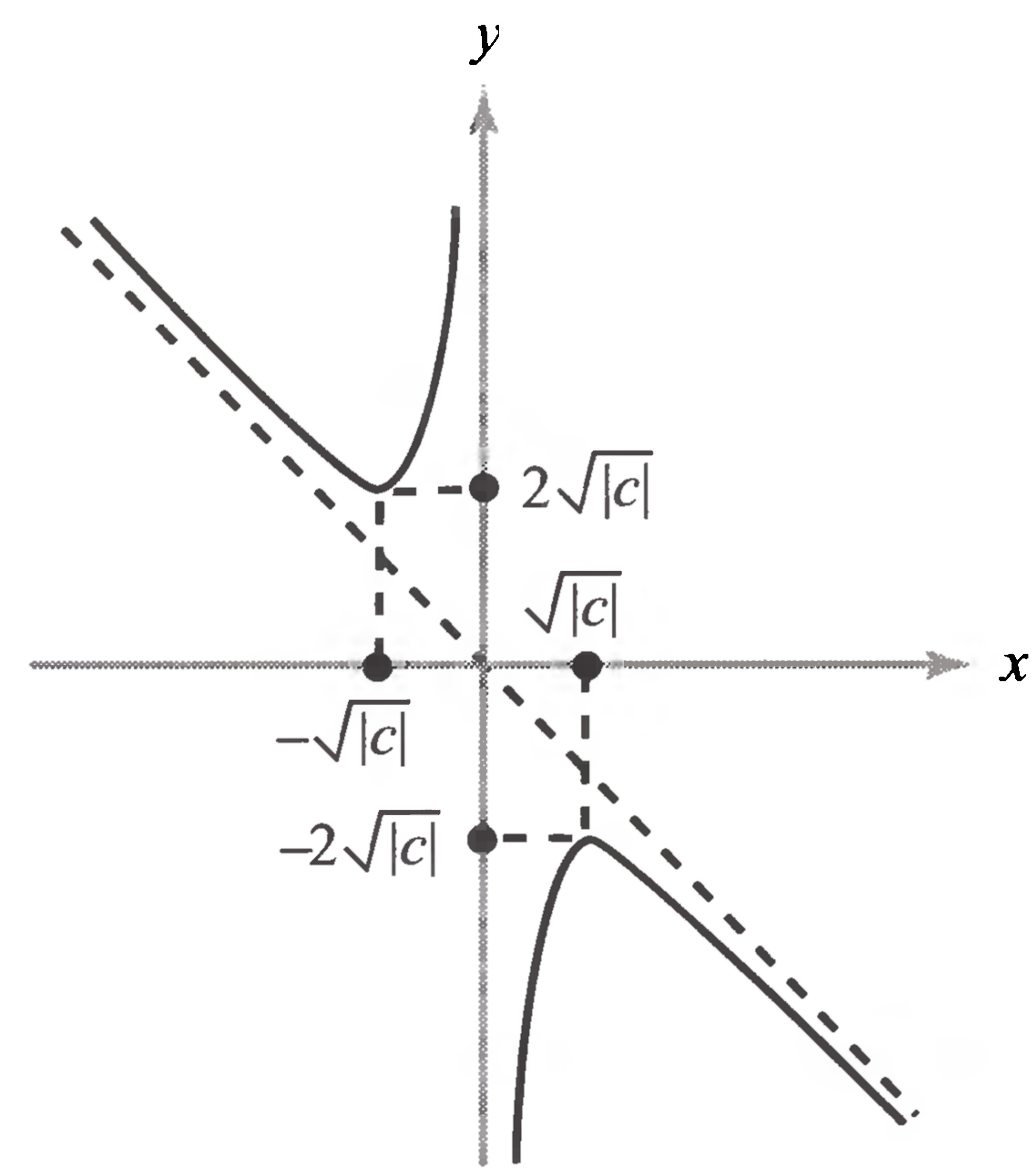
5. The level curves are circles $x^2 + y^2 = 100 - c^2$ when $c \le 10$. The graph is the upper hemisphere of $x^2 + y^2 + z^2 = 100$.



7. The level curves are circles, and the graph is a paraboloid of revolution. See Example 3 of this section.

9. If $c = 0$, the level curve is the straight line $y = -x$ together with the line $x = 0$. If $c \neq 0$, then $y = -x + (c/x)$. The level curve is a hyperbola with the y axis and the line $y = -x$ as asymptotes. The graph is a hyperbolic paraboloid. Sections along the line $y = ax$ are the parabolas $z = (1 + a)x^2$.



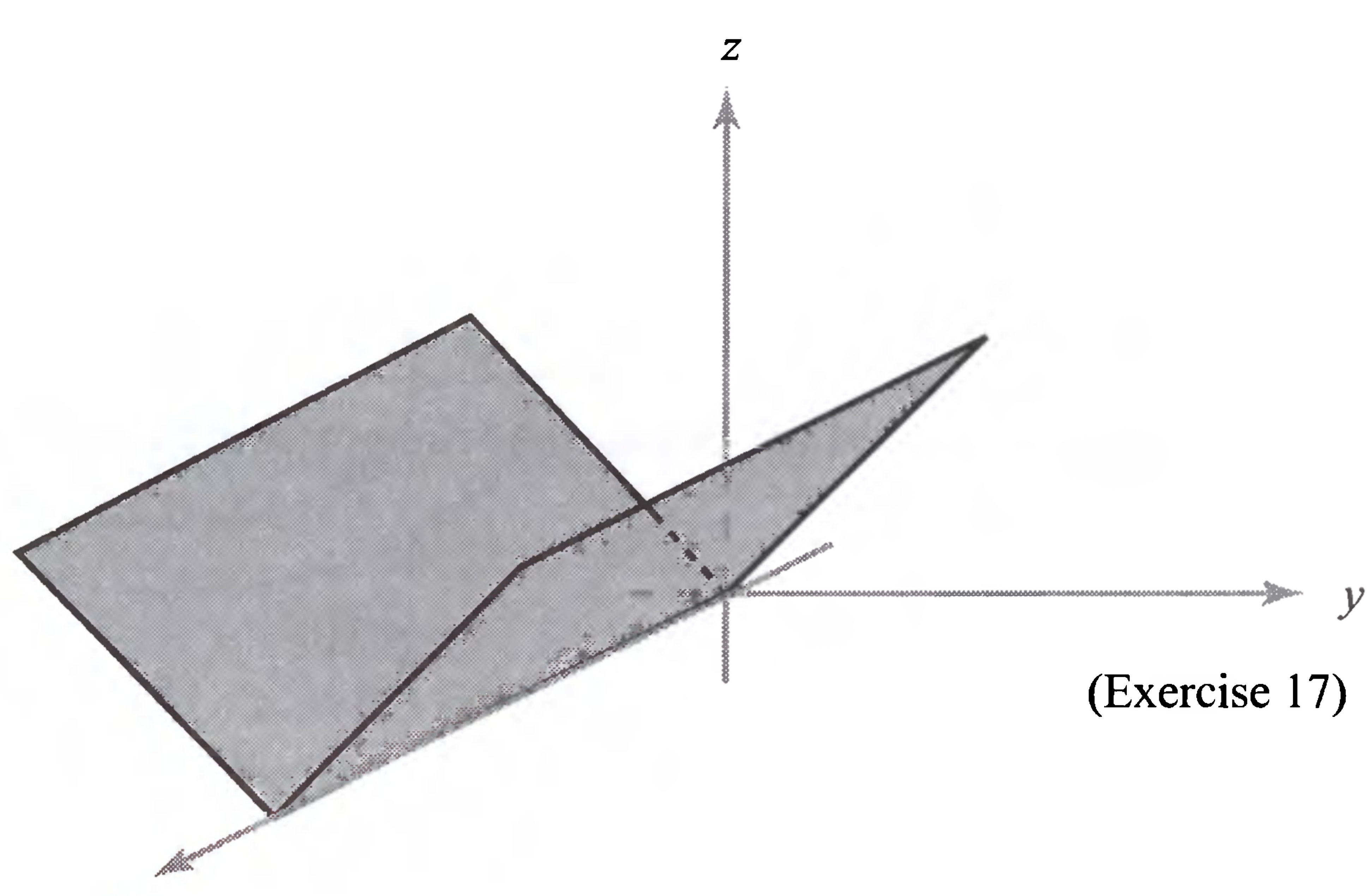


11. If $c > 0$, the level surface $f(x, y, z) = c$ is empty. If $c = 0$, the level surface is the point $(0, 0, 0)$. If $c < 0$, the level surface is the sphere of radius $\sqrt{-c}$ centered at $(0, 0, 0)$. A section of the graph determined by $z = a$ is given by $t = -x^2 - y^2 - a^2$, which is a paraboloid of revolution opening down in xyt space.

13. If $c < 0$, the level surface is empty. If $c = 0$, the level surface is the z axis. If $c > 0$, it is the right-circular cylinder $x^2 + y^2 = c$ of radius $\sqrt{c}$ whose axis is the z axis. A section of the graph determined by $z = a$ is the paraboloid of revolution $t = x^2 + y^2$. A section determined by $x = b$ is a "trough" with parabolic cross section $t(y, z) = y^2 + b^2$.

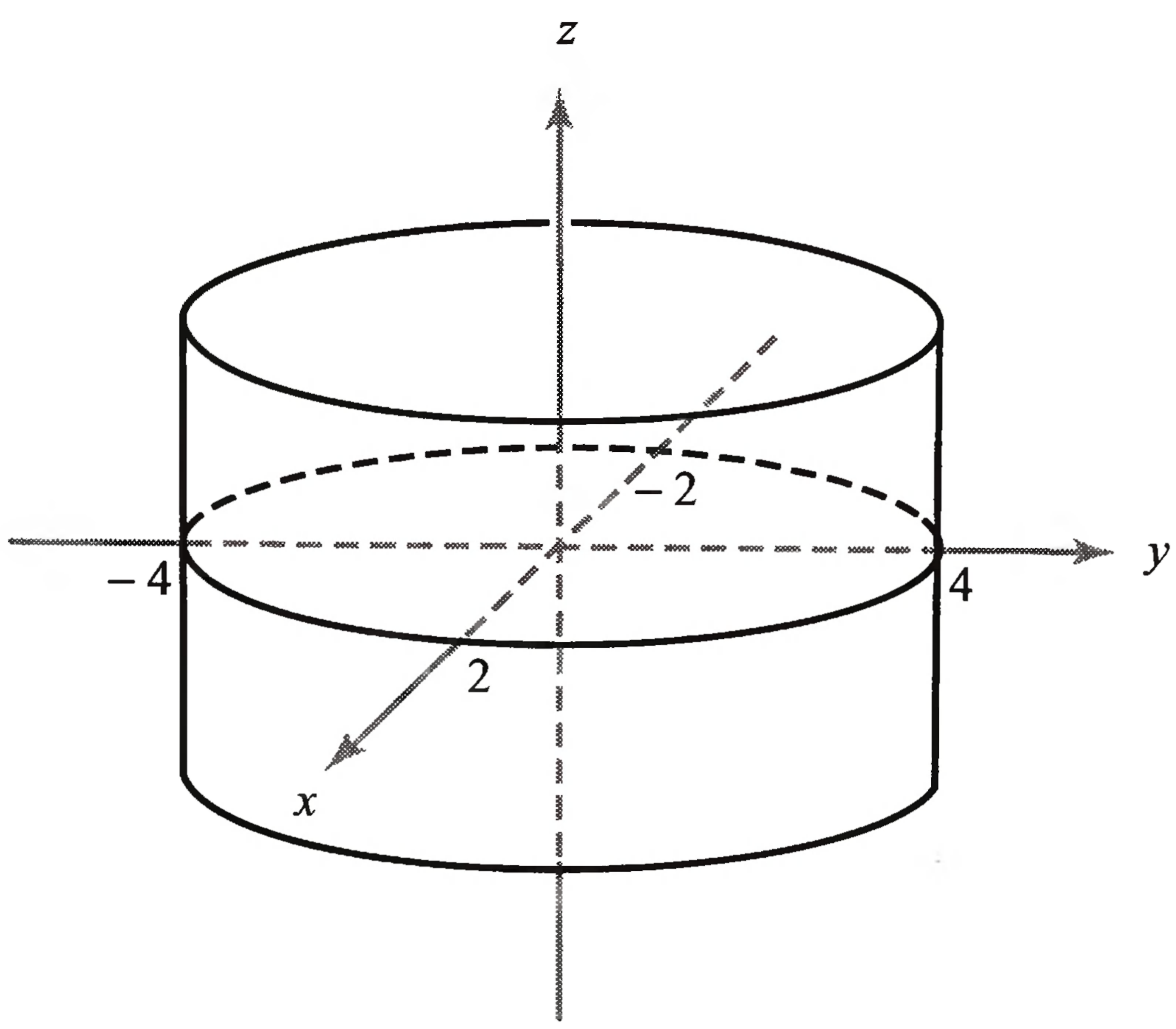
15. Setting $u = (x - z)/\sqrt{2}$ and $v = (x + z)/\sqrt{2}$ gives the u and v axes rotated 45° around the y axis from the x and z axes. Because $f = vy\sqrt{2}$, the level surfaces $f = c$ are "cylinders" perpendicular to the vy plane ($z = -x$) whose cross sections are the hyperbolas $vy = c/\sqrt{2}$, so the section $S_{x=a} \cap$ graph f is the hyperbolic paraboloid $t = (z + a)y$ in yzt space [see Exercise 1(c)]. The section $S_{y=b} \cap$ graph f is the plane $t = bx + bz$ in xzt space. The section $S_{z=b} \cap$ graph f is the hyperbolic paraboloid $t = y(x + b)$ in xyt space.

17. If $c < 0$, the level curve is empty. If $c = 0$, the level curve is the x axis. If $c > 0$, it is the pair of parallel lines $|y| = c$. The sections of graph with x constant are V-shaped curves $z = |y|$ in yz space. The graph is shown in the accompanying figure.



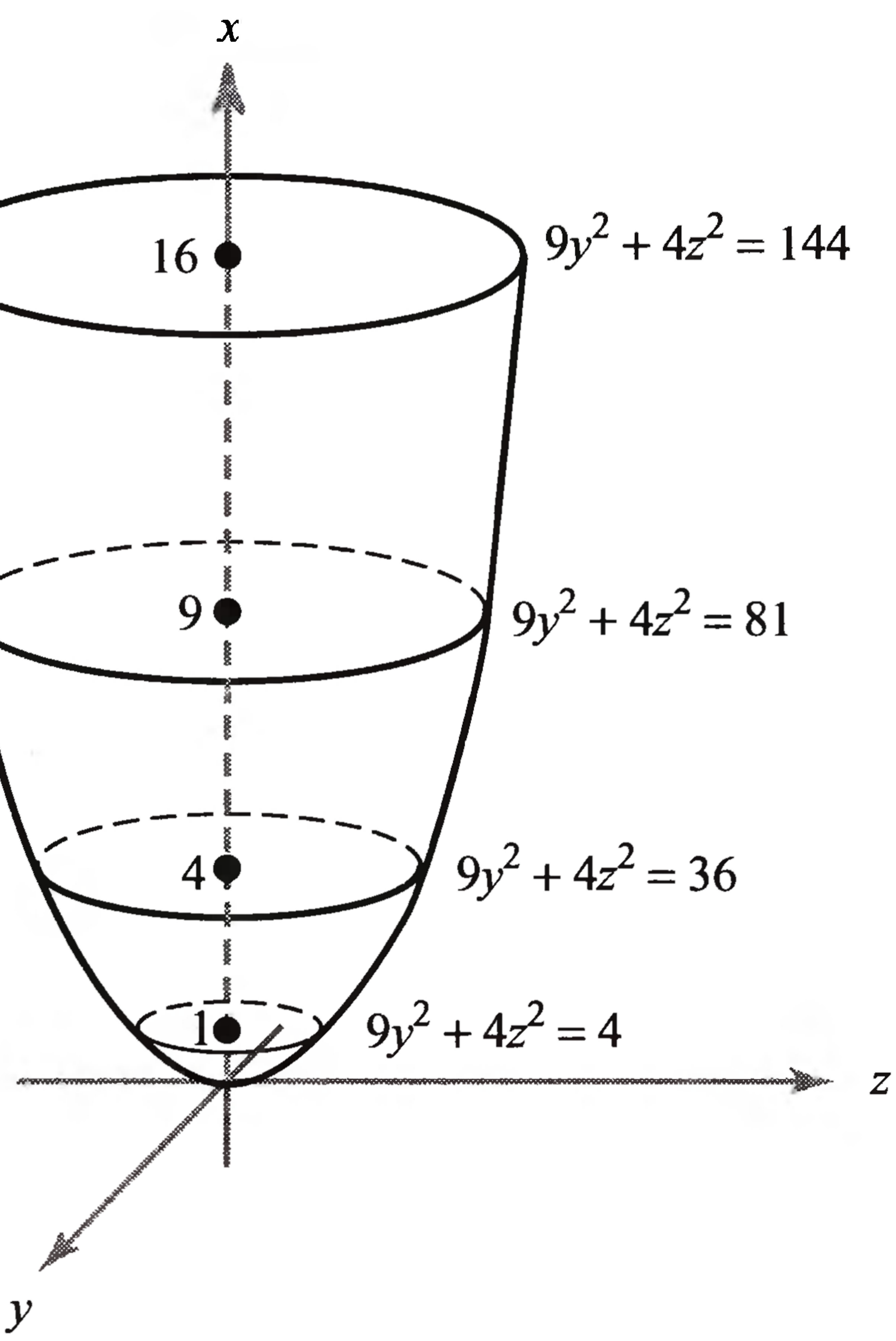
(Exercise 17)

19. The value of z does not matter, so we get a "cylinder" of elliptic cross section parallel to the z axis and intersecting the xy plane in the ellipse $4x^2 + y^2 = 16$.

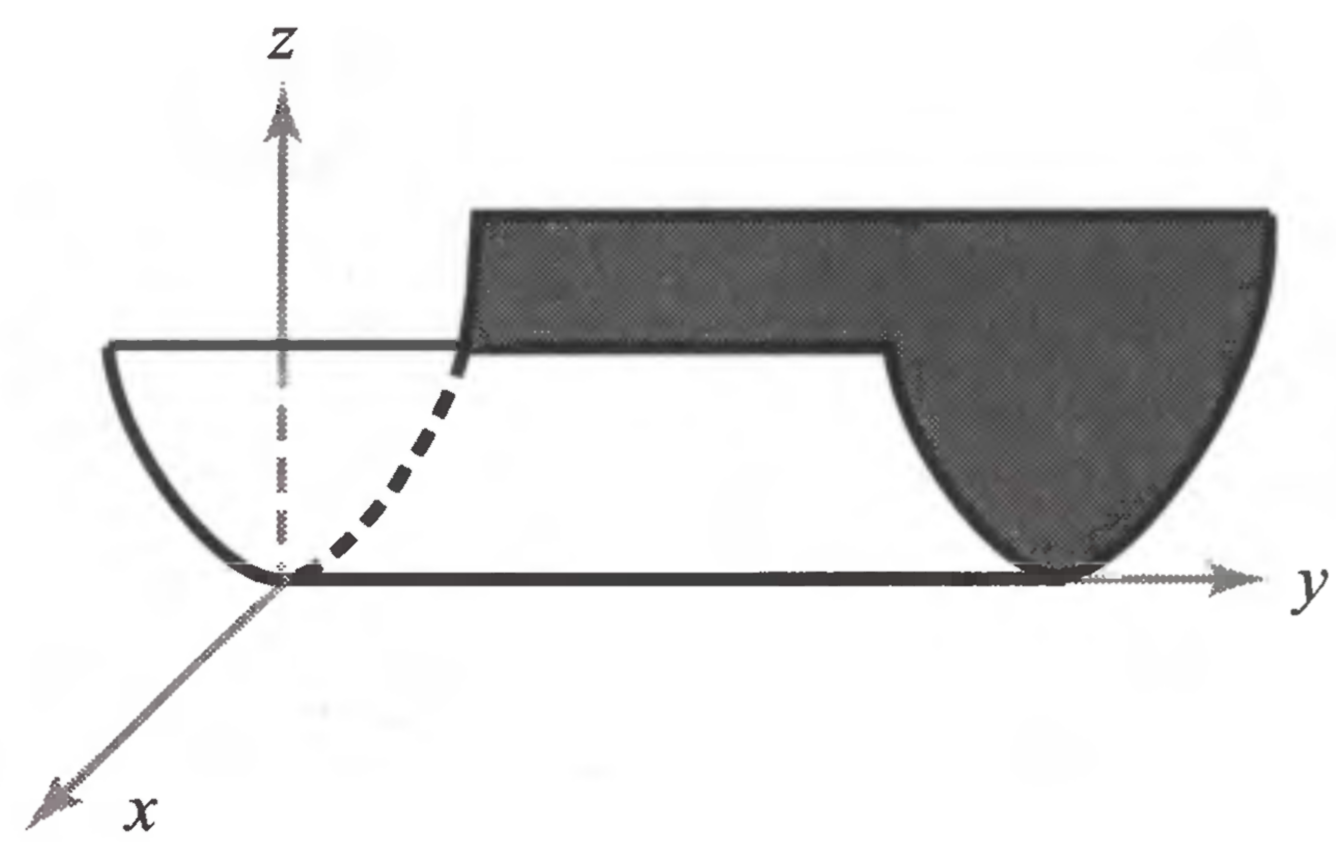


21. The value of x does not matter, so we get a "cylinder" parallel to the x axis of hyberbolic cross section intersecting the yz plane in the hyperbola $z^2 - y^2 = 4$.

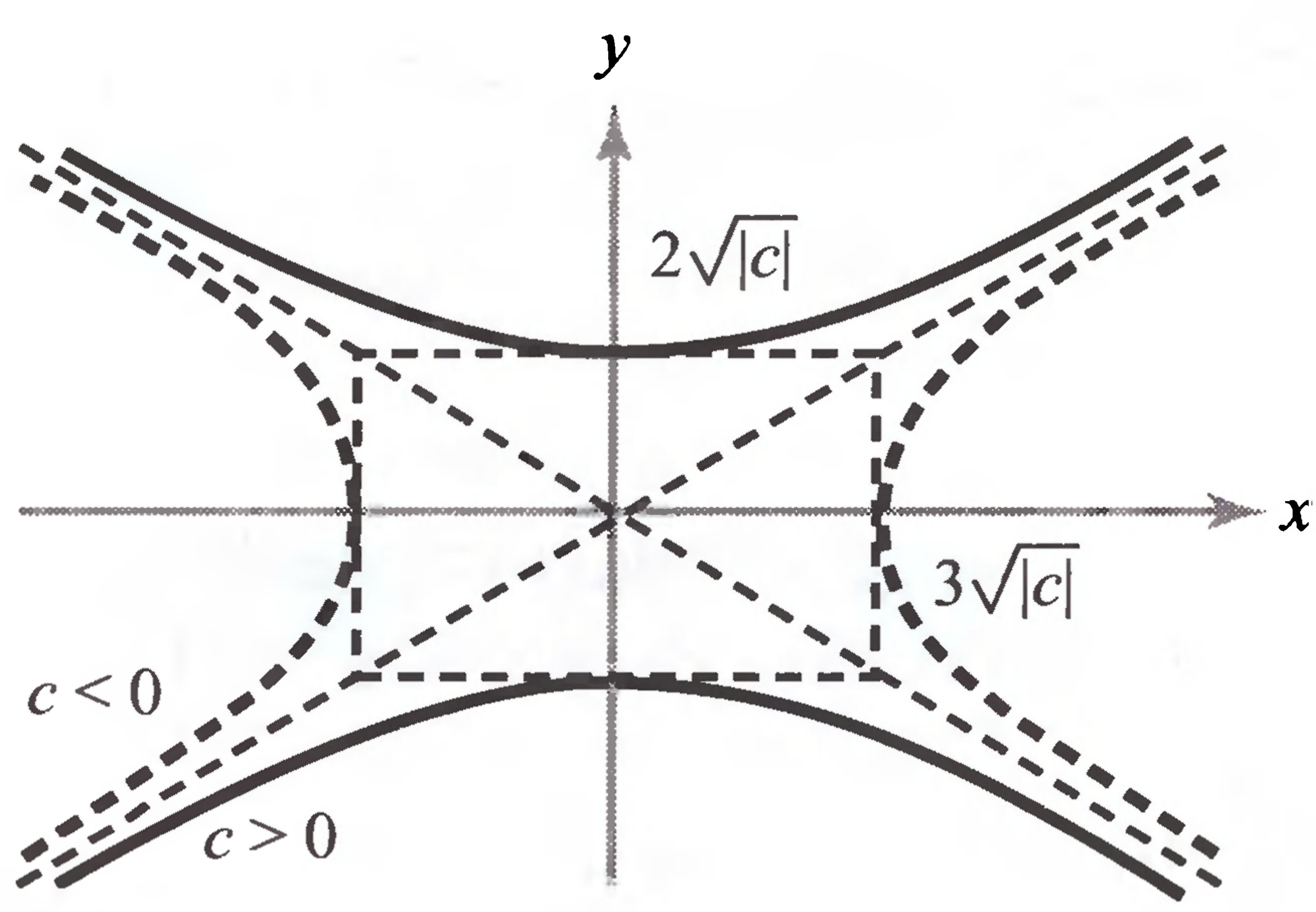
23. An elliptic paraboloid with axis along the x axis.



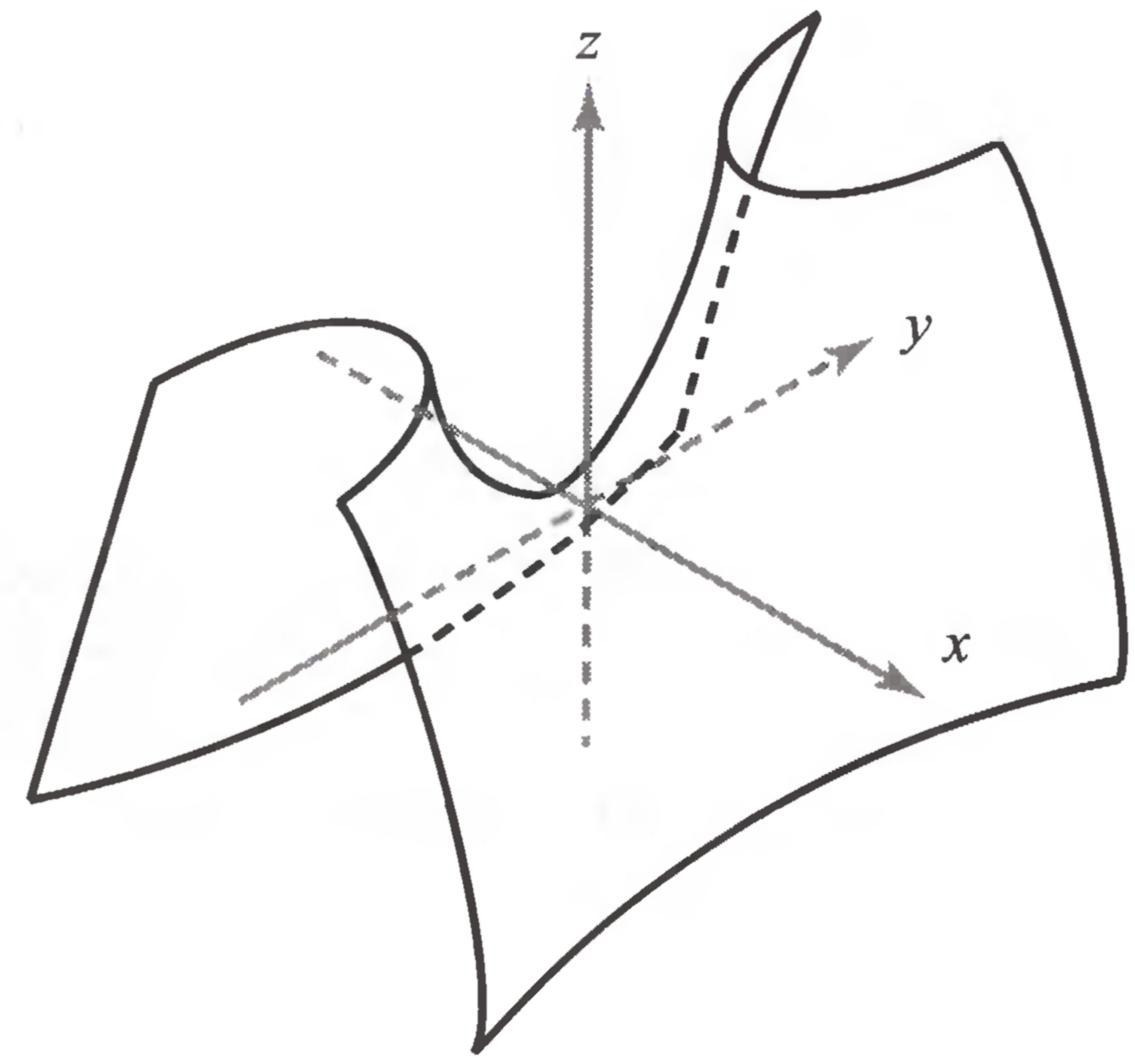
25. The value of y does not matter, so we get a "cylinder" of parabolic cross section.



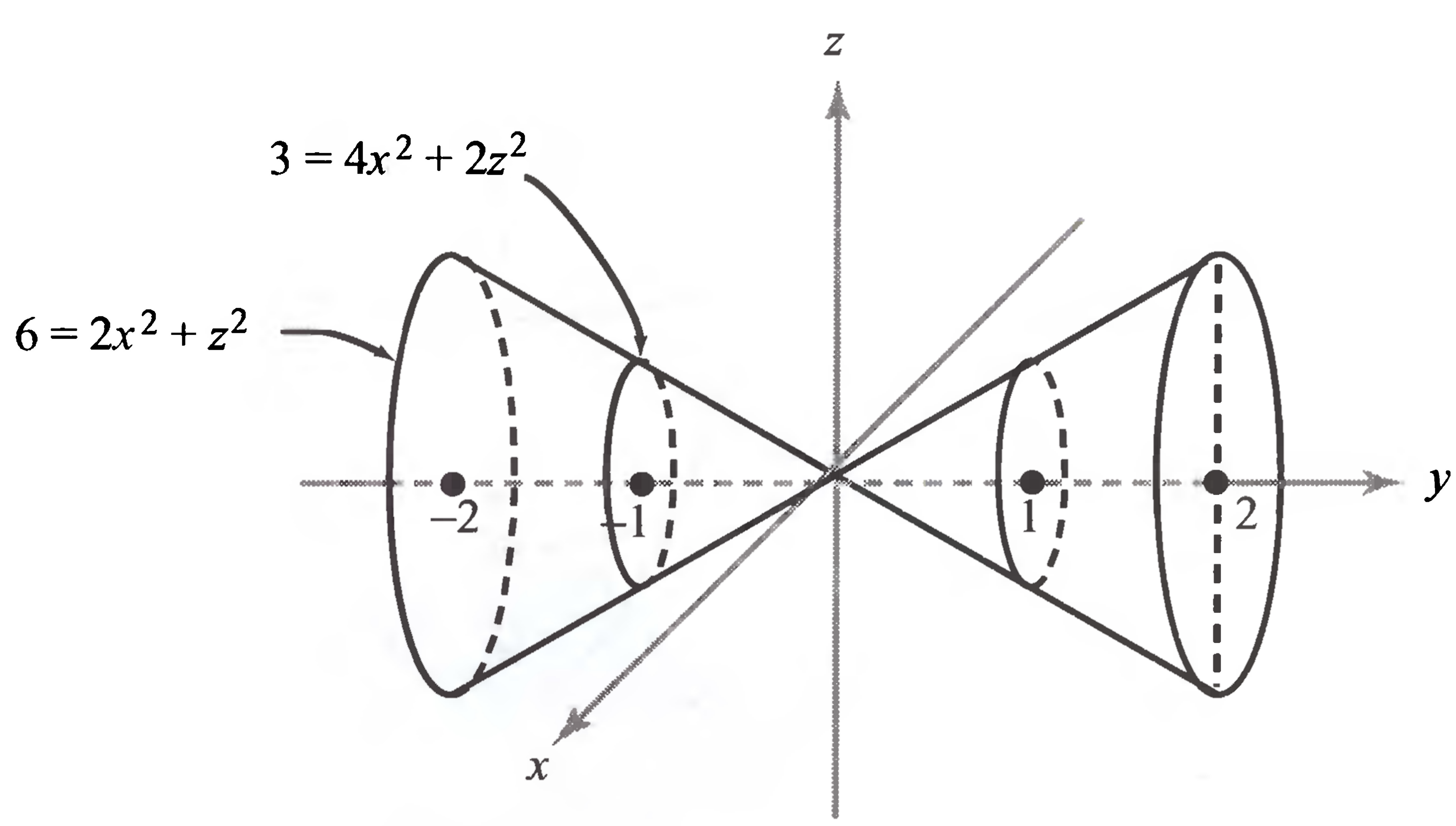
27. This is a saddle surface similar to that of Example 4, but the hyperbolas, which are level curves, no longer have perpendicular asymptotes.



Level curves

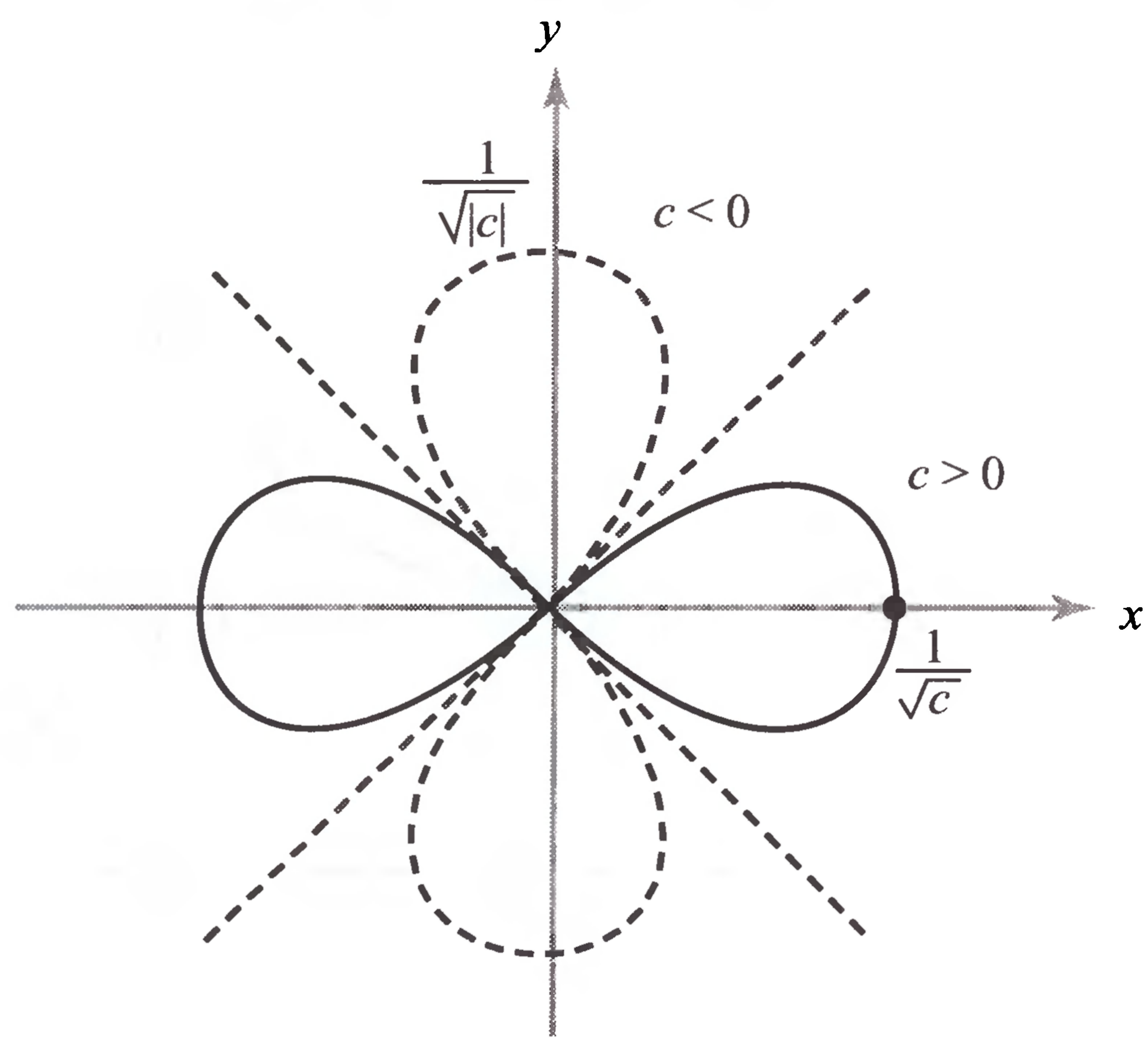


29. A double cone with axis along the y axis and elliptical cross sections



31. Complete the square to get $(x+2)^2 + (y - b/2)^2 + (z + \frac{9}{2})^2 = (b^2 + 4b + 97)/4$. This is an ellipsoid with center at $(-2, b/2, -\frac{9}{2})$ and axes parallel to the coordinate axes.

33. Level curves are described by $\cos 2\theta = cr^2$. If $c > 0$, then $-\pi/4 \le \theta \le \pi/4$ or $3\pi/4 \le \theta \le 5\pi/4$. If $c < 0$, then $\pi/4 \le \theta \le 3\pi/4$ or $5\pi/4 \le \theta \le 7\pi/4$. In either case you get a figure-eight shape, called a *lemniscate*, through the origin. (Such shapes were first studied by Jacques Bernoulli and are sometimes called Bernoulli's lemniscates.)



Section 2.2

1. If $(x_0, y_0) \in A$, then $|x_0| < 1$ and $|y_0| < 1$. Let $r = \min(1 - |x_0|, 1 - |y_0|)$. Prove that $D_r(x_0, y_0) \subset A$ either analytically or by drawing a figure.

3. Let $r = \min(2 - \sqrt{x_0^2 + y_0^2}, \sqrt{x_0^2 + y_0^2} - \sqrt{2})$.

5. (a) 0 (b) $-1/2$ (c) 1

7. (a) 5 (b) 0 (c) $2x$

9. (a) 0 (b) $-1/2$ (c) 0

11. (a) Compose $f(x, y) = xy$ with $g(t) = (\sin t)/t$ for $t \neq 0$ and $g(0) = 1$.
(b) 0 (c) 0

13. (a) 1 (b) $\|\mathbf{x}_0\|$ (c) $(1, e)$
(d) Limit doesn't exist (look at the limits for $x = 0$ and $y = 0$ separately).

15. Use parts (ii) and (iii) of Theorem 4.

17. (a) Let the value of the function be 1 at (0, 0). (b) No.

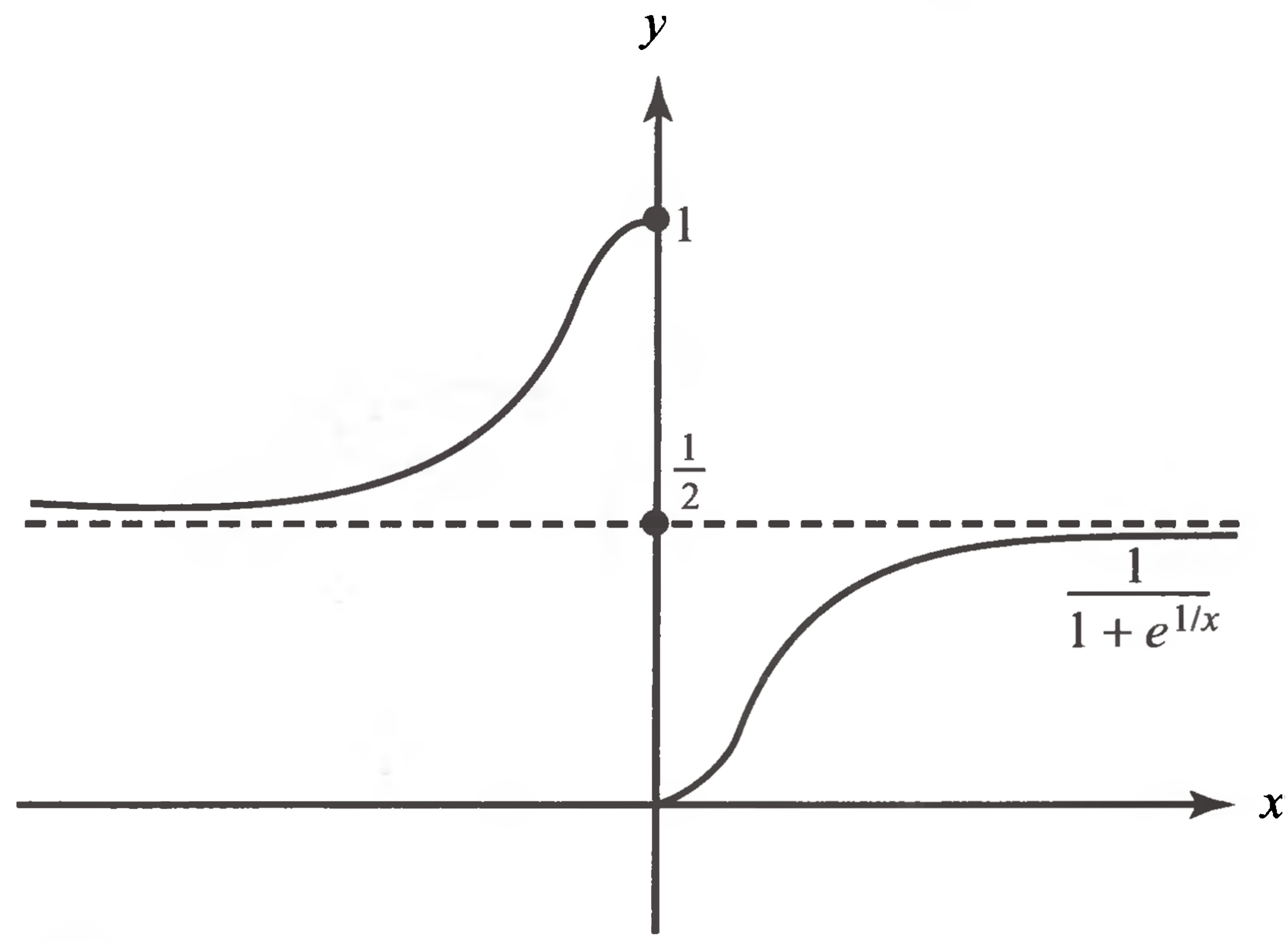
19. For $|x - 2| < \delta = \sqrt{\varepsilon + 4} - 2$, we have $|x^2 - 4| = |x - 2||x + 2| < \delta(\delta + 4) = \varepsilon$. By Theorem 3(iii), $\lim_{x\to 2} x^2 = (\lim_{x\to 2} x)^2 = 2^2 = 4$.

21. Let $r = \|\mathbf{x} - \mathbf{y}\|/2$. If $\|\mathbf{z} - \mathbf{y}\| \le r$, let $f(\mathbf{z}) = \|\mathbf{z} - \mathbf{y}\|/r$. If $\|\mathbf{z} - \mathbf{y}\| > r$, let $f(\mathbf{z}) = 1$.

23. (a) $\lim_{x\to b^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x > b$ and $0 < x - b < \delta$ imply $|f(x) - L| < \varepsilon$.

(b) $\lim_{x\to 0^-} (1/x) = -\infty$, $\lim_{t\to -\infty} e^t = 0$, and so $\lim_{x\to 0^-} e^{1/x} = 0$. Hence $\lim_{x\to 0^-} 1/(1+e^{1/x}) = 1$. The other limit is 0.

(c)



25. If $\varepsilon > 0$ and $\mathbf{x}_0$ are given, let $\delta = (\varepsilon/K)^{1/\alpha}$. Then $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < K\delta^\alpha = \varepsilon$ whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Notice that the choice of δ does not depend on $\mathbf{x}_0$. This means that f is *uniformly continuous.*

27. (a) Choose $\delta < 1/500$. (b) Choose $\delta < 0.002$.

Section 2.3

1. (a) $\partial f/\partial x = y$; $\partial f/\partial y = x$
(b) $\partial f/\partial x = ye^{xy}$; $\partial f/\partial y = xe^{xy}$
(c) $\partial f/\partial x = \cos x \cos y - x \sin x \cos y$; $\partial f/\partial y = -x \cos x \sin y$
(d) $\partial f/\partial x = 2x[1 + \log(x^2 + y^2)]$; $\partial f/\partial y = 2y[1 + \log(x^2 + y^2)]$; $(x, y) \neq (0, 0)$

3. (a) $\partial w/\partial x = (1 + 2x^2)\exp(x^2 + y^2)$; $\partial w/\partial y = 2xy \exp(x^2 + y^2)$
(b) $\partial w/\partial x = -4xy^2/(x^2 - y^2)^2$; $\partial w/\partial y = 4yx^2/(x^2 - y^2)^2$
(c) $\partial w/\partial x = ye^{xy}\log(x^2 + y^2) + 2xe^{xy}/(x^2 + y^2)$;
$\partial w/\partial y = xe^{xy}\log(x^2 + y^2) + 2ye^{xy}/(x^2 + y^2)$
(d) $\partial w/\partial x = 1/y$; $\partial w/\partial y = -x/y^2$
(e) $\partial w/\partial x = -y^2e^{xy}\sin ye^{xy}\sin x + \cos ye^{xy}\cos x$;
$\partial w/\partial y = (xye^{xy} + e^{xy})(-\sin ye^{xy}\sin x)$

5. $z = 6x + 3y - 11$

7. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} e^y & xe^y - \sin y \\ 1 & 0 \\ 1 & e^y \end{bmatrix}$ (d) $\begin{bmatrix} (y + xy^2)e^{xy} & (x + x^2y)e^{xy} \\ \sin y & x\cos y \\ 5y^2 & 10xy \end{bmatrix}$

9. At $z = 1$.

11. Both are xye^{xy}.

13. (a) $\nabla f = (e^{-x^2-y^2-z^2}(-2x^2+1), -2xye^{-x^2-y^2-z^2}, -2xze^{-x^2-y^2-z^2})$
(b) $\nabla f = (x^2+y^2+z^2)^{-2}(yz(y^2+z^2-x^2), xz(x^2+z^2-y^2), xy(x^2+y^2-z^2))$
(c) $\nabla f = (z^2e^x \cos y, -z^2e^x \sin y, 2ze^x \cos y)$

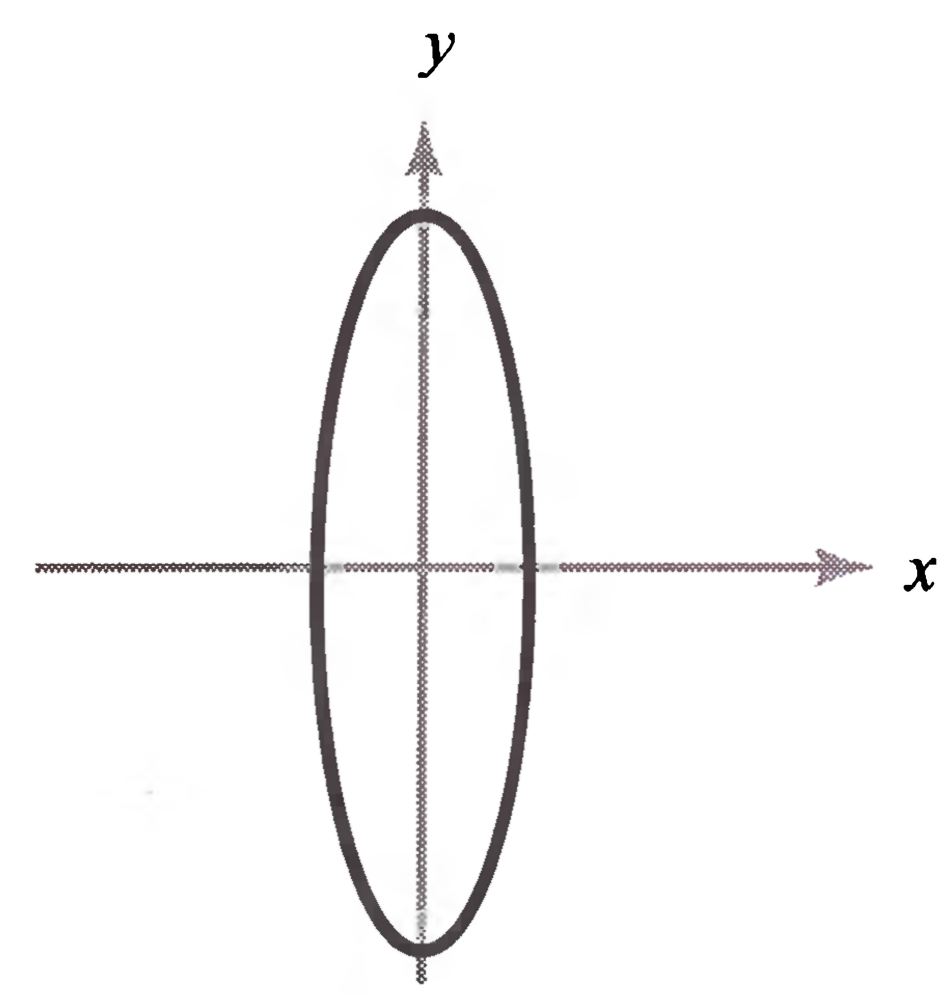
15. $2x + 6y - z = 5$

17. $-2\mathbf{k}$

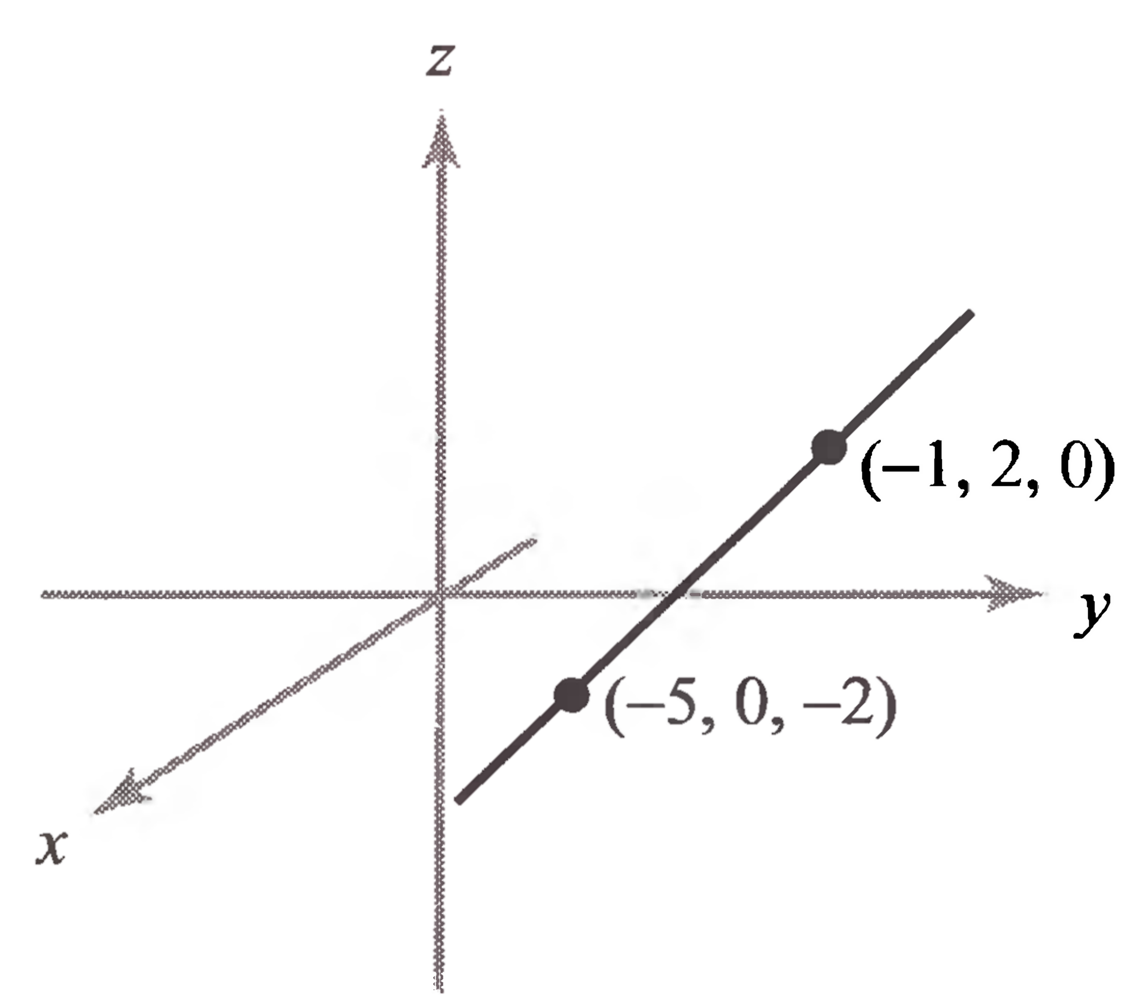
19. They are constant. Show that the derivative is the zero matrix.

Section 2.4

1. This curve is the ellipse $(y^2/16) + x^2 = 1$:



3. This curve is the straight line through $(-1, 2, 0)$ with direction $(2, 1, 1)$:



5. $6\mathbf{i} + 6t\mathbf{j} + 3t^2\mathbf{k}$

7. $(-2\cos t \sin t, 3 - 3t^2, 1)$

9. $\mathbf{c}'(t) = (e^t, -\sin t)$

11. $\mathbf{c}'(t) = (t\cos t + \sin t, 4)$

13. Horizontal when $t = (R/v)n\pi$, n an integer; speed is zero if n is even; speed is $2v$ if n is odd.

15. $(\sin 3, \cos 3, 2) + (3\cos 3, -3\sin 3, 5)(t - 1)$

17. $(8, 8, 0)$

19. $(8, 0, 1)$

Section 2.5

1. Use parts (i), (ii), and (iii) of Theorem 10. The derivative at $\mathbf{x}$ is $2(f(\mathbf{x})+1)\mathbf{D}f(\mathbf{x})$.

3. (a) $h(x,y)=f(x,u(x,y))=f(p(x),u(x,y))$. We use p here solely as notation: $p(x)=x$.

Written out: $$\frac{\partial h}{\partial x}=\frac{\partial f}{\partial p}\frac{dp}{dx}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}=\frac{\partial f}{\partial p}+\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} \quad \text{because} \quad \frac{dp}{dx}=\frac{dx}{dx}=1$$

JUSTIFICATION: Call (p,u) the variables of f. To use the chain rule we must express h as a composition of functions; that is, first find g such that $h(x,y)=f(g(x,y))$. Let $g(x,y)=(p(x),u(x,y))$. Therefore, $\mathbf{D}h=(\mathbf{D}f)(\mathbf{D}g)$. Then

$$\begin{bmatrix}\dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial f}{\partial p} & \dfrac{\partial f}{\partial u}\end{bmatrix}\begin{bmatrix}\dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y}\\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial f}{\partial p} & \dfrac{\partial f}{\partial u}\end{bmatrix}\begin{bmatrix}1 & 0\\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\end{bmatrix}$$
$$=\begin{bmatrix}\dfrac{\partial f}{\partial p}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} & \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}\end{bmatrix},$$

and so $\dfrac{\partial h}{\partial x}=\dfrac{\partial f}{\partial p}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$. You may see $\dfrac{\partial h}{\partial x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$ as an answer. This requires careful interpretation because of possible ambiguity about the meaning of $\partial f/\partial x$, which is why the name p was used.

(b) $\dfrac{\partial h}{\partial x}=\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial u}\dfrac{du}{dx}+\dfrac{\partial f}{\partial v}\dfrac{dv}{dx}$ (c) $\dfrac{\partial h}{\partial x}=\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x}+\dfrac{\partial f}{\partial w}\dfrac{dw}{dx}$

5. Compute each in two ways; the answers are
(a) $(f\circ \mathbf{c})'(t)=e^t(\cos t-\sin t)$
(b) $(f\circ \mathbf{c})'(t)=15t^4\exp(3t^5)$
(c) $(f\circ \mathbf{c})'(t)=(e^{2t}-e^{-2t})[1+\log(e^{2t}+e^{-2t})]$
(d) $(f\circ \mathbf{c})'(t)=(1+4t^2)\exp(2t^2)$

7. Use Theorem 10(iii) and replace matrices by vectors.

9. $(f\circ g)(x,y)=(\tan(e^{x-y}-1)-e^{x-y}, e^{2(x-y)}-(x-y)^2)$ and
$\mathbf{D}(f\circ g)(1,1)=\begin{bmatrix}0 & 0\\ 2 & -2\end{bmatrix}$.

11. $\frac{1}{2}\cos(1)\cos(\log\sqrt{2})$

13. $-2\cos t\sin t e^{\sin t}+\sin^4 t+\cos^3 t e^{\sin t}-3\cos^2 t\sin^2 t$ for both (a) and (b).

15. $(2,0)$

17. (a) $G(x, y(x)) = 0$ and so $\dfrac{\partial G}{\partial x} + \dfrac{\partial G}{\partial y}\dfrac{dy}{dx} = 0.$

(b) $\begin{bmatrix} \dfrac{dy_1}{dx} \\ \dfrac{dy_2}{dx} \end{bmatrix} = -\begin{bmatrix} \dfrac{\partial G_1}{\partial y_1} & \dfrac{\partial G_1}{\partial y_2} \\ \dfrac{\partial G_2}{\partial y_1} & \dfrac{\partial G_2}{\partial y_2} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial G_1}{\partial x} \\ \dfrac{\partial G_2}{\partial x} \end{bmatrix}$ where $^{-1}$ means the inverse matrix.

The first component of this equation reads

$$\frac{dy_1}{dx} = \frac{-\dfrac{\partial G_1}{\partial x}\dfrac{\partial G_2}{\partial y_2} + \dfrac{\partial G_2}{\partial x}\dfrac{\partial G_1}{\partial y_2}}{\dfrac{\partial G_1}{\partial y_1}\dfrac{\partial G_2}{\partial y_2} - \dfrac{\partial G_2}{\partial y_1}\dfrac{\partial G_1}{\partial y_2}}.$$

(c) $\dfrac{dy}{dx} = \dfrac{-2x}{3y^2 + e^y}$

19. Apply the chain rule to $\partial G/\partial T$ where $G(t(T,P), p(T,P), V(T,P)) = P(V-b)e^{a/RVT} - RT$ is identically 0; $t(T,P) = T$; and $p(T,P) = P$.

21. Define $R_1(\mathbf{h}) = f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - [\mathbf{D}f(\mathbf{x}_0)]\mathbf{h}$.

23. Let g_1 and g_2 be C^1 functions from $\mathbb{R}^3$ to $\mathbb{R}$ such that $g_1(\mathbf{x}) = 1$ for $\|\mathbf{x}\| < \sqrt{2}/3$; $g_1(\mathbf{x}) = 0$ for $\|\mathbf{x}\| > 2\sqrt{2}/3$; $g_2(\mathbf{x}) = 1$ for $\|\mathbf{x} - (1,1,0)\| < \sqrt{2}/3$; and $g_2(\mathbf{x}) = 0$ for $\|\mathbf{x} - (1,1,0)\| > 2\sqrt{2}/3$. (See Exercise 22.) Let

$$h_1(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad h_2(\mathbf{x}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

and put $f(\mathbf{x}) = g_1(\mathbf{x})h_1(\mathbf{x}) + g_2(\mathbf{x})h_2(\mathbf{x})$.

25. Proof of rule (iii) follows:

$$\begin{aligned} &\frac{|h(\mathbf{x}) - h(\mathbf{x}_0) - [f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0) + g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \\ &\qquad \le |f(\mathbf{x}_0)|\frac{|g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \\ &\qquad\quad + |g(\mathbf{x}_0)|\frac{|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|} \\ &\qquad\quad + \frac{|f(\mathbf{x}) - f(\mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|}\frac{|g(\mathbf{x}) - g(\mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|}\|\mathbf{x} - \mathbf{x}_0)\|. \end{aligned}$$

As $\mathbf{x} \to \mathbf{x}_0$, the first two terms go to 0 by the differentiability of f and g. The third does so because $|f(\mathbf{x}) - f(\mathbf{x}_0)|/\|\mathbf{x} - \mathbf{x}_0\|$ and $|g(\mathbf{x}) - g(\mathbf{x}_0)|/\|\mathbf{x} - \mathbf{x}_0\|$ are bounded by a constant, say M, on some ball $D_r(\mathbf{x}_0)$. To see this, choose r small enough that $[f(\mathbf{x}) - f(\mathbf{x}_0)]/\|\mathbf{x} - \mathbf{x}_0\|$ is within 1 of $\mathbf{D}f(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)/\|\mathbf{x} - \mathbf{x}_0\|$ if $\|\mathbf{x} - \mathbf{x}_0\| < r$. Then we have

$|f(\mathbf{x}) - f(\mathbf{x}_0)|/\|\mathbf{x}-\mathbf{x}_0\| \le 1 + |\mathbf{D}f(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|/\|\mathbf{x}-\mathbf{x}_0\| = 1 + |\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)|/$ $\|\mathbf{x}-\mathbf{x}_0\| \le 1 + \|\nabla f(\mathbf{x}_0)\|$ by the Cauchy–Schwarz inequality.

The proof of rule (iv) follows from rule (iii) and the special case of the quotient rule, with f identically 1; that is, $\mathbf{D}(1/g)(\mathbf{x}_0) = [-1/g(\mathbf{x}_0)^2]\mathbf{D}g(\mathbf{x}_0)$. To obtain this answer, note that on some small ball $D_r(\mathbf{x}_0)$, $g(\mathbf{x}) > m > 0$. Use the triangle inequality and the Schwarz inequality to show that

$$\frac{\left|\dfrac{1}{g(\mathbf{x})} - \dfrac{1}{g(\mathbf{x})} + \dfrac{1}{g(\mathbf{x}_0)^2}\mathbf{D}g(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)\right|}{\|\mathbf{x}-\mathbf{x}_0\|}$$
$$\le \frac{1}{|g(\mathbf{x})|}\frac{1}{|g(\mathbf{x}_0)|}\frac{|g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|}{\|\mathbf{x}-\mathbf{x}_0\|}$$
$$+\frac{|g(\mathbf{x}) - g(\mathbf{x}_0)|}{|g(\mathbf{x})|g(\mathbf{x}_0)^2}\frac{|\mathbf{D}g(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|}{\|\mathbf{x}-\mathbf{x}_0\|}$$
$$\le \frac{1}{m^2}\frac{|g(\mathbf{x}) - g(\mathbf{x}_0) - \mathbf{D}g(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)|}{\|\mathbf{x}-\mathbf{x}_0\|} + \frac{\|\nabla g(\mathbf{x}_0)\|}{m^3}|g(\mathbf{x}) - g(\mathbf{x}_0)|.$$

These last two terms both go to 0, because g is differentiable and continuous.

27. First find formula for $(\partial/\partial x)(F(x,x))$, using the chain rule. Let $F(x,z) = \int_0^x f(z,y)\,dy$ and use the fundamental theorem of calculus.

29. By Exercise 26 and Theorem 10(iii) (Exercise 25), each component of k is differentiable and $\mathbf{D}k_i(\mathbf{x}_0) = {}'f(\mathbf{x}_0)\mathbf{D}g_i(\mathbf{x}_0) + g_i(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0)$. Because $[\mathbf{D}g_i(\mathbf{x}_0)]\mathbf{y}$ is the ith component of $[\mathbf{D}g(\mathbf{x}_0)]y$ and $[\mathbf{D}f(\mathbf{x}_0)]\mathbf{y}$ is the number $\nabla f(\mathbf{x}_0)\cdot\mathbf{y}$, we get $[\mathbf{D}k(\mathbf{x}_0)]\mathbf{y} = f(\mathbf{x}_0)[\mathbf{D}g(\mathbf{x}_0)]\mathbf{y} + [\mathbf{D}f(\mathbf{x}_0)]\mathbf{y}[g(\mathbf{x}_0)] = f(x_0)[\mathbf{D}g\,(x_0)]\mathbf{y} + [\nabla f(\mathbf{x}_0)\cdot\mathbf{y}]g(\mathbf{x}_0)$.

Section 2.6

1. $\nabla f(1,1,2)\cdot\mathbf{v} = (4,3,4)\cdot(1/\sqrt{5}, 2/\sqrt{5}, 0) = 2\sqrt{5}$

3. (a) $17e^e/13$ (b) $e/\sqrt{3}$ (c) 0

5. (a) $z + 9x = 6y - 6$ (b) $z + y = \pi/2$ (c) $z = 1$

7. (a) $-\dfrac{1}{3\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (b) $2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ (c) $-\frac{2}{9}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

9. $\mathbf{k}$

11. The graph of f is the level surface $0 = F(x,y,z) = f(x,y) - z$. Therefore, the tangent plane is given by

$$0 = \nabla F(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0)$$
$$= \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1\right)\cdot(x-x_0, y-y_0, z-z_0).$$

Because $z_0 = f(x_0, y_0)$, this is $z = f(x_0, y_0) + (\partial f/\partial x)(x_0, y_0)(x - x_0) + (\partial f/\partial y)(x_0, y_0)(y - y_0)$.

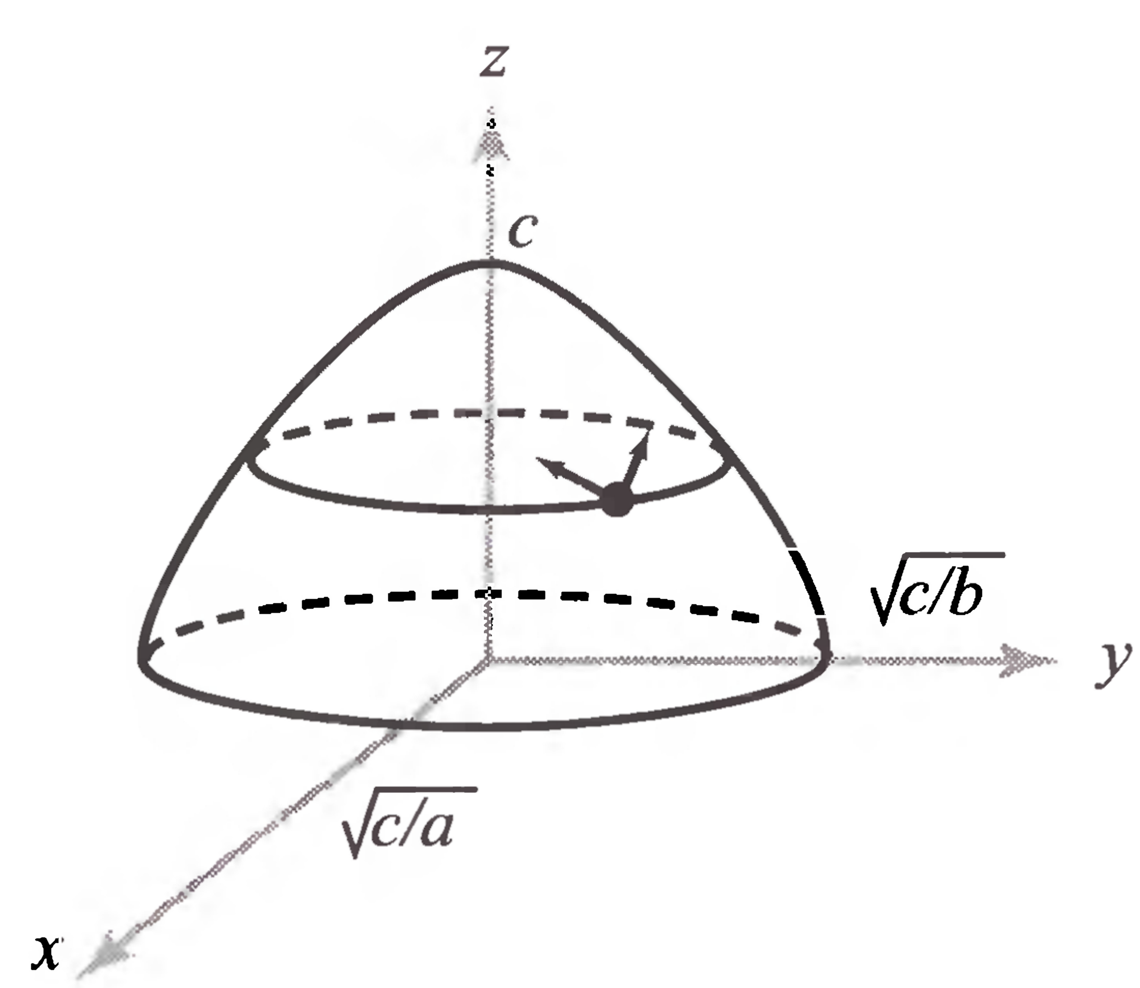
13. (a) $\nabla f = (z + y, z + x, x + y)$, $\mathbf{g}'(t) = (e^t, -\sin t, \cos t)$,
$(f \circ \mathbf{g})'(1) = 2e \cos 1 + \cos^2 1 - \sin^2 1$
(b) $\nabla f = (yze^{xyz}, xze^{xyz}, xye^{xyz})$, $\mathbf{g}'(t) = (6, 6t, 3t^2)$, $(f \circ \mathbf{g})'(1) = 108e^{18}$
(c) $\nabla f = [1 + \log(x^2 + y^2 + z^2)](x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$, $\mathbf{g}' = (e^t, -e^{-t}, 1)$,
$(f \circ \mathbf{g})'(1) = [1 + \log(e^2 + e^{-2} + 1)](e^2 - e^{-2} + 1)$

15. Let $f(x, y, z) = 1/r = (x^2 + y^2 + z^2)^{-1/2}$; $\mathbf{r} = (x, y, z)$. Then we calculate $\nabla f = -(x^2 + y^2 + z^2)^{-3/2}(x, y, z) = -(1/r^3)\mathbf{r}$.

17. $\nabla f = (g'(x), 0)$.

19. $\mathbf{D}f(0, 0, \ldots, 0) = [0, \ldots, 0]$

21. $\mathbf{d}_1 = [-(0.03 + 2by_1)/2a]\mathbf{i} + y_1\mathbf{j}$, $\mathbf{d}_2 = [-(0.03 + 2by_2)/2a]\mathbf{i} + y_2\mathbf{j}$, where y_1 and y_2 are the solutions of $(a^2 + b^2)y^2 + 0.03by + \left(\frac{0.03^2}{4} - a^2\right) = 0$.



23. $$\nabla V = \frac{\lambda}{2\pi\varepsilon_0}\left[\left(\frac{x + x_0}{r_2^2} - \frac{x - x_0}{r_1^2}\right)\mathbf{i} + 2y\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)\mathbf{j}\right]$$

25. Crosses at $(2, 2, 0)$, $\sqrt{5}/10$ seconds later.

Review Exercises for Chapter 2

1. (a) Elliptic paraboloid.
(b) Let $y' = y + 3$ and write $z = xy'$. This is a (shifted) hyperbolic paraboloid.

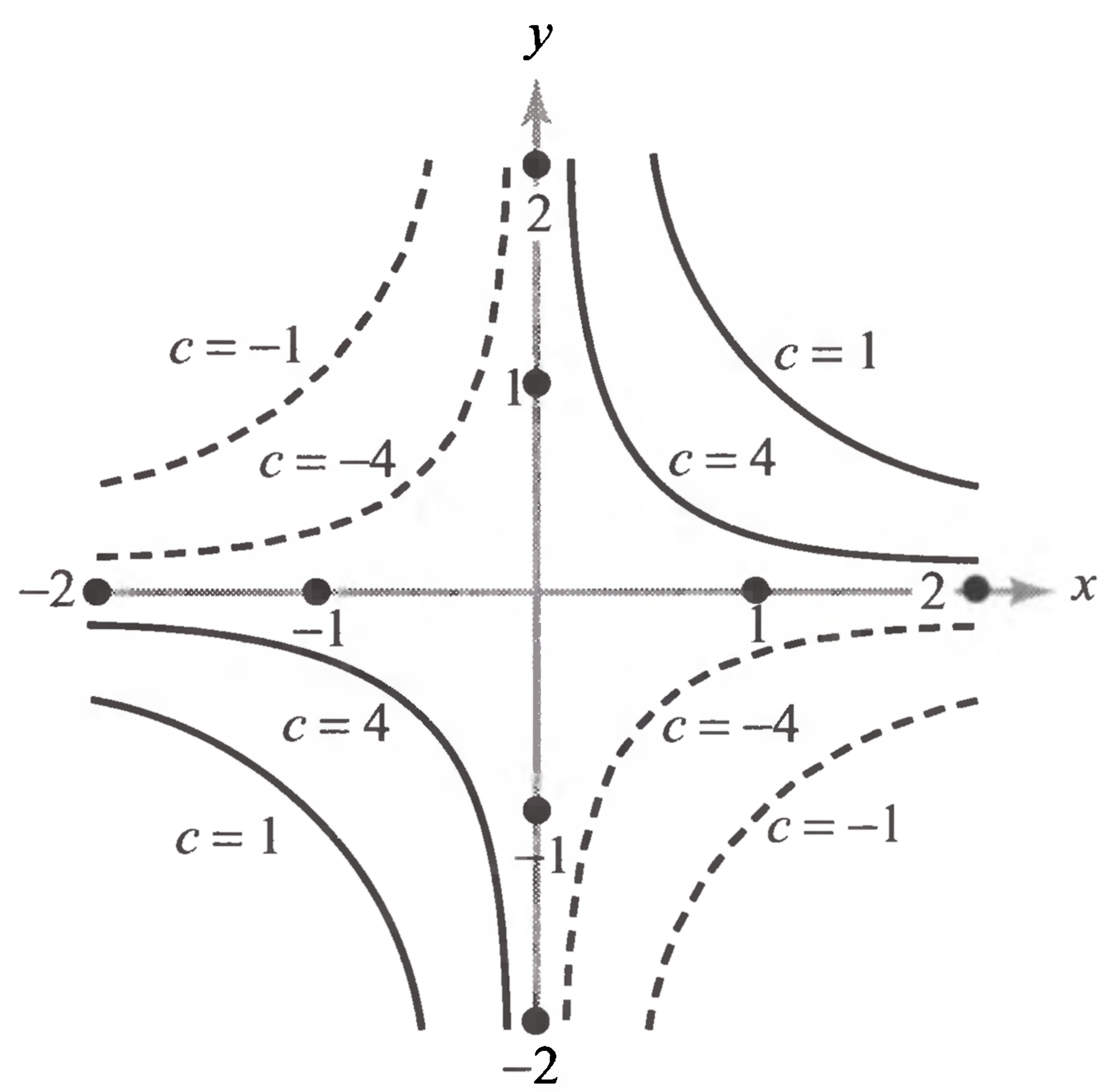
3. (a) $\mathbf{D}f(x, y) = \begin{bmatrix} 2xy & x^2 \\ -ye^{-xy} & -xe^{-xy} \end{bmatrix}$ (c) $\mathbf{D}f(x, y, z)] = [e^x \quad e^y \quad e^z]$

(b) $\mathbf{D}f(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (d) $\mathbf{D}f(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

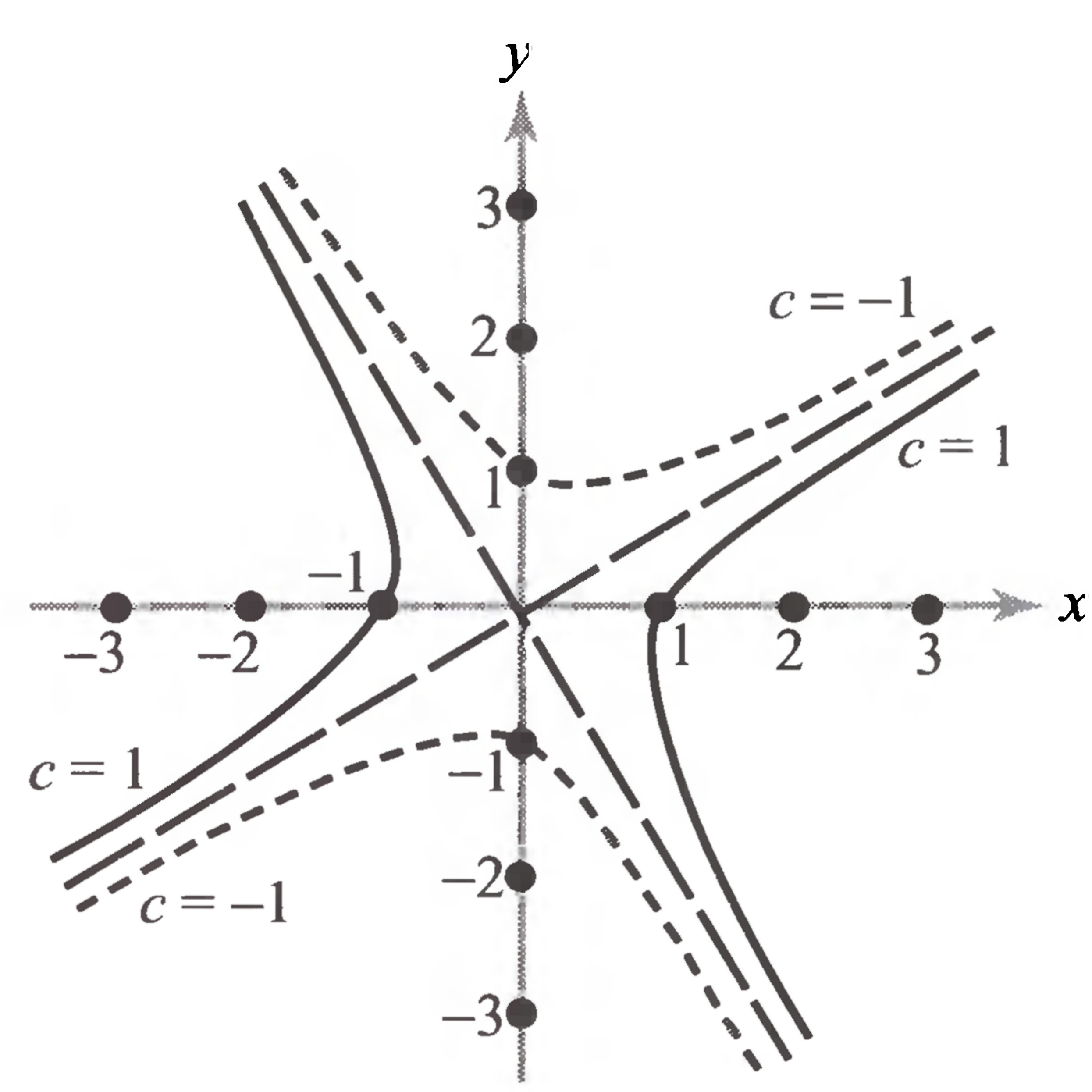
5. The plane tangent to a sphere at (x_0, y_0, z_0) is normal to the line from the center to (x_0, y_0, z_0).

7. (a) $z = x - y + 2$ (d) $10x + 6y - 4z = 6 - \pi$
(b) $z = 4x - 8y - 8$ (e) $2z = \sqrt{2}x + \sqrt{2}y$
(c) $x + y + z = -1$ (f) $x + 2y - z = 2$

9. (a) The level curves are hyperbolas $xy = 1/c$:



(b) $c = x^2 - xy - y^2 = \left(x - \frac{1+\sqrt{5}}{2}y\right)\left(x - \frac{1-\sqrt{5}}{2}y\right)$



11. (a) 0 (b) Limit does not exist.

13. $(1 + 2x^2)\exp(1 + x^2 + y^2)$

15. (a) The line $\mathbf{L}(t) = (x_0, y_0, f(x_0, y_0)) + t(a, b, c)$ lies in the plane $z = f(x_0, y_0)$ if $c = 0$ and is perpendicular to $\nabla f(x_0, y_0)$ if $a(\partial f/\partial x)(x_0, y_0) + b(\partial b/\partial y)(x_0, y_0) = 0$. On $\mathbf{L}$, we have

$$\begin{aligned} f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0) \\ = f(x_0, y_0) + at\left[\frac{\partial f}{\partial x}(x_0, y_0)\right] + bt\left[\frac{\partial f}{\partial y}(x_0, y_0)\right] \\ = f(x_0, y_0) = z \end{aligned}$$

Therefore, $\mathbf{L}$ lies in the tangent plane. An upward unit normal to the tangent plane is $\mathbf{p} = (1 + \|\nabla f\|^{-1/2}(-(\partial f/\partial x)(x_0, y_0), -(\partial f/\partial y)(x_0, y_0), 1)$. Therefore, $\cos\theta = \mathbf{p} \cdot \mathbf{k} = (1 + \|\nabla f\|^2)^{-1/2}$, and $\tan\theta = \sin\theta/\cos\theta = \{\|\nabla f\|^2/(1 + \|\nabla f\|^2)\}^{1/2}/(1 + \|\nabla f\|^2)^{-1/2} = \|\nabla f\|$ as claimed.

(b) The tangent plane contains the horizontal line through $(1, 0, 2)$ perpendicular to $\nabla f(1, 0) = (5, 0)$, that is, parallel to the y axis. It makes an angle of arctan $(\|\nabla f(1, 0)\|) =$ arctan $5 \approx 78.7°$ with respect to the xy plane.

17. $(1/\sqrt{2}, 1/\sqrt{2})$ or $(-1/\sqrt{2}, -1/\sqrt{2})$

19. A unit normal is $(\sqrt{2}/10)(3, 5, 4)$. The tangent plane is $3x + 5y + 4z = 18$.

21. $4\mathbf{i} + 16\mathbf{j}$

23. (a) Because g is the composition $\lambda \mapsto \lambda x \mapsto f(\lambda x)$, the chain rule gives

$$g'(\lambda) = \mathbf{D}f(\lambda\mathbf{x})\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Thus,

$$g'(1) = \mathbf{D}f(\mathbf{x})\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \nabla f(\mathbf{x}) \cdot \mathbf{x}$$

But also $g(\lambda) = \lambda^p f(\mathbf{x})$, so $g'(\lambda) = p\lambda^{p-1} f(\mathbf{x})$ and $g'(1) = pf(\mathbf{x})$.

(b) $p = 1$.

25. Differentiate directly using the chain rule, or use Exercise 23(a) with $p = 0$.

27. (a) If $(x, y) \neq (0, 0)$, then one calculates for (i) that $\partial f/\partial x = (y^3 - yx^2)/(x^2 + y^2)^2$ and $\partial f/\partial y = (x^3 - xy^2)/(x^2 + y^2)^2$. If $x = y = 0$, use the definition directly to find that both partial derivatives are 0. For (ii), if $(x, y) \neq (0, 0)$, then $\partial f/\partial x = 2xy^6/(x^2 + y^4)^2$ and $\partial f/\partial y = (2x^4 y - 2x^2 y^5)/(x^2 + y^4)^2$. The partials at the origin are zero.

(b) The function (i) is not continuous at $(0, 0)$; the function (ii) is differentiable, but the derivative is not continuous.

29. (a) $\sqrt{2}\pi/8$ (b) $-\sin\sqrt{2}$ (c) $-2\sqrt{2}e^{-2}$

31. $(-4e^{-1}, 0)$

33. (a) See Theorem 11.

(b) $g(u) = (\sin 3u)^2 + \cos 8u$
$g'(u) = 6\sin 3u\cos 3u - 8\sin 8u$
$g'(0) = 0$

$\nabla f = (2x, 1)$
$\nabla f(\mathbf{h}(0)) = \nabla f(0, 1) = (0, 1)$
$\mathbf{h}'(u) = (3\cos 3u, -8\sin 8u)$
$g'(0) = \nabla f(\mathbf{h}(0))\cdot\mathbf{h}'(0) = (0, 1)\cdot(3, 0) = 0$

35. $t = \sqrt{14}(-3+\sqrt{359})/70 = (-3+\sqrt{359})/5\sqrt{14}$

37. $\partial z/\partial x = 4(e^{-2x-2y+2xy})(1+y)/(e^{-2x-2y}-e^{2xy})^2$
$\partial z/\partial y = 4(e^{-2y-2x+2xy})(1+x)/(e^{-2x-2y}-e^{2xy})^2$

39. Notice that $y = x^2$, so that if y is constant, x cannot be a variable.

41. $[f'(t)g(t) + f(t)g'(t)]\exp[f(t)g(t)]$

43. $d[f(\mathbf{c}(t))]/dt = 2t/[(1+t^2+2\cos^2 t)(2-2t^2+t^4)]$
$-4t(t^2-1)\ln(1+t^2+2\cos^2 t)/(2-2t^2+t^4)^2$
$-4\cos t\sin t/[(1+t^2+2\cos^2 t)(2-2t^2+t^4)]$

45. Let $x = f(t)$, $y = t$, and use the chain rule to differentiate $u(x, y)$ with respect to t.

47. (a) $n = PV/RT; P = nRT/V; T = PV/nR; V = nRT/P.$

(b) $\partial V/\partial T = nR/P; \partial T/\partial P = V/nR; \partial P/\partial V = -nRT/V^2$. Multiply, remembering that $PV = nRT$.

49. (a) One can solve for any of the variables in terms of the other two.

(b) $\partial T/\partial P = (V-\beta)/R;$
$\partial P/\partial V = -RT/(V-\beta)^2 + 2\alpha/V^3;$
$\partial V/\partial T = R/[(V-\beta)(RT/(V-\beta)^2 - 2\alpha/V^3)]$

(c) Multiply and cancel factors.

51. (a) $(1/\sqrt{2}, 1/\sqrt{2})$

(b) The directional derivative is 0 in the direction

$$(x_0\mathbf{i} + y_0\mathbf{j})/\sqrt{x_0^2+y_0^2}.$$

(c) The level curve through (x_0, y_0) must be tangent to the line through $(0, 0)$ and (x_0, y_0). The level curves are lines or half-lines emanating from the origin.

53. $G(x, y) = x - y$

Chapter 3

Section 3.1

1. $\dfrac{\partial^2 f}{\partial x^2} = 24\dfrac{x^3y - xy^3}{(x^2+y^2)^4}, \dfrac{\partial^2 f}{\partial y^2} = 24\dfrac{-x^3y + xy^3}{(x^2+y^2)^4},$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{-6x^4 + 36x^2y^2 - 6y^4}{(x^2+y^2)^4}$$

3. $\dfrac{\partial^2 f}{\partial x^2} = -y^4\cos(xy^2), \dfrac{\partial^2 f}{\partial y^2} = -2x\sin(xy^2) - 4x^2y^2\cos(xy^2),$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\partial x} = -2y\sin(xy^2) - 2xy^3\cos(xy^2)$$

5. $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2(\cos^2 x + e^{-y})\cos 2x + 2\sin^2 2x}{(\cos^2 x + e^{-y})^3},$

$$\frac{\partial^2 f}{\partial y^2} = \frac{e^{-y} - \cos^2 x}{e^y(\cos^2 x + e^{-y})^3}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{2\sin 2x}{e^y(\cos^2 x + e^{-y})^3}$$

7. (a) $\partial^2 z/\partial x^2 = 6, \partial^2 z/\partial y^2 = 4,$ $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x = 0$ (b) $\partial^2 z/\partial x^2 = 0, \partial^2 z/\partial y^2 = 4x/3y^3,$ $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x = -2/3y^2$

9. $f_{xy} = 2x + 2y, f_{yz} = 2z, f_{zx} = 0, f_{xyz} = 0$

11. Because f and $\partial f/\partial z$ are both of class C^2, we have

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z} = \frac{\partial^2}{\partial x\,\partial y}\frac{\partial f}{\partial z} = \frac{\partial^2}{\partial y\,\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x\,\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial z\,\partial x}\right) = \frac{\partial^3 f}{\partial y\,\partial z\,\partial x}.$$

13. $f_{xzw} = f_{zwx} = e^{xyz}[2xy\cos(xw) + x^2y^2z\cos(xw) - x^2yw\sin(xw)]$

15. (a) $\dfrac{\partial f}{\partial x} = \arctan\dfrac{x}{y} + \dfrac{xy}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{-x^2}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2y^3}{(x^2+y^2)^2}, \frac{\partial^2 f}{\partial y^2} = \frac{2x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{-2xy^2}{(x^2+y^2)^2}$$

(b) $\dfrac{\partial f}{\partial x} = \dfrac{-x \sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}, \dfrac{\partial f}{\partial y} = \dfrac{-y \sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{x^2 \sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}} - \frac{x^2 \cos\sqrt{x^2+y^2}}{x^2+y^2} - \frac{\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{1/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{y^2 \sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}} - \frac{y^2 \cos\sqrt{x^2+y^2}}{x^2+y^2} - \frac{\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{1/2}}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = xy\left[\frac{\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}} - \frac{\cos\sqrt{x^2+y^2}}{x^2+y^2}\right]$$

(c) $\dfrac{\partial f}{\partial x} = -2x\exp(-x^2-y^2), \dfrac{\partial f}{\partial y} = -2y\exp(-x^2-y^2),$

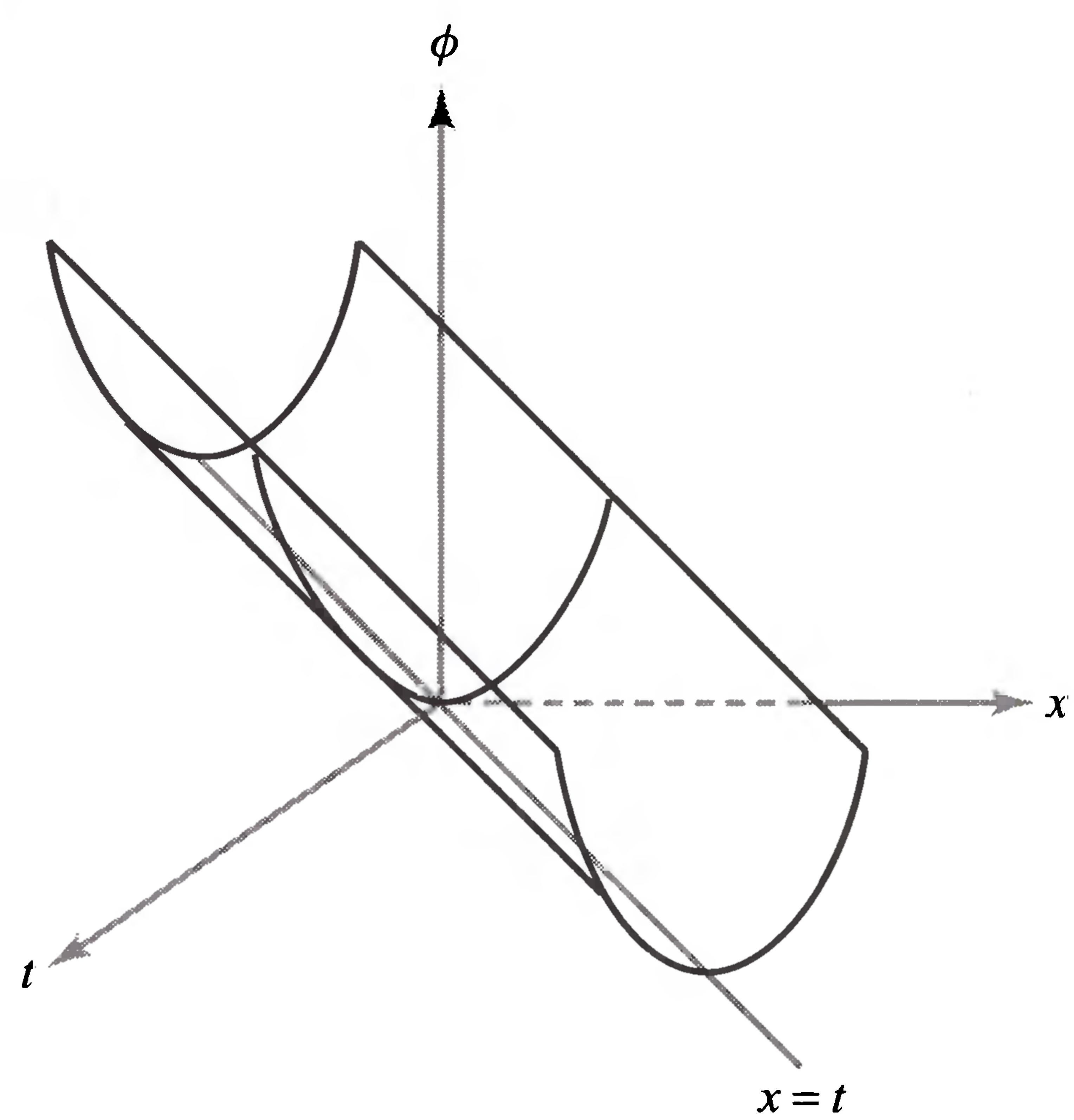
$$\frac{\partial^2 f}{\partial x^2} = (4x^2-2)\exp(-x^2-y^2), \frac{\partial^2 f}{\partial y^2} = (4y^2-2)\exp(-x^2-y^2),$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = 4xy\exp(-x^2-y^2)$$

17. $\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{dx}{dt}\right)^2 + 2\dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{dx}{dt}\dfrac{dy}{dt} + \dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{dy}{dt}\right)^2 + \dfrac{\partial f}{\partial x}\dfrac{d^2x}{dt^2} + \dfrac{\partial f}{\partial y}\dfrac{d^2y}{dt^2},$
where $\mathbf{c}(t) = (x(t), y(t))$

19. Evaluate the derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ and add.

21. (a) Evaluate the derivatives and compare.
(b)



23. $V = -GmM/r = -GmM(x^2+y^2+z^2)^{-1/2}$. Check that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = GmM(x^2+y^2+z^2)^{-3/2}[3 - 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-1}] = 0$$

Section 3.2

1. $f(h_1, h_2) = h_1^2 + 2h_1h_2 + h_2^2$ $\quad [R_2(\mathbf{0}, \mathbf{h}) = 0$ in this case$]$

3. $f(h_1, h_2) = 1 + h_1 + h_2 + \dfrac{h_1^2}{2} + h_1h_2 + \dfrac{h_2^2}{2} + R_2(\mathbf{0}, \mathbf{h})$

5. $f(h_1, h_2) = 1 + h_1h_2 + R_2(\mathbf{0}, \mathbf{h})$

7. (a) Show that $|R_k(x, a)| \le AB^{k+1}/(k+1)!$ for constants A, B, and x in a fixed interval $[a, b]$. Prove that $R_k \to 0$ as $k \to \infty$. (Use convergence of the series $\sum c^k/k! = e^c$ and use Taylor's theorem.)

(b) The only possible trouble is at $x = 0$. Use L'Hôpital's rule to show that

$$\lim_{t\to\infty} p(t)e^t = \infty$$

for every polynomial $p(t)$. Using this, establish that $\lim_{x\to 0^+} p(x)e^{-1/x} = 0$ for every rational function $p(x)$, and conclude that $f^{(k)}(0) = 0$ for every k.

(c) $f\colon \mathbb{R}^n \to \mathbb{R}$ is analytic at $\mathbf{x}_0$ if the series

$$f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{x}_0) + \cdots$$
$$+ \frac{1}{k!}\sum_{i_1,\ldots,i_k=1}^{n} h_{i_1}h_{i_2}\cdots h_{i_k}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(\mathbf{x}_0) + \cdots$$

converges to $f(\mathbf{x}_0 + \mathbf{h})$ for all $\mathbf{h} = (h_1, \ldots, h_n)$ in some sufficiently small disk $\|\mathbf{h}\| < \varepsilon$. The function f is analytic if for every $R > 0$ there is a constant M such that $|(\partial^k f/\partial x_{i_1}\cdots\partial x_{i_k})(\mathbf{x})| < M^k$ for each kth-order derivative at every x satisfying $\|\mathbf{x}\| \le R$.

(d) $f(x, y) = 1 + x + y + \frac{1}{2}(x^2 + 2xy + y^2) + \cdots + \dfrac{1}{k!}\displaystyle\sum_{j=0}^{k}\binom{k}{j}x^j y^{k-j} + \cdots$

Section 3.3

1. $(0, 0)$; saddle point.

3. The critical points are on the line $y = -x$; they are local minima, because $f(x, y) = (x + y)^2 \ge 0$, equaling zero only when $x = -y$.

5. $(0, 0)$; saddle point.

7. $(-\frac{1}{4}, -\frac{1}{4})$; local minimum.

9. $(0, 0)$; local maximum. (The tests fail, but use the fact that $\cos z \le 1$.)
$(\sqrt{\pi/2}, \sqrt{\pi/2})$, local minimum
$(0, \sqrt{\pi})$, local minimum.

11. No critical points.

13. (1, 1) is a local minimum.

15. $(0, n\pi)$; critical points, no local maxima or minima.

17. Minimum at (0, 0) and maxima at $(0, \pm 1)$ [and saddles at $(\pm 1, 0)$].

19. (a) $\partial f/\partial x$ and $\partial f/\partial y$ vanish at (0, 0).
(b) Show that $f(g(t)) = 0$ at $t = 0$ and that $f(g(t)) \geq 0$ if $|t| < |b|/3a^2$.
(c) f is negative on the parabola $y = 2x^2$.

21. The critical points are on the line $y = x$ and they are local minima (see Exercise 3).

23. Minimize $S = 2xy + 2yz + 2xz$ with $z = V/xy$, V the constant volume.

25. 40, 40, 40

27. The only critical point is (0, 0, 0). It is a minimum, because

$$f(x, y, z) \geq \frac{x^2 + y^2}{2} + z^2 + xy = \frac{1}{2}(x + y)^2 + z^2 \geq 0.$$

29. $(1, \frac{3}{2})$ is a saddle point; $(5, \frac{27}{2})$ is a local minimum.

31. $\frac{3}{2}$ is the absolute maximum and 0 is the absolute minimum.

33. -2 is the absolute minimum; 2 is the absolute maximum.

35. $(\frac{1}{2}, 4)$ is a local minimum.

37. If $u_n(x, y) = u(x, y) + (1/n)e^x$, then $\nabla^2 u_n = (1/n)e^x > 0$. Thus, u_n is strictly subharmonic and can have its maximum only on ∂D, say, at $\mathbf{p}_n = (x_n, y_n)$. If $(x_0, y_0) \in D$, check that this implies $u(x_n, y_n) > u(x_0, y_0) - e/n$. Thus, there must be a point $\mathbf{q} = (x_\infty, y_\infty)$ on ∂D such that arbitrarily close to $\mathbf{q}$ we can find an (x_n, y_n) for n as large as we like. Conclude from the continuity of u that $u(x_\infty, y_\infty) \geq u(x_0, y_0)$.

39. Follow the methods of Exercise 37.

41. (a) If there were an x_1 with $f(x_1) < f(x_0)$, then the maximum of f on the interval between x_0 and x would be another critical point.
(b) Verify (i) by the second derivative test; for (ii), f goes to $-\infty$ as $y \to \infty$ and $x = -y$.

Section 3.4

1. Maximum at $\sqrt{\frac{2}{3}}(1, -1, 1)$, minimum at $\sqrt{\frac{2}{3}}(-1, 1, -1)$

3. Maximum at $(\sqrt{3}, 0)$, minimum at $(-\sqrt{3}, 0)$

5. Maximum at $(\frac{9}{\sqrt{70}}, \frac{4}{\sqrt{70}})$, minimum at $(-\frac{9}{\sqrt{70}}, -\frac{4}{\sqrt{70}})$

7. The minimum value 4 is attained at (0, 2). Use a geometric picture rather than Lagrange multipliers.

9. (0, 0, 2) is a minimum of f.

11. $\frac{3}{2}$ is the absolute maximum and 0 is the absolute minimum.

13. The diameter should equal the height, $20/\sqrt[3]{2\pi}$ cm.

15. Maximum value $\sqrt{3}$ at $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$ and minimum value $-\sqrt{3}$ at $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

17. Horizontal length is $\sqrt{qA/p}$, vertical length is $\sqrt{pA/q}$.

19. For Exercise 1, the bordered Hessians required are

$$|\bar{H}_2| = \begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix} = 8\lambda(x^2 + y^2),$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{vmatrix} = -16\lambda(x^2 + y^2 + z^2).$$

At $\sqrt{\frac{2}{3}}(1, -1, 1)$ the Lagrange multiplier is $\lambda = \sqrt{6}/4 > 0$, indicating a maximum at $\sqrt{\frac{2}{3}}(1, -1, 1)$, and $\lambda = -\sqrt{6}/4 < 0$ indicates a minimum at $\sqrt{\frac{2}{3}}(-1, 1, -1)$. In Exercise 5, $|\bar{H}| = 24\lambda(4x^2 + 6y^2)$, and so $\lambda = \sqrt{70}/12 > 0$ indicates a maximum at $(9/\sqrt{70}, 4/\sqrt{70})$ and $\lambda = -\sqrt{70}/12 < 0$ indicates a minimum at $(-9/\sqrt{70}, -4/\sqrt{70})$.

21. 11,664 in^3

23. (a) $\nabla f(\mathbf{x}) = A\mathbf{x}$.

(b) S is defined by the constraint function $g(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 1$. Because $\nabla g(\mathbf{x}) = 2\mathbf{x}$ is not $\mathbf{0}$, Theorem 9 applies. At an $\mathbf{x}$ where f is extreme, there is a $\lambda/2$ such that $\nabla f(\mathbf{x}) = (\lambda/2)/\nabla g(\mathbf{x})$. That is, $A\mathbf{x} = \lambda\mathbf{x}$.

25. Minimum is $(-1/\sqrt{2}, 0)$ maximum is $(\frac{1}{4}, \pm\sqrt{7/8})$, local minimum at $(1/\sqrt{2}, 0)$.

27. No critical points; no maximum or minimum

29. $(-1, 0, 1)$

31. The point $(K, L) = (\alpha B/q, (1 - \alpha)B/p)$ optimizes the profit.

Section 3.5

1. Use the special implicit function theorem with $n = 1$. (See Example 1.) Line (i) is given by $0 = (x - x_0, y - y_0) \cdot \nabla F(x_0, y_0) = (x - x_0)(\partial F/\partial x)(x_0, y_0) + (y - y_0)(\partial F/\partial y)(x_0, y_0)$. For line (ii), Theorem 11 gives $dy/dx = -(\partial F/\partial x)/(\partial F/\partial y)$, and so the lines agree and are given by

$$y = y_0 - \frac{(\partial F/\partial x)(x_0, y_0)}{(\partial F/\partial y)(x_0, y_0)}(x - x_0).$$

3. (a) If $x < -\frac{1}{4}$, we can solve for y in terms of x using the quadratic formula.
(b) $\partial F/\partial y = 2y + 1$ is nonzero for $\{y \mid y < -\frac{1}{2}\}$ and $\{y \mid > -\frac{1}{2}\}$. These regions correspond to the upper and lower halves of a horizontal parabola with vertex at $(-\frac{1}{4}, -\frac{1}{2})$ and to the choice of sign in the quadratic formula. The derivative $dy/dx = -3/(2y + 1)$ is negative on the top half of the parabola, positive on the bottom.

5. Let $F(x, y, z) = x^3z^2 - z^3yx$; $\partial F/\partial z = 2x^3z - 3z^2yx \neq 0$ at $(1, 1, 1)$. Near the origin, with $x = y \neq 0$, we get solutions $z = 0$ and $z = x$, and so there is no unique solution. At $(1, 1)$, $\partial z/\partial x = 2$ and $\partial z/\partial y = -1$.

7. With $F_1 = y + x + uv$ and $F_2 = uxy + v$, the determinant in the general implicit function theorem is

$$\begin{vmatrix} \partial F_1/\partial u & \partial F_1/\partial v \\ \partial F_2/\partial u & \partial F_2/\partial v \end{vmatrix} = v - uxy,$$

which is 0 at $(0, 0, 0, 0)$. Thus, the implicit function theorem does not apply. If we try directly, we find that $v = -uxy$, so $x + y = u^2xy$. For a particular choice of (x, y) near $(0, 0)$, either there are no solutions for (u, v) or else there are two.

9. No. $f(x, y) = (-1, 0)$ has infinitely many solutions, namely, $(x, y) = (0, y)$ for any y.

11. (a) $x_0^2 + y_0^2 \neq 0$.
(b) $f'(z) = -z(x + 2y)/(x^2 + y^2)$; $\quad g'(z) = z(y - 2x)/(x^2 + y^2)$.

13. Multiply and equate coefficients to get a_0, a_1, and a_2 as functions of r_1, r_2, and r_3. Then compute the Jacobian determinant $\partial(a_0, a_1, a_2)/\partial(r_1, r_2, r_3) = (r_3 - r_2)(r_1 - r_2)(r_1 - r_3)$. This is not zero if the roots are distinct. Thus, the inverse function theorem shows that the roots may be found as functions of the coefficients in some neighborhood of any point at which the roots are distinct. That is, if the roots r_1, r_2, r_3 of $x^3 + a_2x^2 + a_1x + a_0$ are all different, then there are neighborhoods V of (r_1, r_2, r_3) and W of (a_0, a_1, a_2) such that the roots in V are smooth functions of the coefficients in W.

Review Exercises for Chapter 3

1. (a) Saddle point.
(b) Saddle point for any C.

3. (a) 1 $\quad$ (b) $\sqrt{83}/6$

5. Use the second derivative test; (0, 0) is a local maximum; (−1, 0) is a saddle point; (2, 0) is a local minumum.

7. Saddle points at $(n, 0)$, $n =$ integer.

9. Maximum ≈ 2.618, minimum ≈ 0.382.

11. Maximum 1, minimum cos 1

13. $z = 1/4$

15. $(0, 0, \pm 1)$

17. If $b \geq 2$, the minimum distance is $2\sqrt{b-1}$; if $b \leq 2$, the minimum distance is $|b|$.

19. Not stable.

21. $f(-\frac{3}{2}, -\sqrt{3}/2) = 3\sqrt{3}/4$

23. $x = (20/3)\sqrt[3]{3}; y = 10\sqrt[3]{3}; z = 5\sqrt[3]{3}$

25. The determinant required in the general implicit function theorem is not zero, and so we can solve for u and v; $(\partial u/\partial x)(2, -1) = 13/32$.

27. A new orthonormal basis may be found with respect to which the quadratic form given by the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

takes diagonal form. This change of basis defines new variables ξ and η, which are linear functions of x and y. Manipulations of linear algebra and the chain rule show that $Lv = \lambda(\partial^2 v/\partial \xi^2) + \mu(\partial^2 v/\partial \eta^2)$. The numbers λ and μ are the eigenvalues of A and are positive, because the quadratic form is positive-definite. At a maximum, $\partial v/\partial \xi = \partial v/\partial \eta = 0$. Moreover, $\partial^2 v/\partial \xi^2 \leq 0$ and $\partial^2 v/\partial \eta^2 \leq 0$, because if either were greater than 0, the cross section of the graph in that direction would have a minimum. Then $Lv \leq 0$, thus contradicting strict subharmonicity.

29. Reverse the inequalities in Exercises 27 and 28.

31. The equations for a critical point, $\partial s/\partial m = \partial s/\partial b = 0$, when solved for m and b give $m = (y_1 - y_2)/(x_1 - x_2)$ and $b = (y_2 x_1 - y_1 x_2)/(x_1 - x_2)$. The line $y = mx + b$ then goes through (x_1, y_1) and (x_2, y_2).

33. At a minimum of s, we have $0 = \partial s/\partial b = -2\sum_{i=1}^{n}(y_i - mx_i - b)$.

35. $y = \frac{9}{10}x + \frac{6}{5}$

Chapter 4

Section 4.1

1. $\mathbf{r}'(t) = -(\sin t)\mathbf{i} + 2(\cos 2t)\mathbf{j}$, $r'(0) = 2\mathbf{j}$, $\mathbf{a}(t) = -(\cos t)\mathbf{i} - 4(\sin 2t)\mathbf{j}$, $\mathbf{a}(0) = -\mathbf{i}$, $\mathbf{l}(t) = \mathbf{i} + 2t\mathbf{j}$

3. $\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$, $\mathbf{r}'(0) = \sqrt{2}\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{a}(t) = e^t\mathbf{j} + e^{-t}\mathbf{k}$, $\mathbf{a}(0) = \mathbf{j} + \mathbf{k}$, $\mathbf{l}(t) = \sqrt{2}t\mathbf{i} + (1+t)\mathbf{j} + (1-t)\mathbf{k}$

5. $(e^t - e^{-t}, \cos t - \sin t, -3t^2)$

7. $[-3t^2(2\sin t + \cos t) - t^3(2\cos t - \sin t)]\mathbf{i} + [3t^2(2e^t + e^{-t}) + t^3(2e^t - e^{-t})]\mathbf{j} + [e^t(\cos t - \sin t) - e^{-t}(-\sin t + \cos t)]\mathbf{k}$

9. $m(0, 6, 0)$

11. $-24\pi^2(\cos(2\pi t/5), \sin(2\pi t/5))/25$

13. $\dfrac{d}{dt}(\|\mathbf{v}\|^2) = \dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\dfrac{d\mathbf{v}}{dt} = 2\mathbf{v}\cdot\mathbf{a} = 0$

15. 6129 seconds

17. $\mathbf{c}(t) = \left(\dfrac{t^2}{2}, e^t - 6, \dfrac{t^3}{3} + 1\right)$

19. (a) $\mathbf{c}(t) = (t, e^t)$, $-\infty < t < \infty$. The image of this path is the graph $y = e^x$.
(b) $\mathbf{c}(t) = (\frac{1}{2}\cos t, \sin t)$, $0 \le t \le 2\pi$, an ellipse.
(c) $\mathbf{c}(t) = (at, bt, ct)$
(d) $\mathbf{c}(t) = (\frac{2}{3}\cos t, \frac{1}{2}\sin t)$, $0 \le t \le 2\pi$, an ellipse.

21. $\mathbf{c}(t) \times \mathbf{c}'(t)$ is normal to the plane of the orbit at time t. As in Exercise 20, its derivative is 0, and so the orbital plane is constant.

Section 4.2

1. $2\sqrt{5}\pi$

3. $2(2\sqrt{2} - 1)$

5. $\dfrac{6-\sqrt{3}}{\sqrt{2}} + \dfrac{1}{2}\log\left[\dfrac{2\sqrt{2}+3}{\sqrt{2}+\sqrt{3}}\right]$

7. $2\pi(\sqrt{5} + \sqrt{2})$

9. $3 + \log 2$

11. (a) Because α is strictly increasing, it maps $[a, b]$ one-to-one onto $[\alpha(a), \alpha(b)]$. By definition, $\mathbf{v}$ is the image of $\mathbf{c}$ if and only if there is a t in $[a, b]$ with $\mathbf{c}(t) = \mathbf{v}$. There is one point s in $[\alpha(a), \alpha(b)]$ with $s = \alpha(t)$, so $\mathbf{d}(s) = \mathbf{c}(t) = \mathbf{v}$. Therefore, the image of $\mathbf{c}$ is contained in that of $\mathbf{d}$. Use α^{-1} similarly for the opposite inclusion.

(b)
$$l_{\mathbf{d}} = \int_{\alpha(a)}^{\alpha(b)} \|\mathbf{d}'(s)\|\, ds = \int_{s=\alpha(a)}^{s=\alpha(b)} \|\mathbf{d}'(\alpha(t))\|\alpha'(t)\, dt$$
$$= \int_{t=a}^{t=b} \|\mathbf{d}'(\alpha(t))\alpha'(t)\| dt = \int_a^b \|\mathbf{c}'(t)\|\, dt = l_{\mathbf{c}}.$$

(c) Differentiate $\mathbf{d}$ using the chain rule.

13. (a) $l_{\mathbf{c}} = \int_a^b \|\mathbf{c}'(s)\|\, ds = \int_a^b ds = b - a$

(b) $\mathbf{T}(s) = \mathbf{c}'(s)/\|\mathbf{c}'(s)\| = \mathbf{c}'(s)$, so $\mathbf{T}'(s) = \mathbf{c}''(s)$. Then $k = \|\mathbf{T}'\| = \|\mathbf{c}''(s)\|$.

(c) Show that if $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{R}^3$, $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{w} - (\mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|^2)\mathbf{v}\| \cdot \|\mathbf{v}\|$. Use this to show that if $\rho(t) = (x(t), y(t), z(t))$ is never $(0, 0, 0)$ and $\mathbf{f}(t) = \rho(t)/\|\rho(t)\|$, then

$$\frac{d\mathbf{f}}{dt} = \frac{1}{\|\rho(t)\|}\left[\rho'(t) - \frac{\rho(t) \cdot \rho'(t)}{\|\rho(t)\|^2}\rho(t)\right] \quad \text{and} \quad \frac{d\mathbf{f}}{dt} = \frac{\|\rho(t) \times \rho'(t)\|}{\|\rho(t)\|^2}.$$

With $\rho(t) = \mathbf{c}'(t)$, this gives

$$\mathbf{T}'(t) = \frac{\mathbf{c}''(t)}{\|\mathbf{c}'(t)\|} - \frac{\mathbf{c}'(t) \cdot \mathbf{c}''(t)}{\|\mathbf{c}'(t)\|^3}\mathbf{c}'(t) \quad \text{and} \quad \|\mathbf{T}'(t)\| = \frac{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|}{\|\mathbf{c}'(t)\|^2}.$$

If s is the arc length of $\mathbf{c}$, $ds/dt = \|\mathbf{c}'(t)\|$, and therefore

$$\left\|\frac{d\mathbf{T}}{dt}\right\| = \left\|\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right\| = k\|\mathbf{c}'(t)\|.$$

Thus,

$$k = \frac{1}{\|\mathbf{c}'(t)\|}\frac{d\mathbf{T}}{dt} = \frac{\|\mathbf{c}'(t) \times \mathbf{c}''(t)\|}{\|\mathbf{c}'(t)\|^3}.$$

(This result is useful in Exercise 15.)

(d) $1/\sqrt{2}$

15. (a) Because $\mathbf{c}$ is parametrized by arc length, $\mathbf{T}(s) = \mathbf{c}'(s)$, and $\mathbf{N}(s) = \mathbf{c}''(s)/\|\mathbf{c}''(s)\|$. Use Exercise 13 to show that

$$\frac{d\mathbf{B}}{ds} = \left(\mathbf{c}'' \times \frac{\mathbf{c}''}{\|\mathbf{c}''\|}\right) + \mathbf{c}' \times \left(\frac{\mathbf{c}'''}{\|\mathbf{c}''\|} - \frac{\mathbf{c}'' \cdot \mathbf{c}'''}{\|\mathbf{c}''\|^3}\mathbf{c}''\right)$$

and

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{(\mathbf{c}' \times \mathbf{c}''') \cdot \mathbf{c}''}{\|\mathbf{c}''\|^2} = \frac{(\mathbf{c}' \times \mathbf{c}'') \cdot \mathbf{c}'''}{\|\mathbf{c}''\|^2}.$$

(b) Obtain $\mathbf{T}'(t)$ and $\|\mathbf{T}'(t)\|$ as in Exercise 13. $\mathbf{B}$ is a unit vector in the direction of $\mathbf{c}' \times \mathbf{T}' = (\mathbf{c}' \times \mathbf{c}'')/\|\mathbf{c}'\|$, so $\mathbf{B} = (\mathbf{c}' \times \mathbf{c}'')/\|\mathbf{c}' \times \mathbf{c}''\|$. Use the solution of Exercise 13 with $\rho = \mathbf{c}' \times \mathbf{c}''$ to obtain

$$d\mathbf{B}/dt = (\mathbf{c}' \times \mathbf{c}''')/\|\mathbf{c}' \times \mathbf{c}''\| - \{[(\mathbf{c}' \times \mathbf{c}'') \cdot (\mathbf{c}' \times \mathbf{c}''')]/\|\mathbf{c}' \times \mathbf{c}''\|^3\}(\mathbf{c}' \times \mathbf{c}''),$$

and the values of $\mathbf{T}'$ and $\|\mathbf{T}'\|$ to get

$$\mathbf{N} = (\|\mathbf{c}'\|/\|\mathbf{c}' \times \mathbf{c}''\|)(\mathbf{c}'' - (\mathbf{c}' \times \mathbf{c}'')/\|\mathbf{c}'\|^2).$$

Finally, use the chain rule and the inner product of these to obtain

$$\tau = -\left[\frac{d\mathbf{B}}{ds}(s(t))\right] \cdot \mathbf{N}(s(t)) = -\frac{1}{|ds/dt|}\frac{d\mathbf{B}}{dt} \cdot \mathbf{N} = \frac{(\mathbf{c}' \times \mathbf{c}'') \cdot \mathbf{c}'''}{\|\mathbf{c}' \times \mathbf{c}''\|^2}.$$

(c) $\sqrt{2}/2$

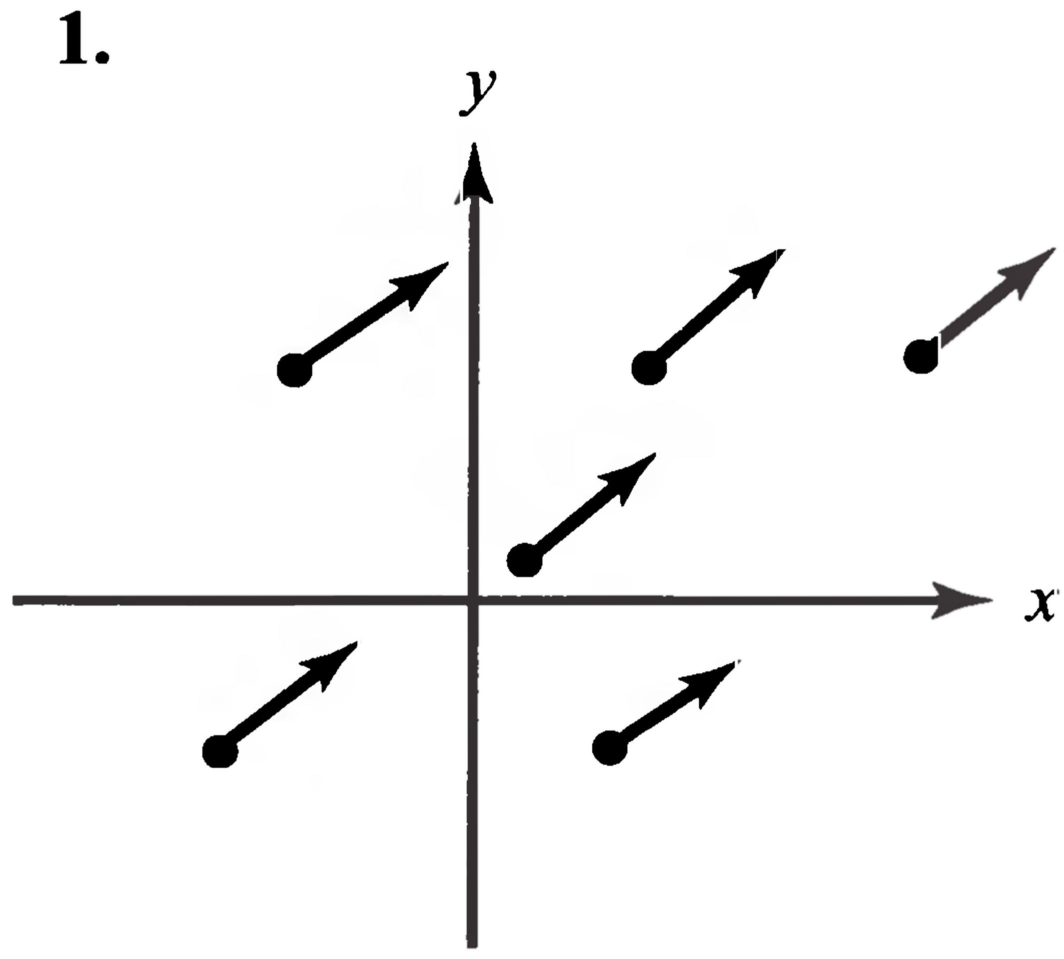
17. (a) $\mathbf{N}$ is defined as $\mathbf{T}'/\|\mathbf{T}'\|$, so $\mathbf{T}' = \|\mathbf{T}'\|\mathbf{N} = k\mathbf{N}$. Because $\mathbf{T} \cdot \mathbf{T}' = 0$, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are an orthonormal basis for $\mathbb{R}^3$. Differentiating $\mathbf{B}(s) \cdot \mathbf{B}(s) = 1$ and $\mathbf{B}(s) \cdot \mathbf{T}(s) = 0$ shows that $\mathbf{B}' \cdot \mathbf{B} = 0$ and $\mathbf{B}' \cdot \mathbf{T} + \mathbf{B} \cdot \mathbf{T}' = 0$. But $\mathbf{T}' \cdot \mathbf{B} = \|\mathbf{T}'\|\mathbf{N} \cdot \mathbf{B} = 0$, so $\mathbf{B}' \cdot \mathbf{T} = 0$ also. Thus, $\mathbf{B}' = (\mathbf{B}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{B}' \cdot \mathbf{N})\mathbf{N} + (\mathbf{B}' \cdot \mathbf{B})\mathbf{B} = (\mathbf{B}' \cdot \mathbf{N})\mathbf{N} = -\tau\mathbf{N}$. Also, $\mathbf{N}' \cdot \mathbf{N} = 0$, because $\mathbf{N} \cdot \mathbf{N} = 1$. Thus, $\mathbf{N}' = (\mathbf{N}' \cdot \mathbf{T})\mathbf{T} + (\mathbf{N}' \cdot \mathbf{B})\mathbf{B}$. But differentiating $\mathbf{N} \cdot \mathbf{T} = 0$ and $\mathbf{N} \cdot \mathbf{B} = 0$ gives $\mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \mathbf{T}' = -k$ and $\mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot \mathbf{B}' = \tau$, and so the middle equation follows.

(b) $\omega = \tau\mathbf{T} + k\mathbf{B}$

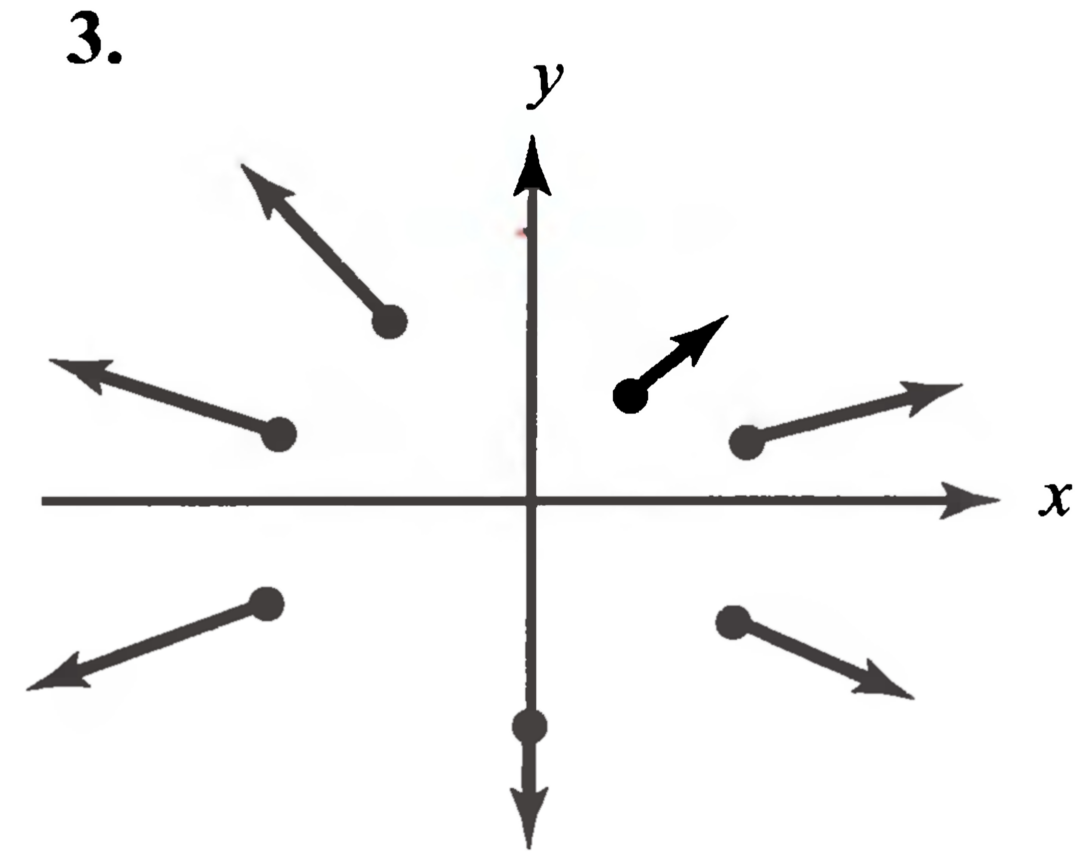
19. Follow the hint in the text.

Section 4.3

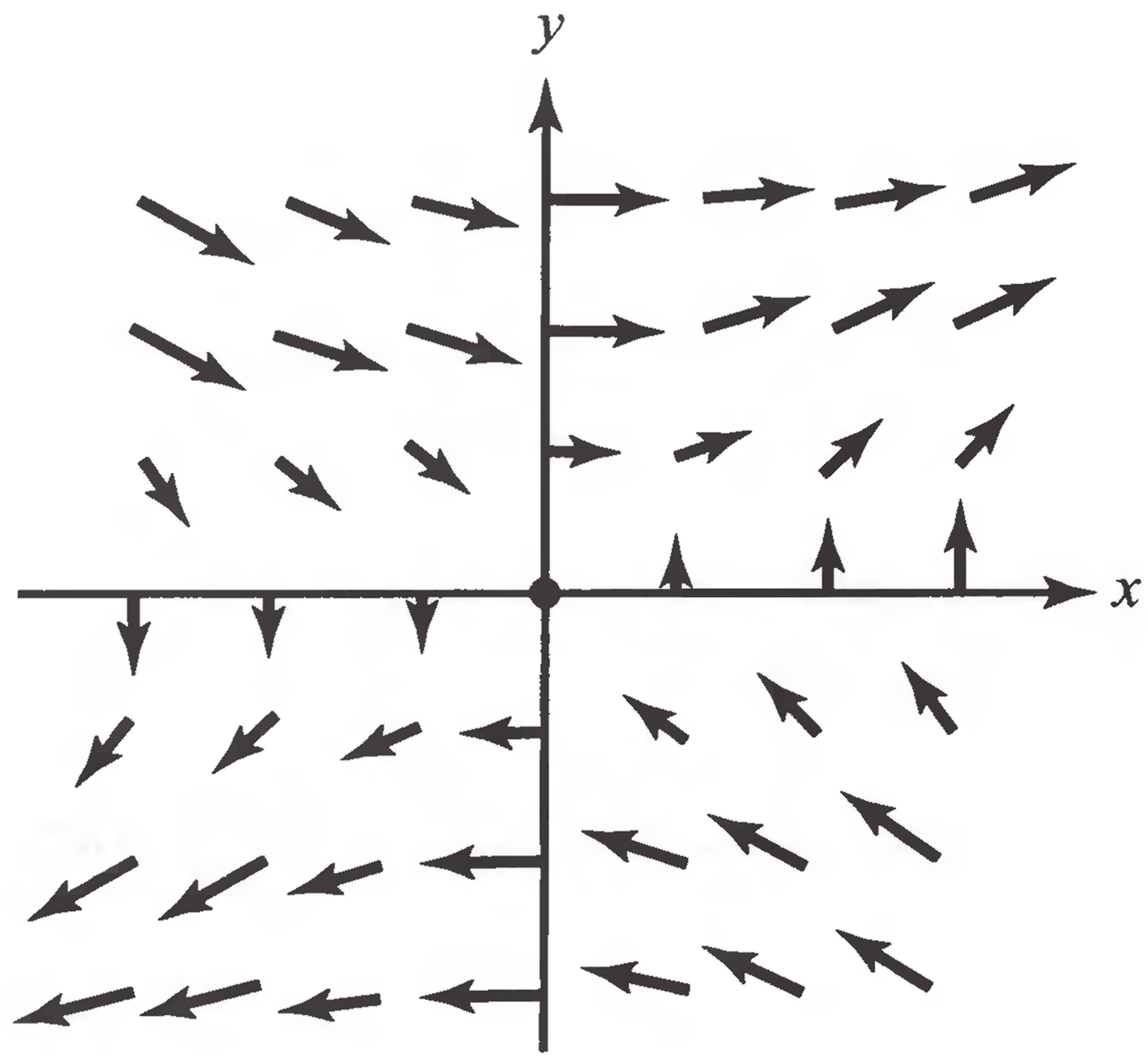
1.



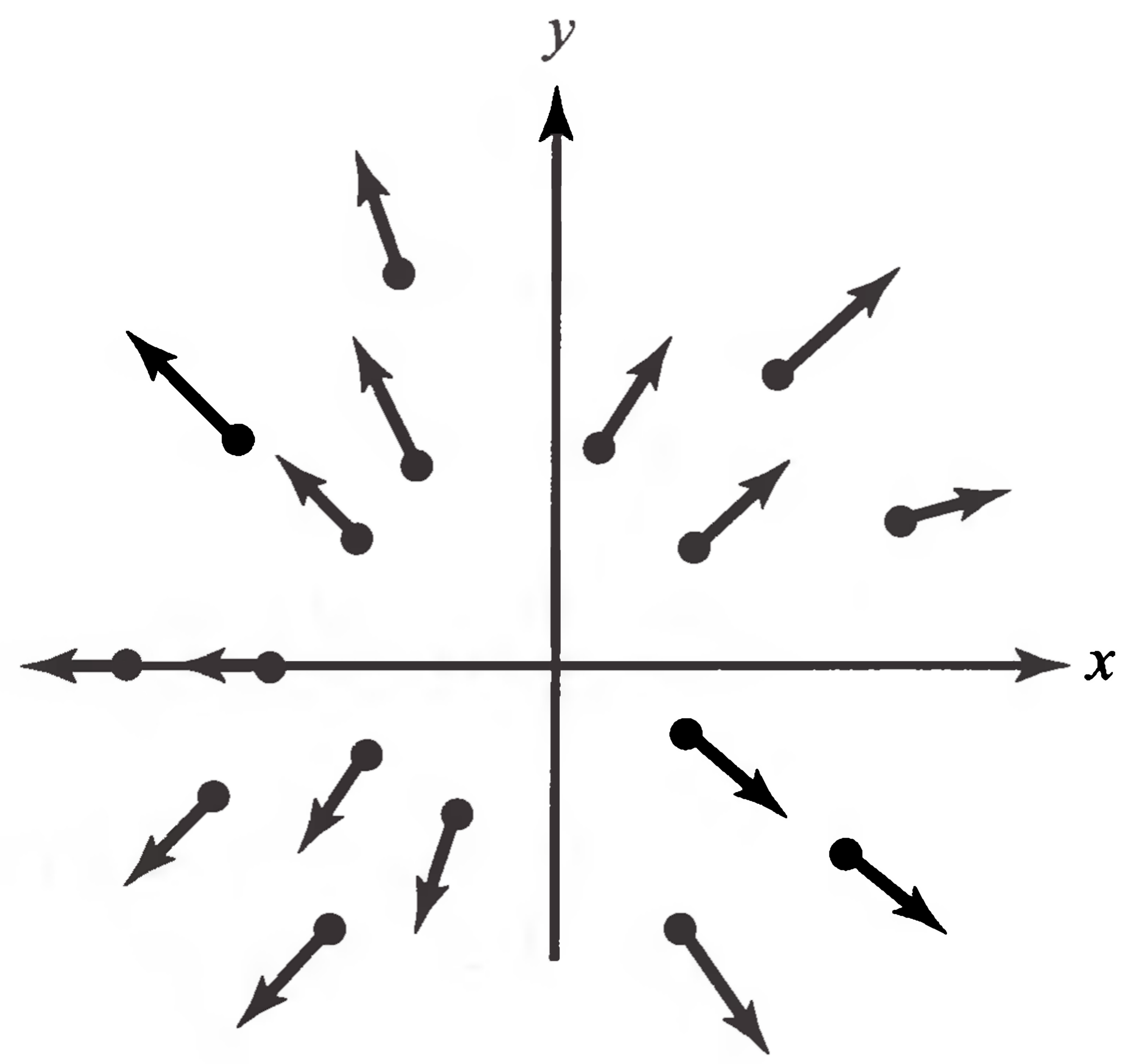
3.



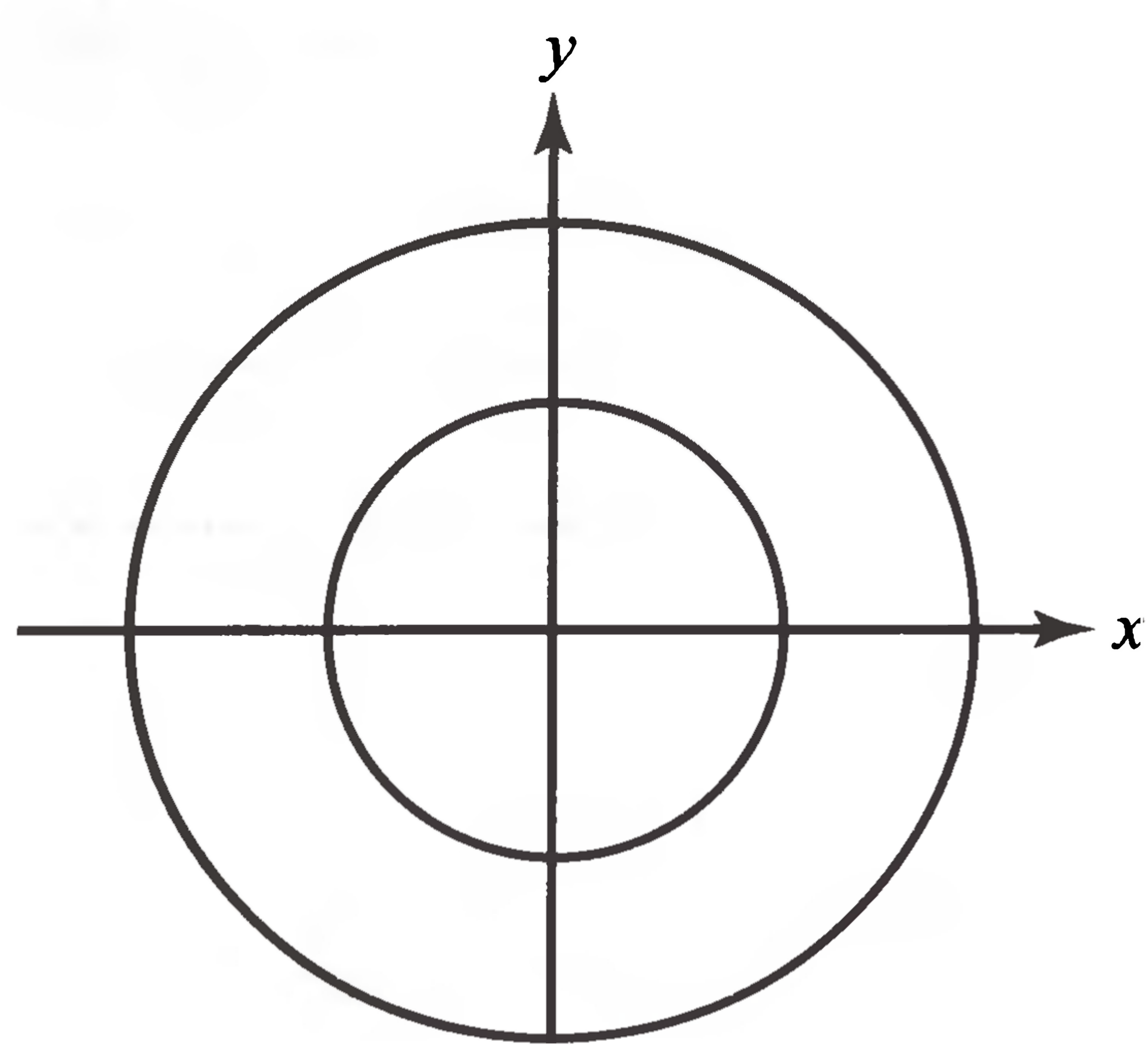
5. $\mathbf{F} = (2y, x)$:



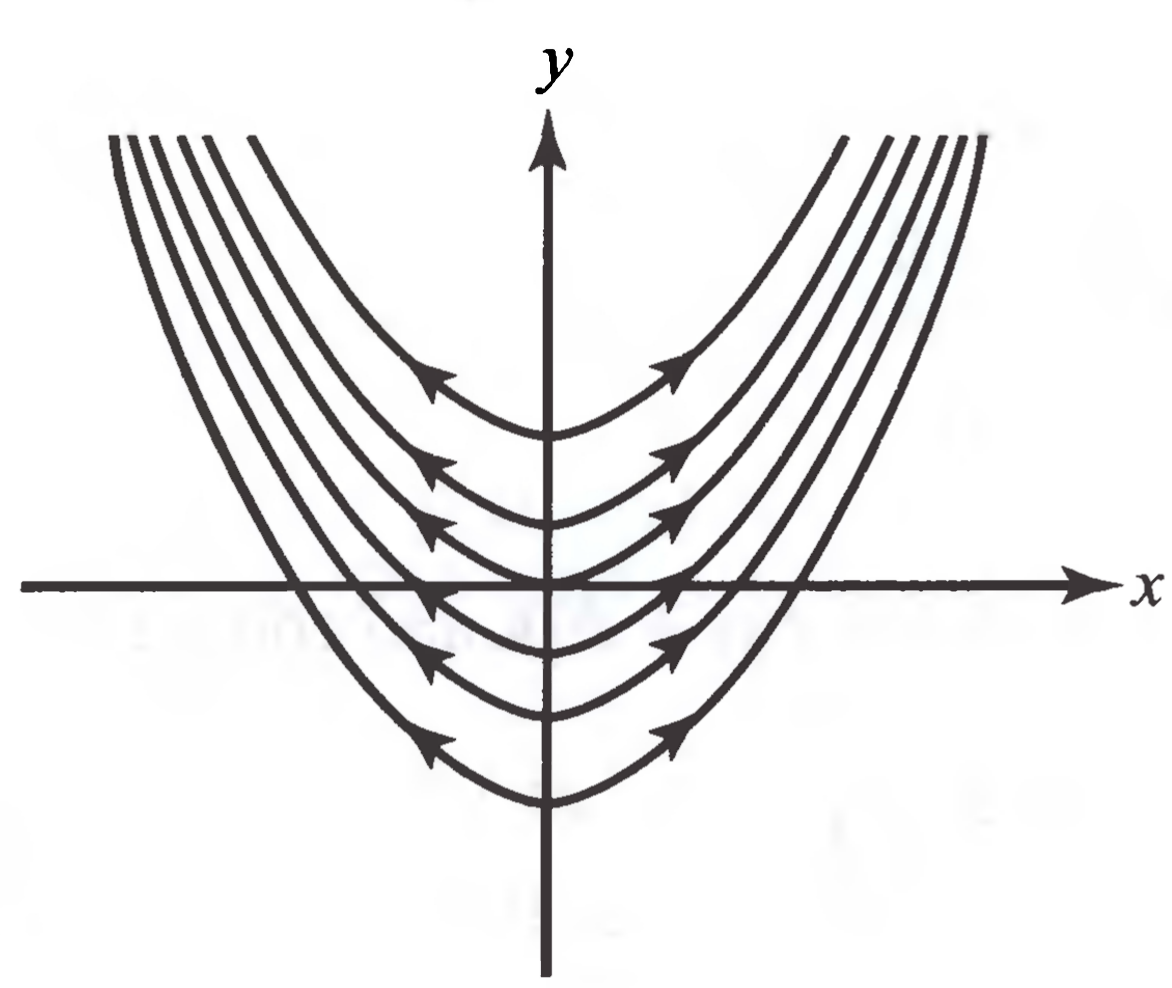
7.



9. The flow lines are concentric circles:



11. The flow lines for $t > 0$:



13. $\mathbf{c}'(t) = (2e^{2t}, 1/t, -1/t^2) = \mathbf{F}(\mathbf{c}(t))$

15. $\mathbf{c}'(t) = (\cos t, -\sin t, e^t) = \mathbf{F}(\mathbf{c}(t))$

17. Compare $\frac{1}{2}mv^2$ for the escape velocity $v_e = \sqrt{2gR_0}$ and the velocity in an orbit of radius R_0 given in Section 4.1. (Ignore the rotation of the earth.)

19. Use the fact that $-\nabla T$ is perpendicular to the surface $T =$ constant.

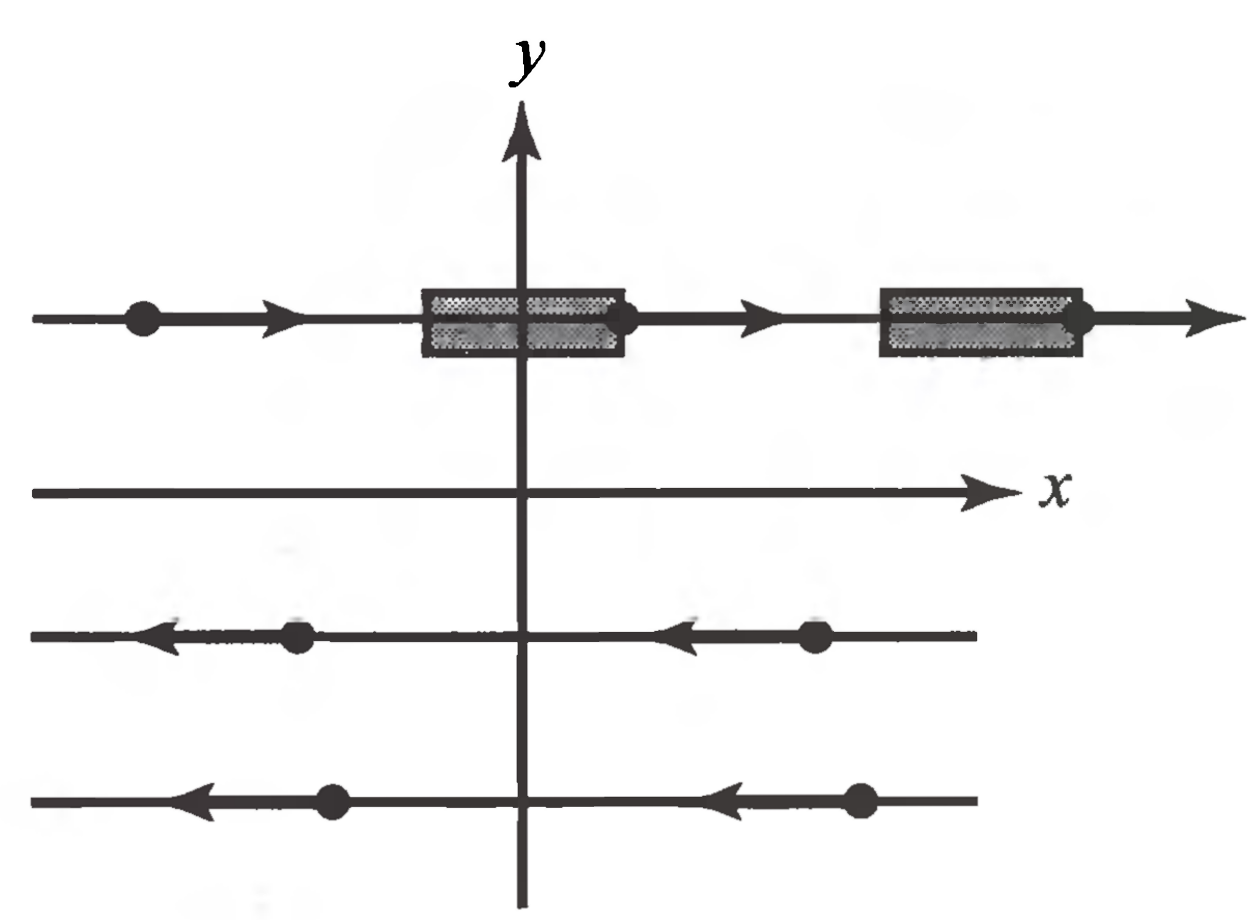
Section 4.4

1. $ye^{xy} - xe^{xy} + ye^{yz}$

3. 3

5. div $\mathbf{V} > 0$ in the first and third quadrants,
div $\mathbf{V} < 0$ in the second and fourth quadrants.

7. $\nabla \cdot \mathbf{F} = 0$; if $\mathbf{F}$ represents a fluid, there is neither expansion nor compression; the area of a small rectangle remains the same.



9. $3x^2 - x^2 \cos(xy)$

11. $y \cos(xy) + x^2 \sin(x^2 y)$

13. $\mathbf{0}$

15. $(10y - 8z)\mathbf{i} + (6z - 10x)\mathbf{j} + (8x - 6y)\mathbf{k}$

17. $-\sin x$

19. x

21. $\nabla \times \nabla f = \mathbf{0}$

23. $\nabla \times \nabla f = \mathbf{0}$

25. $\nabla \times \mathbf{F} \neq \mathbf{0}$

27. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and compute both sides of the identity.

29. (a) $2xy\mathbf{i} + x^2\mathbf{j}$
(b) $(3y^2xz, 4xz - y^3z, 0)$
(c) $(-y^3zx^3, 2x^2y^4z, 2x^3z^2 - 2xy)$
(d) $4x^2yz^2 + x^2$

31. No.

33. Separate each expression into its real and imaginary parts and then treat the resulting quantity as a vector field on $\mathbb{R}^2$. Directly calculate its curl and divergence.

In (a), $\mathbf{F} = (x^2 - y^2)\mathbf{i} - 2xy\mathbf{j}$; in (b), $\mathbf{F} = (x^3 - 3xy^2)\mathbf{i} + (y^3 - 3x^2y)\mathbf{j}$; and in (c), $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j}$. Show that $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = 0$ in each case.

Review Exercises for Chapter 4

1. $\mathbf{v}(1) = (3, -e^{-1}, -\pi/2)$; $\mathbf{a}(1) = (6, e^{-1}, 0)$;
$s(1) = \sqrt{9 + e^{-2} + \frac{\pi^2}{4}}$; $\mathbf{l}(t) = (2, e^{-1}, 0) + (t-1)(3, -e^{-1}, -\pi/2)$

3. $\mathbf{v}(0) = (1, 1, 0)$; $\mathbf{a}(0) = (1, 0, -1)$; $s = \sqrt{2}$; $l(t) = (1, 0, 1) + t(1, 1, 0)$

5. Tangent vector: $\mathbf{v} = -(1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j} + \mathbf{k}$
Acceleration vector: $\mathbf{a} = -(1/\sqrt{2})(\mathbf{i} + \mathbf{j})$

7. $m(2, 0, -1)$

9. $\displaystyle\int_1^4 \sqrt{1 + \tfrac{4}{9}t^{-2/3} + \tfrac{4}{25}t^{-6/5}}\, dt$

11. (a) $\mathbf{v} = (-2t\sin(t^2), 2t\cos(t^2), 0)$; $s = 2t$
(b) $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}, 0\right)$
(c) $\sqrt{5\pi/3}$
(d) $\mathbf{v} = 2\sqrt{5\pi/3}(\sqrt{3}/2, 1/2, 0)$; $s = 2\sqrt{5\pi/3}$
(e) $\left(\dfrac{3}{2} + \dfrac{5\pi}{\sqrt{3}}\right)\Big/\sqrt{5\pi}$

13. $x = 1 + t,\ y = -\dfrac{1}{2} + \dfrac{t}{2},\ z = -\dfrac{2}{3} + \dfrac{t}{3}$

15. Compute $\mathbf{c}'(t)$ and check that it equals $\mathbf{F}(\mathbf{c}(t))$.

17. 9; $\mathbf{0}$

19. 3; $-\mathbf{i} - \mathbf{j} - \mathbf{k}$

21. 0; $-\mathbf{i} - \mathbf{j} - \mathbf{k}$

23. $\nabla f = (ye^{xy} - y\sin xy, xe^{xy} - x\sin xy, 0)$; verify that $\nabla \times \nabla f = 0$ in this case.

25. $\nabla f = (2xe^{x^2} + y^2 \sin xy^2, 2xy\sin xy^2, 0)$; check that $\nabla \times \nabla f = 0$ from this.

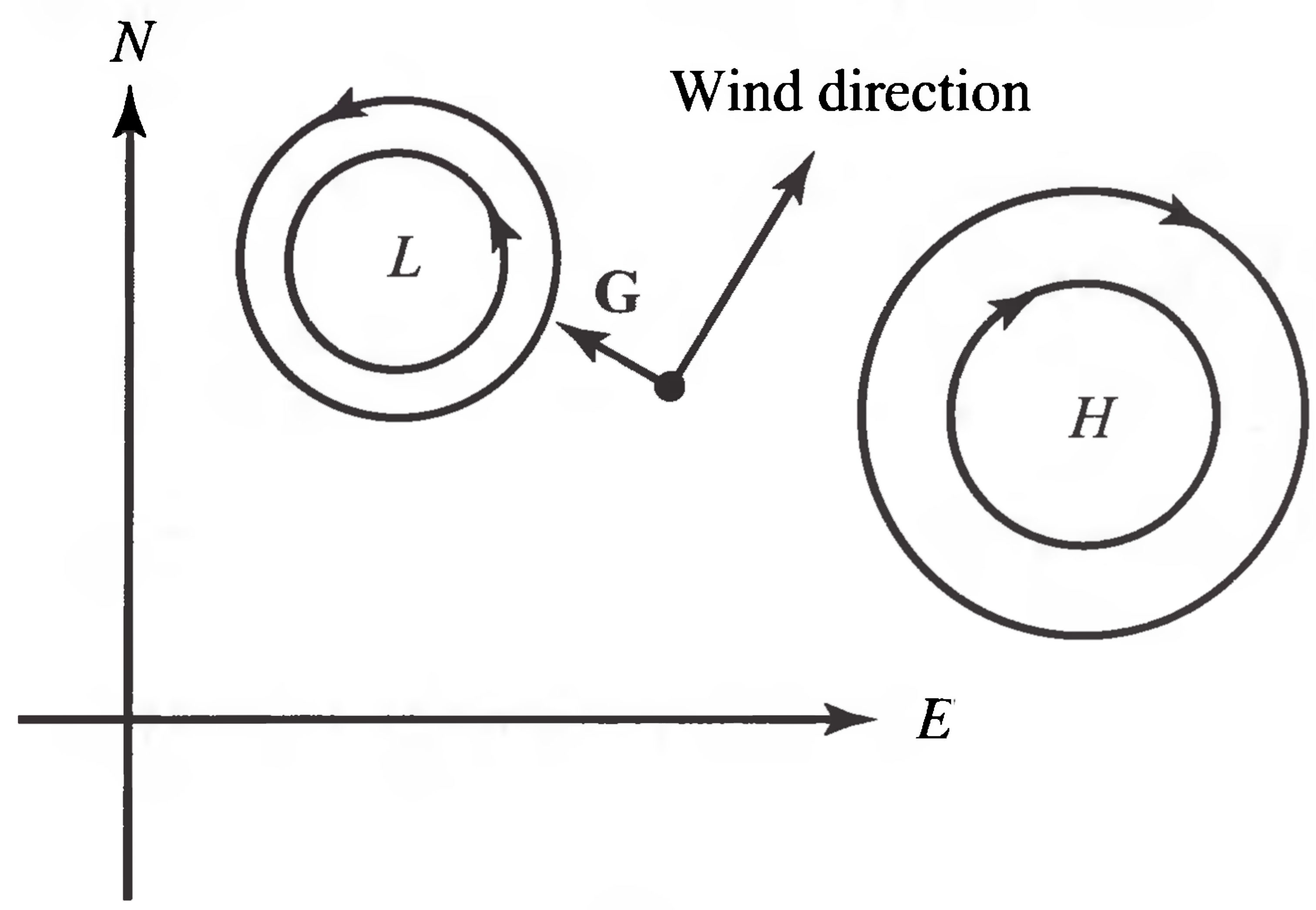
27. (a) $(yz^2, xz^2, 2xyz)$; (b) $(z - y, 0, -x)$
(c) $(2xyz^3 - 3xy^2z^2, 2x^2y^2z - y^2z^3, y^2z^3 - 2x^2yz^2)$

29. $\operatorname{div}\mathbf{F} = 0$; $\operatorname{curl}\mathbf{F} = (0, 0, 2(x^2+y^2)f'(x^2+y^2) + 2f(x^2+y^2))$

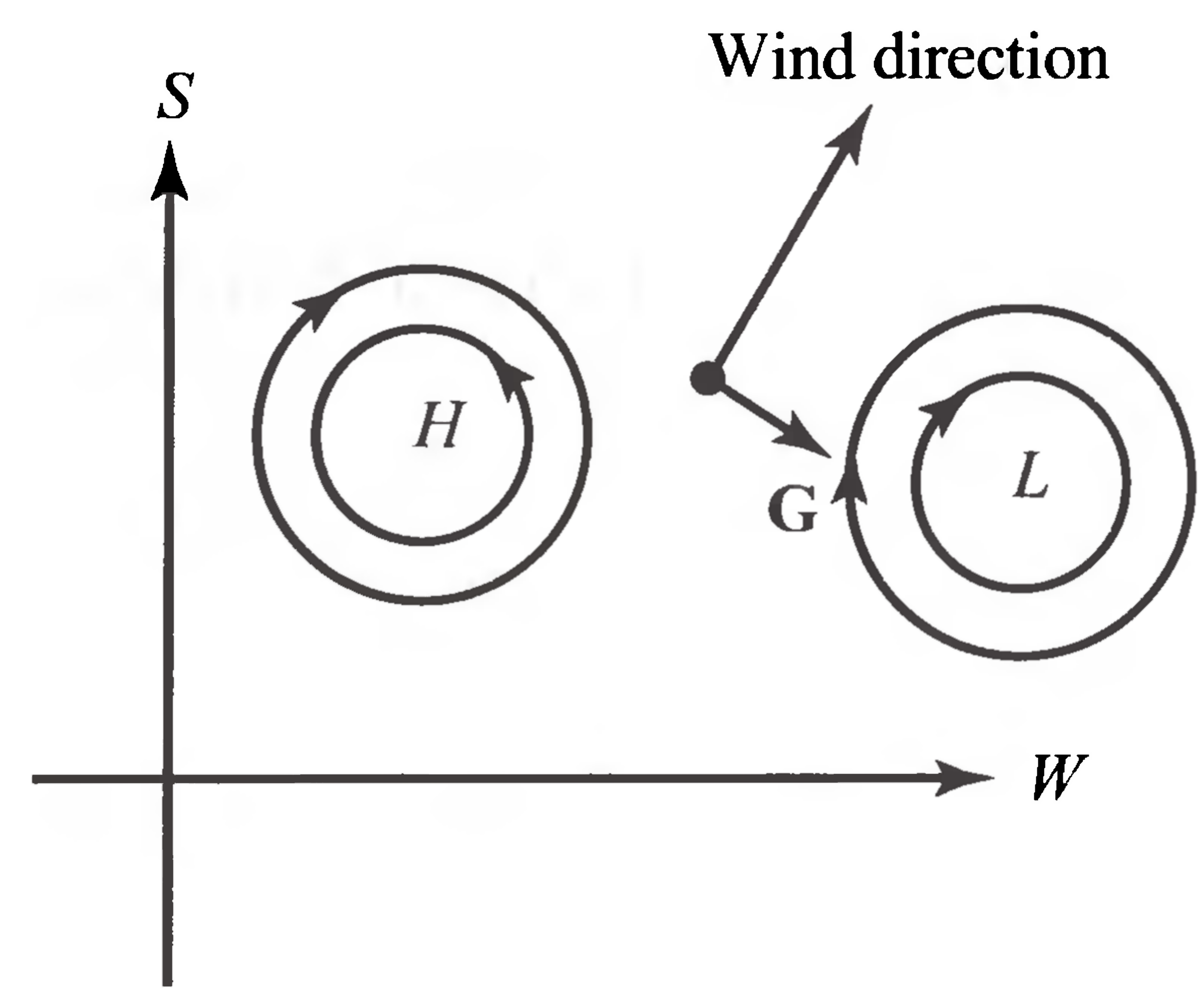
31. (a) A cone about $\mathbf{i}'$ making an angle of $\pi/3$ with $\mathbf{i}'$.
(b) $\nabla g = (3x^2, 5z, 5y + 2z)$

33. (a) $[\partial P/\partial x)^2 + (\partial P/\partial y)^2]^{1/2}$
(b) A small packet of air would obey $\mathbf{F} = m\mathbf{a}$.

(c)



(d)



35. (a) $\dfrac{\sqrt{R^2+\rho^2}}{\rho}(z_0 - z_1)$ (b) $\sqrt{\dfrac{2(R^2+\rho^2)z_0}{g\rho^2}}$

37. 680 miles per hour

Chapter 5

Section 5.1

1. (a) $\frac{13}{15}$ (b) $\pi + \frac{1}{2}$ (c) 1 (d) $\log 2 - \frac{1}{2}$

3. To show that the volumes of the two cylinders are equal, show that their area functions are equal.

5. $2r^3(\tan\theta)/3$

7. $\frac{26}{9}$

9. $(2/\pi)(e^2+1)$

11. $\frac{196}{15}$

Section 5.2

1. (a) $\frac{7}{12}$ (b) $e-2$ (c) $\frac{1}{9}\sin 1$ (d) $2\ln 4 - 2$

3. $1/4$

5. Use Fubini's theorem to write

$$\iint_R [f(x)g(y)]\,dx\,dy = \int_c^d g(y)\left[\int_a^b f(x)\,dx\right]dy,$$

and notice that $\int_a^b f(x)\,dx$ is a constant and so may be pulled out.

7. 11/6

9. Because $\int_0^1 dy = \int_0^1 2y\,dy = 1$, we have $\int_0^1 [\int_0^1 f(x, y)\,dy]\,dx = 1$. In any partition of $R = [0, 1] \times [0, 1]$, each rectangle R_{jk} contains points $\mathbf{c}_{jk}^{(1)}$ with x rational and $\mathbf{c}_{jk}^{(2)}$ with x irrational. If in the regular partition of order n we choose $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(1)}$ in those rectangles with $0 \leq y \leq \frac{1}{2}$ and $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(2)}$ when $y > \frac{1}{2}$, the approximating sums are the same as those for

$$g(x, y) = \begin{cases} 1 & 0 \leq y \leq \frac{1}{2} \\ 2y & \frac{1}{2} < y < 1. \end{cases}$$

Because g is integrable, the approximating sums must converge to $\int_R g\,dA = 7/8$. However, if we had picked all $\mathbf{c}_{ij} = \mathbf{c}_{jk}^{(1)}$, all approximating sums would have the value 1.

11. Fubini's theorem does not apply because the integrand is not continuous nor bounded at (0, 0).

Section 5.3

1. (a) 1/3, both (b) 5/2, both (c) $(e^2 - 1)/4$, both (d) 1/35, both

3. $A = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\,dx = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = r^2[\arcsin 1 - \arcsin(-1)] = \pi r^2$.

5. 28,000 ft^3

7. 0

9. y-simple; $\pi/2$

11. $50\,\pi$

13. $\pi/24$

15. Compute the integral with respect to y first. Split that into integrals over $[-\phi(x), 0]$ and $[0, \phi(x)]$ and change variables in the first integral, or use symmetry.

17. Let $\{R_{ij}\}$ be a partition of a rectangle R containing D and let f be 1 on D. Thus, f^* is 1 on D and 0 on $R \backslash D$. Let $\mathbf{c}_{jk} \in R \backslash D$ if R_{ij} is not wholly contained in D. The approximating Riemann sum is the sum of the areas of those rectangles of the partition that are contained in D.

Section 5.4

1. (a) 1/8 (b) $\pi/4$ (c) 17/12
(d) $G(b) - G(a)$, where $dG/dy = F(y, y) - F(a, y)$ and $\partial F/\partial x = f(x, y)$

3. Note that the maximum value of f on D is e and the minimum value of f on D is $1/e$. Use the ideas in the proof of Theorem 4 to show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint f(x, y)\,dA \leq e.$$

5. The smallest value of $f(x, y) = 1/(x^2 + y^2 + 1)$ on D is $\frac{1}{6}$, at $(1, 2)$, and so

$$\iint_D f(x, y)\,dx\,dy \geq \frac{1}{6} \cdot \text{area } D = 1.$$

The largest value is 1, at $(0, 0)$, and so

$$\iint_D f(x, y)\,dx\,dy \leq 1 \cdot \text{area } D = 6.$$

7. $\dfrac{4}{3}\pi abc$

9. $\pi(20\sqrt{10} - 52)/3$

11. $\sqrt{3}/4$

13. D looks like a slice of pie.

$$\int_0^1 \left[\int_0^x f(x, y)\,dy\right] dx + \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} f(x, y)\,dy\right] dx.$$

15. Use the chain rule and the fundamental theorem of calculus.

Section 5.5

1. $1/3$

3. 10

5. $x^2 + y^2 \leq z \leq \sqrt{x^2 + y^2}, -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2}, -1 \leq y \leq 1$

7. $0 \leq z \leq \sqrt{1 - x^2 - y^2}, -\sqrt{1 - y^2} \leq x \leq \sqrt{1 - y^2}, -1 \leq y \leq 1$

9. $50\pi/\sqrt{6}$

11. $1/2$

13. 0

15. $a^5/20$

17. 0

19. $3/10$

21. $1/6$

23. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} f(x, y, z)\,dz\,dy\,dx$

25. $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} f(x, y, z)\,dz\,dy\,dx$

27. $\iint_D \int_0^{f(x,y)} dz\,dx\,dy = \iint_D f(x, y)\,dx\,dy$

29. Let M_ϵ and m_ϵ be the maximum and minimum of f on B_ϵ. Then we have the inequality $m_\epsilon \operatorname{vol}(B_\epsilon) \leq \iiint_{B_\epsilon} f\,dV \leq M_\epsilon \operatorname{vol}(B_\epsilon)$. Divide by $\operatorname{vol}(B_\epsilon)$, let $\epsilon \to 0$ and use continuity of f.

Review Exercises for Chapter 5

1. 81/2

3. $\frac{1}{4}e^2 - e + \frac{9}{4}$

5. 81/2

7. $\frac{1}{4}e^2 - e + \frac{9}{4}$

9. 7/60

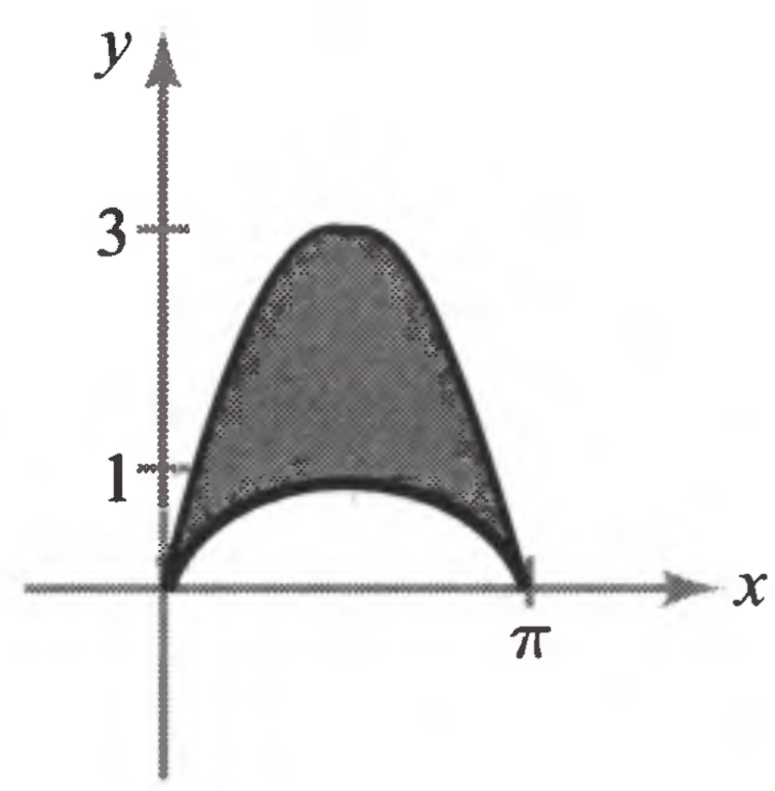
11. 1/2

13. In the notation of Figure 5.3.1,

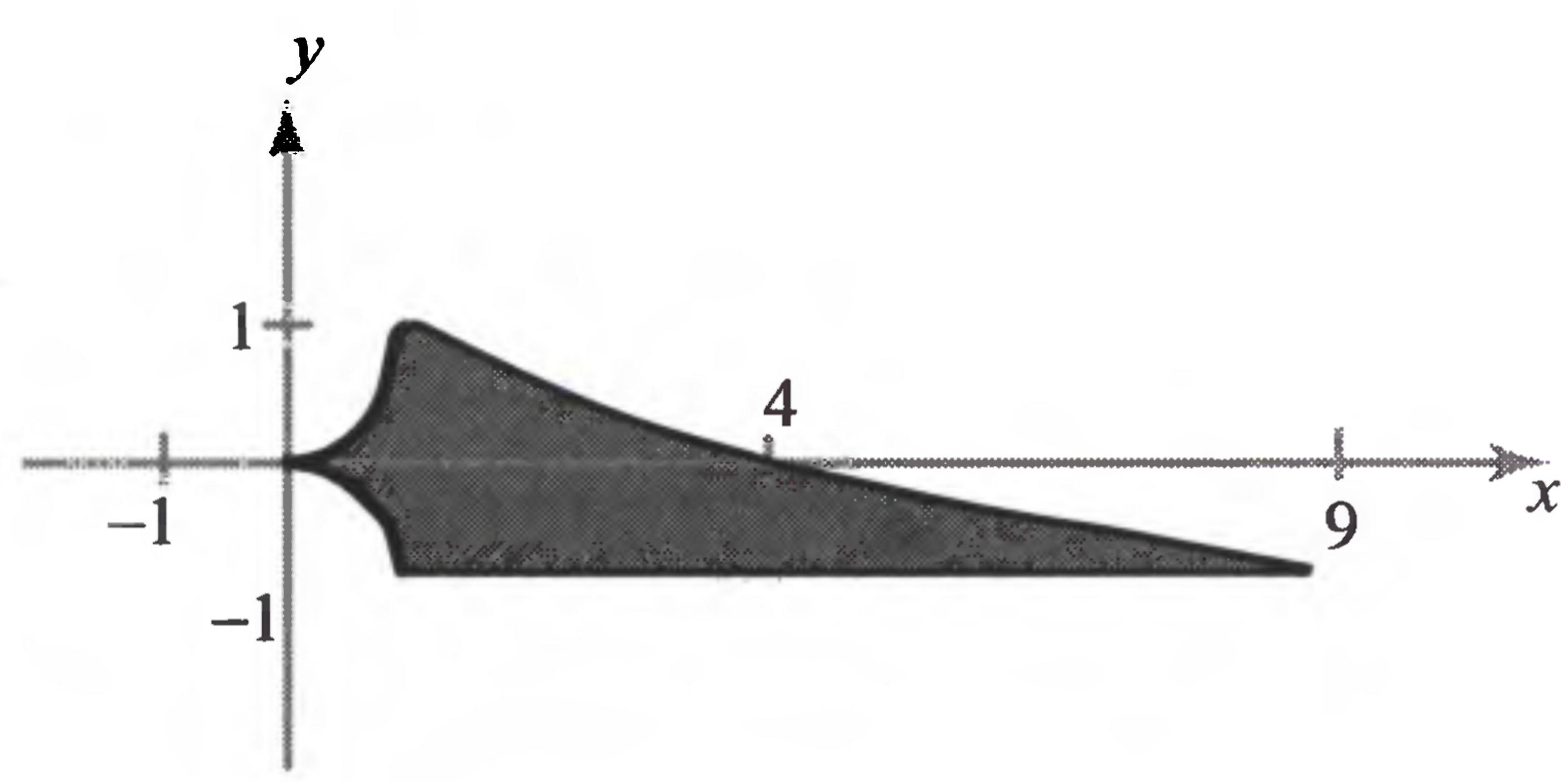
$$\iint_D dx\,dy = \int_a^b [\phi_2(x) - \phi_1(x)]\,dx.$$

15. (a) 0 (b) $\pi/24$ (c) 0

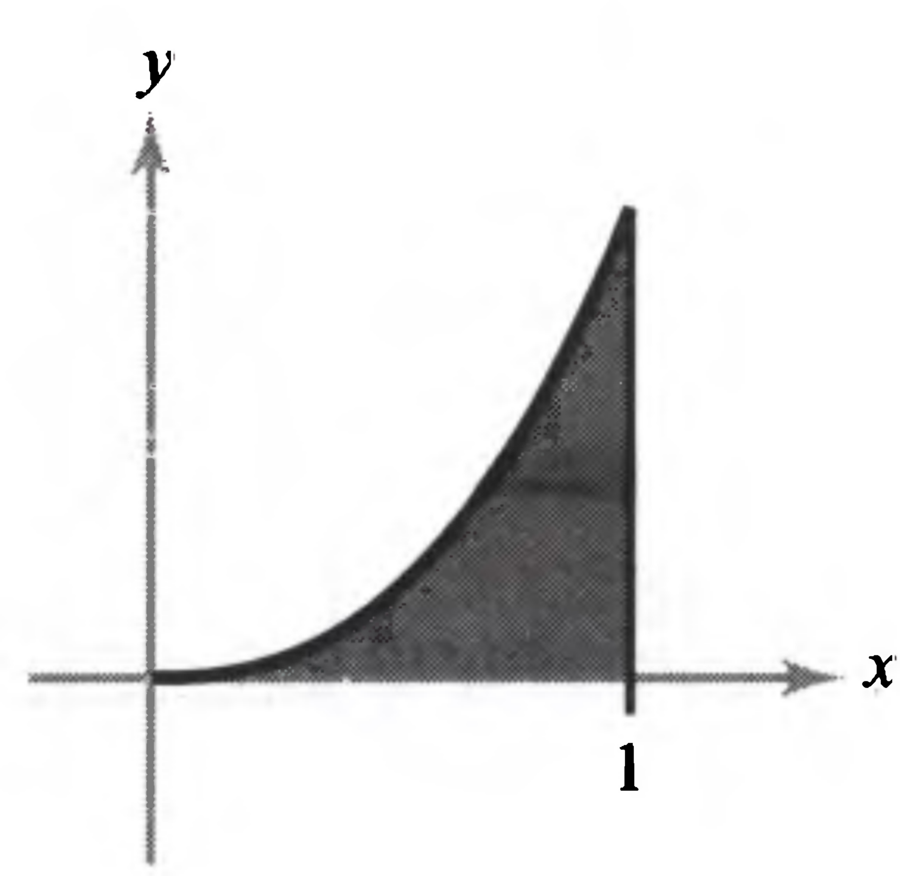
17. y-simple; $2\pi + \pi^2$



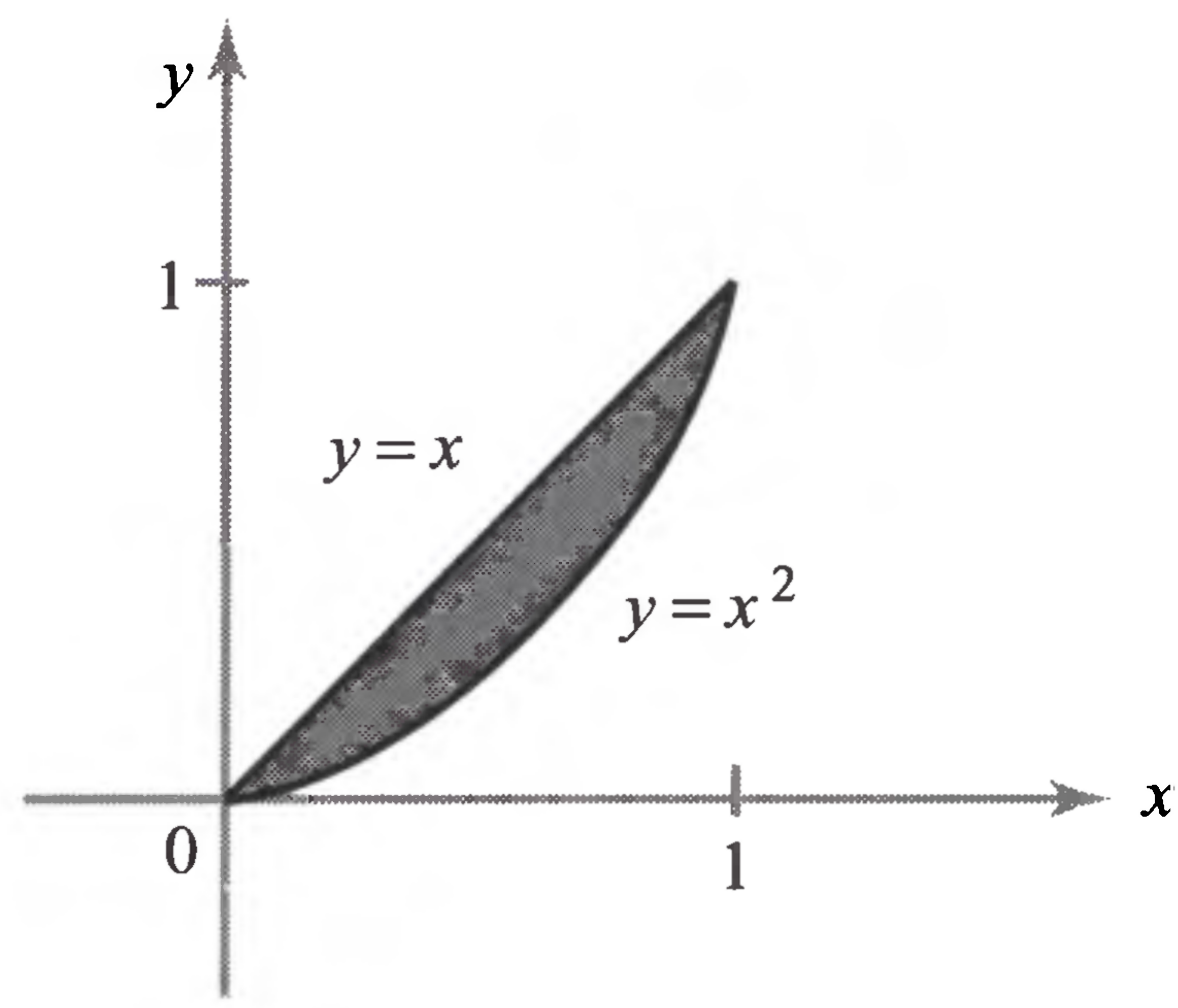
19. x-simple; 73/3



21. y-simple; 33/140



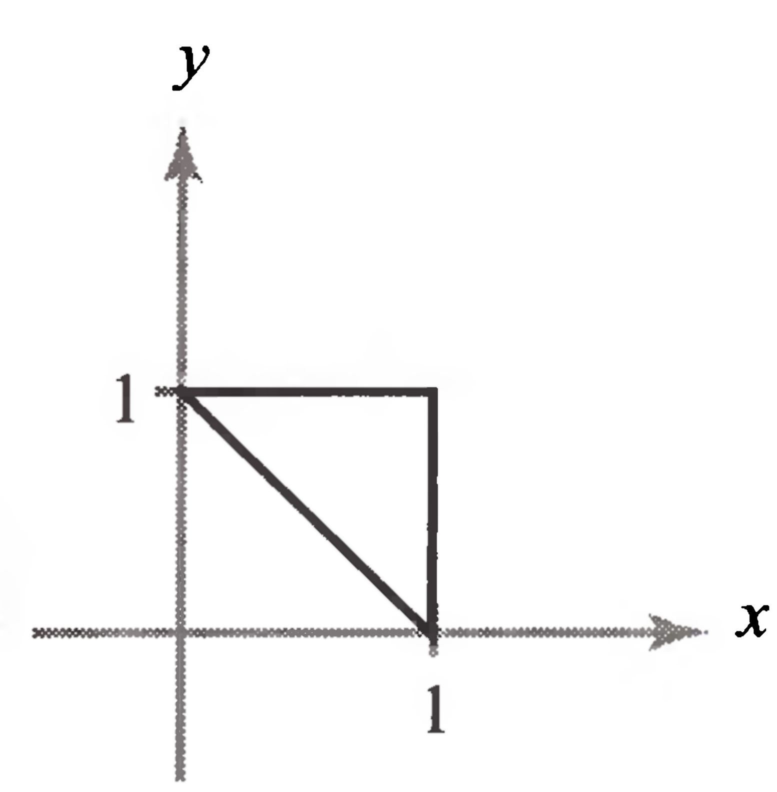
23. y-simple; $71/420$.



25. $1/3$

27. $19/3$

29. $7/12$



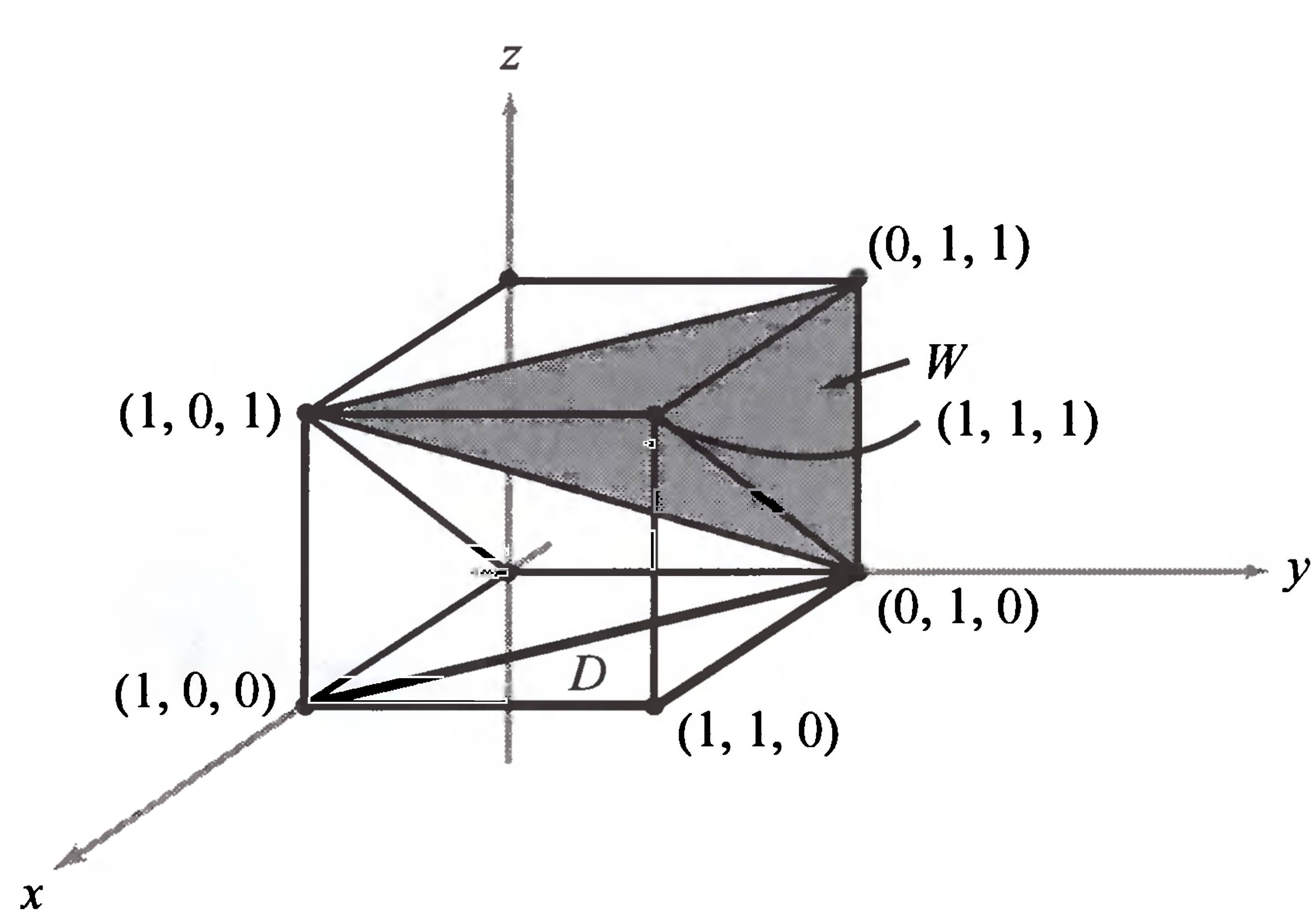
31. The function $f(x, y) = x^2 + y^2 + 1$ lies between 1 and $2^2 + 1 = 5$ on D, and so the integral lies between these values times 4π, the area of D.

33. Interchange the order of integration (the reader should draw a sketch in the (u, t) plane);

$$\int_0^x \int_0^t F(u)\,du\,dt = \int_0^x \int_u^x F(u)\,dt\,du = \int_0^x (x-u)F(u)\,du.$$

35. $\pi/12$

37. The region is the shaded region W in the figure.



The integral in the order $dy\,dx\,dz$, for example, is

$$\int_0^1 \int_z^1 \int_{1-x}^1 f(x, y, z)\,dy\,dx\,dz.$$

Chapter 6

Section 6.1

1. $S =$ the unit disk minus its center.

3. $D = [0, 3] \times [0, 1]$; yes

5. The image is the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$. T is not one-to-one, but becomes so if we eliminate the portion $x^* = 0$.

7. D is the set of (x, y, z) with $x^2 + y^2 + z^2 \leq 1$ (the unit ball). T is not one-to-one, but is one-to-one on $(0, 1] \times (0, \pi) \times (0, 2\pi]$.

9. Showing that T is onto is equivalent in the 2×2 case to showing that the system $ax + by = e$, $cx + dy = f$ can always be solved for x and y, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

When you do this by elimination or by Cramer's rule, the quantity by which you must divide is $\det A$. Thus, if $\det A \neq 0$, the equations can always be solved.

11. Because $\det A \neq 0$, T maps $\mathbb{R}^2$ one-to-one onto $\mathbb{R}^2$. Let T^{-1} be the inverse transformation. Show that T^{-1} has matrix A^{-1} and $\det(A^{-1}) = 1/\det A$, where $\det A \neq 0$. By Exercise 10, $P^* = T^{-1}(P)$ is a parallelogram.

Section 6.2

1. $\pi(e - 1)$

3. D is the region $0 \leq x \leq 4$, $\frac{1}{2}x + 3 \leq y \leq \frac{1}{2}x + 6$. (a) 140 (b) -42

5. D^* is the region $0 \leq u \leq 1, 0 \leq v \leq 2$; $\frac{2}{3}(9 - 2\sqrt{2} - 3\sqrt{3})$.

7. π

9. $\dfrac{64\pi}{5}$

11. $3\pi/2$

13. $\dfrac{5\pi}{2}(e^4 - 1)$

15. $2a^2$

17. $\dfrac{21}{2}\left(e - \dfrac{1}{e}\right)$

19. $100\pi/3$

21. $2\pi[\sqrt{3} - 2\log(1+\sqrt{3}) + \log\sqrt{2}]$

23. $4\pi\log(a/b)$

25. $2\pi[(b^2+1)e^{-b^2} - (a^2+1)e^{-a^2}]$

27. 24 (use the change of variables $x = 3u - v + 1, y = 3u + v$).

29. (a) $\frac{4}{3}\pi abc$ (b) $\frac{4}{5}\pi abc$

31. (a) Check that if $T(u_1, v_1) = T(u_2, v_2)$, then $u_1 = u_2$ and $v_1 = v_2$.
(b) $160/3$

33. $\frac{4}{9}a^{2/3}\iint_{D^*}[f((au^2)^{1/3}, (av^2)^{1/3})u^{-1/3}v^{-1/3}]\,du\,dv$

Section 6.3

1. $[\pi^2 - \sin(\pi^2)]/\pi^3$

3. $\left(\frac{11}{18}, \frac{65}{126}\right)$

5. \$503.64

7. (a) δ, where δ is the (constant) mass density. (b) $37/12$

9. $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

11. $1/4$

13. Letting δ be density, the moment of inertia is $\delta\int_0^k\int_0^{2\pi}\int_0^{a\sec\phi}(\rho^4\sin^3\phi)\,d\rho\,d\theta\,d\phi$.

15. $(1.00\times 10^8)m$

17. (a) The only plane of symmetry for the body of an automobile is the one dividing the left and right sides of the car.

(b) $\bar{z}\cdot\iiint_W \delta(x, y, z)\,dx\,dy\,dz$ is the z coordinate of the center of mass times the mass of W. Rearrangement of the formula for $\bar{z}$ gives the first line of the equation. The next step is justified by the additivity property of integrals. By symmetry, we can replace z by $-z$ and integrate in the region above the xy plane. Finally, we can factor the minus sign outside the second integral, and because $\delta(x, y, z) = \delta(u, v, -w)$, we are subtracting the second integral from itself. Thus, the answer is 0.

(c) In part (b), we showed that $\bar{z}$ times the mass of W is 0. Because the mass must be positive, $\bar{z}$ must be 0.

(d) By part (c), the center of mass must lie in both planes.

19. $V = -(4.71\times 10^{19})Gm/R \approx -(3.04\times 10^9)m/R$, where m is the mass of a test particle at distance R from the planet's center.

Section 6.4

1. 4

3. 3/16

5. (a) 3π (b) $\lambda < 1$

7. Integration of $\iint e^{-xy}\,dx\,dy$ with respect to x first and then y gives log 2. Reversing the order gives the integral on the left side of the equality stated in the exercise.

9. Integrate over $[\varepsilon, 1] \times [\varepsilon, 1]$ and let $\varepsilon \to 0$ to show the improper integral exists and equals 2 log 2.

11. $\dfrac{2\pi}{9}[(1+a^3)^{3/2} - a^{9/2} - 1]$

13. Use the fact that

$$\frac{\sin^2(x-y)}{\sqrt{1-x^2-y^2}} \le \frac{1}{\sqrt{1-x^2-y^2}}.$$

15. Use the fact that $e^{x^2+y^2}/(x-y) \ge 1/(x-y)$ on the given region.

17. Each integral equals 1/4, and Theorem 3 (Fubini's theorem) does apply.

Review Exercises for Chapter 6

1. (a) $T\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}2u+v\\2v\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$
(b) $\iint_P f(x,y)\,dx\,dy = 4\iint_S f(2u+v, 2v)\,du\,dv$

3. 3 (Use the change of variables $u = x^2 - y^2$, $v = xy$.)

5. $\frac{1}{3}\pi(4\sqrt{2} - \frac{7}{2})$

7. $(5\pi/2)\sqrt{15}$

9. $abc/6$

11. Cut with the planes $x + y + z = \sqrt[3]{k/n}$, $1 \le k \le n-1$, k an integer.

13. $(25 + 10\sqrt{5})\pi/3$

15. $(e - e^{-1})/4$ (Use the change of variables $u = y - x$, $v = y + x$.)

17. $(9.92 \times 10^6)\pi$ grams

19. (a) 32
(b) This occurs at the point of the unit sphere $x^2 + y^2 + z^2 = 1$ inscribed in the cube.

21. $(0, 0, 3a^{4/8})$

23. $4\pi \ln(a/b)$

25. $\pi/2$

27. (a) $9/2$ (b) 64π

29. Work the integral with respect to y first on the region $D_{\varepsilon,L} = \{(x, y)|\varepsilon \le x \le L,$ $0 \le y \le x\}$ to obtain $I_{\varepsilon,L} = \iint_{D_{\varepsilon,L}} f\,dx\,dy = \int_\varepsilon^L x^{-3/2}(1 - e^{-x})\,dx$. The integrand is positive, and so $I_{\varepsilon,L}$ increases as $\varepsilon \to 0$ and $L \to \infty$. Bound $1 - e^{-x}$ above by x for $0 < x < 1$ and by 1 for $1 < x < \infty$ to see that $I_{\varepsilon,L}$ remains bounded and so must converge. The improper integral does exist.

31. (a) $1/6$ (b) $16\pi/3$

33. 2π

Chapter 7

Section 7.1

1. $\int_{\mathbf{c}} f(x, y, z)\,ds = \int_I f(x(t), y(t), z(t))\|\mathbf{c}'(t)\|\,dt = \int_0^1 0 \cdot 1\,dt = 0.$

3. (a) 2 (b) $52\sqrt{14}$

5. $-\frac{1}{3}(1 + 1/e^2)^{3/2} + \frac{1}{3}(2^{3/2})$

7. (a) The path follows the straight line from (0, 0) to (1, 1) and back to (0, 0) in the xy plane. Over the path, the graph of f is a straight line from (0, 0, 0) to (1, 1, 1). The integral is the area of the resulting triangle covered twice and equals $\sqrt{2}$.

(b)
$$s(t) = \begin{cases} \sqrt{2}(1 - t^4) & \text{when} \quad -1 \le t \le 0 \\ \sqrt{2}(1 + t^4) & \text{when} \quad 0 < t \le 1. \end{cases}$$

The path is

$$\mathbf{c}(s) = \begin{cases} (1 - s/\sqrt{2})(1, 1) & \text{when} \quad 0 \le s \le \sqrt{2} \\ (s/(\sqrt{2} - 1))(1, 1) & \text{when} \quad \sqrt{2} \le s \le 2\sqrt{2} \end{cases}$$

and $\int_{\mathbf{c}} f\,ds = \sqrt{2}$.

9. $2a/\pi$

11. (a) $[2\sqrt{5} + \log(2 + \sqrt{5})]/4$ (b) $(5\sqrt{5} - 1)/[6\sqrt{5} + 3\log(2 + \sqrt{5})]$

13. The path is a unit circle centered at (0, 0, 0) in the plane $x + y + z = 0$ and so may be parametrized by $\mathbf{c}(\theta) = (\cos\theta)\mathbf{v} + (\sin\theta)\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are orthogonal unit vectors laying in that plane. For example, $\mathbf{v} = (1/\sqrt{2})(-1, 0, 1)$ and $\mathbf{w} = (1/\sqrt{6})(1, -2, 1)$ will do. The total mass is $2\pi/3$ grams.

15. Choosing either $\mathbf{c}(t) = (t^2, 1, 0)$ or $\mathbf{c}(t) = (1, t^2, 0)$, $0 \le t \le 1$ will do.

Section 7.2

1. (a) 3/2 (b) 0 (c) 0 (d) 147

3. 9

5. By the Cauchy–Schwarz inequality, $|\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)| \le \|\mathbf{F}(\mathbf{c}(t))\| \, \|\mathbf{c}'(t)\|$ for every t. Thus,

$$\left| \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} \right| = \left| \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \right| \le \int_a^b |\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)|\, dt$$
$$\le \int_a^b \|\mathbf{F}(\mathbf{c}(t))\| \, \|\mathbf{c}'(t)\|\, dt \le M \int_a^b \|\mathbf{c}'(t)\|\, dt = Ml.$$

7. $\frac{3}{4} - (n-1)/(n+1)$

9. 0

11. The length of $\mathbf{c}$.

13. If $\mathbf{c}'(t)$ is never 0, then the unit vector $\mathbf{T}(t) = \mathbf{c}'(t)/\|\mathbf{c}'(t)\|$ is a continuous function of t and so is a smoothly turning tangent to the curve. The answer is no.

15. 7

17. Use the fact that $\mathbf{F}$ is a gradient to show that the work done is $\frac{1}{R_2} - \frac{1}{R_1}$, independent of the path.

19. (a) $\|\mathbf{c}'(x)\|$

(b) f has a positive derivative; it is one-to-one and onto $[0, L]$ by the mean-value and intermediate-value theorems. It has a differentiable inverse by the inverse function theorem.

(c) $g'(s) = 1/\|\mathbf{c}'(x)\|$ where $s = f(x)$.

(d) By the chain rule, $\mathbf{b}'(s) = \mathbf{c}'(x) \cdot g'(s)$, which has unit length by part (c).

Section 7.3

1. $z = 2(y-1) + 1$

3. $18(z-1) - 4(y+2) - (x-13) = 0$ or $18z - 4y - x - 13 = 0.$

5. The vector $\mathbf{n} = (\cos v \sin u, \sin v \sin u, \cos u) = (x, y, z)$. The surface is the unit sphere centered at the origin.

7. $\mathbf{n} = -(\sin v)\mathbf{i} - (\cos v)\mathbf{k}$; the surface is a cylinder.

9. (a) $x = x_0 + (y - y_0)(\partial h/\partial y)(y_0, z_0) + (z - z_0)(\partial h/\partial z)(y_0, z_0)$ describes the plane tangent to $x = h(y, z)$ at (x_0, y_0, z_0), $x_0 = h(y_0, z_0)$.

(b) $y = y_0 + (x - x_0)(\partial k/\partial x)(x_0, z_0) + (z - z_0)(\partial k/\partial z)(x_0, z_0)$.

11. $z - 6x - 8y + 3 = 0$

13. (a) The surfaces is a helicoid. It looks like a spiral ramp winding around the z axis. (See Figure 7.4.2.) It winds twice around, since θ goes up to 4π.

(b) $\mathbf{n} = \pm(1/\sqrt{1+r^2})(\sin\theta, -\cos\theta, r)$.

(c) $y_0 x - x_0 y + (x_0^2 + y_0^2)z = (x_0^2 + y_0^2)z_0$.

(d) If $(x_0, y_0, z_0) = (r_0 \cos\theta_0, r_0 \sin\theta_0, \theta_0)$, then representing the line segment in the form $\{(r\cos\theta_0, r\sin\theta_0, \theta_0) | 0 \le r \le 1\}$ shows that the line lies in the surface. Representing the line as $\{(x_0, ty_0, z_0) | 0 \le t \le 1/(x_0^2 + y_0^2)\}$ and substituting into the results of part (c) shows that it lies in the tangent plane at (x_0, y_0, z_0).

15. (a) Using cylindrical coordinates leads to the parametrization

$$\mathbf{\Phi}(z, \theta) = (\sqrt{25 + z^2}\cos\theta, \sqrt{25 + z^2}\sin\theta, z), \qquad -\infty < z < \infty, 0 \le \theta \le 2\pi$$

as one possible solution.

(b) $\mathbf{n} = (\sqrt{25+z^2}\cos\theta, \sqrt{25+z^2}\sin\theta, -z)/\sqrt{25+2z^2}$.

(c) $x_0 x + y_0 y = 25$.

(d) Substitute the coordinates along these lines into the defining equation of the surface and the result of part (c).

17. (a) $u \mapsto u$, $v \mapsto v$, $u \mapsto u^3$, and $v \mapsto v^3$ all map $\mathbb{R}$ onto $\mathbb{R}$.

(b) $\mathbf{T}_u \times \mathbf{T}_v = (0, 0, 1)$ for $\mathbf{\Phi}_1$, and this is never $\mathbf{0}$. For the surface $\mathbf{\Phi}_2$, $\mathbf{T}_u \times \mathbf{T}_v = 9u^2v^2(0, 0, 1)$, and this is $\mathbf{0}$ along the u and v axes.

(c) We want to show that any two parametrizations of a surface that are smooth near a point will give the same tangent plane there. Thus, suppose $\mathbf{\Phi}: D \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\mathbf{\Psi}: B \subset \mathbb{R}^2 \to \mathbb{R}^3$ are parametrized surfaces such that

$$\mathbf{\Phi}(u_0, v_0) = (x_0, y_0, z_0) = \mathbf{\Psi}(s_0, t_0) \tag{i}$$

and

$$\left(\mathbf{T}_u^{\Phi} \times \mathbf{T}_v^{\Phi}\right)\Big|_{(u_0,v_0)} \neq \mathbf{0} \quad \text{and} \quad \left(\mathbf{T}_s^{\Psi} \times \mathbf{T}_t^{\Psi}\right)\Big|_{(s_0,t_0)} \neq \mathbf{0}, \tag{ii}$$

so that $\mathbf{\Phi}$ and $\mathbf{\Psi}$ are smooth and one-to-one in neighborhoods of (u_0, v_0) and (s_0, t_0), which we may as well assume are D and B. Suppose further that they "describe the same surface," that is, $\mathbf{\Phi}(D) = \mathbf{\Phi}(B)$. To see that they give the same tangent plane at (x_0, y_0, z_0), show that they have parallel normal vectors. To do this, show that there is an open set C with $(u_0, v_0) \in C \subset D$ and a differentiable map $f: C \to B$ such that $\mathbf{\Phi}(u, v) = \mathbf{\Psi}(f(u, v))$ for $(u, v) \in C$. Once you have done this, computation shows that the normal vectors are related by $\mathbf{T}_u^{\Phi} \times \mathbf{T}_v^{\Phi} = [\partial(s, t)/\partial(u, v)]\mathbf{T}_s^{\Psi} \times \mathbf{T}_t^{\Psi}$.

To see that there is such an f, notice that since $\mathbf{T}_s^{\Psi} \times \mathbf{T}_t^{\Psi} \neq \mathbf{0}$, at least one of the 2×2 determinants in the cross product is not zero. Assume, for example, that

$$\begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\[2mm] \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix} \neq 0.$$

Now use the inverse function theorem to write (s, t) as a differentiable function of (x, y) in some neighborhood of (x_0, y_0).

(d) No.

Section 7.4

1. 4π

3. $\frac{3}{2}\pi[\sqrt{2} + \log(1 + \sqrt{2})]$

5. $\frac{1}{3}\pi(6\sqrt{6} - 8)$

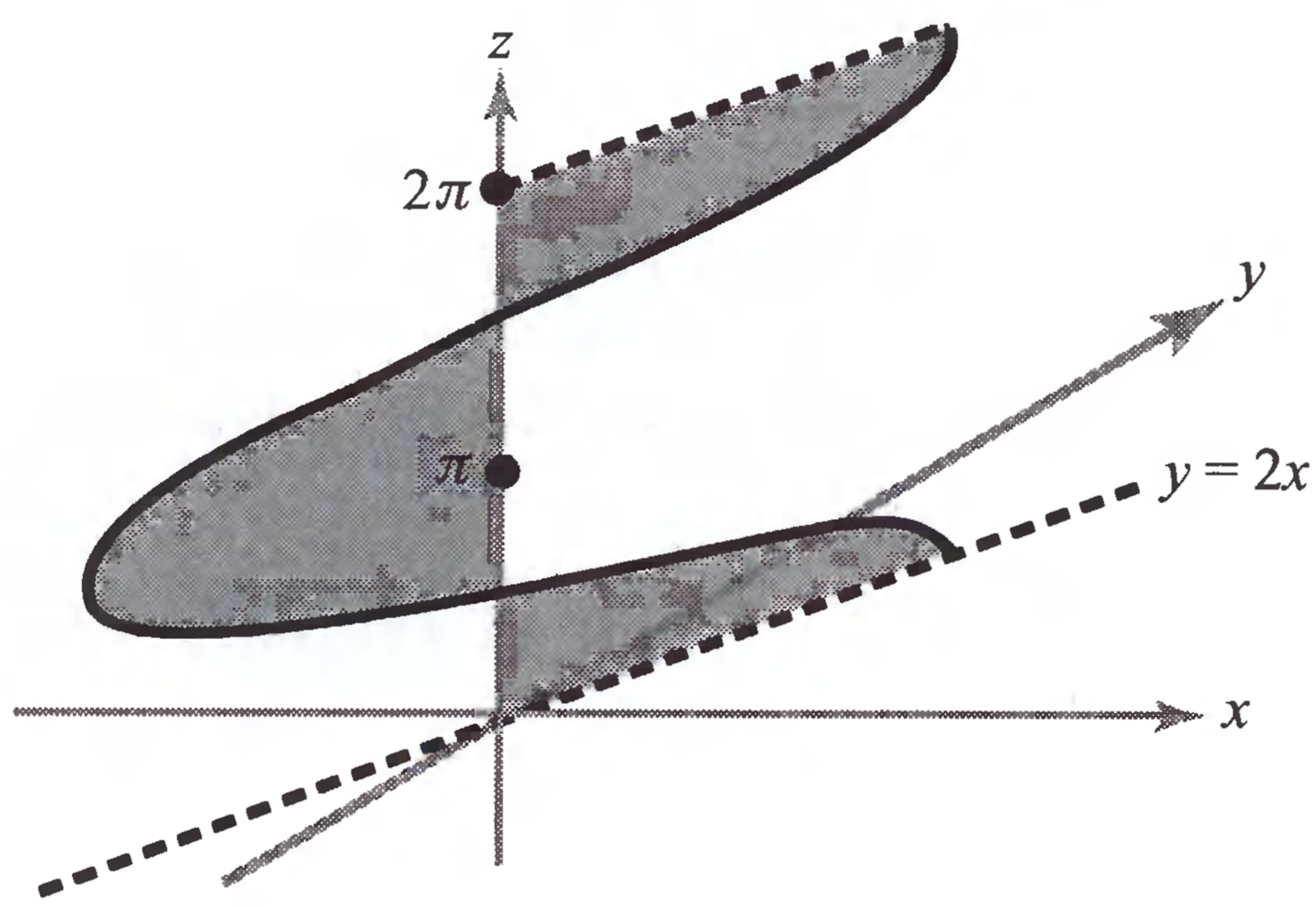
7. The integral for the volume converges, whereas that for the area diverges.

9. $A(E) = \int_0^{2\pi}\int_0^{\pi}\sqrt{a^2b^2\sin^2\phi\cos^2\phi + b^2c^2\sin^4\phi\cos^2\theta + a^2c^2\sin^4\phi\sin^2\theta}\,d\phi\,d\theta$

11. $(\pi/6)(5\sqrt{5} - 1)$

13. $(\pi/2)\sqrt{6}$

15. $4\sqrt{5}$; for fixed θ, (x, y, z) moves along the horizontal line segment $y = 2x, z = \theta$ from the z axis out to a radius of $\sqrt{5}|\cos\theta|$ into quadrant 1 if $\cos\theta > 0$ and into quadrant 3 if $\cos\theta < 0$.



17. $(\pi + 2)/(\pi - 2)$

19. $\pi(a + b)\sqrt{1 + m^2}(b - a)$

21. $\frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1)$

23. With $f(x, y) = \sqrt{R^2 - x^2 - y^2}$, (4) becomes

$$A(S') = \iint_D \sqrt{\frac{x^2 + y^2}{R^2 - x^2 - y^2} + 1}\,dx\,dy$$
$$= \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dx\,dy$$

where D is the disk of radius R. Evaluate using polar coordinates, noting it is improper at the boundary, to get $2\pi R^2$.

Section 7.5

1. $\dfrac{3\sqrt{2}+5}{24}$

3. πa^3

5. (a) $\sqrt{2}\pi R^2$ (b) $2\pi R^2$

7. $\dfrac{\pi}{4}\left(\dfrac{5\sqrt{5}}{3}+\dfrac{1}{15}\right)$

9. $16\pi R^3/3$

11. (a) The sphere looks the same from all three axes, so these three integrals should be the same quantity with different labels on the axes.
(b) $4\pi R^4/3$
(c) $4\pi R^4/3$

13. $(R/2, R/2, R/2)$

15. (a) Directly compute the vector cross product $\mathbf{T}_u \times \mathbf{T}_v$ and then calculate its length and compare your answer to the left-hand side.
(b) In this case, $F = 0$, so $A(s) = \iint_D \sqrt{EG}\, du\, dv$.
(c) $4\pi a^2$.

17. Let $a = \partial x/\partial u$, $b = \partial y/\partial u$, $c = \partial x/\partial v$, and $d = \partial y/\partial v$. The conditions (a) and (b) in Exercise 16 are then $a^2 + b^2 = c^2 + d^2$ and $ac + bd = 0$. Show that $a \neq 0$ and, by a normalization argument, show that you can assume $a = 1$. Now calculate further.

19. $2a^2$

Section 7.6

1. $\pm 48\pi$ (the sign depends on orientation)

3. 4π

5. 2π (or -2π, if you choose a different orientation)

7. 2π

9. $12\pi/5$

11. With the usual spherical coordinate parametrization, $\mathbf{T}_\theta \times \mathbf{T}_\phi = -\sin\phi\, \mathbf{r}$ (see Example 1). Thus,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint \mathbf{F}\cdot(\mathbf{T}_\phi \times \mathbf{T}_\theta)\, d\phi\, d\theta = \iint (\mathbf{F}\cdot\mathbf{r})\sin\phi\, d\phi\, d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} F_r \sin\phi\, d\phi\, d\theta$$

and

$$\iint_S f\,dS = \int_0^{2\pi} \int_0^{\pi} f \sin\phi \, d\phi \, d\theta.$$

13. For a cylinder of radius $R = 1$ and normal component F_r,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_a^b \int_0^{2\pi} F_r \, d\theta \, dz.$$

15. $2\pi/3$

17. $\frac{2}{5}a^3bc\pi$

Section 7.7

1. Apply formula (3) of this section and simplify; $H = 0$ and $K = -b^2/(u^2 + b^2)^2$.

3. Apply formula (3) of this section and simplify.

5. $K = \dfrac{-4a^6b^6}{(a^4b^4 + 4b^4u^2 + 4a^4v^2)^2}$

7. Apply formula (3) of this section and simplify.

9. Apply formula (2) of this section and simplify; $K = -h''/[(1 + (h')^2)^2 h]$.

Review Exercises for Chapter 7

1. (a) $3\sqrt{2}(1 - e^{6\pi})/13$
(b) $-\pi\sqrt{2}/2$
(c) $(236, 158\sqrt{26} - 8)/35 \cdot (25)^3$
(d) $8\sqrt{2}/189$

3. (a) $2/\pi + 1$ (b) $-1/2$

5. $2a^3$

7. (a) A sphere of radius 5 centered at $(2, 3, 0)$; $\mathbf{\Phi}(\theta, \phi) = (2 + 5\cos\theta\sin\phi,$ $3 + 5\sin\theta\sin\phi,\ 5\cos\phi)$; $0 \le \theta \le 2\pi; 0 \le \phi \le \pi$
(b) An ellipsoid with center at $(2, 0, 0)$; $\mathbf{\Phi}(\theta, \phi) = (2 + (1/\sqrt{2})3\cos\theta\sin\phi,$ $3\sin\theta\sin\phi, 3\cos\phi)$; $0 \le \theta \le 2\pi, 0 \le \phi \le \pi$
(c) An elliptic hyperboloid of one sheet; $\mathbf{\Phi}(\theta, z) = \left(\frac{1}{2}\sqrt{8 + 2z^2}\cos\theta,\right.$ $\left.\frac{1}{3}\sqrt{8 + 2z^2}\sin\theta, z\right)$; $0 \le \theta \le 2\pi,\ -\infty < z < \infty$

9. $A(\mathbf{\Phi}) = \frac{1}{2}\int_0^{2\pi} \sqrt{3\cos^2\theta + 5}\, d\theta$; $\mathbf{\Phi}$ describes the upper nappe of a cone with elliptical horizontal cross sections.

11. $11\sqrt{3}/6$

13. $\sqrt{2}/3$

15. $5\sqrt{5}/6$

17. (a) $(e^y \cos \pi z, xe^y \cos \pi z, -\pi xe^y \sin \pi z)$ (b) 0

19. $\frac{1}{2}(e^2 + 1)$

21. $\mathbf{n} = (1/\sqrt{5})(-1, 0, 2)$, $2z - x = 1$

23. 0

25. If $\mathbf{F} = \nabla\phi$, then $\nabla \times \mathbf{F} = \mathbf{0}$ (at least if ϕ is of class C^2; see Theorem 1, Section 3.4). Theorem 3 of Section 7.2 shows that $\int_{\mathbf{c}} \nabla\phi \cdot d\mathbf{s} = 0$ because $\mathbf{c}$ is a closed curve.

27. (a) 24π (b) 24π (c) 60π

29. (a) $[\sqrt{R^2 + p^2}(z_0 - z_1)]/p$ (b) $\sqrt{2z_0(R^2 + p^2)/p^2 g}$

Chapter 8

Section 8.1

1. -8

3. (a) 0 (b) $-\pi R^2$ (c) 0 (d) $-\pi R^2$

5. $3\pi a^2$

7. $3\pi/2$

9. $3\pi(b^2 - a^2)/2$

11. (a) Both sides are 2π. (b) 0

13. 0

15. πab

17. A horizontal line segment divides the region into three regions of which Green's theorem applies; now use Exercise 8 or the technique in Figure 8.1.5.

19. $9\pi/8$

21. If $\varepsilon > 0$, there is a $\delta > 0$ such that $|u(\mathbf{q}) - u(\mathbf{p})| < \varepsilon$ whenever $\|\mathbf{p} - \mathbf{q}\| = \rho < \delta$. Parametrize $\partial B_\rho(\mathbf{p})$ by $\mathbf{q}(\theta) = \mathbf{p} + \rho(\cos\theta, \sin\theta)$. Then

$$|I(\rho) - 2\pi u(\mathbf{p})| \leq \int_0^{2\pi} |u(\mathbf{q}(\theta)) - u(\mathbf{p})|\, d\theta \leq 2\pi\varepsilon.$$

23. Parametrize $\partial B_\rho(\mathbf{p})$ as in Exercise 21. If $\mathbf{p} = (p_1, p_2)$, then $I(\rho) = \int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\, d\theta$. Differentiation under the integral sign gives

$$\frac{dI}{d\rho} = \int_0^{2\pi} \nabla u \cdot (\cos\theta, \sin\theta)\, d\theta = \int_0^{2\pi} \nabla u \cdot \mathbf{n}\, d\theta = \frac{1}{\rho}\int_{\partial B_\rho} \frac{\partial u}{\partial \mathbf{n}}\, ds = \frac{1}{\rho}\iint_{B_\rho} \nabla^2 u\, dA$$

(the last equality uses Exercise 22).

25. Using Exercise 24,

$$\begin{aligned}\iint_{B_R} u\, dA &= \int_0^R \int_0^{2\pi} u[\mathbf{p} + \rho(\cos\theta, \sin\theta)]\rho\, d\theta\, d\rho \\ &= \int_0^R \left(\int_{\partial B_\rho} u\, ds\right) d\rho = \int_0^R 2\pi\rho u(\mathbf{p})\, d\rho = \pi R^2 u(\mathbf{p}).\end{aligned}$$

27. Suppose u is subharmonic. We establish the assertions corresponding to Exercises 26(a) and (b). The argument for superharmonic functions is similar, with inequalities reversed.

Suppose $\nabla^2 u \geq 0$ and $u(\mathbf{p}) \geq u(\mathbf{q})$ for all $\mathbf{q}$ in $B_R(\mathbf{p})$. By Exercise 23, $I'(\rho) \geq 0$ for $0 < \rho \leq R$, and so Exercise 24 shows that $2\pi u(\mathbf{p}) \leq I(\rho) \leq I(R)$ for $0 < \rho \leq R$. If $u(\mathbf{q}) < u(\mathbf{p})$ for some $\mathbf{q} = \mathbf{p} + \rho(\cos\theta_0, \sin\theta_0) \in B_R(\mathbf{p})$, then, by continuity, there is an arc $[\theta_0 - \delta, \theta_0 + \delta]$ on $\partial B_\rho(\mathbf{p})$ where $u < u(\mathbf{p}) - d$ for some $d > 0$. This would mean that

$$\begin{aligned}2\pi u(\mathbf{p}) \leq I(\rho) &= \frac{1}{\rho}\int_0^{2\pi} u[\mathbf{p} + \rho(\cos\theta, \sin\theta)]\rho\, d\theta \\ &\leq (2\pi - 2\delta)u(\mathbf{p}) + 2\delta[u(\mathbf{p}) - d] \leq 2\pi u(\mathbf{p}) - 2\delta d.\end{aligned}$$

This contradiction shows that we must have $u(\mathbf{q}) = u(\mathbf{p})$ for every $\mathbf{q}$ in $B_B(\mathbf{p})$.

If the maximum at $\mathbf{p}$ is absolute for D, the last paragraph shows that $u(\mathbf{x}) = u(\mathbf{p})$ for all $\mathbf{x}$ in some disk around $\mathbf{p}$. If $\mathbf{c}\colon [0, 1) \to D$ is a path from $\mathbf{p}$ to $\mathbf{q}$, then $u(\mathbf{c}(t)) = u(\mathbf{p})$ for all t in some interval $[0, b)$. Let b_0 be the largest $b \in [0, 1]$ such that $u(\mathbf{c}(t)) = u(\mathbf{p})$ for all $t \in [0, b)$. (Strictly speaking, this requires the notion of the least upper bound from a good calculus text.) Because u is continuous, $u(\mathbf{c}(b_0)) = u(\mathbf{p})$. If $b_0 \neq 1$, then the last paragraph would apply at $\mathbf{c}(b_0)$ and u is constantly equal to $u(\mathbf{p})$ on a disk around $\mathbf{c}(b_0)$. In particular, there is a $\delta > 0$ such that $u(\mathbf{c}(t)) = u(\mathbf{c}(b_0)) = u(\mathbf{p})$ on $[0, b_0 + \delta)$. This contradicts the maximality of b_0, so we must have $b_0 = 1$. That is, $\mathbf{c}(\mathbf{q}) = \mathbf{c}(\mathbf{p})$. Because $\mathbf{q}$ was an arbitrary point in D, u is constant on D.

29. Assume $\nabla^2 u_1 = 0$ and $\nabla^2 u_2 = 0$ are two solutions. Let $\phi = u_1 - u_2$. Then $\nabla^2\phi = 0$ and $\phi(x) = 0$ for all $x \in \partial D$. Consider the integral $\iint_D \phi\nabla^2\phi\, dA = -\iint_D \nabla\phi \cdot \nabla\phi\, dA$. Thus, $\iint_D \nabla\phi \cdot \nabla\phi\, dA = 0$, which implies that $\nabla\phi = \mathbf{0}$, and so ϕ is a constant function and hence must be identically zero.

Section 8.2

1. -2π

3. Each integral in Stokes' theorem is zero.

5. 0

7. $-4\pi/\sqrt{3}$

9. 0

11. $\pm 2\pi$

13. Using Faraday's law, $\iint_S [\nabla \times \mathbf{E} + \partial \mathbf{H}/\partial t] \cdot d\mathbf{S} = 0$ for any surface S. If the integrand were a nonzero vector at some point, then by continuity the integral over some small disk centered at that point and lying perpendicular to that vector would be nonzero.

15. The orientations of $\partial S_1 = \partial S_2$ must agree.

17. Suppose C is a closed loop on the surface drawn so that it divides the surface into two pieces S_1 and S_2. For the surface of a doughnut (torus) you must use two closed loops; can you see why? Then C bounds both S_1 and S_2, but with positive orientation with respect to one and negative with respect to the other. Therefore,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} - \int_C \mathbf{F} \cdot d\mathbf{s} = 0.$$

19. (a) If $C = \partial S$, $\int_C \mathbf{v} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{s} = 0$.
(b) $\int_C \mathbf{v} \cdot d\mathbf{s} = \int_a^b \mathbf{v} \cdot \mathbf{c}'(t)dt = \mathbf{v} \cdot \int_a^b \mathbf{c}'(t)dt = \mathbf{v} \cdot (\mathbf{c}(b) - \mathbf{c}(a))$, where $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$ is a parametrization of C. (The vector integral is the vector whose components are the integrals of the component functions.) If C is closed, the last expression is 0.

21. Both integrals give $\pi/4$.

23. (a) 0 (b) π (c) π

25. -20π (or 20π if the opposite orientation is used)

27. One possible answer: The Möbius curve C is also the boundary of an *oriented* surface $\tilde{S}$; the equation in Faraday's law *is* valid for this new surface.

Section 8.3

1. If $\mathbf{F} = \nabla f = \nabla g$ and C is a curve from $\mathbf{v}$ to $\mathbf{w}$, then $(f - g)(\mathbf{w}) - (f - g)(\mathbf{v}) = \int_C \nabla(f - g) \cdot d\mathbf{s} = 0$ and so $f - g$ is constant.

3. $x^2yz - \cos x + C$

5. Yes, it is the gradient of $g(x, y) = F(x) + F(y)$, where $F'(x) = f(x)$.

7. No; $\nabla \times \mathbf{F} = (0, 0, -x) \neq \mathbf{0}$.

9. $e \sin 1 + \frac{1}{3}e^3 - \frac{1}{3}$

11. 3.5×10^{29} ergs

13. (a) $f = x^2/2 + y^2/2 + C$ (c) $f = \frac{1}{3}x^3 + xy^2 + C$
(b) $\mathbf{F}$ is not a gradient field.

15. Use Theorem 7 in each case.
(a) $-3/2$ (b) -1 (c) $\cos(e^2) - \cos(1/e)/e$

17. (a) No. (b) $\left(\frac{1}{2}z^2, xy - z, x^2y\right)$ or $\left(\frac{1}{2}z^2 - 2xyz - \frac{1}{2}y^2, -x^2z - z, 0\right)$

19. $\frac{1}{3}(z^3\mathbf{i} + x^3\mathbf{j} + y^3\mathbf{k})$

21. $(-z\sin y + y\sin x, xz\cos y, 0)$ (Other answers are possible.)

23. (a) $\nabla \times \mathbf{F} = (0, 0, 2) \neq \mathbf{0}$
(b) Let $\mathbf{c}(t)$ be the path of an object in the fluid. Then $\mathbf{F}(\mathbf{c}(t)) = \mathbf{c}'(t)$. Let $\mathbf{c}(t) = (x(t), y(t), z(t))$. Then $x' = -y$, $y' = x$, and $z' = 0$, and so z is constant and the motion is parallel to the xy plane. Also, $x'' + x = 0$, $y'' + y = 0$. Thus, $x = A\cos t + B\sin t$ and $y = C\cos t + D\sin t$. Substituting these values in $x' = -y$, $y' = x$, we get $C = -B$, $D = A$, so that $x^2 + y^2 = A^2 + B^2$ and we have a circle.
(c) Counterclockwise

25. (a) $\mathbf{F} = -\dfrac{GmM}{(x^2 + y^2 + z^2)^{3/2}}(x, y, z)$;

$$\nabla \cdot \mathbf{F} = -GmM\left[\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3y^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{x^2 + y^2 + z^2 - 3z^2}{(x^2 + y^2 + z^2)^{5/2}}\right]$$
$$= 0$$

(b) Let S be the unit sphere, S_1 the upper hemisphere, S_2 the lower hemisphere, and C the unit circle. If $\mathbf{F} = \nabla \times \mathbf{G}$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{G} \cdot d\mathbf{s} - \int_C \mathbf{G} \cdot d\mathbf{s} = 0.$$

But $\iint_S \mathbf{F} \cdot d\mathbf{S} = -GmM \iint_S (\mathbf{r}/\|\mathbf{r}\|^3) \cdot \mathbf{n}\, dS = -4\pi GmM$, because $\|\mathbf{r}\| = 1$ and $\mathbf{r} = \mathbf{n}$ on S. Thus, $\mathbf{F} = \nabla \times \mathbf{G}$ is impossible. This does not contradict Theorem 8 because $\mathbf{F}$ is not smooth at the origin.

Section 8.4

1. 4π

3. 3

5. (a) 0 (b) $4/15$ (c) $-4/15$

7. 6

9. 1

11. Apply the divergence theorem to $f\mathbf{F}$ using $\nabla\cdot(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}$.

13. If $\mathbf{F} = \mathbf{r}/r^2$, then $\nabla\cdot\mathbf{F} = 1/r^2$. If $(0,0,0)\notin\Omega$, the result follows from Gauss' theorem. If $(0,0,0)\in\Omega$, we compute the integral by deleting a small ball $B_\varepsilon = \{(x,y,z)|(x^2+y^2+z^2)^{1/2} < \varepsilon\}$ around the origin and then letting $\varepsilon\to 0$:

$$\begin{aligned}\iiint_\Omega \frac{1}{r^2}\,dV &= \lim_{\varepsilon\to 0}\iiint_{\Omega\setminus B_\varepsilon}\frac{1}{r^2}\,dV = \lim_{\varepsilon\to 0}\iint_{\partial(\Omega\setminus B_\varepsilon)}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS\\ &= \lim_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS - \iint_{\partial B_\varepsilon}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS\right) = \lim_{\varepsilon\to 0}\left(\iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS - 4\pi\varepsilon\right)\\ &= \iint_{\partial\Omega}\frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS.\end{aligned}$$

The integral over ∂B_ε is obtained from Theorem 10 (Gauss' law), because $r=\varepsilon$ everywhere on B_ε.

15. Use the vector identity for div$(f\mathbf{F})$ and the divergence theorem for part (a). Use the vector identity $\nabla\cdot(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$ for part (b).

17. (a) If $\phi(\mathbf{p}) = \iiint_W \rho(\mathbf{q})/(4\pi\|\mathbf{p}-\mathbf{q}\|)\,dV(\mathbf{q})$, then

$$\begin{aligned}\nabla\phi(\mathbf{p}) &= \iiint_W [\rho(\mathbf{q})/4\pi]\nabla_{\mathbf{p}}(1/\|\mathbf{p}-\mathbf{q}\|)\,dV(\mathbf{q})\\ &= -\iiint_W [\rho(\mathbf{q})/4\pi][(\mathbf{p}-\mathbf{q})/\|\mathbf{p}-\mathbf{q}\|^3]\,dV(\mathbf{q}),\end{aligned}$$

where $\nabla_{\mathbf{p}}$ means the gradient with respect to the coordinates of $\mathbf{p}$ and the integral is the vector whose components are the three component integrals. If $\mathbf{p}$ varies in $V\cup\partial V$ and $\mathbf{n}$ is the outward unit normal to ∂V, we can take the inner product using these components and collect the pieces as

$$\nabla\phi(\mathbf{p})\cdot\mathbf{n} = -\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\frac{1}{\|\mathbf{p}-\mathbf{q}\|^3}(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}\,dV(\mathbf{q}).$$

Thus,

$$\iint_{\partial V}\nabla\phi(\mathbf{p})\cdot\mathbf{n}\,dV(\mathbf{p}) = -\iint_{\partial V}\left(\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\frac{1}{\|\mathbf{p}-\mathbf{q}\|^3}(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}\,d\mathbf{q}\right)dV(\mathbf{p}).$$

There are essentially five variables of integration here, three placing $\mathbf{q}$ in W and two placing $\mathbf{p}$ on ∂V. Use Fubini's theorem to obtain

$$\iint_{\partial V}\nabla\phi\cdot\mathbf{n}\cdot d\mathbf{S} = -\iiint_W \frac{\rho(\mathbf{q})}{4\pi}\left[\iint_{\partial V}\frac{(\mathbf{p}-\mathbf{q})\cdot\mathbf{n}}{\|\mathbf{p}-\mathbf{q}\|^3}\,dS(\mathbf{p})\right]dV(\mathbf{q}).$$

If V is a symmetric elementary region, Theorem 10 says that the inner integral is 4π if $\mathbf{q} \in V$ and 0 if $\mathbf{q} \notin V$. Thus,

$$\iint_{\partial V} \nabla\phi \cdot \mathbf{n}\, dS = -\iiint_{W\cap V} \rho(\mathbf{q})\, dV(\mathbf{q}).$$

Because $\rho = 0$ outside W,

$$\iint_{\partial V} \nabla\phi \cdot \mathbf{n}\, dS = -\iiint_{V} \rho(\mathbf{q})\, dV(\mathbf{q}).$$

If V is not a symmetric elementary region, subdivide it into a sum of such regions. The equation holds on each piece, and, upon adding them together, the boundary integrals along appropriately oriented interior boundaries cancel, leaving the desired result.

(b) By Theorem 9, $\iint_{\partial V} \nabla\phi \cdot d\mathbf{S} = \iiint_V \nabla^2\phi\, dV$, and so $\iiint_V \nabla^2\phi\, dV = -\iiint_V \rho\, dV$. Because both ρ and $\nabla^2\phi$ are continuous and this holds for arbitrarily small regions, we must have $\nabla^2\phi = -\rho$.

19. If the charge Q is spread evenly over the sphere S of radius R centered at the origin, the density of charge per unit area must be $Q/4\pi R^2$. If $\mathbf{p}$ is a point not on S and $\mathbf{q} \in S$, then the contribution to the electric field at $\mathbf{p}$ due to charge near $\mathbf{q}$ is directed along the vector $\mathbf{p} - \mathbf{q}$. Because the charge is evenly distributed, the tangential component of this contribution will be canceled by that from a symmetric point on the other side of the sphere at the same distance from $\mathbf{p}$. (Draw the picture.) The total resulting field must be radial. Because S looks the same from any point at a distance $\|\mathbf{p}\|$ from the origin, the field must depend only on radius and be of the form $\mathbf{E} = f(r)\mathbf{r}$.

If we look at the sphere Σ of radius $\|\mathbf{p}\|$, we have

$$\begin{aligned}(\text{charge inside } \Sigma) &= \iint_\Sigma \mathbf{E}\cdot d\mathbf{S} = \iint_\Sigma f(\|\mathbf{p}\|)\mathbf{r}\cdot\mathbf{n}\, dS \\ &= f(\|\mathbf{p}\|)\|\mathbf{p}\| \text{ area } \Sigma = 4\pi\|\mathbf{p}\|^3 f(\|\mathbf{p}\|).\end{aligned}$$

If $\|\mathbf{p}\| < R$, there is no charge inside Σ; if $\|\mathbf{p}\| > R$, the charge inside Σ is Q, and so

$$\mathbf{E}(\mathbf{p}) = \begin{cases} \dfrac{1}{4\pi}\dfrac{Q}{\|\mathbf{p}\|^3}\mathbf{p} & \text{if } \|\mathbf{p}\| > R \\ 0 & \text{if } \|\mathbf{p}\| < R. \end{cases}$$

21. By Theorem 10, $\iint_{\partial M} \mathbf{F}\cdot d\mathbf{S} = 4\pi$ for any surface enclosing the origin. But if $\mathbf{F}$ were the curl of some field, then the integral over such a closed surface would have to be 0.

23. If $S = \partial W$, then $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iiint_W \nabla\cdot\mathbf{r}\, dV = \iiint_W 3\,dV = 3 \text{ volume } (W)$. For the geometric explanation, assume $(0, 0, 0) \in W$ and consider the skew cone with its vertex at $(0, 0, 0)$ with base ΔS and altitude $\|\mathbf{r}\|$. Its volume is $\frac{1}{3}(\Delta S)(\mathbf{r}\cdot\mathbf{n})$.

Section 8.5

1. Write the components of φ as $\xi(\mathbf{x}, t)$, $\eta(\mathbf{x}, t)$, and $\zeta(\mathbf{x}, t)$. First, observe that by definition of φ,

$$\frac{\partial}{\partial t}\varphi(\mathbf{x}, t) = \mathbf{F}(\varphi(\mathbf{x}, t), t).$$

The determinant J can be differentiated by recalling that the determinant of a matrix is multilinear in the columns (or rows). Thus, holding $\mathbf{x}$ fixed,

$$\frac{\partial}{\partial t}J = \begin{bmatrix} \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial}{\partial t}\frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix} + \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} \end{bmatrix}.$$

Now write

$$\frac{\partial}{\partial t}\frac{\partial \xi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial x}F_1(\varphi(\mathbf{x},t),t),$$

$$\frac{\partial}{\partial t}\frac{\partial \xi}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial y}F_2(\varphi(\mathbf{x},t),t),$$

$$\frac{\partial}{\partial t}\frac{\partial \zeta}{\partial z} = \frac{\partial}{\partial z}\frac{\partial \zeta}{\partial t} = \frac{\partial}{\partial z}F_3(\varphi(\mathbf{x},t),t).$$

The components F_1, F_2, and F_3 of $\mathbf{F}$ in this expression are functions of x, y, and z through $\varphi(\mathbf{x}, t)$; therefore,

$$\frac{\partial}{\partial x}F_1(\varphi(\mathbf{x},t),t) = \frac{\partial F_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial F_1}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial F_1}{\partial \zeta}\frac{\partial \zeta}{\partial x},$$

$$\vdots$$

$$\frac{\partial}{\partial z}F_3(\varphi(\mathbf{x},t),t) = \frac{\partial F_3}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial F_3}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial F_3}{\partial \zeta}\frac{\partial \zeta}{\partial z}.$$

When these are substituted into the previous expression for $\partial J/\partial t$, one gets for the respective terms

$$\frac{\partial F_1}{\partial x}J + \frac{\partial F_2}{\partial y}J + \frac{\partial F_3}{\partial z}J = (\operatorname{div}\mathbf{F})J.$$

3. HINTS: By the transport equation from Theorem 12, with $\mathbf{V}$ in place of $\mathbf{F}$,

$$\frac{d}{dt}\iiint_{W_t} \rho\, dx\, dy\, dz = \iiint_{W_t}\left(\frac{D\rho}{Dt} + \rho \operatorname{div}\mathbf{V}\right) dx\, dy\, dz.$$

Now use the fact that

$$\frac{D\rho}{Dt} + \rho \operatorname{div}\mathbf{V} = \operatorname{div}\mathbf{J} + \frac{\partial \rho}{\partial t},$$

where $\mathbf{J} = \rho\mathbf{V}$, as in the text.

5. If v_i is the ith component of a vector $\mathbf{v}$, then by the transport equation (Exercise 2),

$$
\begin{aligned}
\left[\frac{d}{dt}\iiint_{W_t} f\mathbf{F}\,dx\,dy\,dz\right]_i &= \frac{d}{dt}\iiint_{W_t} (f\mathbf{F})_i\,dx\,dy\,dz = \frac{d}{dt}\iiint_{W_t} fF_i\,dx\,dy\,dz \\
&= \iiint_{W_t}\left[\frac{D(fF_i)}{Dt} + (fF_i)\operatorname{div}\mathbf{F}\right]dx\,dy\,dz \\
&= \iiint_{W_t}\left[\frac{\partial}{\partial t}(fF_i) + \mathbf{D}_x(fF_i)\cdot\mathbf{F} + (fF_i)\operatorname{div}\mathbf{F}\right]dx\,dy\,dz \\
&= \iiint_{W_t}\left[\frac{\partial}{\partial t}(f\mathbf{F}_i) + \nabla(fF_i)\cdot\mathbf{F} + (fF_i)\operatorname{div}\mathbf{F}\right]dx\,dy\,dz \\
&= \iiint_{W_t}\left\{\frac{\partial}{\partial t}(f\mathbf{F}_i) + [\mathbf{D}(f\mathbf{F})\mathbf{F}]_i + [(f\mathbf{F})\operatorname{div}\mathbf{F}]_i\right\}dx\,dy\,dz \\
&= \iiint_{W_t}\left[\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{D}(f\mathbf{F})\mathbf{F} + (f\mathbf{F})\operatorname{div}\mathbf{F}\right]_i dx\,dy\,dz \\
&= \left[\iiint_{W_t}\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{D}(f\mathbf{F})\mathbf{F} + (f\mathbf{F})\operatorname{div}\mathbf{F}\,dx\,dy\,dz\right]_i \\
&= \left[\iiint_{W_t}\left(\frac{\partial}{\partial t}(f\mathbf{F}) + (\mathbf{F}\cdot\nabla)(f\mathbf{F}) + (f\mathbf{F})\operatorname{div}\mathbf{F}\right)dx\,dy\,dz\right]_i.
\end{aligned}
$$

7. (a) Because $\mathbf{V} = \nabla\phi$, $\nabla\times\mathbf{V} = \mathbf{0}$, and therefore $(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\|\mathbf{V}\|^2)$, Euler's equation becomes

$$-\frac{\nabla p}{\rho} = \frac{d\mathbf{V}}{dt} + \frac{1}{2}\nabla(\|\mathbf{V}\|^2) = \nabla\left(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\right).$$

If $\mathbf{c}$ is a path from P_1 to P_2, then

$$
\begin{aligned}
-\int_{\mathbf{c}}\frac{1}{\rho}\,dp &= -\int\frac{1}{\rho}\nabla p\cdot\mathbf{c}'(t)\,dt = \int_{\mathbf{c}}\nabla\left(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\right)\cdot\mathbf{c}'(t)\,dt \\
&= \left(\frac{d\phi}{dt} + \frac{1}{2}\|\mathbf{V}\|^2\right)\Bigg|_{P_1}^{P_2}.
\end{aligned}
$$

(b) If $d\mathbf{V}/dt = \mathbf{0}$ and ρ is constant, then $\frac{1}{2}\nabla(\|\mathbf{V}\|^2) = -(\nabla p)/\rho = -\nabla(p/\rho)$, and therefore $\nabla\left(\frac{1}{2}\|\mathbf{V}\|^2 + p/\rho\right) = \mathbf{0}$.

9. By Ampère's law, $\nabla\cdot\mathbf{J} = \nabla\cdot(\nabla\times\mathbf{H}) - \nabla\cdot(\partial\mathbf{E}/\partial t) = -\nabla\cdot(\partial\mathbf{E}/\partial t) = -(\partial/\partial t)(\nabla\cdot\mathbf{E})$. By Gauss' law this is $-\partial\rho/\partial t$. Thus, $\nabla\cdot\mathbf{J} + \partial\rho/\partial t = 0$.

Section 8.6

1. (a) $(2xy^2 - yx^3)\,dx\,dy$
(b) $(x^2 + y^2)\,dx\,dy$
(c) $(x^2 + y^2 + z^2)\,dx\,dy\,dz$
(d) $(xy + x^2)\,dx\,dy\,dz$
(e) $dx\,dy\,dz$

3. (a) $2xy\,dx + (x^2 + 3y^2)\,dy$
(b) $-(x + y^2 \sin x)\,dx\,dy$
(c) $-(2x + y)\,dx\,dy$
(d) $dx\,dy\,dz$
(e) $2x\,dx\,dy\,dz$
(f) $2y\,dy\,dz - 2x\,dz\,dx$
(g) $-\dfrac{4xy}{(x^2 + y^2)^2}\,dx\,dy$
(h) $2xy\,dx\,dy\,dz$

5. (a)
$$\begin{aligned}\mathrm{Form}_2\,(\alpha\mathbf{V}_1 + \mathbf{V}_2) &= \mathrm{Form}_2\,(\alpha A_1 + A_2, \alpha B_1 + B_2, \alpha C_1 + C_2)\\ &= (\alpha A_1 + A_2)\,dy\,dz + (\alpha B_1 + B_2)\,dz\,dx\\ &\quad + (\alpha C_1 + C_2)\,dx\,dy\\ &= \alpha(A_1\,dy\,dz + B_1\,dz\,dx + C_1\,dx\,dy)\\ &\quad + (A_2\,dy\,dz + B_2\,dz\,dx + C_2\,dx\,dy)\\ &= \alpha\,\mathrm{Form}_2\,(\mathbf{V}_1) + \mathrm{Form}_2(\mathbf{V}_2)\end{aligned}$$

(b)
$$\begin{aligned}d\omega &= \left(\frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial y}\,dy + \frac{\partial A}{\partial z}\,dz\right) \wedge dx + A(dx)^2\\ &\quad + \left(\frac{\partial B}{\partial x}\,dx + \frac{\partial B}{\partial y}\,dy + \frac{\partial B}{\partial z}\,dz\right) \wedge dy + B(dy)^2\\ &\quad + \left(\frac{\partial C}{\partial x}\,dx + \frac{\partial C}{\partial y}\,dy + \frac{\partial C}{\partial z}\,dz\right) \wedge dz + C(dz)^2.\end{aligned}$$

But $(dx)^2 = (dy)^2 = (dz)^2 = dx \wedge dx = dy \wedge dy = dz \wedge dz = 0$, $dy \wedge dx = -dx \wedge dy$, $dz \wedge dy = -dy \wedge dz$, and $dx \wedge dz = -dz \wedge dx$. Hence,

$$\begin{aligned}d\omega &= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy\,dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial y}\right) dz\,dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy\\ &= \mathrm{Form}_2\,(\mathrm{curl}\,\mathbf{V}).\end{aligned}$$

7. An oriented 1-manifold is a curve. Its boundary is a pair of points that may be considered a 0-manifold. Therefore, ω is a 0-form or function, and $\int_{\partial M} d\omega = \omega(b) - \omega(a)$ if the curve M runs from a to b. Furthermore, $d\omega$ is the 1-form $(\partial\omega/\partial x)\,dx + (\partial\omega/\partial y)\,dy$. Therefore, $\int_M d\omega$ is the line integral $\int_M (\partial\omega/\partial x)\,d\omega + (\partial\omega/\,dy)\,dy = \int_M \nabla\omega \cdot d\mathbf{s}$. Thus, we obtain Theorem 3 of Section 7.2, $\int_M \nabla\omega \cdot d\mathbf{s} = \omega(b) - \omega(a)$.

9. Put $\omega = F_1\,dx\,dy + F_2\,dy\,dz + F_3\,dz\,dx$. The integral becomes

$$\iint_{\partial T} \omega = \iiint_T d\omega = \iiint_T \left(\frac{\partial F_1}{\partial z} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial y}\right) dx\,dy\,dz.$$

(a) 0 (b) 40

11. Consider $\omega = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$. Compute that $d\omega = 3\,dx\,dy\,dz$, so that $\frac{1}{3}\iint_{\partial R} \omega = \frac{1}{3}\iiint_R d\omega = \iiint_R dx\,dy\,dz = v(R)$.

Review Exercises for Chapter 8

1. (a) $2\pi a^2$ (b) 0

3. 0

5. (a) $f = x^4/4 - x^2y^3$ (b) $-1/4$

7. (a) Check that $\nabla \times \mathbf{F} = \mathbf{0}$ (b) $f = 3x^2 y \cos z + C$ (c) 0

9. $23/6$

11. No: $\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$

13. (a) $\nabla f = 3ye^{z^2}\mathbf{i} + 3xe^{z^2}\mathbf{j} + 6xyze^{z^2}\mathbf{k}$ (b) 0 (c) Both sides are 0.

15. $8\pi/3$

17. $\pi a^2/4$

19. 21

21. (a) $\mathbf{G}$ is conservative; $\mathbf{F}$ is not.
(b) $\mathbf{G} = \nabla\phi$ if $\phi = (x^4/4) + (y^4/4) - \frac{3}{2}x^2y^2 + \frac{1}{2}z^2 + C$, where C is any constant.
(c) $\int_\alpha \mathbf{F}\cdot d\mathbf{s} = 0$; $\int_\alpha \mathbf{G}\cdot d\mathbf{s} = -\frac{1}{2}$; $\int_\beta \mathbf{F}\cdot d\mathbf{s} = \frac{1}{3}$; $\int_\beta \mathbf{G}\cdot d\mathbf{s} = -\frac{1}{2}$

23. Use $(\nabla\cdot\mathbf{F})(x_0, y_0, z_0) = \operatorname{limit}_{\rho\to 0}\frac{1}{V(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}\, dS$ from Section 8.4.