Applications to Business and Economics

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Aims and Objectives

- Introduce e procedure for using definite integration in applications
- Applying the procedure to calculating the amount of an income stream
- Applying the procedure to calculating the net change

Contents

A Procedure for Using Definite Integration

A Procedure for Using Definite Integration in Application

Tu use definite integration to model the "totality" of a quantity $f(x)$ over an interval $a \leq x \leq b$:

1. Divide the interval $a \leq x \leq b$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$.
   For $j = 1, 2, \ldots, n$ chose x_j form the j-th subinterval.
2. Approximate the contribution to the total quantity that comes from the j-th subinterval by $f(x_j)\,\Delta x$.
   Add the individual contributions to estimate the total quantity:
   $$[f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x.$$

A Procedure for Using Definite Integration. (Continued)

A Procedure for Using Definite Integration in Application

3. Take the limit as $n \to \infty$ to pass from the approximation to the exact value of the quantity.
4. Use the fact that

$$\lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x = \int_a^b f(x)\,dx$$

to transform the limit of a sum in step 3 to a definite integral. Evaluate the definite integral.

The Future Value of an Income Stream

Example

Money is transferred contiuously into an account
at the constant rate of 1,200 € per year.
The account earns interest at the annual rate of 8%
compounded continuously.
How much will be in the account by the end of 2 years?

Solution. . .

Recall that P euros invested 8% compounded continuously
will be worth $Ke^{\frac{8}{100}t}$ t years later. □

The Future Value of an Income Stream. (Continued)

...Solution...

To approximate the future value of the income stream
divide the 2-year time interval $0 \leq t \leq 2$
into n equal subintervals of length Δt years
and let t_j denote the beginning of the j-th subinterval.
During the j-th subinterval the money deposited equals $1200\,\Delta t$.
If it were deposited at the begining of the subinterval (at time t_j),
it would remain in the account for $2 - t_j$ years
and therefore would grow to $(1200\,\Delta t)e^{\frac{8}{100}(2-t_j)}$ euros.
Thus, the future value of money deposited
during the j-th subinterval is approximately

$$1200e^{0.08(2-t_j)}\,\Delta t.$$

$\square$

The Future Value of an Income Stream. (Continued)

. . . Solution. . .

The future value of the entire income stream
is approximately the sum of the future value of the money deposited during each of the n subintervals, i.e.,

$$FV \approx \sum_{j=1}^{n} 1200e^{0.08(2-t_j)}\,\Delta t.$$

$\square$

The Future Value of an Income Stream. (Continued)

. . . Solution.

As $n \to \infty$, the length of each subinterval approaches 0 and the approximation approaches the true value:

$$
\begin{aligned}
FV &= \lim_{n\to\infty} \sum_{j=1}^{n} 1{,}200 e^{0.08(2-t_j)}\,\Delta t \\
&= \int_0^2 1{,}200 e^{0.08(2-t)}\,dt = 1{,}200 e^{0.16} \int_0^2 e^{-0.08t}\,dt \\
&= -\frac{1{,}200}{0.08} e^{0.16}\left(e^{-0.08t}\right)\Big|_0^2 = -15{,}000 e^{0.16}\left(e^{-0.16} - 1\right) \\
&= -15{,}000 + 15{,}000 e^{0.16} \approx 2{,}602.66.
\end{aligned}
$$

$\square$

Net Excess Profit

Example

Suppose that t years from now one investment will be generating profit at the rate of $F_1'(t) = 50 + t^2$ hundred euros per year, while a second investment will be generating profit at the rate of $F_2'(t) = 200 + 5t$ hundred euros per year.

1. For how many years does the rate of profitability of the second investment exceed that of the first?
2. Compute the net excess profit for the time period determined in part 1.
   Interpret the net excess profit as an area.

Net Excess Profit. (Continued)

Solution. . .

1. The rate of profitability of the second investment exceeds that of the first until

$$\begin{aligned} F_1'(t) &= F_2'(t) \\ 50 + t^2 &= 200 + 5t \\ t^2 - 5t - 150 &= 0, \end{aligned}$$

yelding $t = 15$ (since $t = -10$ is rejected).

□

Net Excess Profit. (Continued)

. . . Solution.

2. The net excess profit for the period $0 \leq t \leq 15$ is given by the definite integral:

$$NE = \int_0^{15} [F_2'(t) - F_1'(t)]\, dt$$

$$= \int_0^{15} [(200 + 5t) - (50 + t^2)]\, dt = \int_0^{15} (150 + 5t - t^2)\, dt$$

$$= \left(150t + \frac{5}{2}t^2 - \frac{1}{3}t^3\right)\Big|_0^{15} = 1687.50$$

hundred euros, i.e. 168,750 €.
The net excess profit is the area between the rate curves.

□

Net Excess Profit. (Continued)

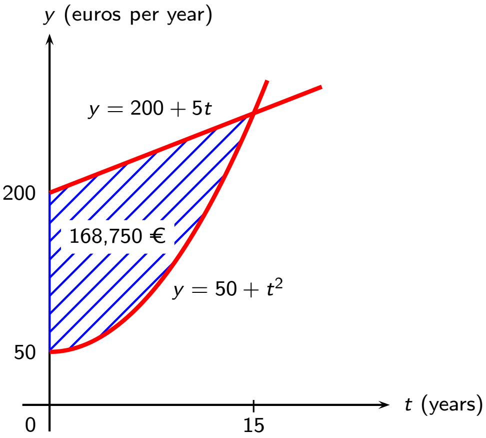

Figure: The net excess profit

Net Earnigns

Example

Suppose that when it is t years old, a particular industrial machine generates revenue at the rate $R'(t) = 5{,}000 - 20t^2$ euros per year and that operating and servicing costs related to the machine accumulate at the rate $C'(t) = 2{,}000 + 10t^2$ euros per year.

1. How many years pass before the profitabilityof the machine begins to decline?
2. Compute the net earnings generated by the machine over the time period determined in part 1.

Net Earnigns. (Continued)

. . . Solution.

1. The profit after t years of operation is $P(t) = R(t) - C(t)$ and the rate of profitability is

$$P'(t) = R'(t) - C'(t) = (5{,}000 - 20t^2) - (2{,}000 + 10t^2)$$
$$= 3{,}000 - 30t^2.$$

The profitability begins do decline when

$$\begin{aligned} P'(t) &= 0 \\ 3{,}000 - 30t^2 &= 0 \\ t^2 &= 100 \\ t &= 10. \end{aligned}$$

$\square$

Net Earnigns. (Continued)

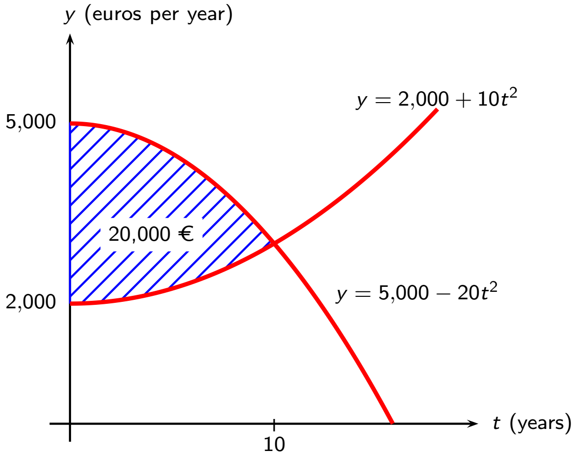

Figure: The net earnings from an industrial machine.

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 442–459.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 254–264.

Summary

- Present value of an income stream
- Net change