Definite Integration

The Definite Integral

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Aims and Objectives

- Definite integral as the area under a curve
  and the limit of a sum.
- The conenction between definite integral
  and antideifferentiation: the funcamental theorem of calculus
- The area between two curves

Contents

Area Under a Curve

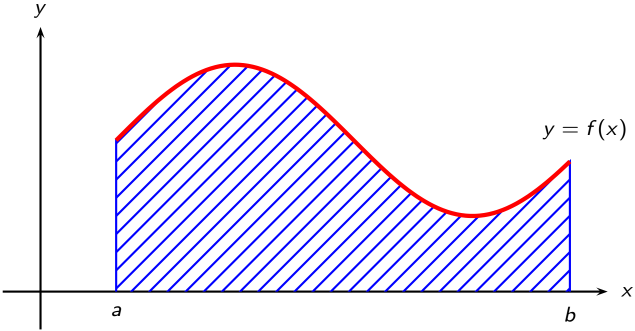

Figure: The area under the curve $y = f(x)$ above the interval $a \leq x \leq b$.

Area Under a Curve. (Continued)

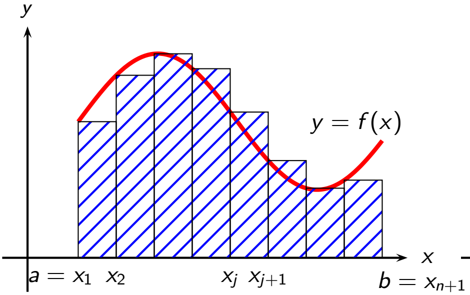

Figure: n subintervals

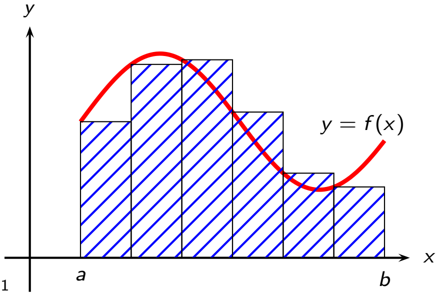

Figure: 6 subintervals

Area Under a Curve. (Continued)

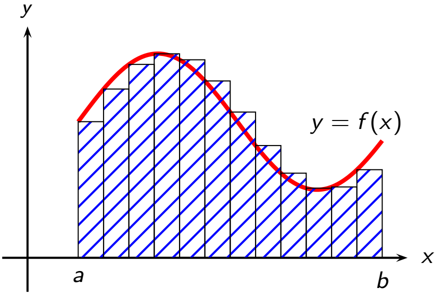

Figure: 12 subintervals

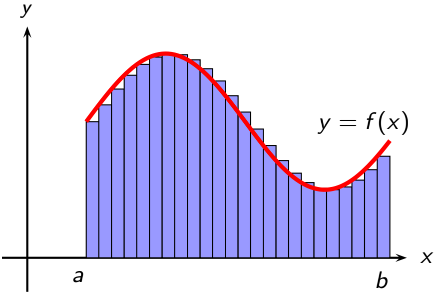

Figure: 24 subintervals

Area Under a Curve. (Continued)

- The area of the j-th rectangle:

$$f(x_j)\Delta x$$

- The sum of the areas of all n rectangles:

$$\begin{aligned} S_n &= f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x \\ &= [f(x_1) + f(x_2) + \cdots + f(x_n)]\,\Delta x \end{aligned}$$

Area Under a Curve. (Continued)

Area Under a Curve

Let $f(x)$ be continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$.
Then the region under the curve $y = f(x)$
above the interval $a \leq x \leq b$ has area

$$A = \lim_{n \to \infty} [f(x_1) + f(x_2) + \cdots + f(x_n)] \, \Delta x,$$

where x_j is the left endpoint of the j-th subinterval
if the interval $a \leq x \leq b$ is divided into n equal parts,
each of length $\Delta x = \frac{b-a}{n}$.

The Definite Integral

The Definite Integral

Let $f(x)$ be continuous on the interval $a \leq x \leq b$.
Subdivide the interval $a \leq x \leq b$ into n equal parts,
each of length $\Delta x = \frac{b-a}{n}$, and
let x_j be a number chosen from the j-th subinterval $(j = 1, \ldots, n)$.
Then, the *definite integral* of $f(x)$ over $a \leq x \leq b$ is given by

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\, \Delta x.$$

The Definite Integral. (Continued)

The Definite Integral and the Area Under the Curve

If $f(x)$ is continuous and $f(x) \geq 0$ on the interval $a \leq x \leq b$, then the region under the curve $y = f(x)$ above the interval has area

$$A = \int_a^b f(x)\, dx.$$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If $f(x)$ is continuous on the interval $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F(x)$ is any antiderivative of $f(x)$ on $a \leq x \leq b$.

- We shall use the notation

$$F(x)\Big|_a^b = F(b) - F(a).$$

- Thus

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

The Fundamental Theorem of Calculus. (Continued)

Example

Find

$$\int_0^1 (x^2 - \sqrt{x})\, dx.$$

Solution...

Since

$$\int (x^2 - \sqrt{x})\, dx = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + C,$$

we conclude that each antiderivative of $f(x) = x^2 - \sqrt{x}$ has the form $F(x) = \frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + C$ for constant C. $\square$

The Fundamental Theorem of Calculus. (Continued)

. . . Solution.

Hence,

$$\begin{aligned}\int_0^1 (x^2 - \sqrt{x})\,dx &= \left(\frac{1}{3}x^3 - \frac{2}{3}x^{3/2} + C\right)\Bigg|_0^1 \\ &= \left(\frac{1}{3}x^3 - \frac{2}{3}x^{3/2}\right)\Bigg|_0^1 \\ &= \left(\frac{1}{3}\cdot 1^3 - \frac{2}{3}\cdot 1^{3/2}\right) - \left(\frac{1}{3}\cdot 0^3 - \frac{2}{3}\cdot 0^{3/2}\right) = -\frac{1}{3}.\end{aligned}$$

$\square$

The Fundamental Theorem of Calculus. (Continued)

Example

Find

$$\int_1^3 4x(x^2-1)^3\,dx.$$

Solution. . .

Substitute $u = x^2 - 1$, i.e. $du = 2x\,dx$, find the indefinite integral

$$\int 4x(x^2-1)^3\,dx = \int 2u^3\,du = \frac{1}{2}u^4.$$

The limits of integration refer to the variable x and not to u.
Therefore either rewrite the antiderivative in terms of x,
or find the values of u that correspond to $x = 1$ and $x = 3$. $\square$

The Fundamental Theorem of Calculus. (Continued)

. . . Solution. . .

The first alternative:

$$\int 4x(x^2-1)^3\,dx = \frac{1}{2}u^4 = \frac{1}{2}(x^2-1)^4,$$

hence

$$\int_1^3 4x(x^2-1)^3\,dx = \left[\frac{1}{2}(x^2-1)^4\right]\Big|_1^3 = 2048 - 0 = 2048.$$

$\square$

The Fundamental Theorem of Calculus. (Continued)

. . . Solution. . .

The second alternative:
use the fact that $u = x^2 - 1$ to conclude that $u = 0$ when $x = 1$ and $u = 8$ when $x = 3$.
Hence,

$$\int_1^3 4x(x^2-1)^3\,dx = \int_0^8 u^3\,du = \left(\frac{1}{2}u^4\right)\Big|_0^8 = 2048 - 0 = 2048.$$

$\square$

The Fundamental Theorem of Calculus. (Continued)

Example

Find the area of the region bounded by the curve $y = -x^2 + x + 2$ and the x axis.

The Fundamental Theorem of Calculus. (Continued)

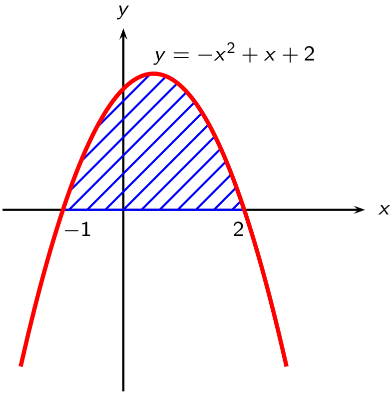

Figure: The region bounded by $y = -x^2 + x + 2$ and the x axis.

The Fundamental Theorem of Calculus. (Continued)

Solution

From the factored form of the polynomial

$$y = -x^2 + x + 2 = -(x+1)(x-2)$$

note that x*-intercepts of the curve are* $(-1, 0)$ *and* $(2, 0)$*. Hence,*

$$A = \int_{-1}^{2} (-x^2 + x + 2)\,dx = \left(-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Bigg|_{-1}^{2}$$

$$= \left(-\frac{1}{3}\cdot 2^3 + \frac{1}{2}\cdot 2^2 + 2\cdot 2\right) - \left(-\frac{1}{3}\cdot(-1)^3 + \frac{1}{2}\cdot(-1)^2 + 2\cdot(-1)\right)$$

$$= \frac{9}{2}.$$

The Area Between Two Curves

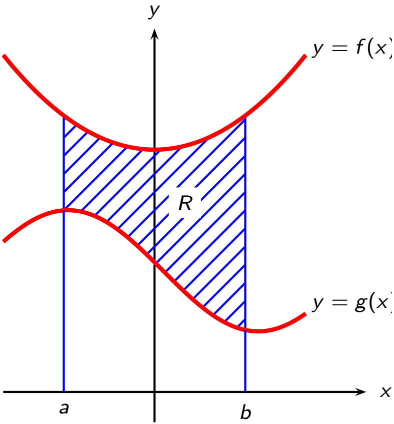

Figure: The region R between $y = f(x)$ and $y = g(x)$

The Area Between Two Curves: $R = R_1 - R_2$

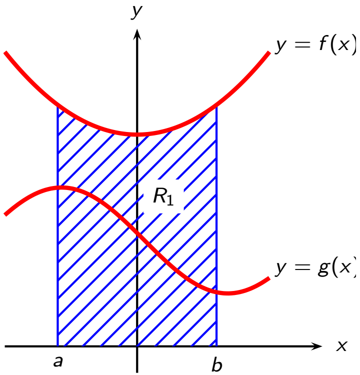

Figure: The region R_1 under the curve $y = f(x)$

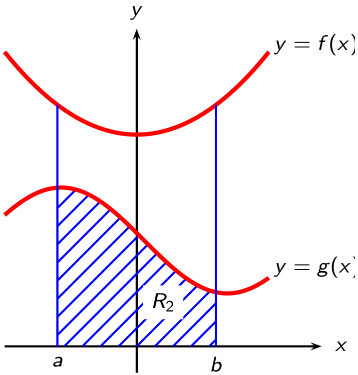

Figure: The region R_2 under the curve $y = g(x)$

The Area Between Two Curves. (Continued)

The Area Between Two Curves

If $f(x)$ and $g(x)$ are continuous on the interval $a \leq x \leq b$ and $f(x) \geq g(x)$, and A is the area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$, then

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

The Area Between Two Curves. (Continued)

Example

Find the area of the region bounded by the curves $y = x^2$ and $y = 2x$.

The Area Between Two Curves. (Continued)

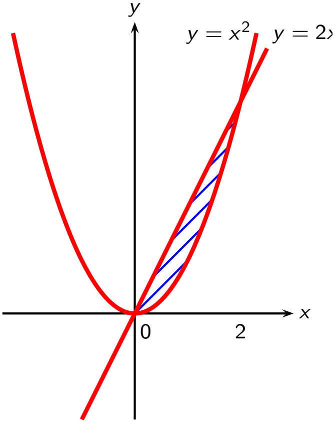

Figure: The region bounded by $y = x^2$ and $y = 2x$.

The Area Between Two Curves. (Continued)

Solution. . .

Find the point of intersection:

$$\begin{aligned} x^2 &= 2x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0, \end{aligned}$$

Intersection points: $(0,0)$ and $(2,4)$.
Notice that for $0 \le x \le 2$ the graph of $y = 2x$ lies above that of $y = x^2$. Hence

$$A = \int_0^2 (2x - x^2)\,dx = \left(x^2 - \frac{1}{3}x^3\right)\Bigg|_0^2 = \frac{4}{3}. \qquad \square$$

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 428–441.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 219–230.

Summary

- Definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}[f(x_1)+f(x_2)+\cdots+f(x_n)]\,\Delta x.$$

- Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b)-F(a), \quad \text{where } F'(x)=f(x)$$

- Area under a curve:

$$A=\int_a^b f(x)\,dx$$

- Area between two curves:

$$A=\int_a^b [f(x)-g(x)]\,dx$$