The Chain Rule

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Aims and Objectives

- Expressing the rate of change of a quantity as a product of other rates
- Differentiation of a composite function
- Using the chain rule to solve problems in business applications.

Contents

Product or rates of change

- Let C, q and t denote the cost in euros, unit produced and time in work-hours, respectively.
- The rate of change of cost with respect to output:

$$\frac{dC}{dq} \quad \text{€ per unit}$$

- The rate of change of output with respect to time:

$$\frac{dq}{dt} \quad \text{units per hour}$$

- The rate of change of cost with respect to time:

$$\frac{dC}{dt} = \frac{dC}{dq}\frac{dq}{dt} \quad \text{€ per hour}$$

The Chain Rule

The Chain Rule

Let y be a differentiable function of u,
and u be a differentiable function of x.
Then y is a composite function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

- To remember, pretend that you cancel:

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}}\frac{\cancel{du}}{dx}$$

Application: The Rate of Change of Cost with Respect to Time

Example

The cost of producing x units of a particular commodity is $C(x) = 0.2x^2 + x + 900$ euros, and the production level t hours into a particular production run is $x(t) = t^2 + 100t$ units. At what rate is cost changing with respect to time after 1 hour?

Solution...

Since

$$\frac{dC}{dx} = 0.4x + 1$$
$$\frac{dx}{dt} = 2t + 100,$$

Application: The Rate of Change of Cost. . . (Continued)

. . . Solution.

according to the chain rule,

$$\frac{dC}{dt} = \frac{dC}{dx}\frac{dx}{dt} = (0.4x + 1)(2t + 100).$$

The goal is to evaluate the derivative when $t = 1$.

$$x(1) = 1^2 + 100 \cdot 1 = 101 \quad \text{units}$$

$$\left.\frac{dC}{dt}\right|_{t=1} = (0.4 \cdot 101 + 1)(2 \cdot 1 + 100) = 4{,}222.8$$

Thus, after 1 hour the cost is increasing
at the rate 4,222.80 € per hour. □

Differentiation of Composite Functions

Differentiation of a Composite Function

If $g(u)$ and $h(x)$ are differentiable functions, then

$$\frac{d}{dx}g[h(x)] = g'[h(x)]h'(x).$$

- Indeed, for

$$y = g(u), \quad u = h(x),$$

by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = g'(u)h'(x) = g'[h(x)]h'(x).$$

Example of Differentiating a Composite Function

Example

Differentiate the function $f(x) = \sqrt{x^2 - x + 3}$.

Solution...

The function is of the form

$$f(x) = \left(\boxed{x^2 - x + 3}\right)^{1/2}.$$

Then

$$\left(\boxed{x^2 - x + 3}\right)' = 2x - 1$$

□

Example of Differentiating a Composite Function. (Continued)

. . . Solution.

and according to the rule of differentiating a composite function,

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\boxed{x^2 - x + 3}\right)^{-1/2}\left(\boxed{x^2 - x + 3}\right)' \\ &= \frac{1}{2}\left(\boxed{x^2 - x + 3}\right)^{-1/2}(2x - 1) = \frac{2x - 1}{2\sqrt{x^2 - x + 3}}. \end{aligned}$$

$\square$

The Chain Rule for Power Functions

The General Power Rule

For any real number n and differentiable function h,

$$\frac{d}{dx}[h(x)]^n = n[h(x)]^{n-1}\frac{d}{dx}[h(x)].$$

Example of Differentiating a Power Function

Example

Differentiate the function $f(x) = \frac{1}{(3x-2)^4}$.

Solution...

$$f(x) = (3x-2)^{-4}$$

$$f'(x) = -4(3x-2)^{-5}\frac{d}{dx}(3x-2) = -4(3x-2)^{-5}\cdot 3 = -\frac{12}{(3x-2)^5}$$

$\square$

The Chain Rule for Logarithmic Functions

The Chain Rule for Logarithmic Functions

If $h(x)$ is a differentiable function, then

$$\frac{d}{dx}[\ln h(x)] = \frac{1}{h(x)}\frac{d}{dx}[h(x)].$$

Application: Marginal Revenue

Example

A manufacturer estimates that x units of a commodity will be sold when the price is $p(x) = \frac{\ln(x+3)}{x+3}$ hundred euros per unit.
What is the marginal revenue when 4 units are sold.

Solution...

The revenue function is

$$R(x) = p(x)x = \frac{\ln(x+3)}{x+3}x = \frac{x\ln(x+3)}{x+3}$$

hundred euros, so the marginal revenue is □

Application: Marginal Revenue. (Continued)

...Solution...

$$\begin{aligned}
\mathrm{MR}(x) = R'(x) &= \frac{[x\ln(x+3)]'(x+3) - [x\ln(x+3)](x+3)'}{(x+3)^2} \\
&= \frac{[1\cdot\ln(x+3) + x(\ln(x+3))'](x+3) - [x\ln(x+3)]\cdot 1}{(x+3)^2} \\
&= \frac{[\ln(x+3) + x\left(\frac{1}{x+3}(x+3)'\right)](x+3) - x\ln(x+3)}{(x+3)^2} \\
&= \frac{[\ln(x+3) + \frac{x}{x+3}](x+3) - x\ln(x+3)}{(x+3)^2} \\
&= \frac{x + 3\ln(x+3)}{(x+3)^2} \qquad \square
\end{aligned}$$

Application: Marginal Revenue. (Continued)

. . . Solution.

When $x = 4$, the marginal revenue is

$$R'(4) = \frac{4 + 3\ln(4+3)}{(4+3)^2} \approx 0.20$$

hundred euros per unit; i.e. approximately 20 € per unit. □

The Chain Rule for Exponential Functions

The Chain Rule for Exponential Functions

If $h(x)$ is a differentiable function, then

$$\frac{d}{dx}[e^{h(x)}] = e^{h(x)} \frac{d}{dx}[h(x)].$$

Application: Total Revenue

Example

A manufacturer estimates that $D(p) = 8{,}000e^{-0.04p}$ units of a particular commodity will be sold when the price is p euros per unit.
What happens to the total revenue when the price is 25 € per unit?

Solution. . .

$$R(p) = pD(p) = 8{,}000pe^{-0.04p}$$

$$R'(p) = 8{,}000(e^{-0.04p} - 0.04pe^{-0.04p}) = 8{,}000(1 - 0.04p)e^{-0.04p}$$

$$R'(25) = 8{,}000(1 - 0.04 \cdot 25)e^{-0.04 \cdot 25} = 0.$$

□

Application: Total Revenue. (Continued)

. . . Solution.

It is not difficult to see that the total revenue
has a positive rate of change when $p < 25$ €
and a negative rate of change when $p > 25$ €.
we conclude that the manufacturer generates maximal revenue
exactly when the price of the commodity is 25 € per unit. □

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 148–160.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 189–198.

Summary

- The chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

- Differentiation of composite functions:

$$\frac{d}{dx}g[h(x)] = g'[h(x)]h'(x)$$

  - The general power rule
  - The chain rule for logarithmic functions.
  - The chain rule for exponential functions.

- The idea of applying the derivative of a function to determine the location of its maximum or minimum.