Marginal Analysis: Approximation by Increments

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Aims and Objectives

- Notions of marginal cost, marginal revenue, and marginal profit
- Applying marginal analysis to approximating additional value of a business function for an additional unit of the product
- Notions of average cost for a production unit and marginal average cost
- Applying approximation by increments in business applications

Contents

Marginal Analysis

- The cost of producing the $(x_0 + 1)$-st unit of a product:

$$C(x_0 + 1) - C(x_0)$$

- *Marginal cost $MC(x)$:* the derivative of the cost $C(x)$:

$$\mathrm{MC}(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

- For "small" values of h:

$$\mathrm{MC}(x_0) \approx \frac{C(x_0 + h) - C(x_0)}{h}$$

- For $h = 1$, we can make the approximation:

$$\boxed{\mathrm{MC}(x_0) \approx C(x_0 + 1) - C(x_0)}$$

Approximating $C(x_0+1) - C(x_0)$ by Means of $\mathrm{MC}(x_0)$

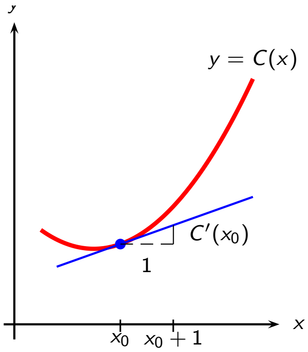

Figure: Marginal cost $\mathrm{MC}(x_0)$ for $x = x_0$ is $C'(x_0)$

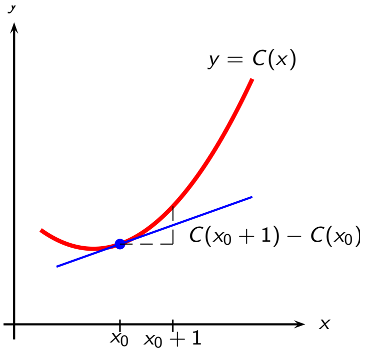

Figure: The cost of producing the $(x_0 + 1)$-st unit is $C(x_0 + 1) - C(x_0)$

Marginal Functions

Marginal Cost, Marginal Revenue and Marginal Profit

If $C(x)$ is the total cost of producing x units of a commodity, $R(x) = px$ is the revenue from selling the commodity for a price p, and $P(x) = R(x) - C(x)$ is the corresponding profit, then

- the *marginal cost function* is
$$\mathrm{MC}(x) = C'(x)$$
- the *marginal revenue function* is
$$\mathrm{MR}(x) = R'(x)$$
- the *marginal profit function* is
$$\mathrm{MP}(x) = P'(x).$$

Example of Applying Marginal Analysis

Example

A manufacturer estimates that when x units of a commodity are produced, the total cost will be $C(x) = \frac{1}{5}x^2 + 4x + 27$ euros and that all x units will be sold when the price is $p(x) = 22 - \frac{1}{4}x$ euros per unit.

1. Use the marginal cost function
   to estimate the cost of producing the fourth unit.
   What is the actual cost of producing the fourth unit?
2. Use the marginal revenue function to estimate the revenue derived from the sale of the fourth unit.
3. Sketch the profit function and determine the level
   of production where profit is maximized.
   What is the marginal profit at this optimal level of production?

Example of Applying Marginal Analysis. (Continued)

Solution...

1. The marginal cost function:

$$\mathrm{MC}(x) = C'(x) = \frac{2}{5}x + 4$$

The additional cost for the fourth unit, approximately:

$$C(4) - C(3) \approx \mathrm{MC}(3) = \frac{2}{5} \cdot 3 + 4 = \frac{26}{5} = 5.2 \quad \text{euros}$$

The exact change in cost as x increases from 3 to 4:

$$C(4) - C(3) = \left(\frac{1}{5} \cdot 4^2 + 4 \cdot 4 + 27\right) - \left(\frac{1}{5} \cdot 3^2 + 4 \cdot 3 + 27\right) = \frac{27}{5} = 5.4$$

$\square$

Example of Applying Marginal Analysis. (Continued)

. . . Solution. . .

2. The totale revenue function:

$$R(x) = p(x)x = \left(22 - \frac{1}{4}x\right)x = -\frac{1}{4}x^2 + 22x$$

The marginal revenue function:

$$\mathrm{MR}(x) = R'(x) = -\frac{1}{2}x + 22$$

The revenue derived from the sale of the fourth unit, approximately:

$$R(4) - R(3) \approx \mathrm{MR}(3) = -\frac{1}{2} \cdot 3 + 22 = \frac{41}{2} = 20.5.$$

□

Example of Applying Marginal Analysis. (Continued)

. . . Solution. . .

3. The profit function:

$$P(x) = R(x) - C(x) = \left(-\frac{1}{4}x^2 + 22x\right) - \left(\frac{1}{5}x^2 + 4x + 27\right)$$
$$= -\frac{9}{20}x^2 + 18x - 27.$$

The graph is a downward opening parabola with its highest point (vertex) above:

$$x = \frac{-b}{2a} = \frac{-18}{2(-\frac{9}{20})} = 20$$

Example of Applying Marginal Analysis. (Continued)

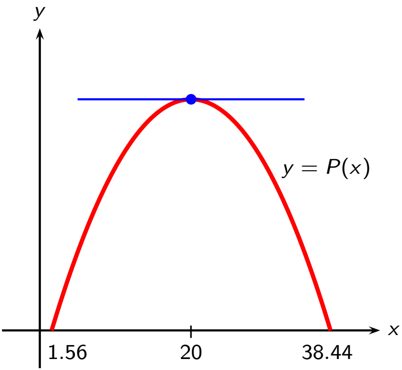

Figure: The graph of the profit function $P(x) = -\frac{9}{20}x^2 + 18x - 27$.

Example of Applying Marginal Analysis. (Continued)

. . . Solution.

The marginal profit function:

$$\mathrm{MP}(x) = P'(x) = -\frac{9}{10}x + 18$$

At optimal level of priduction $x = 20$, the maginal profit is

$$\mathrm{MP}(20) = -\frac{9}{10} \cdot 20 + 18 = 0.$$

$\square$

Average Cost and its Marginal Function

Average Cost and Marginal Average Cost

If $C(x)$ is the total cost of producing x units of a certain commodity,

- the *average cost function* is

$$\text{AC}(x) = \frac{C(x)}{x},$$

- the *marginal average cost function* is

$$\text{MAC}(x) = (\text{AC})'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right].$$

A Business Applciation

Example

Let $C(x) = \frac{1}{5}x^2 + 4x + 27$ be the total cost function for the commodity in the previous example.

1. Find the average cost and the marginal average cost for the commodity.
2. For what level of production is marginal average cost equal to 0?
3. For what level of production does marginal cost equal average cost?

A Business Applciation. (Continued)

Solution. . .

1. The average cost:

$$AC(x) = \frac{C(x)}{x} = \frac{\frac{1}{5}x^2 + 4x + 27}{x} = \frac{1}{5}x + 4 + \frac{27}{x}$$

The marginal average cost:

$$MAC(x) = (AC)'(x) = \frac{d}{dx}\left(\frac{1}{5}x + 4 + \frac{27}{x}\right) = \frac{1}{5} - \frac{27}{x^2}.$$

$\square$

A Business Applciation. (Continued)

. . . Solution. . .

2. Marginal average cost is 0 when

$$\begin{aligned} \mathrm{MAC}(x) &= 0 \\ \frac{1}{5} - \frac{27}{x^2} &= 0 \\ x^2 &= 27 \cdot 5 \\ x^2 &= 135 \\ x &= \pm\sqrt{135}. \end{aligned}$$

Since the production quantity x cannnot be negative,

$$x = \sqrt{135} \approx 11.62.$$

□

A Business Applciation. (Continued)

. . . Solution.

3. The marginal cost $\mathrm{MC}(x) = \frac{2}{5}x + 4$, equals the average cost when kur

$$\begin{aligned} \mathrm{MC}(x) &= \mathrm{AC}(x) \\ \frac{2}{5}x + 4 &= \frac{1}{5}x + 4 + \frac{27}{x} \\ \frac{1}{5}x &= \frac{27}{x} \\ x^2 &= 27 \cdot 5 \\ x &= \sqrt{135} \\ x &\approx 11.62. \end{aligned}$$

$\square$

Approximating the Change of a Function

- The derivative of a function f at x_0:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- For small values of h:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

- The approximate value of the "rise":

$$f(x_0 + h) - f(x_0) \approx f'(x_0)h$$

- The approximate value of the function:

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h$$

Approximation by Increments

Approximation by Increments

If $y = f(x)$ is differentiable at $x = x_0$
and $h = \Delta x$ is a small change in x, then

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x,$$

or, by putting $\Delta y = f(x_0 + \Delta x) - f(x_0)$,

$$\Delta y \approx f'(x_0)\Delta x.$$

Applying Approximation by Increments to an Applciation

Example

Suppose the total cost of manufacturing q units of a commodity is $C(q) = \frac{7}{2}q^2 + 16q + 37$ euros.
If the current level of production is 30 units,
estimate how the total cost will change
if 30.5 units are produced.

Applying Approximation by Increments. (Continued)

Solution. . .

The current level level of production is $q_0 = 30$,
the change in production is $\Delta q = 0.5$.
By the approximation formula, the corresponding change in cost:

$$\Delta C = C(30.5) - C(30) \approx C'(30)\Delta q = C'(30) \cdot 0.5.$$

Since

$$C'(q) = 7q + 16,$$

it follows that

$$\Delta C \approx C'(30) \cdot 0.5 = (7 \cdot 30 + 16) \cdot 0.5 = 113.$$

□

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 133–148.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 178–188.

Summary

- Marginal functions
  - Marginal cost: $MC(x) = C'(x)$
  - Marginal revenue: $MR(x) = R'(x)$
  - Marginal profit: $MP(x) = P'(x)$
- Applying marginal analysis; e.g.,

$$C(x_0 + 1) - C(x_0) \approx MC(x_0)$$

- Average cost per unit and marginal average cost
  - Average cost: $AC(x) = \frac{C(x)}{x}$
  - Marginal average cost: $MAC(x) = (AC)'(x) = \frac{d}{dx}\left[\frac{C(x)}{x}\right]$
- Applying approximation by increments to business applications:

$$\Delta y \approx f'(x_0)\Delta x$$