The Product and Quotient Rules

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Aims and Objectives

- Differentiation of a product of two functions
- Differentiation of a quotient of two functions
- Differentiation of an exponential function
- Differentiation of a logarithmic function
- Applying the rules to business applications.

Contents

1. The Product and Quotient Rules
   - The Product Rule
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2. Differentiation of Exponential and Logarithmic Functions
   - Differentiaion of a Logarithmic Function
   - Differentiaion of an Exponential Function

Differentiation of a Product

The Product Rule

The product of two differentiable functions
is a differentiable function and

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$$

or

$$(fg)' = f'g + fg'$$

In words, the derivative of a product of two functions
is the derivative of the first times the second plus the first times
the derivative of the second.

A Numeric Example

Example

By applying the product rule, differentiate the function

$$P(x) = (x^2 + 1)(3x - 1).$$

Solution.

$$\begin{aligned} P'(x) &= [(x^2 + 1)(3x - 1)]' \\ &= [(x^2 + 1)]'(3x - 1) + (x^2 + 1)[(3x - 1)]' \\ &= (2x)(3x - 1) + (x^2 + 1) \cdot 3 = 9x^2 - 2x + 3 \end{aligned}$$

$\square$

Differentiation of a Quotient

The Quotient Rule

The quotient of two differentiable functions (nominator non-zero) is a differentiable function and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{g(x)^2}$$

or

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

In words, the derivative of a quotient of two functions is the derivative of the first times the second minus the first times the derivative of the second, all over the second squared.

An Application: Profit

Example

The profit derived from the sale of q units of a certain product is

$$P(q) = \frac{-q^3 + 27q^2 + 160q + 5}{q + 4} \quad \text{thousand €}.$$

At what rate is the profit changing with respect to sales when $q = 2$?

An Application Example: Profit. (Continued)

Solution.

We need to compute $P'(2)$.

$$P'(q)$$

$$= \frac{(-q^3 + 27q^2 + 160q + 5)'(q+4) - (-q^3 + 27q^2 + 160q + 5)(q+4)'}{(q+4)^2}$$

$$= \frac{(-3q^2 + 54q + 160)(q+4) - (-q^3 + 27q^2 + 160q + 5) \cdot 1}{(q+4)^2}$$

$$= \frac{-2q^3 + 15q^2 + 216q + 635}{(q+4)^2}$$

$$P'(2) = \frac{-2 \cdot 2^3 + 15 \cdot 2^2 + 216 \cdot 2 + 635}{(2+4)^2} \approx 30.861$$

thousand € per unit; i.e. 30,861 € per unit. □

Note on Using the Quotient Rule

Remember!

Quotient rule is cumbersome. So, don't use it unnecessarily.

Example

Differentiate the function

$$y = \frac{3}{2x^2} - \frac{x}{3} + \frac{5}{4} + \frac{x-1}{x}$$

The Logarithmic Rule

The Derivative of $\ln x$

For $x > 0$ holds

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

An Application: Revenue

Example

A manufacturer estimates that q units of a particular commodity will be sold when the price is

$$p(q) = 112 - q \ln q^3 \quad \text{hundred euros per unit.}$$

At what rate does the total revenue associated with the commodity change when 4 units are sold?

An Application: Revenue. (Continued)

Solution. . .

The total revenue is

$$R(q) = p(q)q = (112 - q\ln q^3)q$$
$$= 112q - q^2(3\ln q) = 112q - 3q^2\ln q$$

hundred euros, so the derivative is

$$R'(q) = 112 - 3[(q^2)'\ln q + q^2(\ln q)']$$
$$= 112 - 3\left(2q\ln q + q^2\frac{1}{q}\right) = 112 - 6q\ln q - 3q.$$

$\square$

An Application: Revenue. (Continued)

. . . Solution.

When $q = 4$, the rate of change is

$$R'(4) = 112 - 6 \cdot 4 \cdot \ln 4 - 3 \cdot 4 \approx 66.73$$

hundred euros per unit; i.e., 6,673 € per unit. □

The Exponential Rule

The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

or

$$(e^x)' = e^x$$

In words, e^x is its own derivative.

Graphical Interpretation

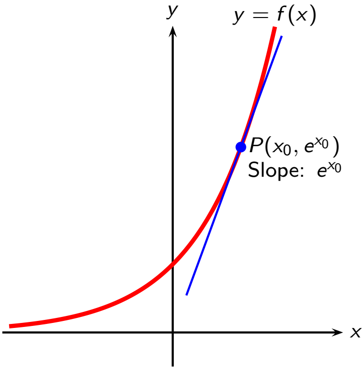

Figure: At each point on the curve, the slope equals the value

An Application: Optimization of Revenue

Example

A manufacturer estimates that that the demand for a particular commodity is

$$D(p) = 5,000e^{-p} \quad \text{units}$$

when the price is p hundred euros per unit.
What happens to the total revenue when the price is 90 €?
What happens when the price is 110 €?
What can you conclude about the revenue when the price is 100 €?

An Application: Optimization of Revenue. (Continued)

Solution. . .

$$R(p) = pD(p) = 5,000pe^{-p} \quad \text{hundred euros}$$

$$R'(p) = (5,000pe^{-p})' = 5,000\left(\frac{p}{e^p}\right)'$$

$$= 5{,}000\frac{(p)'e^p - p(e^p)'}{(e^p)^2} = 5,000\frac{1-p}{e^p} \quad \text{hundred euros per 100 €}$$

$\square$

An Application: Optimization of Revenue. (Continued)

. . . Solution.

When the price is 90 €, we have $p = 0.9$:

$$R'(0.9) = 5,000\frac{1-0.9}{e^{0.9}} \approx 203.29 > 0$$

When the price is 110 €, we have $p = 1.1$:

$$R'(1.1) = 5,000\frac{1-1.1}{e^{1.1}} \approx -166.44 < 0$$

Finally, for the price 100 €, we have $p = 1$:

$$R'(1) = 5,000\frac{1-1}{e^{1}} = 0,$$

which tells us that maximal revenue occurs at this price. □

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 122–133.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 168–178.

Summary

- The product rule:

$$(fg)' = f'g + fg'$$

- The quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

- Logarithmic rule:

$$(\ln x)' = \frac{1}{x}$$

- Exponential rule:

$$(e^x)' = e^x$$

- Investigating increasing and decreasing functions, and peeks on the graph of a function by using the derivative.