Ordinary Annuities

F. M. Berisha

South East European University, Tetovo

Aims and Objectives

- Understanding relationship between the present value of compound interest and ordinary annuities
- Calculating the present value of an ordinary annuity with periodic payments
- Applying in practical applications

Contents

Present Value of Ordinary Annuities

- Suppose that a lump sum of money is invested in an *ordinary annuity* that earns interest at rate $p\%$, compounded annually, in order to receive (withdraw) payments of size R at the end of each year.
- Given:
    - P – present value of the ordinary annuity,
    - R – value of a payment (withdrawal),
    - p – annual compound interest rate,
    - n – number of years of duration of the ordinary annuity.

Present Value of Ordinary Annuities. (Continued)

- The present value of the first payment (payable after 1 period):

$$R \cdot \frac{1}{r}, \qquad \text{where is} \quad r = 1 + \frac{p}{100}$$

- The present value of the second payment (payable after 2 periods):

$$R \cdot \frac{1}{r^2}$$

- ...
- The present value of the last payment (payable after n periods):

$$R \cdot \frac{1}{r^n}$$

Present Value of Ordinary Annuities. (Continued)

- Hence, the present value of the ordinary annuity is

$$\begin{aligned} P = R \cdot \frac{1}{r} + R \cdot \frac{1}{r^2} + \cdots + R \cdot \frac{1}{r^n} \\ = R \cdot \frac{1}{r^n}(r^{n-1} + r^{n-2} + r^{n-3} + \cdots + 1). \end{aligned}$$

- Recall the sum of a geometric sequence:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$$

Present Value of an Ordinary Annuity with Annual Payments

Present Value of an Ordinary Annuity

If a payment of R € is to be made at the end of each year for n years from an account that earns interest at a rate $p\%$, compounded annually, then the account is an ordinary annuity, and its *present value* is

$$\boxed{P = R\frac{r^n - 1}{r^n(r-1)},}$$

Present Value with Annual Payments

Example

What is the lump sum that one must invest in an annuity in order to receive 5,000 € at the end of each year for the next 12 years, if the annuity pays 10%, compounded annually?

Present Value with Annual Payments. (Continued)

Solution.

We have $R = 5{,}000$, $p = 10$, $n = 12$.

$$P = R\frac{r^n - 1}{r^n(r-1)}$$

$$r = 1 + \frac{10}{100} = 1.1$$

$$P = 5{,}000 \cdot \frac{1.1^{12} - 1}{1.1^{12} \cdot (1.1 - 1)}$$

$$\approx 5{,}000 \cdot 6.813692 \approx 34{,}068.46.$$

$\square$

Present Value with Periodic Payments

Example

What is the present value of an annuity of 4,000 €
payable at the end of each 6-month period for 12 years
if money is worth 8%, compounded semiannually?

Present Value of an Ordinary Annuity with Periodic Payments

Present Value of an Ordinary Annuity with Periodic Payments

If a payment of R € is to be made at the end of each period m times per year for n years from an account that earns interest at a rate $p\%$, compounded m times per year, then the *present value* of the ordinary annuity is

$$\boxed{P = R\frac{r^{mn} - 1}{r^{mn}(r-1)},}$$

where now is

$$\boxed{r = 1 + \frac{p}{100m}.}$$

Present Value with Periodic Payments. (Continued)

Solution.

Given $R = 4{,}000$, $p = 8$, $n = 12$, $m = 2$,

$$r = 1 + \frac{8}{100 \cdot 2} = 1.04$$

$$P = 4{,}000 \cdot \frac{1.04^{2 \cdot 12} - 1}{1.04^{2 \cdot 12} \cdot (1.04 - 1)}$$

$$\approx 4{,}000 \cdot 15.246963 \approx 60{,}987.85.$$

$\square$

Duration Time of an Ordinary Annuity

Example

An amount of 87,700 € is invested in a bank
at 6%, compounded semiannually.
Calculate how many payments of 5,000 € can be received
at the end of each half-year.

Duration Time of an Ordinary Annuity. (Continued)

Solution.

We have $P = 87{,}700$, $R = 5{,}000$, $p = 6$, $m = 2$,

$$r = 1 + \frac{p}{100m} = 1 + \frac{6}{100 \cdot 2} = 1.03.$$

$$87{,}700 = 5{,}000 \frac{1.03^{2n} - 1}{1.03^{2n}(1.03 - 1)},$$

$$\frac{877}{50} \cdot 0.03 = \frac{1.03^{2n} - 1}{1.03^{2n}},$$

$$0.5262 \cdot 1.03^{2n} = 1.03^{2n} - 1.$$

□

Duration Time of an Ordinary Annuity. (Continued)

Solution.

$$1.03^{2n}(1-0.5262)=1$$
$$1.03^{2n}=\frac{1}{0.4738}$$
$$2n\log 1.03 \approx \log 2.110595$$
$$2n \approx \frac{\log 2.110595}{\log 1.03} \approx 25.27$$

Since $25 < 2n < 26$, the annuity will generate 25 complete payments of 5,000 €. □

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 421–432.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 7-1–7-84.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 85–90.

Summary

- Relationship between compound interest
  and calculating the present value of an ordinary annuity.
- Applications of ordinary annuities
  - with a single payment period per year
  - with multiple payment periods per year