Compound Interest

Continuous Compounding

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Aims and Objectives

- Understanding the relationship between percentual calculation and compound interest
- Calculating compound interest with different compounding periods
- Continuous compounding

Contents

Compound Interest

- A *compound interest* investment is one in which interest is added to the account periodically and in effect the deposit earns "interest on interest."
- Given:
  - P – principal (present value),
  - p – annual compound interest rate,
  - n – number of years of duration of the investment,
  - P_n – future value.

Compound Interest. (Continued)

- Future value after the first *compounding:*

$$P_1 = P + \frac{Pp}{100} = P\left(1 + \frac{p}{100}\right)$$

- Future value after the second compounding:

$$P_2 = P_1 + \frac{P_1 p}{100} = P_1\left(1 + \frac{p}{100}\right) = P\left(1 + \frac{p}{100}\right)^2$$

- Future value after the third compounding:

$$P_3 = P_2 + \frac{P_2 p}{100} = P_2\left(1 + \frac{p}{100}\right) = P\left(1 + \frac{p}{100}\right)^3,$$

Compound Interest with Annual Compounding

Future Value with Annual Compounding

If P € are invested at annual compound interest rate $p\%$, compounded annually, the *future value* after n years will be

$$P_n = P\left(1 + \frac{p}{100}\right)^n.$$

Compound Interest Factor with Annual Compounding

The value

$$r = 1 + \frac{p}{100}$$

is called *compound interest factor*.

Compound Interest with Periodic Compounding

Future Value with Periodic Compounding

If P € are invested at annual compound interest rate $p\%$,
compounded m times per year,
then the total number of compounding periods is mn,
the interest rate per compounding period is $\frac{p}{m}\%$
and the future value after n years is

$$P_{mn} = P\left(1 + \frac{p}{100m}\right)^{mn}.$$

Compound Interest with Periodic Compounding. (Cont.)

Compound Interest Factor with Periodic Compounding

If the interest is compounded m times per year,
then the compound interest factor is

$$r = 1 + \frac{p}{100m}.$$

Future Value with Annual Compounding

Example

If 10,000 € were invested in a bank in 1994 at 8%, compounded annually, what will the future value be in 2015?

Solution.

Since $P = 10{,}000$, $p = 8$, $n = 2015 - 1994 = 21$, we have

$$P_{21} = 10{,}000 \left(1 + \frac{8}{100}\right)^{21} \approx 50{,}338.34.$$

□

Present Value with Annual Compounding

Example

What present value should be deposited in a bank at 5%, compounded annually, to grow to 1,000 € in 5 years?

Solution.

We have $p = 5$, $P_5 = 1{,}000$ and $n = 5$. P is required.

$$P = P_n \left(1 + \frac{p}{100}\right)^{-n}$$

$$P = 1{,}000 \left(1 + \frac{5}{100}\right)^{-5} \approx 783.53.$$

□

Annual Compound Interest Rate

Example

What compound interest rate has the bank payed
if 1,000 € have earned an interest of 400 € in 8 years?

Solution.

We have $P = 1{,}000$, $p = ?$,

$$P_n = P + I = 1{,}000 + 400 = 1{,}400.$$

$$p = 100 \left(\sqrt[n]{\frac{P_n}{P}} - 1 \right) = 100 \left(\sqrt[8]{\frac{1{,}400}{1{,}000}} - 1 \right) \approx 4.3.$$

□

Number of Semiannual Compounding Periods

Example

In what time will 50,000 € earn 25,000 € at 6%, compounded semiannually?

Number of Semiannual Compounding Periods. (Continued)

Solution.

We have $P = 50{,}000$, $p = 6$, $m = 2$,

$$P_{2n} = P + I = 50{,}000 + 25{,}000 = 75{,}000.$$

$$75{,}000 = 50{,}000\left(1 + \frac{6}{100 \cdot 2}\right)^{2n}$$

$$1.03^{2n} = 1.5.$$

$$2n \log 1.03 = \log 1.5$$

$$n = \frac{\log 1.5}{2 \log 1.03} \approx 6.86.$$

Thus, the given interest will be earned (and exceeded) in 7 years. □

Continuous Compounding

- In more advanced applications (macroeconomic, usually), where changes happen at a high frequency continuous compounding of interest is used.
- Given:
  - P – principal (present value),
  - p – annual compound interest rate, compounded continuously,
  - n – number of years,
  - $P_{\infty n}$ – future value.

Continuously Compounded Interest

Continuously Compounded Interest

If P € are invested at annual compound interest rate $p\%$, compounded continuously,
then the future value after n years will be

$$\boxed{P_{\infty n} = P e^{n \frac{p}{100}}}$$

Present Value with Continuous Compounding

Example

Miss. Grace wants to enrolle for studies of business informatics. After graduation in 4 years, she wants to make a trip to the USA which she estimates to cost 5,000 €.
How much should she invest today at 7%,
compounded continuously, in order to have enough for the trip.

Solution.

We have $n = 4$, $p = 7$, $P_{\infty 4} = 5{,}000$.

$$P = P_{\infty n} e^{-n \frac{p}{100}} = 5{,}000 e^{-4 \frac{7}{100}} \approx 3{,}778.92.$$

□

Doubling Time

Example

In what time will a sum double if invested
at the annual rate of 8%, compounded continuously?

Doubling Time. (Continued)

Solution.

We have $p = 8$.

$$P_{\infty n} = 2P$$

$$Pe^{n\frac{p}{100}} = 2P$$

$$e^{n\frac{p}{100}} = 2$$

$$e^{0.08n} = 2$$

$$0.08n = \ln 2$$

$$n = \frac{\ln 2}{0.08} \approx 8.66$$

□

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 411-421.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 7-1–7-84.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 73–81.

Summary

- Calculating compound interest, compounded
  - annually ($m = 1$),
  - m times per year,
  - continuously.