Annuities Due

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Aims and Objectives

- Understanding the relationship between percentual calculus, sum of a geometric sequence, and calculating the future value of an annuity due
- Calculating the future value of an annuity due with periodic payments

Contents

1. Annuities Due
   - Examples of Applications
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Future Value of an Annuity Due

- Suppose that a depositor makes equal periodic payments (deposits) D at the beginning of each year, thus contributing to an *annuity due,* which earns interest at $p\%$ compounded annually.
- Given:
  - D – value of a periodic deposit,
  - p – annual compound interest rate,
  - n – number of years of duration of the annuity due,
  - S_n – future value of the annuity due.

Future Value of an Annuity Due. (Continued)

- Future value of the first deposit after n compounding periods:

$$D\left(1+\frac{p}{100}\right)^{n}$$

- Future value of the second deposit (after $n-1$ periods):

$$D\left(1+\frac{p}{100}\right)^{n-1}$$

- ...
- Future value of the last deposit (after 1 period):

$$D\left(1+\frac{p}{100}\right)$$

Future Value of an Annuity Due. (Continued)

- Hence, the future value at the end on n-th year will be

$$\begin{aligned} S_n &= D\left(1+\frac{p}{100}\right)^n + D\left(1+\frac{p}{100}\right)^{n-1} + \cdots + D\left(1+\frac{p}{100}\right) \\ &= D\left(1+\frac{p}{100}\right)\left\{\left(1+\frac{p}{100}\right)^{n-1} + \left(1+\frac{p}{100}\right)^{n-2} + \cdots + 1\right\}. \end{aligned}$$

- Put $r = 1 + \frac{p}{100}$:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}$$

Future Value of an Annuity Due with Annual Payments

Future Value of an Annuity Due

If D € are deposited at the beginning of each year
at an annual compound interest rate of $p\%$, compounded annually,
the *future value* after n years is

$$S_n = D\frac{r(r^n - 1)}{r - 1}.$$

Future Value with Annual Payments

Example

At the beginning of each year, 1,000 € are deposited in a bank at 7.5% compounded annually.
What will the value be by the end of 7-th year?

Future Value with Annual Payments. (Continued)

Solution.

Given $D = 1{,}000$, $p = 7.5$ and $n = 7$.

$$S_n = D\frac{r(r^n - 1)}{r - 1}$$

$$r = 1 + \frac{p}{100} = 1 + \frac{7.5}{100} = 1.075.$$

$$S_7 = 1{,}000 \cdot \frac{1.075(1.075^n - 1)}{1.075 - 1} \approx 1{,}000 \cdot 9.446371 \approx 9{,}446.37$$

Duration Time of an Annuity Due

Example

How long (in years) should 10,000 € be deposited each year at 6%, compounded annually, to amount to 100,000 €?

Duration Time of an Annuity Due. (Continued)

Solution.

We have $D = 10{,}000$, $S_n = 100{,}000$,

$$r = 1 + \frac{p}{100} = 1 + \frac{6}{100} = 1.06.$$

$$100{,}000 = 10{,}000\frac{1.06(1.06^n - 1)}{1.06 - 1}$$

$$1.06^n = 10 \cdot \frac{0.06}{1.06} + 1$$

$$1.06^n \approx 1.56604.$$

$$n \log 1.06 \approx \log 1.56604,$$

$$n \approx \frac{\log 1.56604}{\log 1.06} \approx 7.70.$$

Thus, the deposits will amount to the sum (and exceed it) after 8 years. □

Future Value with Periodic Payments

Example

A person deposits 500 € at the beginning of each semester for 30 years at 6% compounded semiannually.
Calculate the future value.

Future Value of an Annuity Due with Periodic Payments

Future Value of an Annuity Due with Periodic Payments

If D € are deposited at the beginning of each period,
m times per year, at an annual compound interest rate of $p\%$,
compounded m times per year,
the *future value* after mn periods is

$$S_{mn} = D\frac{r(r^{mn}-1)}{r-1},$$

where now is

$$r = 1 + \frac{p}{100m}.$$

Future Value with Periodic Payments. (Continued)

Solution.

Given $D = 500,\ n = 30,\ m = 2,\ p = 6,$

$$r = 1 + \frac{p}{100m} = 1 + \frac{6}{100 \cdot 2} = 1.03,$$

$$S_{2 \cdot 30} = 500 \cdot \frac{1.03(1.03^{2 \cdot 30} - 1)}{1.03 - 1},$$

or

$$S_{60} \approx 500 \cdot 167.94504 \approx 83{,}972.52.$$

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 421–432.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 7-1–7-84.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 82–85.

Summary

- Relationship between the future value of compound interest and the future value of annuities due
- Calculating the future value of an annuity due
  - with a single payment period per year
  - with periodic payment periods