Loans and Amortization

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Aims and Objectives

- Understanding the relationship between ordinary annuities and amortization of loans
- Calculating the regular periodic payments required to amortize a debt
- Developing an amortization schedule
- Finding the unpaid balance of a loan

Contents

Amortization by Equal Size Payments

- Let a loan of P €, with interest at an annual rate of $p\%$ compounded annually, be amortized by n equal payments of size R € each being made at the end of a year.
- Given:
  - P – present value of the loan,
  - R – value of a single payment,
  - p – annual compound interest rate,
  - n – number of years of the amortization period.
- Note that, amortization of a loan is, in fact, an ordinary annuity (with switched roles of the debtor and the creditor).

Amortization by Equal Size Payments. (Continued)

- By making use of the formula for ordinary annuities, the present value of the loan is:

$$P = R\frac{r^n - 1}{r^n(r-1)}, \qquad \text{where is} \quad r = 1 + \frac{p}{100}.$$

- Solve the equation for the size R of a payment.

Amortization by Equal Size Payments. (Continued)

The Size of a Payment

If a loan of P €, with interest at an annual rate of p%, compounded annually, is amortized by n equal payments, being made at the end of each year, the size of each payment is

$$\boxed{R = P\frac{r^n(r-1)}{r^n-1}.}$$

The Size of an Annual Payment

Example

A loan of 50,000 €, with interest at 5%, compounded annually, is amortized over 12 years. What is the size of an annual payment?

The Size of an Annual Payment. (Continued)

Solution.

We have $P = 50{,}000$, $p = 5$, $n = 12$,

$$R = P\frac{r^n(r-1)}{r^n - 1}$$

$$r = 1 + \frac{p}{100} = 1 + \frac{5}{100} = 1.05.$$

$$R = 50{,}000\frac{1.05^{12}(1.05-1)}{1.05^{12}-1} \approx 5{,}641.27.$$

$\square$

The Present Value with Periodic Payments

Example

What mortgage with interest at 6%, compounded monthly, can payed back over 10 years by monthly payments of 200 €?

The Size of a Periodic Payment

Amortization Formula

If a loan of P €, with interest at an annual rate of $p\%$, compounded m times per year, is amortized over n years by m equal payments per year, being made at the end of each period, the size of each payment is

$$\boxed{R = P\frac{r^{mn}(r-1)}{r^{mn}-1},}$$

where now is

$$\boxed{r = 1 + \frac{p}{100m}.}$$

The Present Value with Periodic Payments. (Continued)

Solution.

We have $R = 200,\ p = 6,\ n = 10,\ m = 12,$

$$r = 1 + \frac{p}{100m} = 1 + \frac{6}{100 \cdot 12} = 1.005.$$

$$P = R\frac{r^{mn} - 1}{r^{mn}(r-1)}$$

$$= 200\frac{1.005^{12\cdot 10} - 1}{1.005^{12\cdot 10}(1.005 - 1)} \approx 18{,}014.69.$$

$\square$

Balance Reductions of a Loan

- Each payment is the sum of interest on the the *unpaid balance* and the amount used to reduce the principal (*balance reduction*).
- First payment $R = B_1 + I_1 = B_1 + \frac{Pp}{100}$

$$B_1 = R - \frac{Pp}{100}.$$

- Second payment $R = B_2 + I_2 = B_2 + \frac{(P-B_1)p}{100}$

$$\begin{aligned} B_2 = R - \frac{(P-B_1)p}{100} &= R - \frac{Pp}{100} + \frac{B_1 p}{100} \\ &= B_1 + \frac{B_1 p}{100} = B_1\left(1 + \frac{p}{100}\right) = B_1 r. \end{aligned}$$

Balance Reductions of a Loan. (Continued)

- Third payment $R = B_3 + I_3 = B_3 + \frac{(B-B_1-B_2)p}{100}$

$$B_3 = R - \frac{(P - B_1 - B_2)p}{100} = R - \frac{(P - B_1)p}{100} + \frac{B_2 p}{100}$$
$$= B_2 + \frac{B_2 p}{100} = B_2 \left(1 + \frac{p}{100}\right) = B_2 r = B_1 r^2.$$

- ...
- The balance reduction from the n-th payment:

$$\boxed{B_n = B_1 r^{n-1}.}$$

Balance Reductions of a Loan. (Continued)

Example

A mortgage of 300,000 €, with interest at 10%,
compounded annually, is amortized over 15 years.
Determine the balance reduction from the last annual payment.

Balance Reductions of a Loan. (Continued)

Solution.

Given $P = 300{,}000$, $n = 15$, $p = 10$,

$$r = 1 + \frac{p}{100} = 1 + \frac{10}{100} = 1.1.$$

$$R = 300{,}000 \frac{1.1^{15}(1.1 - 1)}{1.1^{15} - 1} \approx 39{,}442.13,$$

$$B_1 = R - \frac{Pp}{100} \approx 39{,}442.13 - \frac{300{,}000 \cdot 10}{100} \approx 9{,}442.13.$$

$$B_{15} = B_1 r^{14} \approx 9{,}442.13 \cdot 1.1^{14} \approx 35{,}856.48.$$

□

Amortization Schedule

- Denote
  - I_i interest on the unpaid balance for the i-th period,
  - B_i balance reduction from the i-th payment,
  - U_i unpaid balance after i payments have been made.

1. The interest:

$$\boxed{I_i = \frac{U_{i-1}p}{100}}$$

  - Note that when the frequency of periods is m per year, i.e. when $m \neq 1$

$$\boxed{I_i = \frac{U_{i-1}p}{100m}}$$

Amortization Schedule. (Continued)

1. The balance reduction:

$$\boxed{B_i = R - I_i}$$

2. The unpaid balance:

$$\boxed{U_i = U_{i-1} - B_i}$$

Amortization Schedule. (Continued)

i	I_i	B_i	R	U_i
0				P
1	$\frac{U_0 p}{100m}$	$R - I_1$	R	$U_0 - B_1$
2	$\frac{U_1 p}{100m}$	$R - I_2$	R	$U_1 - B_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
mn	$\frac{U_{n-1} p}{100m}$	$R - I_n$	R	$U_{n-1} - B_n$

Amortization Schedule with Annual Payments

Example

A debt of 100,000 €, with interest at 7%, compounded annually, is amortized over 5 years by annual payments.
Develop the amortization schedule.

Amortization Schedule with Annual Payments. (Continued)

Solution...

We have $P = 100{,}000$, $n = 5$, $p = 7$,

$$r = 1 + \frac{p}{100} = 1 + \frac{7}{100} = 1.07$$

$$R = P\frac{r^n(r-1)}{r^n - 1} = 100{,}000\frac{1.07^5(1.07-1)}{1.07^5 - 1} \approx 24{,}389.07$$

□

Amortization Schedule with Annual Payments. (Continued)

. . . Solution.

i	I_i	B_i	R	U_i
0				100,000
1	7,000	17,389.07	24,389.07	82,610.93
2	5,782.77	18,606.30	24,389.07	64,004.63
3	4,480.32	19,908.75	24,389.07	44,095.88
4	3,086.71	21,302.36	24,389.07	22,793.52
5	1,595.55	22,793.52	24,389.07	0

□

Unpaid Balance of a Loan

Unpaid Balance (Payoff Amount, Outstanding Principal) of a Loan

For a loan of mn payments of R € per period
at an annual interest rate of $p\%$, compounded m times per year,
the unpaid balance after i payments have been made
is the present value of an ordinary annuity with $mn - i$ payments.
That is,

$$\boxed{U_i = R\frac{r^{mn-i} - 1}{r^{mn-i}(r-1)}}$$

Unpaid Balance of a Loan. (Continued)

Example

A loan of 400,000 € at 8%, compounded semiannually,
is amortized over 20 years by six-month payments.
Determine the unpaid balance after 30 payments have been made.

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 432–439.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 7-1–7-84.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 90–101.

Summary

- Relationship between amortization of a loan by equal periodic payments and the ordinary annuity
- Calculating the size of periodic payments
- Developing the amortization schedule of a loan by
  - annual payments
  - periodic payments