Limits of Functions

F. M. Berisha

UNIVERSITETI I EJL
ЈИЕ УНИВЕРЗИТЕТ
SEE UNIVERSITY

South East European University, Tetovo

Aims and Objectives

- Introducing tha concept of limit of a function, through an intuitive approach
- Using the algebraical rules for compute a limit
- Notion of continuous function

Contents

Notion of Limit of a Function

- Roughly speaking, the limit process involves examining the behaviour of a function $f(x)$ as x approaches to a number c
  that may be or may not be in the domain of f.
- The „behaviour“ of the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

as x approaches 1:

				x approaches 1 from the left	$\rightarrow \leftarrow$	x approaches 1 from the right			
x	0.9	0.95	0.99	0.999	1	1.001	1.01	1.05	1.1
$f(x)$	1.9	1.95	1.99	1.999	$\times$	2.001	2.01	2.05	2.1

- Symbolically, we put

$$\lim_{x \rightarrow 1} f(x) = 2.$$

Notion of Limit of a Function. (Continued)

Limit of a Function

If $f(x)$ gets closer and closer to a number L
as x gets closer and closer to c from either side,
then L is the *limit of $f(x)$ as x approaches to c.*
This behaviour is expressed by writing

$$\lim_{x \longrightarrow c} f(x) = L.$$

Geometric Interpretation of a Limit

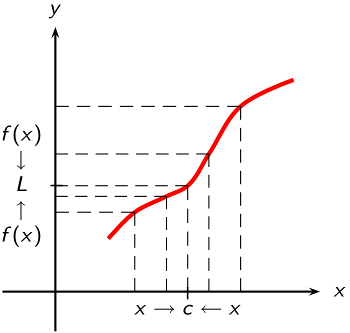


Figure: If $\lim_{x \to c} f(x) = L$, the height of the graph $y = f(x)$ approaches L as x approaches c.

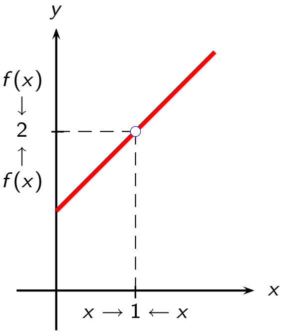
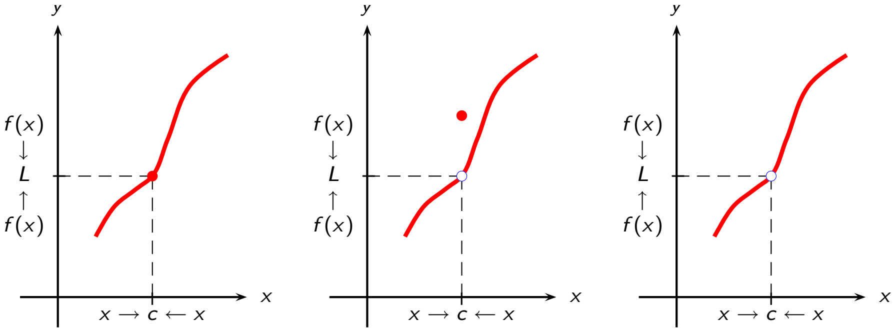


Figure: Geometric interpretation of the limit $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Three functions having a limit $\lim_{x \longrightarrow c} f(x) = L$



Two functions that without a limit $\lim_{x \to c} f(x)$

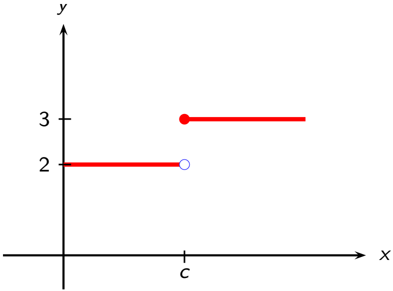


Figure: $f(x)$ does not approach to the same value as x tends towards c from both sides.

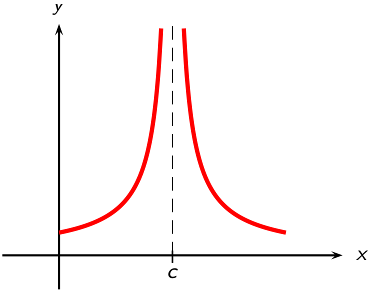


Figure: $f(x)$ increase without bound as x approaches c.

Algebraical Properties of Limits

Algebraical Properties of Limits

If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$, exist, then

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x),$$

$$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x),$$

$$\lim_{x\to c}[kf(x)] = k \lim_{x\to c} f(x) \quad \text{for any constant } k,$$

$$\lim_{x\to c}[f(x)g(x)] = [\lim_{x\to c} f(x)][\lim_{x\to c} g(x)],$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \quad \text{if } \lim_{x\to c} g(x) \neq 0,$$

$$\lim_{x\to c} f(x)^p = \left(\lim_{x\to c} f(x)\right)^p \quad \text{if } \lim_{x\to c} f(x)^p \text{ exists.}$$

Algebraical Properties of Limits. (Continued)

Remember!

The limit of a sum, a difference, a multiple, a product,
a quotient, or a power
is the sum, difference, multiple, product, quotient, or power
of individual limits, as long as all expressions involved are defined.

Limits of Two Elementary Linear Functions

Limits of Two Elementary Linear Functions

For any constant k,

$$\lim_{x \longrightarrow c} k = k$$

and

$$\lim_{x \longrightarrow c} x = c.$$

That is, the limit of a constant is the constant itself,
and the limit of $f(x) = x$ as x tends towards c is c.

Geometric Interpretations

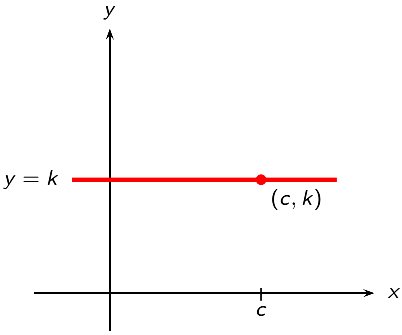


Figure: $\lim_{x \to c} k = k$

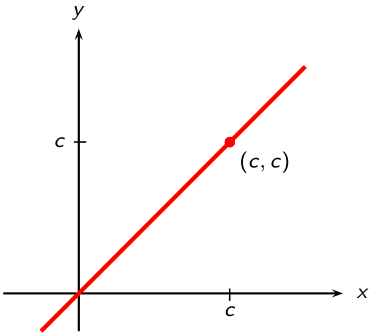


Figure: $\lim_{x \to c} x = c$

Examples of Calculating Limits of Functions

Example

Find

$$\lim_{x \to 1} \frac{3x^2 + 1}{x - 2}.$$

Solution.

Apply the quotient rule:

$$\lim_{x \to 1} \frac{3x^2 + 1}{x - 2} = \frac{\lim\limits_{x \to 1} (3x^2 + 1)}{\lim\limits_{x \to 1} (x - 2)} = \frac{3 \lim\limits_{x \to 1} x^2 + \lim\limits_{x \to 1} 1}{\lim\limits_{x \to 1} x - \lim\limits_{x \to 1} 2} = \frac{3 + 1}{1 - 2} = -4.$$

$\square$

Examples of Computing Limits of Functions. (Continued)

Example

Find

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution.

For $x = 1$ the function is not defined,
while for all other values of x the fraction can be simplified:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2.$$

□

Continuity of a Function

- A continuous function is one whose graph can be drawn without the pen leaving the paper;
  that is it has no holes of gaps.

Continuity

A function f is continuous at c if

1. $f(c)$ is defined;
2. $\lim\limits_{x \longrightarrow c} f(x)$ exists;
3. $\lim\limits_{x \longrightarrow c} f(x) = f(c)$.

Continuity of a Function on an Interval

Continuity on an Interval

A function f is said to be continuous on an interval $a \leq x \leq b$ if it is continuous at each point x in the interval.

Example of a Continuous Function

Example

Show that the polynomial $p(x) = 2x^3 - 3x + 1$
is continuous at $x = 1$.

Solution.

The value $p(1)$ is defined: $p(1) = 0$. Moreover,

$$\lim_{x \to 1} p(x) = \lim_{x \to 1} (2x^3 - 3x + 1) = 0;$$

thus,

$$\lim_{x \to 1} p(x) = p(1),$$

i.e. p(x) is continuous at $x = 1$. $\square$

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 61–74.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 143–152.

Summary

- Limit of a function
- Algebraical properties of limits
- Continuous function