Introduction to Differential Equations

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Aims and Objectives

- Introducing the notion of a differential equation
- Finding the general solution and a particular solution of a differential equation
- Solving a separable differential equation

Contents

Differential Equations

- A *differential equation* is an equation that involves a derivative or differential.
- Examples of differential equations:
  - $\dfrac{dy}{dx} = 3x^2 + 5$
  - $\dfrac{dP}{dt} = kP$
  - $\left(\dfrac{dy}{dx}\right)^2 + 3\dfrac{dy}{dx} + 2y = e^x$
- *General solution* is a complete characterization of all possible solutions of a differential equation.
- A *particular solution* is a solution that satisfies specified side conditions.

Differential Equations. (Continued)

Example

The resale value of a certain industrial machine decreases over a 10-year period at a rate that depends on its age. When the machine is t years old, the rate at which its value is changing is $220(t-10)$ euros per year. Express the value as a function of its age and initial value. If the machine was originally worth 12,000 €, how much will it be worth when it is 10 years old?

Solution. . .

Denote by $V(t)$ the value of the machine when it is t years old.

$$\frac{dV}{dt} = 220(t-10)$$

□

Differential Equations. (Continued)

... Solution...

$$V(t) = \int 220(t-10)\,dt = \int (220t - 2{,}200)\,dt$$
$$= 110t^2 - 2{,}200t + C.$$

The initial value:

$$V(0) = 110 \cdot 0^2 - 2{,}200 \cdot 0 + C = C.$$

The general solution:

$$V(t) = 110t^2 - 2{,}200t + V(0).$$

$\square$

Differential Equations. (Continued)

. . . Solution.

If $V(0) = 12{,}000$, the particular solution:

$$V(t) = 110t^2 - 2{,}200t + 12{,}000.$$

The value of the machine when it is 10 years old:

$$V(10) = 110 \cdot 10^2 - 2{,}200 \cdot 10 + 12{,}000 = 1{,}000$$

euros. □

Differential Equations. (Continued)

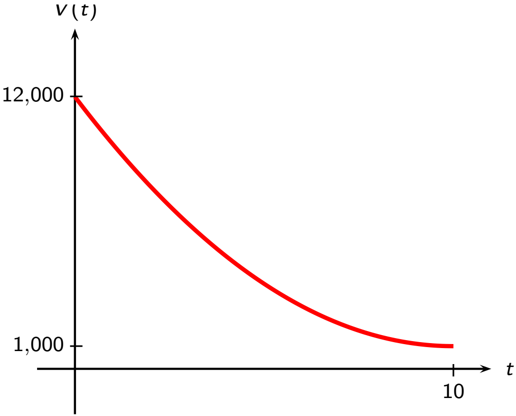

Figure: The value of the machine $V(t) = 110t^2 - 2{,}200t + 12{,}000$.

Separable Differential Equations

Separable Differential Equations

- A differentiable equation that can be written in the form

$$g(y)\,dy = h(x)\,dx$$

  is said to be *separable.*

- Its general solution is obtained by integrating both sides of the equation; i.e.,

$$\int g(y)\,dy = \int h(x)\,dx + C.$$

Learning Models

Example

The rate at which people hear about a new increase in postal rates is proportional to the number of people in the country who have not heard about it.
Express the number of people who have heard about the increase as a function of time.

Solution...

Denote by $Q(t)$ the number of people who have heard about the increase at time t, and by B the number of people in the country.

$$\frac{dQ}{dt} = k(B - Q) \qquad \square$$

Learning Models. (Continued)

. . . Solution. . .

$$\frac{1}{B-Q}dQ = k\,dt$$

$$\int \frac{1}{B-Q}dQ = \int k\,dt$$

$$-\ln|B-Q| = kt + C$$

$\square$

Learning Models. (Continued)

. . . Solution.

Since $Q(t) \leq B$,

$$
\begin{aligned}
-\ln(B-Q) &= kt + C \\
\ln(B-Q) &= -kt - C \\
B - Q &= e^{-kt}e^{-C} \\
Q &= B - e^{-C}e^{-kt}
\end{aligned}
$$

Put $A = e^{-C}$,

$$Q = B - Ae^{-kt},$$

which is the general equation of a *learning curve.* □

Learning Models. (Continued)

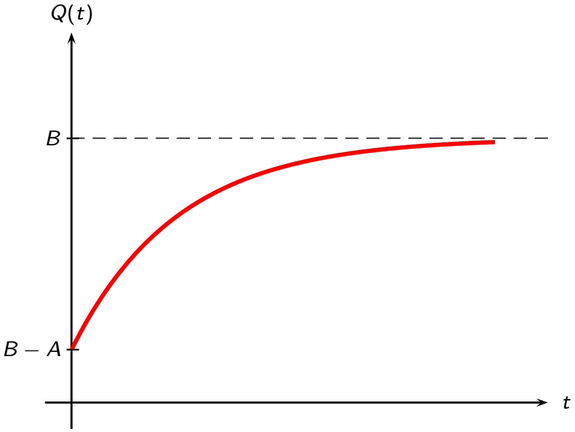

Figure: A learning curve: $Q(t) = B - Ae^{-kt}$.

A Price Adjustment Model

Example

Suppose the price $p(t)$ of a particular commodity varies in such a way that its rate of change with respect to time is proportional to the shortage $D - S$,
where $D(p)$ and $S(p)$ are demand and supply functions:
$D(p) = 8 - 2p$ and $S(p) = 2 + p$.

1. If the price is 5 € when $t = 0$ and 3 € when $t = 2$, find $p(t)$.
2. Determine what happens to $p(t)$ in the "long run", (as $t \longrightarrow +\infty$).

A Price Adjustment Model. (Continued)

Solution. . .

1. The rate of change:

$$\frac{dp}{dt} = k(D - S) = k[(8 - 2p) - (2 + p)]$$

$$= k(6 - 3p)$$

$$\frac{dp}{(6 - 3p)} = k\,dt$$

$$\int \frac{dp}{(6 - 3p)} = \int k\,dt$$

$\square$

A Price Adjustment Model. (Continued)

. . . Solution. . .

$$\begin{aligned}
-\frac{1}{3}\ln|6-3p| &= kt + C_1 \\
\ln|6-3p| &= -3kt - 3C_1 \\
6 - 3p &= e^{-3kt-3C_1} \\
&= e^{-3kt}e^{-3C_1} \\
&= Ce^{-3kt}, \quad \text{where } C = e^{-3C_1} \\
p(t) &= 2 - \frac{1}{3}Ce^{-3kt}
\end{aligned}$$

□

A Price Adjustment Model. (Continued)

. . . Solution. . .

To evaluate C:

$$p(0) = 5$$

$$2 - \frac{1}{3}Ce^{-3k\cdot 0} = 5$$

$$2 - \frac{1}{3}C = 5$$

$$C = -9$$

$$p(t) = 2 + 3e^{-3kt}$$

□

A Price Adjustment Model. (Continued)

...Solution...

To evaluate k:

$$\begin{aligned} p(2) &= 3 \\ 2 + 3e^{-3k\cdot 2} &= 3 \\ e^{-6k} &= \frac{3-2}{3} = \frac{1}{3} \\ -6k &= \ln\frac{1}{3} \\ k &= -\frac{1}{6}\ln\frac{1}{3} \approx 0.1831 \end{aligned}$$

$$p(t) = 2 + 3e^{-3\cdot 0.1831t} = 2 + 3e^{-0.5493t}$$

$\square$

A Price Adjustment Model. (Continued)

. . . Solution.

2. As $t \to \infty$,

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} (2 + 3e^{-0.5493t}) = 2,$$

which is the price at which supply equal demand. $\square$

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 398–410.

Summary

- Differential equation
- General solution; particular solution
- Separable differential equation: If

$$g(y)\,dy = h(x)\,dx,$$

then

$$\int g(y)\,dy = \int h(x)\,dx + C.$$