Integration

Antidifferentiation: The Indefinite Integral

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Aims and Objectives

- Introducing the notions of an antiderivative and the indefinite integral)
- Establishing rules for integrating common functions
- Applying algebraical rules for indefinite integration
- Applying indefinite integration to practical applications

Contents

Antiderivative of a Function

Antiderivative

A function $F(x)$ for which

$$F'(x) = f(x)$$

for every x in the domain of f is said to be an *antiderivative* of $f(x)$.

Example of an Antiderivative

Example

Verify that $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an antiderivative of $f(x) = x^2 + 5$.

Solution.

Differentiate F:

$$F'(x) = \left(\frac{1}{3}x^3 + 5x + 2\right)'$$

$$= \frac{1}{3}(3x^2) + 5 = x^2 + 5 = f(x),$$

as required. □

The General Antiderivative of a Function

- A function has more than one antiderivative:
  - For $f(x) = 3x^2$
  - $F(x) = x^3$ is an antiderivative, since

$$F'(x) = (x^3)' = 3x^2 = f(x),$$

  - but so are $x^3 + 10$, $x^3 - 4$ and $x^3 + \pi$, since

$$\frac{d}{dx}(x^3 + 10) = 3x^2, \quad \frac{d}{dx}(x^3 - 4) = 3x^2, \quad \frac{d}{dx}(x^3 + \pi) = 3x^2.$$

The General Antiderivative of a Function. (Continued)

Fundamental Property of Antiderivatives

If $F(x)$ is an antiderivative of a continuous function $f(x)$, then any other derivative of $f(x)$ has the form $G(x) = F(x) + C$ for some constant C.

Geometric Interpretation of the Fundamental Property

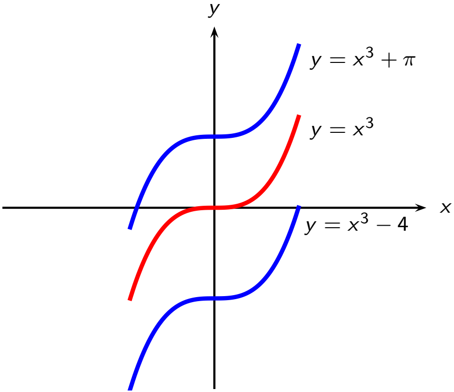

Figure: Some antiderivatives of $f(x) = 3x^2$.

The Indefinite Integral

- We will represent the family of all antiderivatives of $f(x)$ by using the symbolism:

  $$\int f(x)\,dx = F(x) + C,$$

  which is called the *indefinite integral* of f.
- For instance,

  $$\int 3x^2\,dx = x^3 + C.$$

Rules for Integrating Common Functions

Rules for Integrating Common Functions

- The *constant rule:* $\int k\,dx = kx + C$ for constant k.
- The *power rule:* $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for all $n \neq -1$.
- The *logarithmic rule:* $\int \frac{1}{x}\,dx = \ln|x| + C$ for all $x \neq 0$.
- The *exponential rule:*
  $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$ for constant $k \neq 0$.

Rules for Integrating Common Functions. (Continued)

Example

Find the following integrals:

1. $\int 5\, dx$
2. $\int x^{11}\, dx$
3. $\int \sqrt{x}\, dx$

Solution...

1. Use the constant rule with $k = 5$:

$$\int 5\, dx = 5x + C.$$

□

Rules for Integrating Common Functions. (Continued)

. . . Solution.

2. Use the power rule with $n = 11$:

$$\int x^{11}\,dx = \frac{1}{12}x^{12} + C.$$

3. Use the power rule with $n = \frac{1}{2}$:

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C$$

$$= \frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{x^3} + C.$$

□

Algebraic Rules for Indefinite Integration

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- The *constant multiple rule:*

$$\int kf(x)\,dx = k\int f(x)\,dx \quad \text{for constant } k.$$

- The *sum rule:*

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

- The *difference rule:*

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

An Application: Total Cost

Example

A manufacturer has found that marginal cost is $3x^2 - 80x + 500$ euros per unit when x units have been produced. The total cost of producing the first 2 units is 1,000 €. What is the total cost of producing the first 5 units?

Solution...

Recall that the marginal cost is the derivative:

$$C'(x) = 3x^2 - 80x + 500,$$

$\square$

An Application: Total Cost. (Continued)

. . . Solution. . .

and so $C(x)$ must be an antiderivative:

$$C(x) = \int C'(x)\,dx = \int (3x^2 - 80x + 500)\,dx$$
$$= x^3 - 40x^2 + 500x + K$$

for some constant K.
The value of K is determined by the fact that

$$C(2) = 1000.$$

□

An Application: Total Cost. (Continued)

. . . Solution.

In particular,

$$2^3 - 40 \cdot 2^2 + 500 \cdot 2 + K = 1000,$$

or

$$K = 152.$$

Hence

$$C(x) = x^3 - 40x^2 + 500x + 152,$$

and the cost of producing the first 5 units is

$$C(5) = 5^3 - 40 \cdot 5^2 + 500 \cdot 5 + 152 = 1777.$$

$\square$

A Practical Application

Example

A retailer receives a shipment of $12,000$ kilograms of flower, that will be used up at the constant rate of 300 kg per week. If storage cots are 0.5 cent per kilogram per week, how much will the retailer pay in storage costs over the next 40 weeks?

Solution...

Let $S(t)$ denote the storage cost (in euros) over t weeks. The number of kilograms of flower in storage after t weeks:

$$q(t) = 12,000 - 300t.$$

$\square$

A Practical Application. (Continued)

... Solution...

The rate of change of storage cost with respect to time:

$$\frac{dS}{dt} = q(t) \cdot 0.005 = 0.005(12,000 - 300t) = 60 - 1.5t.$$

That is,

$$S(t) = \int \frac{dS}{dt}\,dt = \int (60 - 1.5t)\,dt = 60t - 0.75t^2 + C.$$

To determine C, use the fact that at the time the shipment arrives there is no cost:

$$S(0) = 0;$$

$\square$

A Practical Application. (Continued)

. . . Solution.

i.e.,

$$60 \cdot 0 - 0.75 \cdot 0^2 + C = 0,$$

or

$$C = 0.$$

Hence,

$$S(t) = 60t - 0.75t^2,$$

and the total storage cost over the next 40 weeks:

$$S(40) = 60 \cdot 40 - 0.75 \cdot 40^2 = 1200.$$

$\square$

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 372–386.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 249–258.

Summary

- Antiderivative; indefinite integral:

$$\int f(x)\,dx = F(x) + C \quad \text{if and only if} \quad F'(x) = f(x)$$

- Rules for integrating common functions
  - the power rule: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
  - The logarithmic rule: $\int \frac{1}{x}\,dx = \ln|x| + C$
  - The exponential rule: $\displaystyle\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$
- Algebraic rules for indefinite integration
  - The constant multiple rule: $\int kf(x)\,dx = k\int f(x)\,dx$
  - The sum rule: $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$