Integration by Parts

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Aims and Objectives

- Introducing a technique for integrating certain products $f(x)g(x)$.

Contents

1. Integration by Parts and Why It Works
2. How and When to Use Integration by Parts

Integration by Parts

Integration by Parts

If G is an antiderivative of g, then

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx.$$

Why Integration by Parts Works

- Use the product rule for differentiation:

$$\frac{d}{dx}[f(x)G(x)] = f'(x)G(x)+f(x)G'(x) = f'(x)G(x)+f(x)g(x).$$

- Express in terms of integrals:

$$f(x)G(x) = \int f'(x)G(x)\,dx + \int f(x)g(x)\,dx.$$

- Herefrom

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx,$$

which is precisely the formula for integration by parts.

When to Use Integration by Parts

- Use intgration by parts to integrate products $f(x)g(x)$, in which:
    - one of the factors, say $g(x)$, can be easily integrated
    - and the other, $f(x)$, becomes simplier when differentiated.

How to Use Integration by Parts

How to Use Integration by Parts

1. Select one of the factors to be integrated and the other to be differentiated.
2. Integrate dhe designated factor and multiply it by the other factor.
3. Differentiate the designated factor, multiply it by the integrated factor from step 2, and subtract the integral of this product from the result of step 2.
4. Complete the procedure by finding the new integral that was formed in step 3.

Examples

Example

Find $\int xe^{2x}dx$.

Solution...

The process of differentiation simplifies x:

$$g(x) = e^{2x} \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = \frac{1}{2}e^{2x} \quad \text{and} \quad f'(x) = 1.$$

$\square$

Examples. (Continued)

. . . Solution.

So

$$\int xe^{2x}\,dx = x\left(\frac{1}{2}e^{2x}\right) - \int 1\cdot\left(\frac{1}{2}e^{2x}\right)dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$$
$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C = \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} + C.$$

$\square$

Examples. (Continued)

Example

Find $\int x\sqrt{x+5}\,dx$.

Solution...

$$g(x) = \sqrt{x+5} \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = \int \sqrt{x+5}\,dx = \int (x+5)^{\frac{1}{2}}\,dx = \frac{2}{3}(x+5)^{\frac{3}{2}} \quad \text{and} \quad f'(x) = 1.$$

□

Examples. (Continued)

. . . Solution.

So

$$\begin{aligned}\int x\sqrt{x+5}\,dx &= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{2}{3}\int (x+5)^{\frac{3}{2}}\,dx \\ &= \frac{2}{3}x(x+5)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(x+5)^{\frac{5}{2}} + C.\end{aligned}$$

□

Examples. (Continued)

Example

Find $\int \ln x \, dx$.

Solution. . .

Write $\ln x$ as the product $1 \cdot \ln x$, where 1 is easy to integrate and $\ln x$ is simplified by differentiation:

$$g(x) = 1 \quad \text{and} \quad f(x) = \ln x.$$

Then

$$G(x) = x \quad \text{and} \quad f'(x) = \frac{1}{x}.$$

$\square$

Examples. (Continued)

. . . Solution.

So

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx = x \ln x - x + C = x(\ln x - 1) + C.$$

$\square$

Examples. (Continued)

Example

Find the particular solution of the differential equation

$$\frac{dy}{dx} = xe^{x-y}$$

that satisfies the condition $y = \ln 2$ when $x = 0$.

Solution...

Separate the variables:

$$\frac{dy}{dx} = \frac{xe^x}{e^y}$$

$$e^y\,dy = xe^x\,dx$$

$\square$

Examples. (Continued)

. . . Solution. . .

$$\int e^y\,dy = \int xe^x\,dx$$
$$e^y = \int xe^x\,dx$$

Integrate by parts:

$$g(x) = e^x \quad \text{and} \quad f(x) = x.$$

Then

$$G(x) = e^x \quad \text{and} \quad f'(x) = 1.$$

□

Examples. (Continued)

. . . Solution.

So

$$\int xe^x dx = xe^x - \int 1 \cdot e^x dx$$
$$= xe^x - e^x + C = (x-1)e^x + C.$$

The general solution of the differential equation:

$$e^y = (x-1)e^x + C$$

□

Examples. (Continued)

. . . Solution.

Determine C:

$$\begin{aligned} y(0) &= \ln 2 \\ e^{\ln 2} &= (0-1)e^0 + C \\ 2 &= -1 \cdot 1 + C \\ C &= 3 \end{aligned}$$

The particular solution:

$$e^y = (x-1)e^x + 3.$$

□

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 410–418.

Summary

- Integration by parts: If G is an antiderivative of g, then

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx.$$