Integration by Substitution

F. M. Berisha

UNIVERSITETI I EJL
ЈИЕ УНИВЕРЗИТЕТ
SEE UNIVERSITY

South East European University, Tetovo

Aims and Objectives

- Applying integration by substitution to find an antiderivative
- Integral form of the chain rule as a technique
  for simplifying an integral by changing variables.

Contents

Integral Form of the Chain Rule

- Let $G(u)$ be an antiderivative of $g(u)$
- By the chain rule

$$\frac{d}{dx}[G(u)] = G'(u)\frac{du}{dx} = g(u)\frac{du}{dx}$$

- Hence

$$\int g(u)\frac{du}{dx}\,dx = G(u) + C,$$

Integral Form of the Chain Rule. (Continued)

Integral Form of the Chain Rule

If g is a continuous function of u and $u(x)$ is a differentiable function of x, then

$$\int g(u)\frac{du}{dx}\,dx = \int g(u)\,du.$$

That is, to integrate a product of the form $g(u)\frac{du}{dx}$, in which one of the factors $\frac{du}{dx}$ is the derivative of an expression u that appears in the other factor:

1. Find the antiderivative $\int g(u)\,du$ of the factor $g(u)$ with respect to u.
2. Replace u in the answer by its expression in terms of x.

Integral Form of the Chain Rule. (Continued)

Example

Find

$$\int 6(x^2 - 5x - 3)^5(2x - 5)\, dx.$$

Solution. . .

The function under the integral sign $6(x^2 - 5x - 3)^5(2x - 5)$ is a product of the form

$$6(x^2 - 5x - 3)^5(2x - 5) = g(u)\frac{du}{dx},$$

where

$$g(u) = 6u^5 \quad \text{and} \quad u = x^2 - 5x - 3.$$

□

Integral Form of the Chain Rule. (Continued)

. . . Solution.

Hence, by the integral form of the chain rule,

$$\begin{aligned}\int 6(x^2 - 5x - 3)^5(2x - 5)\,dx \\ &= \int g(u)\frac{du}{dx}\,dx = \int 6u^5\,du \\ &= u^6 + C = (x^2 - 5x - 3)^6 + C.\end{aligned}$$

$\square$

Change of Variables

Integration by substitution

1. Substitute by u some expression in x that is chosen to simplify the integral.
2. Rewrite the integral in terms of u.
   To rewrite dx, compute $\frac{du}{dx}$ and solve algebraically as if the symbol $\frac{du}{dx}$ were a quotient.
3. Find the resulting integral and then replace u by its expression in terms of x in the answer.

Change of Variables. (Continued)

Remember!

If the integrand is a product or quotient of two terms
and one term is a constant multiple of the derivative
of an expression that appears in the other,
then this expression is probably a good choice for u.

An Example of Integration by Substitution

Example

Find

$$\int 6(x^2 - 5x - 3)^5(2x - 5)\,dx.$$

Solution. . .

The integrand is a product in which one of the factors $2x - 5$ is the derivative of an expression $6(x^2 - 5x - 3)^5$ that appears in the other factor.
Let $u = x^2 - 5x - 3$. Then,

$$\frac{du}{dx} = 2x - 5, \quad \text{i.e.} \quad du = (2x - 5)\,dx.$$

$\square$

An Example of Integration by Substitution. (Continued)

Solution...

Substitute $u = x^2 - 5x - 3$ and $du = (2x - 5)\,dx$

$$\int 6(x^2 - 5x - 3)^5(2x - 5)\,dx = \int 6u^5\,du$$
$$= u^6 + C = (x^2 - 5x - 3)^6 + C.$$

$\square$

An Application Involving Substitution

Example

The price p (in euros) of each unit of a particular commodity is estimated to be changing at the rate

$$\frac{dp}{dx} = -\frac{217x}{\sqrt{16 + x^2}},$$

where x (hundred) units is the consumer demand.
Suppose 300 units ($x = 3$) are demanded
when the price is 240 € per unit.

1. Find the price function $p(x)$.
2. At what price will 400 units be demanded?
   At what price will no units be demanded?
3. How many units are demanded at a price of 30 € per unit?

An Application Involving Substitution. (Continued)

Solution. . .

1. $p(x)$ is found by integrating $p'(x)$ with respect to x:

$$p(x) = \int p'(x)\,dx = \int -\frac{217x}{\sqrt{16+x^2}}dx.$$

Substitute

$$u = 16 + x^2, \quad du = 2x\,dx, \quad \frac{1}{2}\,du = x\,dx,$$

An Application Involving Substitution. (Continued)

. . . Solution. . .

to get

$$\begin{aligned} p(x) &= \int -\frac{217x}{\sqrt{16+x^2}}\,dx = \int -\frac{217}{\sqrt{u}} \cdot \frac{1}{2}\,du \\ &= -\frac{217}{2}\int u^{-1/2}\,du = -\frac{217}{2} \cdot \frac{1}{\frac{1}{2}} u^{1/2} + C \\ &= -217\sqrt{16+x^2} + C. \end{aligned}$$

□

An Application Involving Substitution. (Continued)

. . . Solution. . .

Since $p = 240$ when $x = 3$, we find

$$\begin{aligned} p(3) &= 240 \\ -217\sqrt{16+3^2} + C &= 240 \\ C &= 240 + 217\sqrt{25} \\ C &= 1325, \end{aligned}$$

so

$$p(x) = -217\sqrt{16+x^2} + 1325.$$

An Application Involving Substitution. (Continued)

Solution...

2. When 400 units are demanded, $x = 4$, and the corresponding price is

$$p(4) = -217\sqrt{16 + 4^2} + 1325 \approx 97.46.$$

No units are demanded when $x = 0$,

$$p(0) = -217\sqrt{16 + 0^2} + 1325 = 457.$$

□

An Application Involving Substitution. (Continued)

Solution. . .

3. Solve the equation

$$
\begin{aligned}
p(x) &= 30 \\
-217\sqrt{16+x^2}+1325 &= 30 \\
\sqrt{16+x^2} &= \frac{1295}{217} \\
x &= \sqrt{\left(\frac{1295}{217}\right)^2 - 16} \\
x &\approx 4.43.
\end{aligned}
$$

Roughly, 443 units are demanded when the price is 30 € p.u. □

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 386–398.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 258–267.

Summary

- Integration by substitution:

$$\int g(u)\frac{du}{dx}\,dx = \int g(u)\,du = G(u) + C,$$

where G is an antiderivative of g