Determinants of Matrices of Higher Order

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Aims and Objectives

- Computing the values of determinants of square matrices of order four or higher by expanding them along a row or a column.
- Using higher order determinants for solving systems of several linear equations by Cramer's rule.

Contents

An Example of a System of Four Linear Equations

Example (...)

A corporation is made of four departments.
Suppose that some of the goods being produced
are consumed in the production process.
Formulate the equations for computing the production volume
of each department if...

An Example of a System of Four Equations. (Continued)

Example

...the following table shows the amount of good of Department i used to produce one unit of good of Department j,
and the amount of final product for each department.

Dept.	**Coefficients**				**Final Product**
	1	2	3	4	
1	0.1	0.1	0.2	0.1	500
2	0.2	0.3	0.3	0.2	100
3	0.2	0.1	0.3	0.1	200
4	0.2	0	0	0.1	1000

Table: Relationship between the units produced by departments

An Example of a System of Four Equations. (Continued)

Solution...

Denote by q_1, q_2, q_3, and q_4 the (unknown) amounts of goods produced by department 1, 2, 3, and 4. Then, the production volumes for the departments can be expressed by the following equations:

$$\begin{aligned}
q_1 &= 0.1q_1 + 0.1q_2 + 0.2q_3 + 0.1q_4 + 500 \\
q_2 &= 0.2q_1 + 0.3q_2 + 0.3q_3 + 0.2q_4 + 100 \\
q_3 &= 0.2q_1 + 0.1q_2 + 0.3q_3 + 0.1q_4 + 200 \\
q_4 &= 0.2q_1 + 0q_2 + 0q_3 + 0.1q_4 + 1000.
\end{aligned}$$

$\square$

An Example of a System of Four Equations. (Continued)

. . . Solution.

After regrouping unknowns on the left hand side,
we obtain a system of four linear equations

$$\begin{aligned}
0.9q_1 - 0.1q_2 - 0.2q_3 - 0.1q_4 &= 500\\
-0.2q_1 + 0.7q_2 - 0.3q_3 - 0.2q_4 &= 100\\
-0.2q_1 - 0.1q_2 + 0.7q_3 - 0.1q_4 &= 200\\
-0.2q_1 \qquad\qquad\qquad\qquad + 0.9q_4 &= 1000.
\end{aligned}$$

$\square$

Minors and Cofactors of a Matrix

Let A be a square matrix of order n

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}.$$

Minors and Cofactors of a Matrix. (Continued)

Minors and Cofactors

- If we remove from matrix A its i-th row and j-th column, the determinant of the remaining square matrix of order $n-1$ is called a *minor* of matrix A, and it is denoted by M_{ij}.
- The product
$$\alpha_{ij} = (-1)^{i+j} M_{ij},$$
of a minor with the appropriate sign,
is called the *cofactor* of a_{ij}.

Minors and Cofactors of a Matrix. (Continued)

Example

Compute the cofactors of the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. . .

$$\alpha_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1,$$

$$\alpha_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0,$$

Minors and Cofactors of a Matrix. (Continued)

. . . Solution. . .

$$\alpha_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0,$$

$$\alpha_{21} = (-1)^{2+1} M_{21} = (-1)^3 M_{21} = -\begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = -2,$$

$$\alpha_{22} = (-1)^{2+2} M_{22} = (-1)^4 M_{22} = \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = 1,$$

$$\alpha_{23} = (-1)^{2+3} M_{23} = (-1)^5 M_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$\square$

Minors and Cofactors of a Matrix. (Continued)

. . . Solution.

$$\alpha_{31} = (-1)^{3+1} M_{31} = (-1)^4 M_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7,$$

$$\alpha_{32} = (-1)^{3+2} M_{32} = (-1)^5 M_{32} = -\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = -2,$$

$$\alpha_{33} = (-1)^{3+3} M_{33} = (-1)^6 M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1.$$

□

Determinants of Matrices of Higher Order

The Determinant of a Matrix of Order n ...

The determinant of a matrix A of order n ($n > 1$) is
the sum of the products of the elements of a row or a column
by their cofactors;

Determinants of Matrices of Higher Order. (Continued)

. . . The Determinant of a Matrix of Order n

i.e., along the i-th row

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \cdots + a_{in}\alpha_{in},$$

or along the j-th column

$$\det A = a_{1j}\alpha_{1j} + a_{2j}\alpha_{2j} + \cdots + a_{nj}\alpha_{nj}.$$

Determinants of Matrices of Higher Order. (Continued)

Example

Compute the determinant of the following matrix of order 4

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}.$$

Solution...

$$\det A = 0 \cdot \alpha_{31} + 1 \cdot \alpha_{32} + 0 \cdot \alpha_{33} + 1 \cdot \alpha_{34}$$

□

Determinants of Matrices of Higher Order. (Continued)

. . . Solution.

$$= 1 \cdot (-1)^{3+2} \begin{vmatrix} 0 & 2 & 3 \\ 2 & 2 & 5 \\ 3 & 1 & 2 \end{vmatrix} + 1 \cdot (-1)^{3+4} \begin{vmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= (-1)^5 \cdot 10 + (-1)^7 \cdot 12 = -22.$$

$\square$

Solving a System of Four Linear Equations

Example

Solve the system of four linear equations

$$\begin{aligned} 0.9q_1 - 0.1q_2 - 0.2q_3 - 0.1q_4 &= 500 \\ -0.2q_1 + 0.7q_2 - 0.3q_3 - 0.2q_4 &= 100 \\ -0.2q_1 - 0.1q_2 + 0.7q_3 - 0.1q_4 &= 200 \\ -0.2q_1 \qquad\qquad\qquad\qquad + 0.9q_4 &= 1000. \end{aligned}$$

Solution...

By Cramer's rule we have

$$q_1 = \frac{d_1}{\det A}, \quad q_2 = \frac{d_2}{\det A}, \quad q_3 = \frac{d_3}{\det A}, \quad q_4 = \frac{d_4}{\det A},$$

□

Solving a System of Four Linear Equations. (Continued)

. . . Solution. . .

where are

$$\det A = \begin{vmatrix} 0.9 & -0.1 & -0.2 & -0.1 \\ -0.2 & 0.7 & -0.3 & -0.2 \\ -0.2 & -0.1 & 0.7 & -0.1 \\ -0.2 & 0 & 0 & 0.9 \end{vmatrix}$$

$$= -0.2 \cdot \alpha_{41} + 0 \cdot \alpha_{42} + 0 \cdot \alpha_{43} + 0.9 \cdot \alpha_{44}$$

$$= \cdots = 0.3096,$$

□

Solving a System of Four Linear Equations. (Continued)

...Solution...

$$d_1 = \begin{vmatrix} 500 & -0.1 & -0.2 & -0.1 \\ 100 & 0.7 & -0.3 & -0.2 \\ 200 & -0.1 & 0.7 & -0.1 \\ 1000 & 0 & 0 & 0.9 \end{vmatrix}$$

$$= 1000 \cdot \alpha_{41} + 0 \cdot \alpha_{42} + 0 \cdot \alpha_{43} + 0.9 \cdot \alpha_{44}$$

$$= \cdots = 326.7,$$

□

Solving a System of Four Linear Equations. (Continued)

. . . Solution. . .

$$d_2 = \begin{vmatrix} 0.9 & 500 & -0.2 & -0.1 \\ -0.2 & 100 & -0.3 & -0.2 \\ -0.2 & 200 & 0.7 & -0.1 \\ -0.2 & 1000 & 0 & 0.9 \end{vmatrix}$$

$$= -0.2 \cdot \alpha_{41} + 1000 \cdot \alpha_{42} + 0 \cdot \alpha_{43} + 0.9 \cdot \alpha_{44}$$

$$= \cdots = 383.5,$$

□

Solving a System of Four Linear Equations. (Continued)

. . . Solution. . .

$$d_3 = \begin{vmatrix} 0.9 & -0.1 & 500 & -0.1 \\ -0.2 & 0.7 & 100 & -0.2 \\ -0.2 & -0.1 & 200 & -0.1 \\ -0.2 & 0 & 1000 & 0.9 \end{vmatrix}$$

$$= -0.2 \cdot \alpha_{41} + 0 \cdot \alpha_{42} + 1000 \cdot \alpha_{43} + 0.9 \cdot \alpha_{44}$$

$$= \cdots = 296.1,$$

□

Solving a System of Four Linear Equations. (Continued)

. . . Solution. . .

$$d_4 = \begin{vmatrix} 0.9 & -0.1 & -0.2 & 500 \\ -0.2 & 0.7 & -0.3 & 100 \\ -0.2 & -0.1 & 0.7 & 200 \\ -0.2 & 0 & 0 & 1000 \end{vmatrix}$$

$$= -0.2 \cdot \alpha_{41} + 0 \cdot \alpha_{42} + 1000 \cdot \alpha_{43} + 1000 \cdot \alpha_{44}$$

$$= \cdots = 416.6.$$

□

Solving a System of Four Linear Equations. (Continued)

...Solution...

Thus,

$$q_1 = \frac{d_1}{\det A} = \frac{326.7}{0.3096} \approx 1{,}055.23,$$
$$q_2 = \frac{d_2}{\det A} = \frac{383.5}{0.3096} \approx 1{,}238.70,$$
$$q_3 = \frac{d_3}{\det A} = \frac{296.1}{0.3096} \approx 956.40,$$
$$q_4 = \frac{d_4}{\det A} = \frac{416.6}{0.3096} \approx 1{,}345.61.$$

□

Solving a System of Four Linear Equations. (Continued)

. . . Solution.

Finally, since in our application q_1, q_2, q_3, q_4
represent amounts of products expressed in units, we have

$$q_1 \approx 1{,}055, \quad q_2 \approx 1{,}239, \quad q_3 \approx 956, \quad q_4 \approx 1{,}346.$$

$\square$

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 282-301.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 3-1–3-36.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 19-28.

Summary

- Minors and cofactors of a matrix
- Using cofactors to expand a determinant along a row or a column
- Computing determinants of order 4
- Using determinants of order 4 for solving a system of four linear equations by Cramer's rule.