Matrices and Determinants. Solving Systems of Three Equations by Cramer's Rule

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Aims and Objectives

- Learning the notion of a matrix as a rectangular array and operations with matrices.
- Computing the value of the determinant of a square matrix of orders two and three.
- Applying determinants for solving systems of linear equations by Cramer's rule.

Contents

Example of a system of three equations

Example

In an automobile dealership, the most popular passenger cars are Brand A, B and C. Because buyers bargain for the best price, the sales price for each brand is not the same.
The table shows the sales and revenues for a 3-month period.
Compute the average sales price for each of these brands of cars.

Month	Brand A	Brand B	Brand C	Revenue
1	25	62	54	$2,756,000$ €
2	28	42	58	$2,695,000$ €
3	45	53	56	$3,124,000$ €

Table: Sales and Revenues from Selling Cars

Example of a system of three equations. (Continued)

Solution.

Denote by x, y and z the average sales price for Brand A, B, and C respectively.
Then, revenues for each of the periods can be represented by the following system of equations.

$$
\begin{aligned}
25x + 62y + 54z &= 2{,}756{,}000 \\
28x + 42y + 58z &= 2{,}695{,}000 \\
45x + 53y + 56z &= 3{,}124{,}000.
\end{aligned}
$$

$\square$

Matrices

A *matrix* is a rectangular array of numbers, such as those shown below.

$$\begin{bmatrix} 1 & 2 & 6 \\ 2 & -3 & 5 \end{bmatrix} \quad \text{ose} \quad \begin{bmatrix} 1 & 3 & 1 \\ -5 & 21 & -3 \\ 1 & -4 & 6 \end{bmatrix}.$$

Matrices. (Continued)

Matrix

In general form, a matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}].$$

- The numbers a_{ij} are the *elements* (or *entries*) of the matrix, where the index i indicates the row and j indicates the column.
- A matrix of m rows and n cols is said to be of $m \times n$ *order*.
- If $m = n$, the matrix is said to be a *square* matrix of order n.

First operations with matrices

Addition and Subtraction of Matrices

The *sum* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of a same order is the matrix $C = A + B = [c_{ij}]$ of the same order, whose entries are the sums of corresponding entries:

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for each } i \text{ and } j.$$

The *difference* of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of a same order is the matrix $C = A - B = [c_{ij}]$ of the same order, whose entries are the differences of corresponding entries:

$$c_{ij} = a_{ij} - b_{ij} \quad \text{for each } i \text{ and } j.$$

First operations with matrices. (Continued)

Example

Compute $A + B$ and $A - B$ given that

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & -3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 3 & 5 \end{bmatrix}.$$

Solution.

$$A + B = \begin{bmatrix} 1+3 & 2-1 & 6+0 \\ 2-2 & -3+3 & 5+5 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 6 \\ 0 & 0 & 10 \end{bmatrix},$$

$$A - B = \begin{bmatrix} 1-3 & 2+1 & 6-0 \\ 2+2 & -3-3 & 5-5 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 6 \\ 4 & -6 & 0 \end{bmatrix}.$$

$\square$

Determinants

Determinant of Order Two

If A is a square matrix of order 2, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

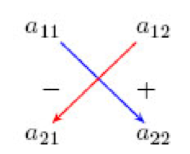
then the *determinant* of A is

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Determinants. (Continued)

Remember!

The following scheme helps remembering the way of computing the determinant of a square matrix of order 2.



$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Determinants. (Continued)

Example

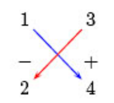
Compute $\det A$ for matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Determinants. (Continued)

Solution.

Following the scheme



we have

$$\det A = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 1 \cdot 4 - 3 \cdot 2 = -2.$$

$\square$

Determinants. (Continued)

Determinant of Order Three

If A is a square matrix of order 3, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

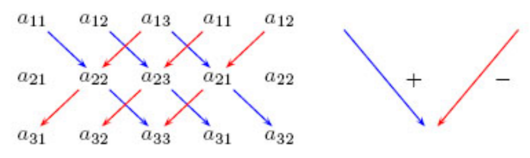
then the *determinant* of A is

$$\begin{aligned} \det A = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}. \end{aligned}$$

Determinants. (Continued)

Remember!

The following scheme helps remembering the way of computing the determinant of a square matrix of order 3.



$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32}$$

Determinants. (Continued)

Example

Compute $\det A$ for matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & -8 \\ 3 & -6 & 9 \end{bmatrix}.$$

Determinants. (Continued)

Solution.

Following the scheme

$$
\begin{array}{ccccc}
1 & 4 & 7 & 1 & 4 \\
2 & 5 & -8 & 2 & 5 \\
3 & -6 & 9 & 3 & -6
\end{array}
$$

$$
\begin{aligned}
\det A = {} & 1 \cdot 5 \cdot 9 + 4 \cdot (-8) \cdot 3 + 7 \cdot 2 \cdot (-6) \\
& - 7 \cdot 5 \cdot 3 - 1 \cdot (-8) \cdot (-6) - 4 \cdot 2 \cdot 9 = -360.
\end{aligned}
$$

□

Cramer's Rule

Consider a system of three equations

$$
\begin{aligned}
a_{11}x + a_{12}y + a_{13}z &= b_1\\
a_{21}x + a_{22}y + a_{23}z &= b_2\\
a_{31}x + a_{32}y + a_{33}z &= b_3.
\end{aligned}
$$

Cramer's Rule. (Continued)

Cramer's Rule...

The unknowns x, y and z can be found from the relations

$$x = \frac{d_1}{\det A}, \quad y = \frac{d_2}{\det A}, \quad z = \frac{d_3}{\det A},$$

where

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is the determinant of the *system matrix A,*

Cramer's Rule. (Continued)

. . . Cramer's Rule

while

$$d_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad d_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \quad d_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

are the determinants formed from the determinant of matrix A by substituting the right hand side column b_1, b_2 and b_3 for the appropriate coefficients column.

Cramer's Rule. (Continued)

Example

Now, solve the system form the example at the beginning:

$$\begin{aligned}
25x + 62y + 54z &= 2{,}756{,}000 \\
28x + 42y + 58z &= 2{,}695{,}000 \\
45x + 53y + 56z &= 3{,}124{,}000.
\end{aligned}$$

Cramer's Rule. (Continued)

Solution...

$$\det A = \begin{vmatrix} 25 & 62 & 54 \\ 28 & 42 & 58 \\ 45 & 53 & 56 \end{vmatrix}$$

$$= 25 \cdot 42 \cdot 56 + 62 \cdot 58 \cdot 45 + 54 \cdot 28 \cdot 53$$

$$- 54 \cdot 42 \cdot 45 - 25 \cdot 58 \cdot 53 - 62 \cdot 28 \cdot 56 = 24{,}630,$$

$$d_1 = \begin{vmatrix} 2{,}756{,}000 & 62 & 54 \\ 2{,}695{,}000 & 42 & 58 \\ 3{,}125{,}000 & 53 & 56 \end{vmatrix} = \cdots = 514{,}890{,}000,$$

$\square$

Cramer's Rule. (Continued)

. . . Solution.

$$d_2 = \begin{vmatrix} 25 & 2{,}756{,}000 & 54 \\ 28 & 2{,}695{,}000 & 58 \\ 45 & 3{,}125{,}000 & 56 \end{vmatrix} = \cdots = 289{,}590{,}000$$

and

$$d_3 = \begin{vmatrix} 25 & 62 & 2{,}756{,}000 \\ 28 & 42 & 2{,}695{,}000 \\ 45 & 53 & 3{,}125{,}000 \end{vmatrix} = \cdots = 686{,}175{,}000.$$

□

Cramer's Rule. (Continued)

. . . Solution.

Thus,

$$x = \frac{d_1}{\det A} = \frac{514{,}890{,}000}{24{,}630} \approx 20{,}904.99,$$
$$y = \frac{d_2}{\det A} = \frac{289{,}590{,}000}{24{,}630} \approx 11{,}757.61,$$
$$z = \frac{d_3}{\det A} = \frac{686{,}175{,}000}{24{,}630} \approx 27{,}859.32.$$

□

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 222-269.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 3-1–3-36.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 9-19.

Summary

- A matrix $A = [a_{ij}]$
- The determinant $\det A = |a_{ij}|$ of a matrix A
- The ways of computing the determinants of matrices of orders 2 and 3
- Cramer's rule

$$x_i = \frac{d_i}{\det A}$$