Matrix Multiplication. The Inverse of a Matrix

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Aims and Objectives

- Solving a system of several linear equations by Gauss' Elimination.
- Introducing two new matrix operations: scalar multiplication and matrix multiplication.
- Learning the notions of an identity matrix, non-singular matrix, the inverse and the adjoint of a matrix.

Contents

Gauss' Elimination

Example

Solve the following system

$$\begin{aligned} \boxed{x_1} + x_2 + 3x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

Solution...

Copy the first equation,
multiply the first equation by 2 and subtract it from the second,
multiply the first equation by 3 and subtract it from the third,
add the first equation to the fourth one. □

Gauss' Elimination. (Continued)

. . . Solution. . .

$$\begin{aligned}(E_2 - 2E_1) &\to (E_2)\\ (E_3 - 3E_1) &\to (E_3)\\ (E_4 + E_1) &\to (E_4)\end{aligned}$$

$$\begin{array}{rrrrl} x_1 + & x_2 & & +\,3x_4 = & 4\\ & \boxed{-x_2} & -\; x_3 & -\,5x_4 = & -7\\ & -4x_2 & -\; x_3 & -\,7x_4 = & -15\\ & 3x_2 + & 3x_3 & +\,2x_4 = & 8.\end{array}$$

$\square$

Gauss' Elimination. (Continued)

...Solution...

$$(E_3 - 4E_2) \to (E_3)$$
$$(E_4 + 3E_2) \to (E_4)$$

$$\begin{aligned}
x_1 + x_2 \qquad + 3x_4 &= 4 \\
-x_2 - x_3 - 5x_4 &= -7 \\
+ 3x_3 + 13x_4 &= 13 \\
- 13x_4 &= -13.
\end{aligned}$$

□

Gauss' Elimination. (Continued)

. . . Solution.

The resulting system can be solved by *backward substitution:*

$$x_4 = \frac{-13}{-13} = 1$$
$$x_3 = \frac{13 - 13x_4}{3} = \frac{1}{3}(13 - 13 \cdot 1) = 0$$
$$x_2 = -(-7 + 5x_4 + x_3) = -(-7 + 5 \cdot 1 + 0) = 2$$
$$x_1 = 4 - 3x_4 - x_2 = 4 - 3 - 2 = -1.$$

$\square$

Gauss' Elimination. (Continued)

Remember!

Easier yet, without writing the equations variables, the method could be applied by using the *augmented matrix* of the system, consisting of the system coefficients matrix augmented by the values on the right hand side of the equations.

Scalar Multiplication

Scalar Multiplication

If k is a number (or, a *scalar*) and A is a matrix,
then the *scalar multiple* kA is the matrix B of the same order as A,
whose entries are the corresponding entries of A multiplied by k;
i.e., $B = kA = [b_{ij}]$, where $b_{ij} = k \cdot a_{ij}$.

Scalar Multiplication. (Continued)

Example

Find the scalar multiple of the matrix

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

by scalar 5.

Solution. . .

$$kA = 5 \cdot \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 5 \cdot (-1) & 5 \cdot 2 \\ 5 \cdot 3 & 5 \cdot (-4) \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ 15 & -20 \end{bmatrix}.$$

$\square$

Matrix Multiplication

If A is a $m \times p$ matrix and B is a $p \times n$ matrix, then the matrix product AB is a $m \times n$ matrix.

Shows that A can be multiplied by B

$$\begin{array}{ccc} A & \quad & B \\ m \times p & \quad & p \times n \end{array}$$

Shows the order of the product AB

Matrix Multiplication. (Continued)

Matrix Multiplication

If A is a $m \times p$ matrix and B is a $p \times n$ matrix,
then the *matrix product* AB is the matrix C of order $m \times n$,
whose entries are sums of products of the entries of a row of A
by the corresponding entries of a column of B;
i.e., $C = AB = [c_{ij}]$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

Matrix Multiplication. (Continued)

Example

Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \\ 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 3 & 0 & -3 \end{bmatrix}.$$

Find the product AB.

Matrix Multiplication. (Continued)

Solution.

Since the matrices A and B are of orders 3×3 and 3×3,
the multiplication AB is possible,
and the result will be the following 3×3 matrix:

$$AB = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \\ 0 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 11 & -4 & 11 \\ -13 & 2 & -13 \end{bmatrix}.$$

□

Identity Matrix

Identity Matrix

Identity matrix I is a square matrix whose
all entries in the main diagonal are 1,
and all other entries are 0.

Examples of Identity Matrices

Example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identity Matrix. (Continued)

Example

Find the product of a square matrix of order 2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with the identity matrix I of order 2×2. Then, find the product IA.

Identity Matrix. (Continued)

Solution.

$$AI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} & 1 \cdot a_{12} + 0 \cdot a_{22} \\ 0 \cdot a_{11} + 1 \cdot a_{21} & 0 \cdot a_{12} + 1 \cdot a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A. \qquad \square$$

Identity Matrix. (Continued)

Property of Identity Matrices

It can be shown that for any matrix A and the identity matrices I for which the products AI or IA exist, we have

$$AI = A \quad \text{and} \quad IA = A.$$

The Inverse of a Matrix

The Inverse of a matrix

- A square matrix A of order n is called *non-singular* if $\det A \neq 0$.
- If a matrix A is non-singular, then *the inverse* of A is the matrix A^{-1} such that

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

The Inverse of a Matrix. (Continued)

The Inverse of a Matrix

If a matrix A is non-singular, then its inverse A^{-1} is

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A,$$

where $\operatorname{adj} A$ is *the adjoint* of A, whose entries along the columns are the cofactors of entries of the rows of A:

$$\operatorname{adj} A = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \dots & \alpha_{n3} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix}.$$

The Inverse of a Matrix. (Continued)

System of linear equations in the matrix form...

A system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

is identical to the matrix equation

$$AX = B,$$

where A is the coefficients matrix, X is the column of unknowns, and B is the column of the values on the right-hand side;

The Inverse of a Matrix. (Continued)

. . . System of linear equations in the matrix form

that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

The Inverse of a Matrix. (Continued)

The Inverse Matrix Method of Solution

$$X = A^{-1}B$$

is the solution of the linear system $AX = B$.

Indeed,

$$AX = A(A^{-1}B) = (AA^{-1})B = IB = B.$$

The Inverse of a Matrix. (Continued)

Remember!

- Computing the inverse of a matrix involves a huge number of operations, hence it is not usually used in practice.
- In general, the least demanding method for solving a system of linear equations is Gauss' Elimination.

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- D. P. Maki, M. Thompson, *Finite mathematics,* pp. 253-281.
- S. T. Karris, *Mathematics for business, science and technology,* pp. 3-1–3-36.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 28-40.

Summary

- The technique of solving a system of linear equations by Gauss' Elimination
- Computing the products of a matrix by a scalar and by a matrix
- Understanding the relation between a system of linear equations and its matrix equation.