Chapter 2

Elementary Algebra

This chapter is an introduction to algebra and algebraic equations. It is aimed for readers who feel that they need an algebra review for understanding and working with simple algebraic equations. It also introduces some financial functions that are used with Excel.

2.1 Introduction

Algebra is the branch of mathematics in which letters of the alphabet represent numbers or a set of numbers. *Equations* are equalities that indicate how some quantities are related to others. For example, the equation

$$°C = \frac{5}{9}(°F - 32) \tag{2.1}$$

is a relation or formula that enables us to convert degrees Fahrenheit, $°F$, to degrees Celsius, $°C$. For instance, if the temperature is $77°F$, the equivalent temperature in $°C$ is

$$°C = \frac{5}{9}(77 - 32) = \frac{5}{9} \times 45 = 25$$

We observe that the mathematical operation in parentheses, that is, the subtraction of 32 from 77, was performed first. This is because numbers within parentheses have *precedence* (priority) over other operations.

In algebra, the *order of precedence,* where 1 is the highest and 4 is the lowest, is as follows:

1. Quantities inside parentheses ( )
2. Exponentiation
3. Multiplication and Division
4. Addition or Subtraction

Example 2.1

Simplify the expression

$$a = \frac{-2^3}{(8-3)^2 \times 4}$$

Solution:

This expression is reduced in steps as follows:

Step 1. Subtracting 3 from 8 inside the parentheses yields 5. Then, $a = \dfrac{-2^3}{5^2 \times 4}$

Step 2. Performing the exponentiation operations we get $a = \dfrac{-8}{25 \times 4}$

Step 3. Multiplying 25 by 4 we get $a = \dfrac{-8}{100}$

Step 4. Dividing -8 by 100 we get $a = -0.08$ (Simplest form)

An *equality* is a mathematical expression where the left side is equal to the right side. For example, $7 + 3 = 10$ is an equality since the left and right sides are equal to each other.

2.2 Algebraic Equations

An *algebraic equation* is an equality that contains one or more unknown quantities, normally represented by the last letters of the alphabet such as x, y, and z. For instance, the expressions $x + 5 = 15$, $3 \times y = 24$, and $x \times y = 12$ are algebraic equations. In the last two equations, the multiplication sign between 3 and y and x and y is generally omitted; thus, they are written as $3y = 24$ and $xy = 12$. Henceforth, terms such as $3y$ will be interpreted as $3 \times y$, xy as $x \times y$, and, in general, XY as $X \times Y$.

Solving an equation means finding the value of the unknown quantity that will make the equation an equality with no unknowns. The following properties of algebra enable us to find the numerical value of the unknown quantity in an equation.

Property 1

The same number may be added to, or subtracted from both sides of an equation.

Example 2.2

Given the equality

$$7 + 5 = 12$$

prove that Property 1 holds if we

a. add 3 to both sides

b. subtract 2 from both sides

Proof:

a. Adding 3 to both sides we get

$$7 + 5 + 3 = 12 + 3 \text{ or } 15 = 15$$

and thus, the equality holds.

b. Subtracting 2 from both sides we get

$$7 + 5 - 2 = 12 - 2 \text{ or } 10 = 10$$

and thus, the equality holds.

Example 2.3

Solve the following equation, that is, find the value of x.

$$x - 5 = 15$$

Solution:

We need to find a value for x so that the equation will still hold after the unknown x has been replaced by the value that we have found. For this type of equations, a good approach is to find a number that when added to or subtracted from both sides of the equation, the left side will contain only the unknown x. For this example, this will be accomplished if we add 5 to both sides of the given equation. When we do this, we get

$$x - 5 + 5 = 15 + 5$$

and after simplification, we obtain the value of the unknown x as

$$x = 20$$

We can check this answer by substitution of x into the given equation. Thus, $20 - 5 = 15$ or $15 = 15$. Therefore, our answer is correct.

Example 2.4

Solve the following equation, that is, find the value of x.

$$x + 5 = 15$$

Solution:

Again, we need to find a number such that when added to or subtracted from both sides of the equation, the left side will have the unknown x only. In this example, this will be accomplished if we subtract 5 from both sides of the given equation. Doing this, we get

$$x + 5 - 5 = 15 - 5$$

and after simplification, we obtain the value of the unknown x as

$$x = 10$$

To verify that this is the correct answer, we substitute this value into the given equation and we get $10 + 5 = 15$ or $15 = 15$.

Property 2

Each side of an equation can be multiplied or divided* by the same number.

Example 2.5

Solve the following equation, that is, find the value of y.

$$\frac{y}{3} = 24 \tag{2.2}$$

Solution:

We need to eliminate the denominator 3 on the left side of the equation. This is done by multiplying both sides of the equation by 3. Then,

$$\frac{y}{3} \times 3 = 24 \times 3$$

and after simplification we get

$$y = 72 \tag{2.3}$$

To verify that this is correct, we divide 72 by 3; this yields 24, and this is equal to the right side of the given equation.

Note 2.1

In an algebraic term such as $4x$ or $(a+b)x$, the number or symbol multiplying a variable or an unknown quantity is called the *coefficient* of that term. Thus, in the term $4x$, the number 4 is the coefficient of that term, and in $(a+b)x$, the coefficient is $(a+b)$. Likewise, the coefficient of y in (2.2) is $1/3$, and the coefficient of y in (2.3) is 1 since $1y = y$. In other words, every algebraic term has a coefficient, and if it is not shown, it is understood to be 1 since, in general, $1x = x$.

Example 2.6

Solve the following equation, that is, find the value of z.

$$3z = 24$$

Solution:

We need to eliminate the coefficient 3 of z on the left side of the equation. This is done by dividing both sides of the equation by 3. Then,

$$\frac{3z}{3} = \frac{24}{3}$$

and after simplification

$$z = 8$$

* *Division by zero is meaningless; therefore, it must be avoided.*

Other algebraic equations may contain exponents and logarithms. For those equations we may need to apply the laws of exponents and the laws of logarithms. These are discussed next.

2.3 Laws of Exponents

In Chapter 1, we learned that for any number *a* and for a positive integer *m*, the exponential number a^m is defined as

$$\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m\ =\ number\ of\ a's} \tag{2.4}$$

where the dots between the *a's* in (2.4) denote multiplication.

The *laws of exponents* state that

$$a^m \cdot a^n = a^{m+n} \tag{2.5}$$

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & if \quad m > n \\ 1 & if \quad m = n \\ \dfrac{1}{a^{n-m}} & if \quad m < n \end{cases} \tag{2.6}$$

$$(a^m)^n = a^{mn} \tag{2.7}$$

The *nth* root is defined as the inverse of the *nth* exponent, that is, if

$$b^n = a, \quad then \quad b = \sqrt[n]{a} \tag{2.8}$$

If, in (2.11) n is an odd positive integer, there will be a unique number satisfying the definition for $\sqrt[n]{a}$ for any value of a*. For instance, $\sqrt[3]{512} = 8$ and $\sqrt[3]{-512} = -8$.

If, in (2.11) n is an even positive integer, for positive values of a there will be two values, one positive and one negative. For instance, $\sqrt{121} = \pm 11$. For additional examples, the reader may refer to the section on square and cubic roots in Chapter 1.

If, in (2.8) n is even positive integer and a is negative, $\sqrt[n]{a}$ cannot be evaluated. For instance, $\sqrt{-3}$ cannot be evaluated; it results in a complex number.

It is also useful to remember the following definitions from Chapter 1.

* *In advance mathematics, there is a restriction. However, since in this text we are only concerned with real numbers, no restriction is imposed.*

$$a^0 = 1 \tag{2.9}$$

$$a^{p/q} = \sqrt[q]{a^p} \tag{2.10}$$

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a} \tag{2.11}$$

These relations find wide applications in business, science, technology, and engineering. We will consider a few examples to illustrate their application.

Example 2.7

The equation for calculating the present value (PV) of an ordinary annuity is

$$PV = Payment \times \frac{1-(1+Interest)^{-n}}{Interest} \tag{2.12}$$

where

PV = cost of the annuity at present

Payment = amount paid at the end of the year

Interest = Annual interest rate which the deposited money earns

n = number of years the *Payment* will be received.

Example 2.8

Suppose that an insurance company offers to pay us $\$12{,}000$ at the end of each year for the next 25 years provided that we pay the insurance company $\$130{,}000$ now, and we are told that our money will earn 8% per year. Should we accept this offer?

Solution:

$Payment = \$12{,}000$, $Interest = 0.08\ (8\%)$, and $n = 25\ years$. Let us calculate PV using (2.12).

$$PV = 12000 \times \frac{1-(1+0.08)^{-25}}{0.08} \tag{2.13}$$

We will use Excel to compute (2.13). We select any cell and we type in the following expression where the *asterisk* (*) denotes multiplication, the *slash* (/) division, and the *caret* (^) exponentiation.

```
=12000*(1–(1+0.08)^(–25))/0.08
```

Excel returns $\$128{,}097$ and this is the fair amount that if deposited now at 8% interest, will earn $\$12{,}000$ per year for 25 years. This is less than $\$130{,}000$ which the insurance company asks for, and therefore, we should reject the offer.

Excel has a "build-in" function, called *PV* that computes the present value directly, that is, without the formula of (2.12). To use it, we perform the sequential steps ***f_x>Financial>PV>Rate=0.08>Nper=25>Pmt=12000>OK.*** We observe that the numerical value is the same as before, but it is displayed in red and within parentheses, that is, as negative value. It is so indicated because Excel interprets outgoing money as negative cash.

Note 2.2

Excel has several financial functions that apply to annuities. It is beyond the scope of this text to discuss all of them. The interested reader may invoke the *Help* feature in Excel to get the description of these functions.

***Property* 3**

Each side of an equation can be raised to the same power

Example 2.9

Solve the following equation, that is, find the value of x.

$$\sqrt{x} = 5 \tag{2.14}$$

Solution:

As a first step, we use the alternate designation of the square root; this was discussed in Chapter 1. Then,

$$x^{1/2} = 5 \tag{2.15}$$

Next, we square (raise to power 2) both sides of (2.15), and we get

$$(x^{1/2})^2 = 5^2 \tag{2.16}$$

Now, multiplication of the exponents on the left side of (2.19) using (2.7) yields

$$x^1 = 5^2$$

and after simplification, the answer is

$$x = 25$$

Example 2.10

Solve the following equation, that is, find the value of z.

$$\sqrt[3]{z} = 8 \tag{2.17}$$

Solution:

As a first step, we use the alternate designation of the cubic root. This was discussed in Chapter 1. Then,

$$z^{1/3} = 8 \tag{2.18}$$

Next, we cube (raise to power 3) both sides of (2.18) and we get

$$(z^{1/3})^3 = 8^3 \tag{2.19}$$

Now, multiplication of the exponents on the left side using (2.19) yields

$$z^1 = 8^3$$

and after simplification, the answer is

$$z = 512$$

Other equations can be solved using combinations of Properties (2.5 through (2.7). As another example, let us consider the temperature conversion from degrees Fahrenheit to degrees Celsius.

We start with

$$°C = \frac{5}{9}(°F - 32) \tag{2.20}$$

Next, we multiply both sides by $9/5$. Then,

$$\left(\frac{9}{5}\right) \cdot °C = \left(\frac{9}{5}\right) \cdot \frac{5}{9}(°F - 32) \tag{2.21}$$

and after simplifying the right side, we get

$$\frac{9}{5}°C = °F - 32 \tag{2.22}$$

Now, we add 32 to both sides to eliminate -32 from the right side. Then,

$$\frac{9}{5}°C + 32 = °F - 32 + 32 \tag{2.23}$$

Finally, after interchanging the positions of the left and right sides, we get

$$°F = \frac{9}{5}°C + 32 \tag{2.24}$$

2.4 Laws of Logarithms

In Chapter 1, we learned that, if

$$x = a^y, \quad then \quad y = log_a x \tag{2.25}$$

where *a* is the base and *y* is the exponent (power).

The *laws of logarithms* state that

$$log_a(xy) = log_a x + log_a y \tag{2.26}$$

$$log_a\left(\frac{x}{y}\right) = log_a x - log_a y \tag{2.27}$$

$$log_a(x^n) = n\,log_a x \tag{2.28}$$

Logarithms are very useful because the laws of (2.26) through (2.28), allow us to replace multiplication, division and exponentiation by addition, subtraction and multiplication respectively. We used logarithms in Chapter 1 to define the decibel.

As we mentioned in Chapter 1, the ***common (base 10) logarithm*** and the ***natural (base e) logarithm,*** where e is an irrational (endless) number whose value is $2.71828\ldots\ldots$, are the most widely used. Both occur in many formulas of probability, statistics, and financial formulas. For instance, the formula below computes the number of periods required to accumulate a specified future value by making equal payments at the end of each period into an interest-bearing account.

$$n = \frac{ln(1 + (Interest \times FutureValue)/Payment)}{ln(1 + Interest)} \tag{2.29}$$

where

n = number of periods (months or years)

ln = natural logarithm

Interest = earned interest

Future Value = desired amount to be accumulated

Payment = equal payments (monthly or yearly) that must be made.

Example 2.11

Compute the number of months required to accumulate $Future\ Value = \$100{,}000$ by making a monthly payment of $\$500$ into a savings account paying 6% annual interest compounded monthly.

Solution:

The monthly interest rate is $6\%/12$ or 0.5%, and thus in (2.29), $Interest = 0.5\% = 0.005$, $Future\ Value = \$100{,}000$, and $Payment = \$500$. Then,

$$n = \frac{ln(1 + (0.005 \times 100,000)/500)}{ln(1 + 0.005)} \tag{2.30}$$

We will use Excel to find the value of n. In any cell, we enter the following formula:

=LN(1+(0.005*100000)/500)/LN(1+0.005)

Excel returns *138.9757*; we round this to *139*. This represents the number of the months that a payment of $*500* is required to be deposited at the end of each month. This time is equivalent to *11* years and *7* months.

Excel has a build-in function named NPER and its *syntax* (orderly arrangement) is

NPER(rate,pmt,pv,fv,type) where, for this example,

rate = 0.005

pmt = -500 (minus because it is a cash outflow)

pv = 0 (present value is zero)

fv = 100000

type = 0 which means that payments are made at the end of each period.

Then, in any cell we enter the formula

=NPER(0.005,-500,0,100000,0)

and we observe that Excel returns *138.9757*. This is the same as the value that we found with the formula of (2.29).

The formula of (2.29) uses the natural logarithm. Others use the common logarithm.

Although we can use a *PC*, or a calculator to find the log of a number, it is useful to know the following facts that apply to common logarithms.

I. Common logarithms consist of an integer, called the *characteristic* and an endless decimal called the *mantissa**.

II. If the decimal point is located immediately to the right of the msd of a number, the characteristic is zero; if the decimal point is located after two digits to the right of the msd, the characteristic is *1*; if after three digits, it is *2* and so on. For instance, the characteristic of *1.9* is *0*, of *58.3* is *1*, and of *476.5* is *2*.

III. If the decimal point is located immediately to the left of the msd of a number, the characteristic is -1, if located two digits to the left of the msd, it is -2, if after three digits, it is -3 and so on. Thus, the characteristic of *0.9* is -1, of *0.0583* is -2, and of *0.004765* is -3.

IV. The mantissa cannot be determined by inspection; it must be extracted from tables of common logarithms.

* *Mantissas for common logarithms appear in books of mathematical tables. No such tables are provided here since we will not use them in our subsequent discussion.*

V. Although the common logarithm of a number less than one is negative, it is written with a *negative* characteristic and a *positive* mantissa. This is because the mantissas in tables are given as positive numbers.

VI. Because mantissas are given in math tables as positive numbers, the negative sign is written above the characteristic. This is to indicate that the negative sign does not apply to the mantissa. For instance, $\log 0.00319 = \overline{3}.50379$, and since $-3 = 7 - 10$, this can be written as $\log 0.00319 = 7.50379 - 10 = -2.4962$.

A convenient method to find the characteristic of logarithms is to first express the given number in scientific notation; the characteristic then is the exponent. For negative numbers, the mantissa is the 9's complement* of the mantissa given in math tables

Example 2.12

Given that $x = 73,000,000$ and $y = 0.00000000073$, find

a. $\log x$ b. $\log y$

Solution:

a.

$$x = 73,000,000 = 7.3 \times 10^7$$
$$\log x = 7.8633$$

b.

$$y = 0.00000000073 = 7.3 \times 10^{-10}$$
$$\log y = -10.8633 = 0.8633 - 10 = -9.1367$$

2.5 Quadratic Equations

Quadratic equations are those that contain equations of second degree. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad (2.31)$$

where a, b and c are real constants (positive or negative). Let x_1 and x_2 be the *roots*† of (2.31). These can be found from the formulas

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad (2.32)$$

* *The 9's complement of a number is obtained by subtraction of that number from a number consisting of 9's with the same number of digits as the number. For the example cited above, we subtract 50379 from 99999 and we obtain 49620 and this is the mantissa of the number.*

† *Roots are the values which make the left and right sides of an equation equal to each other.*

The quantity $b^2 - 4ac$ under the square root is called the *discriminant* of a quadratic equation.

Example 2.13

Find the roots of the quadratic equation

$$x^2 - 5x + 6 = 0 \tag{2.33}$$

Solution:

Using the quadratic formulas of (2.32), we get

$$\begin{aligned} x_1 &= \frac{-(-5)+\sqrt{(-5)^2-4\times1\times6}}{2\times1} = \frac{5+\sqrt{25-24}}{2} = \frac{5+\sqrt{1}}{2} = \frac{5+1}{2} = 3 \\ x_2 &= \frac{-(-5)-\sqrt{(-5)^2-4\times1\times6}}{2\times1} = \frac{5-\sqrt{25-24}}{2} = \frac{5-\sqrt{1}}{2} = \frac{5-1}{2} = 2 \end{aligned} \tag{2.34}$$

We see that this equation has two unequal positive roots.

Example 2.14

Find the roots of the quadratic equation

$$x^2 + 4x + 4 = 0 \tag{2.35}$$

Solution:

Using the quadratic formulas of (2.32), we get

$$\begin{aligned} x_1 &= \frac{-4+\sqrt{4^2-4\times1\times4}}{2\times1} = \frac{-4+\sqrt{16-16}}{2} = \frac{-4+0}{2} = -2 \\ x_2 &= \frac{-4-\sqrt{4^2-4\times1\times4}}{2\times1} = \frac{-4-\sqrt{16-16}}{2} = \frac{-4-0}{2} = -2 \end{aligned} \tag{2.36}$$

We see that this equation has two equal negative roots.

Example 2.15

Find the roots of the quadratic equation

$$2x^2 + 4x + 5 = 0 \tag{2.37}$$

Solution:

Using (2.32), we get

$$\begin{aligned} x_1 &= \frac{-4+\sqrt{4^2-4\times1\times5}}{2\times2} = \frac{-4+\sqrt{16-20}}{4} = \frac{-4+\sqrt{-4}}{4} \quad (\text{value cannot be determined}) \\ x_2 &= \frac{-4-\sqrt{4^2-4\times1\times5}}{2\times2} = \frac{-4-\sqrt{16-20}}{4} = \frac{-4-\sqrt{-4}}{4} \quad (\text{value cannot be determined}) \end{aligned} \tag{2.38}$$

Here, the square root of -4, i.e., $\sqrt{-4}$ is undefined. It is an imaginary number and, as stated earlier, imaginary numbers will not be discussed in this text.

In general, if the coefficients a, b and c are real constants (known numbers), then:

I. If $b^2 - 4ac$ is positive, as in Example 2.13, the roots are real and unequal.

II. If $b^2 - 4ac$ is zero, as in Example 2.14, the roots are real and equal.

III. If $b^2 - 4ac$ is negative, as in Example 2.15, the roots are imaginary.

2.6 Cubic and Higher Degree Equations

A *cubic equation* has the form

$$ax^3 + bx^2 + cx + d = 0 \tag{2.39}$$

and higher degree equations have similar forms. Formulas and procedures for solving these equations are included in books of mathematical tables. We will not discuss them here since their applications to business, basic science, and technology are limited. They are useful in higher mathematics and engineering.

2.7 Measures of Central Tendency

Measures of central tendency are very important in probability and statistics. They will be discussed in more detail in Chapters 7 through 9. The intent here is to become familiar with terminologies used to describe data.

When we analyze data, we begin with the calculation of a single number, which we assume represents all the data. Because data often have a central cluster, this number is called a *measure of central tendency*. The most widely used are the *mean, median*, and *mode*. These are described below.

The *arithmetic mean is* the value obtained by dividing the sum of a set of quantities by the number of quantities in the set. It is also referred to as the *average*.

The arithmetic mean or simply mean, is denoted with the letter x with a bar above it, and is computed from the equation

$$\bar{x} = \frac{\sum x}{n} \tag{2.40}$$

where the symbol Σ stands for summation and n is the number of data, usually called *sample*.

Example 2.16

The ages of 15 college students in a class are

$$24, 26, 27, 23, 31, 29, 25, 28, 21, 23, 32, 25, 30, 24, 26$$

Compute the mean (average) age of this group of students.

Solution:

Here, the sample is $n = 15$ and using (2.40) we get

$$\bar{x} = \frac{24+26+27+23+31+29+25+28+21+23+32+25+30+24+26}{15}$$

$$\bar{x} = \frac{389}{25} = 26.27 \approx 26$$

where the symbol ≈ stands for approximately equal to.

We can check out answer with the Excel AVERAGE function to become familiar with it.

We start with a blank worksheet and we enter the given numbers in Cells A1 through A15. In A16 we type =AVERAGE(A1:A15). Excel displays the answer 26.26667. We will use these values for the next example; therefore, it is recommended that they should not be erased.

The *median* of a sample is the value that separates the lower half of the data, from the upper half. To find the median, we arrange the values of the sample in increasing (ascending) order. If the number of the sample is odd, the median is in the middle of the list; if even, the median is the mean (average) of the two values closest to the middle of the list. We will denote the median as M_d.

Example 2.17

Given the sample of Example 2.16, find the median.

Solution:

The given sample is repeated here for convenience.

$$24, 26, 27, 23, 31, 29, 25, 28, 21, 23, 32, 25, 30, 24, 26$$

We can arrange this sample in ascending (increasing) order with pencil and paper; however, we will let Excel do the work for us. Unless this list has been erased, it still exists in A1:A15. Now, we erase the value in A16 by pressing the *Delete* key. We highlight the range A1:A15 and click on ***Data>Sort>Column A>Ascending>OK***. We observe that the numbers now appear in ascending order, and the median appears in A8 and has the value of 26, thus, for this example, $M_d = 26$.

Excel can find the median without first sorting the data. To illustrate the procedure, we undo sort by clicking on ***Edit>Undo Sort*** and we observe that the list now appears as entered the first time.

We select any cell, we type =MEDIAN(A1:A15), and we observe that Excel displays 26. Again, we will use these values for the next example; therefore, it is recommended that they should not be erased.

The *mode* is the value in a sample that occurs most often. If, in a sequence of numbers, no number appears two or more times, the sample has no mode. The mode, if it exists, may or may not be unique. If two such values exist, we say that the sample is *bimodal*, and if three values exist, we call it *trimodal*. We will denote the mode as M_o.

Example 2.18

Find the mode for the sample of Example 2.16.

Solution:

We assume that the data appear in the original order, that is, as

$$24, 26, 27, 23, 31, 29, 25, 28, 21, 23, 32, 25, 30, 24, 26$$

Let us sort these values as we did in Example 2.17. When this is done, we observe that the values 24, 25, and 26 each appear twice in the sample. Therefore, we say that this sample is trimodal.

Excel has also a function that computes the mode; however, if the sample has no unique mode, it displays only the first, and gives no indication that the sample is bimodal or trimodal. To verify this, we select any cell, and we type =MODE(A1:A15). Excel displays the value 24.

Note 2.3

Textbooks in statistics provide formulas for the computation of the median and mode. We do not provide them here because, for our purposes, these are not as important as the arithmetic mean. In Chapters 8 and 9 we will discuss other important quantities such as the expected value, variance, standard deviation, and probability distributions. We will also present numerous practical applications.

2.8 Interpolation and Extrapolation

Let us consider the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ shown on Figure 2.1 where, in general, the designation $P_n(x_n, y_n)$ is used to indicate the intersection of the lines parallel to the $x-axis$ and $y-axis$.

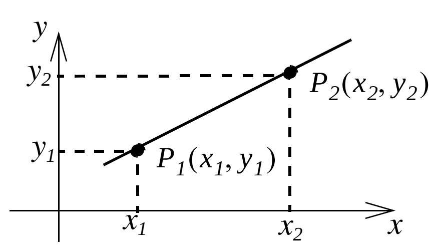


Figure 2.1. Graph to define interpolation and extrapolation

Let us assume that the values x_1, y_1, x_2, and y_2 are known. Next, let us suppose that a known value x_i lies between x_1 and x_2 and we want to find the value y_i that corresponds to the known value of x_i. We must now make a decision whether the unknown value y_i lies on the straight line segment that connects the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ or not. In other words, we must decide whether the new point $P_i(x_i, y_i)$ lies on the line segment P_1P_2, above it, or below it. *Linear interpolation* implies that the point $P_i(x_i, y_i)$ lies on the segment P_1P_2 between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, and *linear extrapolation* implies that the point $P_i(x_i, y_i)$ lies to the left of point $P_1(x_1, y_1)$ or to the right of $P_2(x_2, y_2)$ but on the same line segment which may be extended either to the left or to the right. Interpolation and extrapolation methods other than linear are discussed in numerical analysis* textbooks where polynomials are used very commonly as functional forms. Our remaining discussion and examples will be restricted to linear interpolation.

Linear interpolation and extrapolation can be simplified if the first calculate the *slope* of the straight line segment.

The *slope,* usually denoted as *m,* is the *rise* in the vertical *(y-axis)* direction over the *run* in the abscissa *(x-axis)* direction. Stated mathematically, the slope is defined as

$$slope = m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.41}$$

Example 2.19

Compute the slope of the straight line segment that connects the points $P_1(3, 2)$ and $P_2(7, 4)$ shown in Figure 2.2.

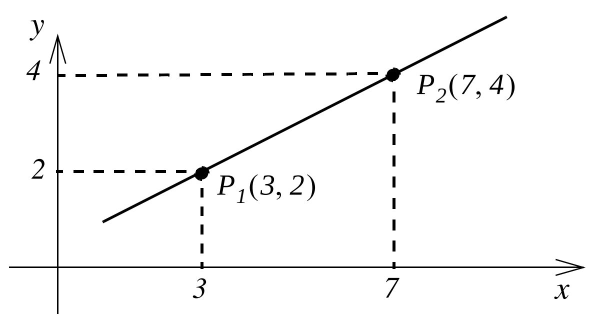


Figure 2.2. Graph for example 2.19

Solution:

$$slope = m = \frac{rise}{run} = \frac{4-2}{7-3} = \frac{2}{4} = 0.5$$

* *Refer, for example, to Numerical Analysis Using MATLAB and Spreadsheets, ISBN 0-9709511-1-6, by Orchard Publications.*

In the graph of Figure 2.1, if we know the value x_i and we want to find the value y_i, we use the formula

$$\frac{y_i - y_1}{x_i - x_1} = slope \tag{2.42}$$

or

$$y_i = slope \times (x_i - x_1) + y_1 = m(x_i - x_1) + y_1 \tag{2.43}$$

and if we know the value of y_i and we want to find the value x_i, we solve (2.46) for x_i and we get

$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1 \tag{2.44}$$

We observe that, if in (2.47) we let $x_1 = 0$, $x_i = x$, $y_i = y$, and $y_1 = b$, we obtain the equation of any straight line

$$y = mx + b \tag{2.45}$$

which will be introduced on Chapter 3.

Example 2.20

Given the graph of Figure 2.3, perform linear interpolation to compute the value of *y* that corresponds to the value $x = 7.5$

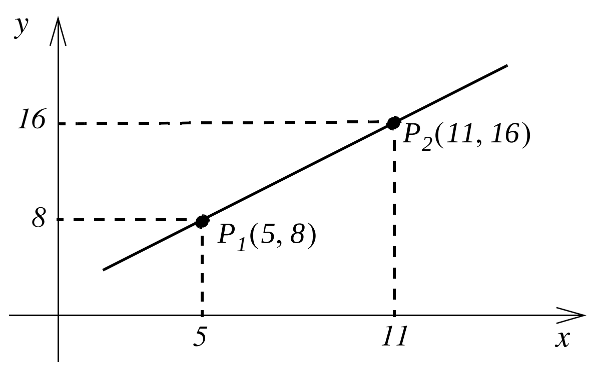


Figure 2.3. Graph for example 2.20

Solution:

Using (2.43) we get

$$y_{x = 7.5} = slope \times (x_i - x_1) + y_1 = \frac{16-8}{11-5} \times (7.5 - 5) + 8 = \frac{8}{6} \times 2.5 + 8 = \frac{10}{3} + 8 = 11.33$$

Note 2.4

The smaller the interval, the better the approximation will be obtained by linear interpolation.

Note 2.5

It is highly recommended that the data points are plotted so that we can assess how reasonable our approximation will be.

Note 2.6

We must exercise good judgement when we use linear interpolation since we may obtain unrealistic values. As as example, let us consider the following table where x represents the indicated numbers and y represents the square of x, that is, $y = x^2$.

x	y
1	1
....	....
6	36

If we use linear interpolation to find the square of 5 with the data of the above table we will find that

$$y_{x=5} = slope \times (x_i - x_1) + y_1 = \frac{36-1}{6-1} \times (5-1) + 1 = \frac{35}{5} \times 4 + 1 = 29$$

and obviously this is gross error since the square of 5 is 25, not 29.

2.9 Infinite Sequences and Series

An infinite sequence is a function whose domain is the set of positive integers. For example, when x is successively assigned the values 1, 2, 3, ... , the function defined as

$$f(x) = \frac{1}{1+x}$$

yields the infinite sequence $1/2$, $1/3$, $1/4$, ... and so on. This sequence is referred to as infinite sequence to indicate that there is no last term. We can create a sequence $\{s_n\}$ by addition of numbers. Let us suppose that the numbers to be added are

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

We let

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ &\ldots \\ s_n &= u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k \end{aligned} \tag{2.46}$$

An expression such as (2.46) is referred to as an *infinite series*. There are many forms of infinite series with practical applications. In this text, we will discuss only the arithmetic series, geometric series, and harmonic series.

2.10 Arithmetic Series

An *arithmetic series* (or *arithmetic progression*) is a sequence of numbers such that each number differs from the previous number by a constant amount, called the *common difference*.

If a_1 is the first term, a_n is the *nth* term, d is the common difference, n is the number of terms, and s_n is the sum of n terms, then

$$a_n = a_1 + (n-1)d \tag{2.47}$$

$$s_n = \frac{n}{2}(a_1 + a_n) \tag{2.48}$$

$$s_n = \frac{n}{2}[2a_1 + (n-1)d] \tag{2.49}$$

The expression

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) \tag{2.50}$$

is known as the *sum identity*.

Example 2.21

Compute the sum of the integers from 1 to 100

Solution:

Using the sum identity, we get

$$\sum_{k=1}^{100} k = \frac{1}{2}100(100+1) = 50 \times 101 = 5050$$

2.11 Geometric Series

A *geometric series* (or *geometric progression*) is a sequence of numbers such that each number bears a constant ratio, called the *common ratio*, to the previous number.

If a_1 is the first term, a_n is the nth term, r is the common ratio, n is the number of terms, and s_n is the sum of n terms, then

$$a_n = a_1 r^{n-1} \tag{2.51}$$

and for $r \neq 1$,

$$\begin{aligned} s_n &= a_1 \frac{1-r^n}{1-r} \\ &= \frac{a_1 - r a_n}{1-r} \\ &= \frac{r a_n - a_1}{r-1} \end{aligned} \tag{2.52}$$

The first sum equation in (2.52) is derived as follows:

The general form of a geometric series is

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + \ldots \tag{2.53}$$

The sum of the first n terms of (2.57) is

$$s_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} \tag{2.54}$$

Multiplying both sides of (2.58) by r we get

$$r s_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + a_1 r^n \tag{2.55}$$

If we subtract (2.59) from (2.58) we will find that all terms on the right side cancel except the first and the last leaving

$$(1-r) s_n = a_1 (1-r^n) \tag{2.56}$$

Provided that $r \neq 1$, division of (2.60) by $1-r$ yields

$$s_n = a_1 \frac{1-r^n}{1-r} \tag{2.57}$$

It is shown in advanced mathematics textbooks that if $|r| < 1$, the geometric series

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + \ldots$$

converges to the sum

$$s_n = \frac{a_1}{1-r} \tag{2.58}$$

Example 2.22

A ball is dropped from x feet above a flat surface. Each time the ball hits the ground after falling a distance h, it rebounds a distance rh where $r < 1$. Compute the total distance the ball travels.

Solution:

The path and the distance the ball travels is shown on the sketch of Figure 2.4.

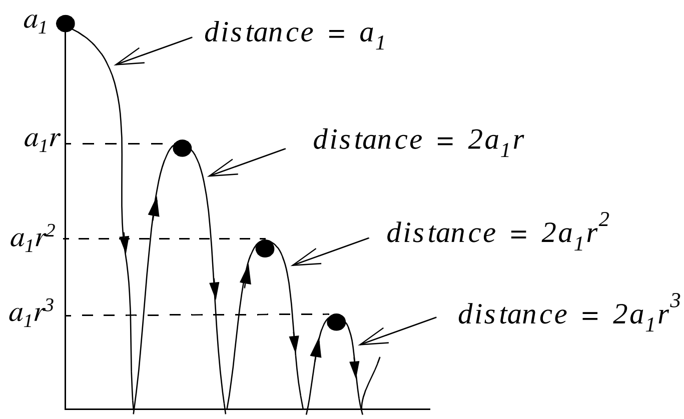


Figure 2.4. Sketch for Example 2.22

The total distance *s* is computed by the geometric series

$$s = a_1 + 2a_1r + 2a_1r^2 + 2a_1r^3 + \dots \tag{2.59}$$

By analogy to equation of (2.62), the second and subsequent terms in (2.63) can be expressed as the sum of

$$\frac{2a_1r}{1-r} \tag{2.60}$$

Adding the first term of (2.63) with (2.64) we form the total distance as

$$s = a_1 + \frac{2a_1r}{1-r} = a_1\frac{1+r}{1-r} \tag{2.61}$$

For example, if $a_1 = 6\ ft$ and $r = 2/3$, the total distance the ball travels is

$$a = 6 \times \frac{1+2/3}{1-2/3} = 30\ ft$$

2.12 Harmonic Series

A sequence of numbers whose reciprocals form an arithmetic series is called an *harmonic series* (or *harmonic progression*). Thus

$$\frac{1}{a_1}, \frac{1}{a_1+d}, \frac{1}{a_1+2d}, \dots, \frac{1}{a_1+(n-1)d}, \dots \tag{2.62}$$

where

$$\frac{1}{a_n} = \frac{1}{a_1 + (n-1)d} \tag{2.63}$$

forms an harmonic series.

Harmonic series provide quick answers to some very practical applications such as the example that follows.

Example 2.23

Suppose that we have a list of temperature numbers on a particular day and time for 50 years. Let us assume the numbers are random, that is, the temperature on a particular day and time of one year has no relation to the temperature on the same day and time of any subsequent year.

The first year was undoubtedly a record year. In the second year, the temperature could equally likely have been more than, or less than, the temperature of the first year. So there is a probability of $1/2$ that the second year was a record year. The expected number of record years in the first two years of record-keeping is therefore $1 + 1/2$. For the third year. the probability is $1/3$ that the third observation is higher than the first two, so the expected number of record temperatures in three years is $1 + 1/2 + 1/3$. Continuing this line of reasoning leads to the conclusion that the expected number of records in the list of n observations is

$$1 + 1/2 + 1/3 + \ldots + 1/n$$

For our example, $n = 50$ and thus the sum of the first 50 terms of harmonic series is 4.499 or 5 record years. Of course, this is the expected number of record temperatures, not the actual.

The partial sums of this harmonic series are plotted on Figure 2.5.

1	1.000000
2	1.500000
3	1.833333
4	2.083333
5	2.283333
6	2.450000
7	2.592857
8	2.717857
9	2.828968
10	2.928968
11	3.019877
12	3.103211

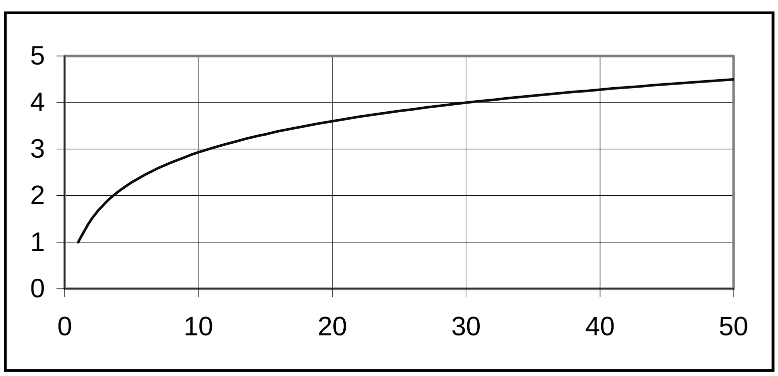


Figure 2.5. Plot for Example 2.23

2.13 Proportions

A *proportion* is defined as two ratios being equal to each other. If

$$\frac{a}{b} = \frac{c}{d} \tag{2.64}$$

where *a*, *b*, *c*, and *d* are non-zero values, then

$$\frac{a+b}{b} = \frac{c+d}{d} \tag{2.65}$$

$$\frac{a-b}{b} = \frac{c-d}{d} \tag{2.66}$$

$$\frac{a-b}{a+b} = \frac{c-d}{c+d} \tag{2.67}$$

Consider a line segment with a length of one unit. Let us divide this segment into two unequal segments and let

x = shorter segment

$1-x$ = longer segment

The *golden proportion* is the one where the ratio of the shorter segment to the longer segment, that is, $x/(1-x)$ is equal to the ratio of the longer segment to the whole segment, that is, $(1-x)/1$. In other words, the golden proportion states that

$$\frac{x}{(1-x)} = \frac{(1-x)}{1} \tag{2.68}$$

Solving for x we get

$$\begin{aligned}(1-x)^2 &= x\\ 1-2x+x^2 &= x\\ x^2-3x+1 &= 0\end{aligned}$$

Solution of this quadratic equation yields

$$x_1, x_2 = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = \frac{3 \pm 2.236}{2}$$

and since x cannot be more than unity, we reject the largest value and we accept x as $x = 0.382$. Then, $1-x = 0.618$ and thus the golden proportion is

$$\frac{0.382}{0.618} = \frac{0.618}{1} \tag{2.69}$$

The *golden rectangle* is defined as one whose length is 1 and its width is 0.618 as shown in Figure 2.6.

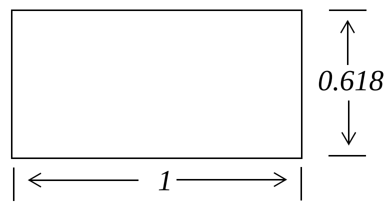


Figure 2.6. The golden rectangle

2.14 Summary

- *Algebra* is the branch of mathematics in which letters of the alphabet represent numbers or a set of numbers.
- *Equations* are equalities that indicate how some quantities are related to others.
- In algebra, the *order of precedence,* where 1 is the highest and 4 is the lowest, is as follows:
  1. Quantities inside parentheses ( )
  2. Exponentiation
  3. Multiplication and Division
  4. Addition or Subtraction
- An *algebraic equation* is an equality that contains one or more unknown quantities, normally represented by the last letters of the alphabet such as x, y, and z.
- Solving an equation means finding the value of the unknown quantity that will make the equation an equality with no unknowns.
- The same number may be added to, or subtracted from both sides of an equation.
- Each side of an equation can be multiplied or divided by the same number. Division by zero must be avoided.
- The *laws of exponents* state that

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & if \quad m > n \\ 1 & if \quad m = n \\ \dfrac{1}{a^{n-m}} & if \quad m < n \end{cases}$$

$$(a^m)^n = a^{mn}$$

- The *nth* root is defined as the inverse of the *nth* exponent, that is, if

$$b^n = a, \quad then \quad b = \sqrt[n]{a}$$

- The *laws of logarithms* state that

$$log_a(xy) = log_a x + log_a y$$

$$log_a\left(\frac{x}{y}\right) = log_a x - log_a y$$

$$log_a(x^n) = n log_a x$$

- Logarithms consist of an integer, called the *characteristic* and an endless decimal called the *mantissa*.
- If the decimal point is located immediately to the right of the msd of a number, the characteristic is zero; if the decimal point is located after two digits to the right of the msd, the characteristic is 1; if after three digits, it is 2 and so on.
- If the decimal point is located immediately to the left of the msd of a number, the characteristic is -1, if located two digits to the left of the msd, it is -2, if after three digits, it is -3 and so on.
- The mantissa cannot be determined by inspection; it must be extracted from tables of common logarithms.
- Although the common logarithm of a number less than one is negative, it is written with a *negative* characteristic and a *positive* mantissa. This is because the mantissas in tables are given as positive numbers.
- Because mantissas are given in math tables as positive numbers, the negative sign is written above the characteristic. This is to indicate that the negative sign does not apply to the mantissa.
- A convenient method to find the characteristic of logarithms is to first express the given number in scientific notation; the characteristic then is the exponent.

- *Quadratic equations* are those that contain equations of second degree. The general form of a quadratic equation is

$$ax^2 + bx + c = 0$$

where a, b and c are real constants (positive or negative).

- The roots of a quadratic equations can be found from the formulas

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- The quantity $b^2 - 4ac$ under the square root is called the *discriminant* of a quadratic equation.
- A *cubic equation* has the form

$$ax^3 + bx^2 + cx + d = 0$$

and higher degree equations have similar forms. Formulas and procedures for solving these equations are included in books of mathematical tables.

- The *arithmetic mean is* the value obtained by dividing the sum of a set of quantities by the number of quantities in the set. It is also referred to as the *average*. The arithmetic mean or simply mean, is denoted with the letter x with a bar above it, and is computed from the equation

$$\bar{x} = \frac{\sum x}{n}$$

where the symbol Σ stand for summation and n is the number of data, usually called *sample*. The mean can also be found with the Excel =AVERAGE function.

- The *median* of a sample is the value that separates the lower half of the data, from the upper half. To find the median, we arrange the values of the sample in increasing (ascending) order. If the number of the sample is odd, the median is in the middle of the list; if even, the median is the mean (average) of the two values closest to the middle of the list. The median can also be found with the Excel =MEDIAN function.
- The *mode* is the value in a sample that occurs most often. If, in a sequence of numbers, no number appears two or more times, the sample has no mode. The mode, if it exists, may or may not be unique. If two such values exist, we say that the sample is *bimodal*, and if three values exist, we call it *trimodal*. The mode can also be found with the Excel =MODE function.
- *Linear interpolation* implies that the point $P_i(x_i, y_i)$ lies on the segment P_1P_2 between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a straight line

- Linear extrapolation implies that the point $P_i(x_i, y_i)$ lies to the left of point $P_1(x_1, y_1)$ or to the right of $P_2(x_2, y_2)$ but on the same line segment which may be extended either to the left or to the right.
- An infinite sequence is a function whose domain is the set of positive integers.
- An expression such as the one shown below is referred to as an *infinite series*.

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

- An *arithmetic series* (or *arithmetic progression*) is a sequence of numbers such that each number differs from the previous number by a constant amount, called the *common difference*.
- A *geometric series* (or *geometric progression*) is a sequence of numbers such that each number bears a constant ratio, called the *common ratio*, to the previous number.
- A sequence of numbers whose reciprocals form an arithmetic series is called an *harmonic series* (or *harmonic progression*).
- A *proportion* is defined as two ratios being equal to each other.
- The *golden proportion* states that

$$\frac{x}{(1-x)} = \frac{(1-x)}{1}$$

- The *golden rectangle* is defined as one whose length is 1 and its width is 0.618.

2.15 Exercises

1. In the following equations V, I, R and P are the unknowns. Solve the following equations, that is, find the value of each unknown.

$$\text{a. } R + 50 = 150 \quad \text{b. } V - 25 = 15 \quad \text{c. } 5I - 10 = 45$$

$$\text{d. } \frac{1}{R} = 0.1 \quad \text{e. } \sqrt{5P} = 10 \quad \text{f. } (100 + R)10^{-3} = 1$$

2. The notation $\sqrt[r]{x}$ stands for the *rth* root of a number x. Thus $\sqrt[4]{256}$ stands for the fourth root of 256, $\sqrt[5]{3125}$ for the fifth root of 3125 and so on. Solve the following equations.

$$\text{a. } \sqrt[6]{262144} = x \quad \text{b. } \sqrt[9]{1000000000} = y$$

3. Mathematical tables show that the mantissa of the common logarithm of 0.000417 is 0.62014. However, Excel displays the number -3.37986 rounded to five decimal places when we use the function =LOG(0.000417). Is there something wrong with the formula that we used, or is Excel giving us the wrong answer? Explain.

4. The formula below computes the terms of an annuity with an initial payment and periodic payments.

$$n = \frac{log\left(\frac{1 + fv \times (int)/(pmt)}{1 + pv \times (int)/(pmt)}\right)}{log(1 + int)}$$

fv =future value, that is, the cash value we want to receive after the last payment is made

int = interest rate per period

pmt = payment made at the end of each period

pv = initial payment.

Use this formula to compute the number of months required to accumulate $\$1,000,000$ by making an initial payment of $\$10,000$ now and monthly payments of $\$2,000$ into a savings account paying 6% annual interest compounded monthly. Verify your answer with Excel's NPER function remembering that the initial and monthly payments are cash outflows and, as such, must be entered as negative numbers. However, in the formula above, these must be entered as positive values.

5. Solve the following quadratic equations.

$$\text{a. } x^2 + 5x + 6 \quad \text{b. } x^2 - 6x + 9$$

6. Twenty houses were sold for the prices listed below where, for simplicity, the dollar ($) sign has been omitted.

547,000	*670,000*	*458,000*	*715,000*	*812,000*
678,000	*595,000*	*760,000*	*490,000*	*563,000*
805,000	*715,000*	*635,000*	*518,000*	*645,000*
912,000	*873,000*	*498,000*	*795,000*	*615,000*

Compute the mean, median, and mode.

2.16 Solutions to Exercises

1.

a. $R + 50 = 150$, $R + 50 - 50 = 150 - 50$, $R = 50$

b. $V - 25 = 15$, $V - 25 + 25 = 15 + 25$, $V = 40$

c. $5I - 10 = 45$, $5I - 10 + 10 = 45 + 10$, $5I = 55$, $(5I)/5 = 55/5$, $I = 11$

d. $\frac{1}{R} = 0.1$, $\frac{1}{R} \times R = 0.1 \times R$, $1 = 0.1 \times R$, $1/0.1 = R$, $R = 10$

e. $\sqrt{5P} = 10$, $(5P)^{(1/2)} = 10$, $(5P)^{(1/2) \times 2} = 10^2$, $5P = 100$, $(5P)/5 = 100/5$, $P = 20$

f. $(100 + R)10^{-3} = 1$, $(100 + R)10^{-3} \times 10^3 = 1 \times 10^3$, $(100 + R)10^0 = 1000$,

$(100 + R) \times 1 = 1000$, $100 + R - 100 = 1000 - 100$, $R = 900$

2.

a. $\sqrt[6]{262144} = x$, $262144^{(1/6)} = x$, and in Excel we enter the formula =262144^(1/6) which yields 8 and thus $x = 8$

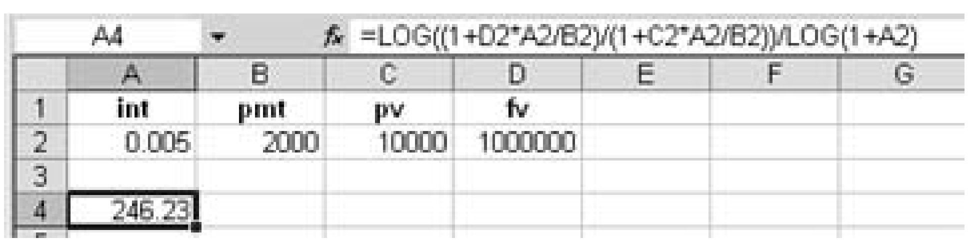
b. $\sqrt[9]{1000000000} = y$, $1000000000^{(1/9)} = y$, and in Excel we enter the formula =1000000000^(1/9) which yields 10 and thus $y = 10$

3.

Mathematical tables show only the mantissa which is the fractional part of the logarithm, and as explained in Section 2.4, mantissas in tables are given as positive numbers. Review Example 2.14 to verify that the answer displayed by Excel is correct.

4.

We construct the spreadsheet below


A4 =LOG((1+D2*A2/B2)/(1+C2*A2/B2))/LOG(1+A2)

	A	B	C	D	E	F	G
1	int	pmt	pv	fv			
2	0.005	2000	10000	1000000			
3							
4	246.23						

and in Cell A4 we type the formula =LOG((1+D2*A2/B2)/(1+C2*A2/B2))/LOG(1+A2)

This yields 246.23 months.

The same result is obtained with Excel's NPER function as shown below.

=NPER(0.06/12,–2000,–10000,1000000,0)

yields 246.23 months.

5.

a.

$$x_1 = \frac{-5+\sqrt{5^2-4\times1\times6}}{2\times1} = \frac{-5+\sqrt{25-24}}{2} = \frac{-5+1}{2} = -2$$

$$x_2 = \frac{-5-\sqrt{5^2-4\times1\times6}}{2\times1} = \frac{-5-\sqrt{25-24}}{2} = \frac{-5-1}{2} = -3$$

b.

$$x_1 = \frac{6+\sqrt{6^2-4\times1\times9}}{2\times1} = \frac{6+\sqrt{36-36}}{2} = \frac{6+0}{2} = 3$$

$$x_2 = \frac{6-\sqrt{6^2-4\times1\times9}}{2\times1} = \frac{6-\sqrt{36-36}}{2} = \frac{6-0}{2} = 3$$

6.

We find the mean, median, and mode using the procedures of Examples 2.16, 2.17, and 2.18 respectively. Then we verify the answers with the spreadsheet below where the values were entered in Cells A1 through A20 in the order given, and they were copied in B1 through B20 and were sorted.

The mean was found with the formula =AVERAGE(B1:B20) entered in D1.

The median was found with the formula =MEDIAN(B1:B20) entered in D2.

The mode was found with the formula =MODE(B1:B20) entered in D3.

	A	B	C	D
1	547,000	458,000	Mean=	664,950
2	678,000	490,000	Median=	657,500
3	805,000	498,000	Mode=	715,000
4	912,000	518,000		
5	670,000	547,000		
6	595,000	563,000		
7	715,000	595,000		
8	873,000	615,000		
9	458,000	635,000		
10	760,000	645,000		
11	635,000	670,000		
12	498,000	678,000		
13	715,000	715,000		
14	490,000	715,000		
15	518,000	760,000		
16	795,000	795,000		
17	812,000	805,000		
18	563,000	812,000		
19	645,000	873,000		
20	615,000	912,000		