Chapter 3

Intermediate Algebra

This chapter is a continuation of Chapter 2. It is intended for readers who need the knowledge for more advanced topics in mathematics. It serves as a prerequisite for probability and statistics. Readers with a strong mathematical background may skip this chapter. Others will find it useful, as well as a convenient source for review.

3.1 Systems of Two Equations

Quite often, we are faced with problems that seem impossible to solve because they involve two or more unknowns. However, a solution can be found if we write two or more linear independent equations* with the same number of unknowns, and solve these simultaneously. Before we discuss generalities, let us consider a practical example.

Example 3.1

Jeff Simpson plans to start his own business, manufacturing and selling bicycles. He wants to compute the *break-even* point; this is defined as the point where the revenues are equal to costs. In other words, it is the point where Jeff neither makes nor loses money.

Jeff estimates that his *fixed costs* (rent, electricity, gas, water, telephone, insurance etc.), would be around *$1,000* per month. Other costs such as material, production and payroll are referred to as *variable costs* and will increase linearly (in a straight line fashion). Preliminary figures show that the variable costs for the production of *500* bicycles will be *$9,000* per month. Then,

$$Total\ Costs = Fixed + Variable = 1000 + 9000 = 10,000 \qquad (3.1)$$

Let us plot cost (*y-axis*), versus number of bicycles sold (*x-axis*) as shown in Figure 3.1.

* *Linear equations are those in which the unknowns, such as x and y, have exponent 1. Thus, the equation $7x + 5y = 24$ is linear since the exponents of x and y are both unity. However, the equation $2x^2 + y = 8$ is non-linear because the exponent of x is 2. By independent equations we mean that these they do not depend on each other. For instance, the equations $3x + 2y = 5$ and $6x + 4y = 10$ are not independent since the second can be formed from the first by multiplying both sides by 2.*

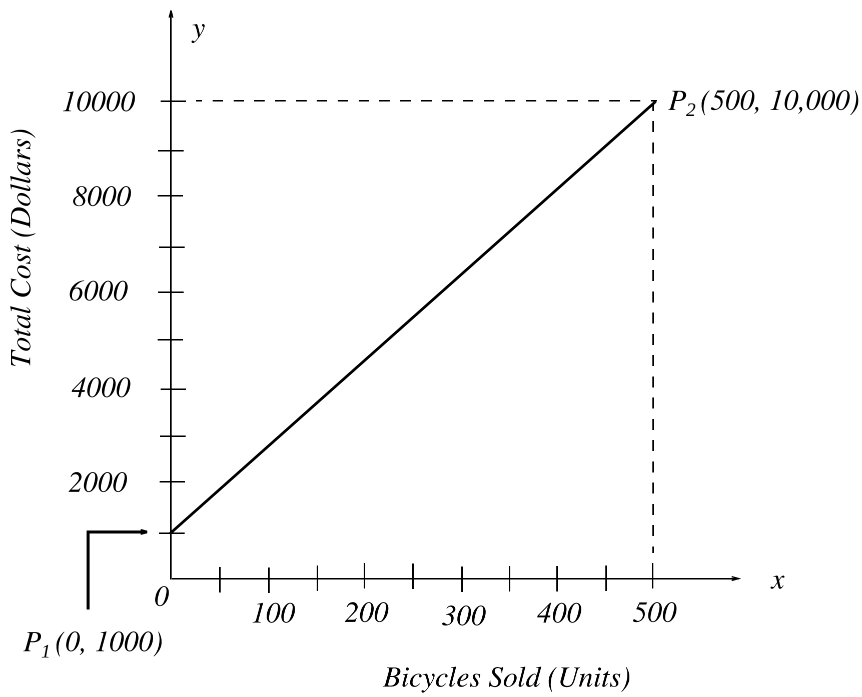


Figure 3.1. Total cost versus bicycles sold.

In Figure 3.1, the *x-axis* is the *abscissa* and the *y-axis* is the *ordinate*. Together, these axes constitute a *Cartesian coordinate system*; this was also mentioned in Chapter 1. The straight line drawn from point P_1 to point P_2, is a graphical presentation of the total costs for the production of the bicycles. It starts at the $\$1,000$ point because this represents a fixed cost; it occurs even though no bicycles are sold. We denote this point as $P_1(0, 1000)$ where the first number 0, within the parentheses, denotes the *x-axis* value at this point, and the second number (1000) denotes the *y-axis* value at that point. For simplicity, the dollar sign has been omitted. Thus, we say that the coordinates of point P_1 are $(0, 1000)$. Likewise, the coordinates of the end point P_2 are denoted as $P_2(500, 10000)$ since the total costs to produce 500 bicycles is $\$10,000$. We will now derive the equation that describes this straight line.

In general, a straight line is represented by the equation

$$y = mx + b \tag{3.2}$$

where m is the *slope*, x is the abscissa, y is the ordinate, and b is the *y-intercept* of the straight line, that is, the point where the straight line crosses the y-axis.

As we indicated in Chapter 2, the *slope* m is the *rise* in the vertical *(y-axis)* direction over the *run* in the abscissa *(x-axis)* direction. We recall from Chapter 2 that the slope is defined as

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.3}$$

In Figure 3.1, $y_2 = 10000$, $y_1 = 1000$, $x_2 = 500$ and $x_1 = 0$. Therefore, the slope is

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10000 - 1000}{500 - 0} = \frac{9000}{500} = 18$$

and, by inspection, the *y*-intercept is *1000*. Then, in accordance with (3.2), the equation of the line that represents the total costs is

$$y = 18x + 1000 \tag{3.4}$$

It is customary, and convenient, to show the unknowns of an equation on the left side and the known values on the right side. Then, (3.4) is written as

$$y - 18x = 1000^{*} \tag{3.5}$$

This equation has two unknowns x and y and thus, no unique solution exists; that is, we can find an infinite number of x and y combinations that will satisfy this equation. We need a second equation, and we will find it by making use of additional facts. These are stated below.

Jeff has determined that if the sells 500 bicycles at $\$25$ each, he will generate a revenue of $\$25 \times 500 = \$12500$. This fact can be represented by another straight line which is added to the previous figure. It is shown in Figure 3.2.

The new line starts at $P_3(0, 0)$ because there will be no revenue if no bicycles are sold. It ends at $P_4(500, 12500)$ which represents the condition that Jeff will generate $\$12500$ when 500 bicycles are sold.

The intersection of the two lines shown as a small circle, establishes the *break-even point* and this is what Jeff is looking for. The projection of the break-even point on the *x-axis*, shown as a broken line, indicates that approximately 140 bicycles must be sold just to break even. Also, the projection on the *y*-axis shows that at this break-even point, the generated revenue is approximately $\$3,500$. This procedure is known as *graphical solution*. A graphical solution gives *approximate* values.

We will now proceed to find the so-called *analytical solution*. This solution will produce *exact* values.

* *We assume that the reader has become proficient with the properties which we discussed in the previous chapter, and henceforth, the details for simplifying or rearranging an equation will be omitted.*

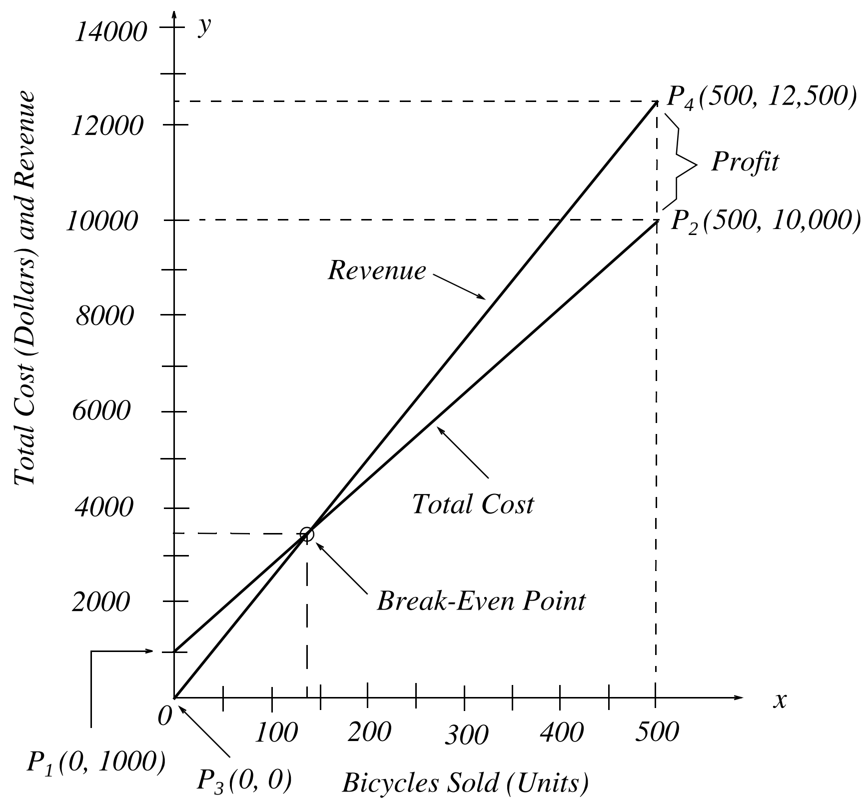


Figure 3.2. Graph showing the intersection of two straight lines.

As stated above, we need a second equation. This is obtained from the straight line in Figure 3.2 which represents the revenue. To obtain the equation of this line, we start with the equation of (3.2) which is repeated here for convenience.

$$y = mx + b \tag{3.6}$$

The slope m of the revenue line is

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12500 - 0}{500 - 0} = \frac{125}{5} = 25 \tag{3.7}$$

and, by inspection, b, that is, the *y-intercept,* is 0. Therefore, the equation that describes the revenue line is

$$y = 25x \tag{3.8}$$

By grouping equations (3.5) and (3.8), we get the system of equations

$$\boxed{\begin{aligned} y - 18x &= 1000 \\ y &= 25x \end{aligned}} \tag{3.9}$$

Now, we must solve the equations of (3.9) simultaneously. This means that we must find unique values for x and y, so that both equations of (3.9) will be satisfied at the same time. An easy way to solve these equations is by substitution of the second into the first. This eliminates y from the first equation which now becomes

$$25x - 18x = 1000 \tag{3.10}$$

or

$$7x = 1000$$

or

$$x = \frac{1000}{7}* \tag{3.11}$$

To find the unknown y, we substitute (3.11) into either the first, or the second equation of (3.9). Thus, by substitution into the second equation, we get

$$y = 25 \times \frac{1000}{7} = \frac{25000}{7} \tag{3.12}$$

Therefore, the *exact* solutions for x and y are

$$\begin{aligned} x &= \frac{1000}{7} \\ y &= \frac{25000}{7} \end{aligned} \tag{3.13}$$

Since no further substitutions are needed, we simplify the values of (3.13) by division and we get $x = 1000/7 = 142.86$ and $y = 25000/7 = 3571.40$. Of course, bicycles must be expressed as whole numbers and therefore, the break-even point for Jeff's business is 143 bicycles which, when sold, will generate a revenue of

$$y = \$25 \times 143 = \$3575$$

Recall that for the break-even point, the graphical solution produced approximate values of 140 bicycles which would generate a revenue of $\$3,500$. Although the graphical solution is not as accurate as the analytical solution, it nevertheless provides us with some preliminary information, which can be used as a check for the answers found using the analytical solution.

Note 3.1

The system of equations of (3.9) was solved by the so-called *substitution method*, also known as *Gauss's elimination method*. In Section 3.3, we will discuss this method in more detail, and two additional methods, the *solution by matrix inversion*, and the *solution by Cramer's rule*.

* *It is not advisable to divide 1000 by 7 at this time. This division will produce an irrational (endless) number which can only be approximated and, when substituted to find the unknown y, it will produce another approximation. Usually, we perform the division at the last step, that is, after the solutions are completed.*

3.2 Systems of Three Equations

Systems of three or more equations also appear in practical applications. We will now consider another example consisting of three equations with three unknowns. It is imperative to remember that *we must have the same number of equations as the number of unknowns.*

Example 3.2

In an automobile dealership, the most popular passenger cars are *Brand A*, *B* and C. Because buyers normally bargain for the best price, the sales price for each brand is not the same. Table 3.1 shows the sales and revenues for a 3-month period.

TABLE 3.1 Sales and Revenues for Example 3.2

Month	*Brand A*	*Brand B*	*Brand C*	*Revenue*
1	25	60	50	$2,756,000
2	30	40	60	2,695,000
3	45	53	58	3,124,000

Compute the average sales price for each of these brands of cars.

Solution:

In this example, the unknowns are the average sales prices for the three brands of automobiles. Let these be denoted as x for *Brand* A, y for *Brand B,* and z for *Brand* C. Then, the sales for each of the three month period can be represented by the following system of equations*.

$$\begin{aligned} 25x + 62y + 54z &= 2756000 \\ 28x + 42y + 58z &= 2695000 \\ 45x + 53y + 56z &= 3124000 \end{aligned} \tag{3.14}$$

The next task is to solve these equations simultaneously, that is, find the values of the unknowns x, y, and z at the same time. In the next section, we will discuss matrix theory and methods for solving systems of equations of this type. We will return to this example at the conclusion of the next section.

3.3 Matrices and Simultaneous Solution of Equations

A *matrix*† is a rectangular array of numbers such as those shown below.

* *The reader should observe that these equations are consistent with each side, that is, both the left and the right side in all equations represent dollars. This observation should be made always when writing equations, since it is a very common mistake to add and equate unrelated quantities.*

† *To some readers, this material may seem too boring and uninteresting. However, it should be read at least once to become familiar with matrix terminology. We will use it for some practical examples in later chapters.*

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \qquad or \qquad \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -5 \\ 4 & -7 & 6 \end{bmatrix}$$

In general form, a matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \tag{3.15}$$

The numbers a_{ij} are the *elements* of the matrix where the index i indicates the row and j indicates the column in which each element is positioned. Thus, a_{43} represents the element positioned in the fourth row and third column.

A matrix of m rows and n columns is said to be of $m \times n$ *order matrix*.

If $m = n$, the matrix is said to be a *square matrix of order* m (or n). Thus, if a matrix has five rows and five columns, it is said to be a square matrix of *order* 5.

In a square matrix, the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called the *diagonal elements*. Alternately, we say that the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are located on the *main diagonal*.

A matrix in which every element is zero, is called a *zero matrix*.

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, that is, $A = B$, if, and only if $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.

Two matrices are said to be *conformable for addition* (*subtraction*) if they are of the same order, that is, both matrices must have the same number of rows and columns.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are conformable for addition (subtraction), their sum (difference) will be another matrix C with the same order as A and B, where each element of C represents the sum (difference) of the corresponding elements of A and B, that is, $C = A \pm B = [a_{ij} \pm b_{ij}]$

Example 3.3

Compute $A + B$ and $A - B$ given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0-1 & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0+1 & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

If k is any scalar (a positive or negative number) and not $[k]$ which is a 1×1 matrix, then multiplication of a matrix A by the scalar k is the multiplication of every element of A by k.

Example 3.4

Multiply the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \text{ by } k = 5$$

Solution:

$$k \cdot A = 5 \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 \times 1 & 5 \times (-2) \\ 5 \times 2 & 5 \times 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 10 & 15 \end{bmatrix}$$

Two matrices A and B are said to be *conformable for multiplication* $A \cdot B$ in that order, only when the number of columns of matrix A is equal to the number of rows of matrix B. The product $A \cdot B$, which is not the same as the product $B \cdot A$, is conformable for multiplication only if A is an $m \times p$ and matrix B is an $p \times n$ matrix. The product $A \cdot B$ will then be an $m \times n$ matrix. A convenient way to determine whether two matrices are conformable for multiplication, and the dimension of the resultant matrix, is to write the dimensions of the two matrices side-by-side as shown below.

Shows that A and B are conformable for multiplication

$$\begin{matrix} A & & B \\ m \times p & & p \times n \end{matrix}$$

Indicates the dimension of the product A · B

For the product $B \cdot A$ we have:

Here, B and A are not conformable for multiplication since n ≠ m

$$\begin{matrix} B & & A \\ p \times n & & m \times p \end{matrix}$$

For matrix multiplication, the operation is row by column. Thus, to obtain the product $A \cdot B$, we multiply each element of a row of A by the corresponding element of a column of B, and then we add these products.

Example 3.5

It is given that

$$C = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Compute the products $C \cdot D$ and $D \cdot C$

Solution:

The dimensions of matrices C and D are 1×3 and 3×1 respectively; therefore, the product $C \cdot D$ is feasible, and results in a 1×1 matrix as shown below.

$$C \cdot D = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2) \cdot (1) + (3) \cdot (-1) + (4) \cdot (2) \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$$

The dimensions for D and C are 3×1 and 1×3 respectively and therefore, the product $D \cdot C$ is also feasible. However, multiplication of these results in a 3×3 matrix as shown below.

$$D \cdot C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} (1) \cdot (2) & (1) \cdot (3) & (1) \cdot (4) \\ (-1) \cdot (2) & (-1) \cdot (3) & (-1) \cdot (4) \\ (2) \cdot (2) & (2) \cdot (3) & (2) \cdot (4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \end{bmatrix}$$

Division of one matrix by another is not defined. However, an equivalent operation exists. It will become apparent later when we discuss the inverse of a matrix.

An *identity matrix* I is a square matrix where all the elements on the main diagonal are ones, and all other elements are zero. Shown below are 2×2, 3×3, and 4×4 identity matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let matrix A be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \qquad (3.16)$$

then, the *determinant of* A, denoted as $detA$, is defined as

$$\begin{aligned} detA = {} & a_{11}a_{22}a_{33}\dots a_{nn} + a_{12}a_{23}a_{34}\dots a_{n1} + a_{13}a_{24}a_{35}\dots a_{n2} + \dots \\ & -a_{n1}\dots a_{22}a_{13}\dots -a_{n2}\dots a_{23}a_{14} - a_{n3}\dots a_{24}a_{15} - \dots \end{aligned} \qquad (3.17)$$

The determinant of a square matrix of order n is referred to as *determinant of order* n.

Let A be a *determinant of order* 2, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad (3.18)$$

Then, by (3.17), (3.18) reduces to

$$detA = a_{11}a_{22} - a_{21}a_{12} \qquad (3.19)$$

Example 3.6

Compute $detA$ and $detB$ given that

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

Solution:

Using (3.19), we get

$$\begin{aligned} detA &= 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2 \\ detB &= 2 \cdot 0 - 2 \cdot (-1) = 0 - (-2) = 2 \end{aligned}$$

Let A be a *determinant of order* 3, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then,

$$\begin{aligned} detA = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} \\ & -a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} \end{aligned} \tag{3.20}$$

A convenient method to evaluate the determinant of order 3 is to write the first two columns to the right of the 3×3 matrix, and add the products formed by the diagonals from upper left to lower right; then subtract the products formed by the diagonals from lower left to upper right as shown below. When this is done properly, we obtain (3.20) above.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

This method works only with second and third order determinants. To evaluate higher order determinants, we must first compute the *cofactors*; these will be defined shortly.

Example 3.7

Compute $detA$ and $detB$ given that

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{bmatrix}$$

Solution:

$$detA = \begin{matrix} 2 & 3 & 5 & 2 & 3 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 2 & 1 \end{matrix}$$

or

$$\begin{aligned} detA = {} & (2 \times 0 \times 0) + (3 \times 1 \times 1) + (5 \times 1 \times 1) \\ & -(2 \times 0 \times 5) - (1 \times 1 \times 2) - (0 \times 1 \times 3) = 11 - 2 = 9 \end{aligned}$$

Likewise,

$$detB = \begin{matrix} 2 & -3 & -4 & 2 & -3 \\ 1 & 0 & -2 & 1 & -2 \\ 0 & -5 & -6 & 2 & -6 \end{matrix}$$

or

$$\begin{aligned} detB = {} & ([2 \times 0 \times (-6)] + [(-3) \times (-2) \times 0] + [(-4) \times 1 \times (-5)]) \\ & -[0 \times 0 \times (-4)] - [(-5) \times (-2) \times 2] - [(-6) \times 1 \times (-3)] \\ = {} & 20 - 38 = -18 \end{aligned}$$

Let matrix A be defined as the square matrix of order n as shown below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

If we remove the elements of its ith row and jth column, the determinant of the remaining $(n-1)$ square matrix is called a *minor of determinant* A, and it is denoted as $[M_{ij}]$.

The signed minor $(-1)^{i+j}[M_{ij}]$ is called the *cofactor* of a_{ij}, and it is denoted as α_{ij}.

Example 3.8

Compute the minors $[M_{11}]$, $[M_{12}]$, $[M_{13}]$, and the cofactors α_{11}, α_{12}, and α_{13} given that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Solution:

$$[M_{11}] = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix},\ [M_{12}] = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix},\ [M_{11}] = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}[M_{11}] = [M_{11}],\quad \alpha_{12} = (-1)^{1+2}[M_{12}] = -[M_{12}]$$
$$\alpha_{13} = (-1)^{1+3}[M_{13}] = [M_{13}]$$

The remaining minors $[M_{21}], [M_{22}], [M_{23}], [M_{31}], [M_{32}], [M_{33}]$, and the remaining cofactors $\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{31}, \alpha_{32}, \alpha_{33}$ are defined similarly.

Example 3.9

Compute the cofactors of

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$$

Solution:

$$\alpha_{11} = (-1)^{1+1}\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} = 20 \quad \alpha_{12} = (-1)^{1+2}\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} = 10$$

$$\alpha_{13} = (-1)^{1+3}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 0 \quad \alpha_{21} = (-1)^{2+1}\begin{bmatrix} 2 & -3 \\ 2 & -6 \end{bmatrix} = 6$$

$$\alpha_{22} = (-1)^{2+2}\begin{bmatrix} 1 & -3 \\ -1 & -6 \end{bmatrix} = -9 \quad \alpha_{23} = (-1)^{2+3}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = -4$$

$$\alpha_{31} = (-1)^{3+1}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = -8 \quad \alpha_{32} = (-1)^{3+2}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = -8$$

and

$$\alpha_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -8$$

It is useful to remember that the signs of the cofactors follow the pattern

$$\begin{matrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{matrix}$$

that is, the cofactors on the diagonals, have the same sign as their minors.

It was shown earlier, that the determinant of a 2×2 or 3×3 matrix can be found by the method of Example 3.7. This method cannot be used for matrices of higher order. In general, *the determinant of a matrix A of any order, is the sum of the products obtained by multiplying each element of any row or any column by its cofactor.*

Example 3.10

Compute the determinant of A from the elements of the first row and their cofactors given that

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$$

Solution:

$$detA = 1\begin{bmatrix}-4 & 2\\ 2 & -6\end{bmatrix} - 2\begin{bmatrix}2 & 2\\ -1 & -6\end{bmatrix} - 3\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix} = 1\times 20 - 2\times(-10) - 3\times 0 = 40$$

The determinant of a matrix of order 4 or higher, must be evaluated using the above procedure. Thus, the determinant of a fourth-order matrix can first be expressed as the sum of the products of the elements of its first row, by its cofactor as shown in (3.21) below.

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix} = a_{11}\begin{bmatrix}a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\end{bmatrix} - a_{21}\begin{bmatrix}a_{12} & a_{13} & a_{14}\\ a_{32} & a_{33} & a_{34}\\ a_{42} & a_{43} & a_{44}\end{bmatrix} + a_{31}\begin{bmatrix}a_{12} & a_{13} & a_{14}\\ a_{22} & a_{23} & a_{24}\\ a_{42} & a_{43} & a_{44}\end{bmatrix} - a_{41}\begin{bmatrix}a_{12} & a_{13} & a_{14}\\ a_{22} & a_{23} & a_{24}\\ a_{32} & a_{33} & a_{34}\end{bmatrix} \quad (3.21)$$

Example 3.11

Compute the determinant of

$$A = \begin{bmatrix}2 & -1 & 0 & -3\\ -1 & 1 & 0 & -1\\ 4 & 0 & 3 & -2\\ -3 & 0 & 0 & 1\end{bmatrix}$$

Solution:

Using the procedure of (3.21), we multiply each element of the first column by its cofactor. Then,

$$\underbrace{A = 2\begin{bmatrix}1 & 0 & -1\\ 0 & 3 & -2\\ 0 & 0 & 1\end{bmatrix}}_{[a]} \underbrace{-(-1)\begin{bmatrix}-1 & 0 & -3\\ 0 & 3 & -2\\ 0 & 0 & 1\end{bmatrix}}_{[b]} \underbrace{+4\begin{bmatrix}-1 & 0 & -3\\ 1 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}}_{[c]} \underbrace{-(-3)\begin{bmatrix}-1 & 0 & -3\\ 1 & 0 & -1\\ 0 & 3 & -2\end{bmatrix}}_{[d]}$$

Next, using the procedure of Example 3.7 or Example 3.10, we find

$$[a] = 6\ ,\ [b] = -3\ ,\ [c] = 0\ ,\ [d] = -36$$

and thus

$$detA = [a] + [b] + [c] + [d] = 6 - 3 + 0 - 36 = -33$$

We can also use the MATLAB function **det(A)** to compute the determinant. For this example,

```
A=[2 −1 0 −3; −1 1 0 −1; 4 0 3 −2; −3 0 0 1]; det(A)
ans =
   −33
```

The determinants of matrices of order five or higher can be evaluated similarly.

Some useful properties of determinants are given below.

1. *If all elements of one row or one column of a square matrix are zero, the determinant is zero.* An example of this is the determinant of the cofactor $[c]$ in Example 3.11.

2. *If all the elements of one row (column) of a square matrix* A *are* m *times the corresponding elements of* another row (column), $detA$ is zero. For instance, if

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

then,

$$detA = \left|\begin{matrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{matrix}\right| \begin{matrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{matrix} = 12 + 4 + 6 - 6 - 4 - 12 = 0$$

Here, $detA$ is zero because the second column in A is 2 times the first column.

We can use the MATLAB function **det(A)** to compute the determinant.

Consider the systems of the three equations below.

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= A \\ a_{21}x + a_{22}y + a_{23}z &= B \\ a_{31}x + a_{32}y + a_{33}z &= C \end{aligned} \tag{3.22}$$

Cramer's rule states that the unknowns x, y, and z can be found from the relations

$$x = \frac{D_1}{\Delta} \qquad y = \frac{D_2}{\Delta} \qquad z = \frac{D_3}{\Delta} \tag{3.23}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad D_1 = \begin{vmatrix} A & a_{12} & a_{13} \\ B & a_{22} & a_{23} \\ C & a_{32} & a_{33} \end{vmatrix} \quad D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix} \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

provided that the determinant Δ (delta) is not zero.

We observe that the numerators of (3.23) are determinants that are formed from Δ by the substitution of the known values A, B, and C for the coefficients of the desired unknown.

Cramer's rule applies to systems of two or more equations.

If (3.22) is a homogeneous set of equations, that is, if $A = B = C = 0$, then D_1, D_2, and D_3 are all zero as we found in Property 1 above. In this case, $x = y = z = 0$ also.

Example 3.12

Use Cramer's rule to find the unknowns v_1, v_2, and v_3 if

$$\begin{aligned} 2v_1 - 5 - v_2 + 3v_3 &= 0 \\ -2v_3 - 3v_2 - 4v_1 &= 8 \\ v_2 + 3v_1 - 4 - v_3 &= 0 \end{aligned}$$

Solution:

Rearranging the unknowns v and transferring known values to the right side we get

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 5 \\ -4v_1 - 3v_2 - 2v_3 &= 8 \\ 3v_1 + v_2 - v_3 &= 4 \end{aligned}$$

Now, by Cramer's rule,

$$\Delta = \left| \begin{array}{ccc|cc} 2 & -1 & 3 & 2 & -1 \\ -4 & -3 & -2 & -4 & -3 \\ 3 & 1 & -1 & 3 & 1 \end{array} \right. = 6 + 6 - 12 + 27 + 4 + 4 = 35$$

Also,

$$D_1 = \left| \begin{array}{ccc|cc} 5 & -1 & 3 & 5 & -1 \\ 8 & -3 & -2 & 8 & -3 \\ 4 & 1 & -1 & 4 & 1 \end{array} \right. = 15 + 8 + 24 + 36 + 10 - 8 = 85$$

$$D_2 = \left| \begin{array}{ccc|cc} 2 & 5 & 3 & 2 & 5 \\ -4 & 8 & -2 & -4 & 8 \\ 3 & 4 & -1 & 3 & 4 \end{array} \right. = -16 - 30 - 48 - 72 + 16 - 20 = -170$$

and

$$D_3 = \left| \begin{array}{ccc|cc} 2 & -1 & 5 & 2 & -1 \\ -4 & -3 & 8 & -4 & -3 \\ 3 & 1 & 4 & 3 & 1 \end{array} \right. = -24 - 24 - 20 + 45 - 16 - 16 = -55$$

Therefore,

$$v_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7}$$

$$v_2 = \frac{D_2}{\Delta} = -\frac{170}{35} = -\frac{34}{7}$$

$$v_3 = \frac{D_3}{\Delta} = -\frac{55}{35} = -\frac{11}{7}$$

We can verify the first equation as follows:

$$2v_1 - v_2 + 3v_3 = 2 \times \frac{17}{7} - \left(-\frac{34}{7}\right) + 3 \times \left(-\frac{11}{7}\right) = \frac{34 + 34 - 33}{7} = \frac{35}{7} = 5$$

We can also find the unknowns in a system of two or more equations by the G*aussian Elimination Method*. With this method, the objective is to eliminate one unknown at a time. This is done by multiplying the terms of any of the given equations of the system, by a number such that we can add (or subtract) this equation to (from) another equation in the system, so that one of the unknowns is eliminated. Then, by substitution to another equation with two unknowns, we can find the second unknown. Subsequently, substitution of the two values that were found, can be made into an equation with three unknowns from which we can find the value of the third unknown. This procedure is repeated until all unknowns are found. This method is best illustrated with the following example which consists of the same equations as the previous example.

Example 3.13

Use the Gaussian elimination method to find v_1, v_2, and v_3 of

$$\begin{aligned} 2v_1 - v_2 + 3v_3 &= 5 \\ -4v_1 - 3v_2 - 2v_3 &= 8 \\ 3v_1 + v_2 - v_3 &= 4 \end{aligned} \tag{3.24}$$

Solution:

As a first step, we add the first equation of (3.24) with the third, to eliminate the unknown v_2. When this is done, we obtain the equation

$$5v_1 + 2v_3 = 9 \tag{3.25}$$

Next, we multiply the third equation of (3.24) by 3 and we add it with the second to eliminate v_2. When this step is done, we obtain the following equation.

$$5v_1 - 5v_3 = 20 \tag{3.26}$$

Subtraction of (3.26) from (3.25) yields

$$7v_3 = -11 \ \ or \ \ v_3 = -\frac{11}{7}^{*} \tag{3.27}$$

Now, we can find the unknown v_1 from either (3.25) or (3.26). Thus, by substitution of (3.27) into (3.25), we get

$$5v_1 + 2 \cdot \left(-\frac{11}{7}\right) = 9 \ \ or \ \ v_1 = \frac{17}{7} \tag{3.28}$$

Finally, we can find the last unknown v_2 from any of the three equations of (3.24). Then, by substitution into the first equation, we get

$$v_2 = 2v_1 + 3v_3 - 5 = \frac{34}{7} - \frac{33}{7} - \frac{35}{7} = -\frac{34}{7}$$

These are the same values as those we found in Example 3.12.

Let A be an n square matrix, and α_{ij} be the cofactor of a_{ij}. Then, *the adjoint of* A, denoted as $adjA$, is defined as the n square matrix of (3.29) below.

$$adjA = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix} \tag{3.29}$$

We observe that the cofactors of the elements of the ith row (column) of A, are the elements of the ith column (row) of $adjA$.

Example 3.14

Given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

compute $adjA$.

* *As stated earlier, it is advisable to leave the values in rational form until the last step.*

Solution:

$$adjA = \begin{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ -\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} & -\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} & -\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

An n square matrix A is called *singular* if $detA = 0$; if $detA \neq 0$, it is called *non-singular.*

Example 3.15

Given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

determine whether this matrix is singular or non-singular.

Solution:

$$detA = \left| \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{matrix} \right| \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{matrix} = 21 + 24 + 30 - 27 - 20 - 28 = 0$$

Therefore, matrix A is singular.

If A and B are n square matrices such that $AB = BA = I$ where I is the identity matrix, B is called the *inverse* of A, denoted as $B = A^{-1}$, and likewise, A is called the *inverse* of B, that is, $A = B^{-1}$.

If a matrix A is non-singular, we can compute its inverse from the relation

$$\boxed{A^{-1} = \frac{1}{detA} adjA} \qquad (3.30)$$

Example 3.16

Given that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

compute its inverse, that is, find A^{-1}.

Solution:

For this example,

$$detA = 9+8+12-9-16-6 = -2$$

and since $detA \neq 0$, it is possible to compute the inverse of A using (3.30).

From Example 3.14,

$$adjA = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{detA}adjA = \frac{1}{-2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3.5 & -3 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{bmatrix}$$

Multiplication of a matrix A by its inverse A^{-1} produces the identity matrix I, that is,

$$AA^{-1} = I \quad or \quad A^{-1}A = I \tag{3.31}$$

Example 3.17

Prove the validity of (3.31) for

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$$

Proof:

$$detA = 8-6 = 2 \quad and \quad adjA = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

then,

$$A^{-1} = \frac{1}{detA}adjA = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}$$

Therefore,

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Consider the relation

$$AX = B \tag{3.32}$$

where A and B are matrices whose elements are known, and X is a matrix whose elements are the unknowns. Here, we assume that A and X are conformable for multiplication. Multiplication of both sides of (3.32) by A^{-1} yields:

$$A^{-1}AX = A^{-1}B = IX = A^{-1}B$$

and since $IX = X$, we obtain the important matrix relation

$$\boxed{X = A^{-1}B} \qquad (3.33)$$

We can use (3.33) to solve any set of simultaneous equations that have solutions. We call this method, the *inverse matrix method of solution*.

Example 3.18

For the system of equations

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 9 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 + 2x_3 &= 8 \end{aligned}$$

compute the unknowns x_1, x_2, and x_3 using the inverse matrix method.

Solution:

In matrix form, the given set of equations is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

and from (3.33),

$$X = A^{-1}B$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Next, we compute the determinant and the adjoint of A. Following the procedures of Examples 3.7 or 3.10, and 3.14 we find that

$$detA = 18 \quad and \quad adjA = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{detA}adjA = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Now, using (3.33) we obtain the solution shown below.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix} = \begin{bmatrix} 1.94 \\ 1.61 \\ 0.28 \end{bmatrix}$$

We can also use Microsoft Excel's MINVERSE (Matrix Inversion) and MMULT (Matrix Multiplication) functions to obtain the values of the three unknowns of Example 3.18.

The procedure is as follows:

1. We start with a blank spreadsheet, and in a block of cells, say B4:D6, we enter the elements of matrix A as shown on the spreadsheet of Figure 3.3. Then, we enter the elements of matrix B in G4:G6.

2. Next, we compute and display the inverse of A, that is, A^{-1}. We choose B8:D10 for the elements of this inverted matrix. We format this block for number display with three decimal places. With this range highlighted, and making sure that the selected cell is B8, we type the formula

   =MINVERSE(B4:D6)

   and we press the *Crtl-Shift-Enter* keys simultaneously. We observe that A^{-1} appears in these cells.

3. Now, we choose the block of cells G8:G10 for the values of the unknowns. As before, we highlight them, and with the cell marker positioned in G8, we type the formula

   =MMULT(B8:D10,G4:G6)

   and we press the *Crtl-Shift-Enter* keys simultaneously. The values of X then appear in G8:G10.

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2	Example 3.17							
3								
4		2	3	1			9	
5	A=	1	2	3		B=	6	
6		3	1	2			8	
7								
8		0.056	-0.278	0.389			1.9444	
9	A^{-1}=	0.389	0.056	-0.278		X=	1.6111	
10		-0.278	0.389	0.056			0.2778	
11								

Figure 3.3. Spreadsheet for Example 3.18

We should not erase this spreadsheet; we will use it to finish Example 3.2.

To continue with Example 3.2, we type-in (overwrite) the existing entries as follows:

1. Replace the contents of matrix A, in A4:D6 with the coefficients of the unknowns in (3.14), which is repeated here for convenience. These are shown in the spreadsheet of Figure 3.4.

$$\begin{aligned} 25x + 62y + 54z &= 2756000 \\ 28x + 42y + 58z &= 2695000 \\ 45x + 53y + 56z &= 3124000 \end{aligned} \tag{3.34}$$

2. In G4:G6, we enter the values that appear on the right side of (3.34). The values of the unknowns x, y, and z appear in G8:G10*.

   The updated spreadsheet is shown in Figure 3.4.

   The range G8:G10 shows that *Brand A* car was sold at an average price of *$20,904.99*, *Brand B* at *$11,757.61*, and *Brand C* at *$27,589.32*.

The last step is to verify that these values satisfy all three equations in (3.34). This can be done easily with Excel as follows:

In any cell, say A11, we type the formula

=B4*G8+C4*G9+D4*G10

* *Make sure that Excel is configured for automatic recalculation. This can be done with the sequential commands* ***Tool>Options>Calculation>Automatic****.*

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and Matrix Multiplication							
2	Example 3.2							
3								
4		25	62	54			2756000	
5	A=	28	42	58		B=	2695000	
6		45	53	56			3124000	
7								
8		-0.029	-0.025	0.054			20904.99	
9	A^{-1}=	0.042	-0.042	0.003		X=	11757.61	
10		-0.016	0.059	-0.028			27859.32	

Figure 3.4. Spreadsheet for Example 3.2

This formula instructs Excel to multiply the contents of B4 by the contents of G8, C4 by G9, D4 by G10, and add these. We observe that A11 displays *2756000* ; this verifies the first equation. The second and third equations can be verified similarly.

MATLAB*, which is discussed in Appendix A, offers a more convenient method for matrix operations. To verify our results, we could use the MATLAB **inv(A)** function, and then multiply A^{-1} by *B*. However, it is easier to use the ***matrix left division*** operation $X = A \backslash B$; this is MATLAB's solution of $A^{-1}B$ for the matrix equation $A \cdot X = B$, where matrix *X* is the same size as matrix *B*. For this example, we enter

```
A=[25  62  54; 28  42  58; 45  53  56]; B=[2756000  2695000  3124000]';
X=A\B
```

and MATLAB displays $x_1 = 2.0905 \times 10^4$, $x_2 = 1.1758 \times 10^4$, and $x_3 = 2.7859 \times 10^4$

* *For readers not familiar with it, it is highly recommended that it is studied next.*

3.4 Summary

- In general, a straight line is represented by the equation

$$y = mx + b$$

where m is the *slope*, x is the abscissa, y is the ordinate, and b is the *y-intercept* of the straight line.

- The *slope* m is the *rise* in the vertical *(y-axis)* direction over the *run* in the abscissa *(x-axis)* direction. In other words,

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

- A *matrix* is a rectangular array of numbers. The general form of a matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

- The numbers a_{ij} are the *elements* of the matrix where the index i indicates the row and j indicates the column in which each element is positioned. Thus, a_{43} represents the element positioned in the fourth row and third column.
- A matrix of m rows and n columns is said to be of $m \times n$ *order matrix*. If $m = n$, the matrix is said to be a *square matrix of order* m (or n).
- In a square matrix, the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called the *diagonal elements*.
- A matrix in which every element is zero, is called a *zero matrix*.
- Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, that is, $A = B$, if, and only if $a_{ij} = b_{ij}$ for $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n$.
- Two matrices are said to be *conformable for addition* (*subtraction*) if they are of the same order, that is, both matrices must have the same number of rows and columns.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ are conformable for addition (subtraction), their sum (difference) will be another matrix C with the same order as A and B, where each element of C represents the sum (difference) of the corresponding elements of A and B, that is, $C = A \pm B = [a_{ij} \pm b_{ij}]$

- If k is any scalar (a positive or negative number) and not $[k]$ which is a 1×1 matrix, then multiplication of a matrix A by the scalar k is the multiplication of every element of A by k.
- Two matrices A and B are said to be *conformable for multiplication* $A \cdot B$ in that order, only when the number of columns of matrix A is equal to the number of rows of matrix B. That is, matrix product $A \cdot B$ is conformable for multiplication only if A is an $m \times p$ and matrix B is an $p \times n$ matrix. The matrix product $A \cdot B$ will then be an $m \times n$ matrix.
- The matrix product $A \cdot B$, is not the same as the matrix product $B \cdot A$.
- For matrix multiplication, the operation is row by column. Thus, to obtain the product $A \cdot B$, we multiply each element of a row of A by the corresponding element of a column of B, and then we add these products.
- Division of one matrix by another is not defined. An equivalent operation is performed with the inverse of a matrix.
- An *identity matrix* I is a square matrix where all the elements on the main diagonal are ones, and all other elements are zero.
- If a matrix A be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

the *determinant of* A, denoted as $detA$, is defined as

$$\begin{aligned} detA = {} & a_{11}a_{22}a_{33}\ldots a_{nn} + a_{12}a_{23}a_{34}\ldots a_{n1} + a_{13}a_{24}a_{35}\ldots a_{n2} + \ldots \\ & -a_{n1}\ldots a_{22}a_{13}\ldots -a_{n2}\ldots a_{23}a_{14} - a_{n3}\ldots a_{24}a_{15} - \ldots \end{aligned}$$

- The determinant of a square matrix of order n is referred to as *determinant of order* n.
- If a matrix A be defined as the square matrix of order n as shown below

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

and we remove the elements of its ith row and jth column, the determinant of the remaining

$(n-1)$ square matrix is called a *minor of determinant* A, and it is denoted as $[M_{ij}]$.

- The signed minor $(-1)^{i+j}[M_{ij}]$ is called the *cofactor* of a_{ij}, and it is denoted as α_{ij}.
- In general, *the determinant of a matrix* A *of any order, is the sum of the products obtained by multiplying each element of any row or any column by its cofactor.*
- *If all elements of one row or one column of a square matrix are zero, the determinant is zero.*
- *If all the elements of one row (column) of a square matrix* A *are* m *times the corresponding elements of* another row (column), $detA$ is zero.
- *Cramer's rule* states that the unknowns x, y, and z can be found from the relations

$$x = \frac{D_1}{\Delta} \qquad y = \frac{D_2}{\Delta} \qquad z = \frac{D_3}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad D_1 = \begin{vmatrix} A & a_{12} & a_{13} \\ B & a_{22} & a_{23} \\ C & a_{32} & a_{33} \end{vmatrix} \quad D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix} \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

provided that the determinant Δ (delta) is not zero.

- We can also find the unknowns in a system of two or more equations by the G*aussian Elimination Method*. With this method we eliminate one unknown at a time. This is done by multiplying the terms of any of the given equations of the system, by a number such that we can add (or subtract) this equation to (from) another equation in the system, so that one of the unknowns is eliminated. Then, by substitution to another equation with two unknowns, we can find the second unknown. Subsequently, substitution of the two values that were found, can be made into an equation with three unknowns from which we can find the value of the third unknown. This procedure is repeated until all unknowns are found.
- If a matrix A is an n square matrix, and α_{ij} is the cofactor of a_{ij}, *the adjoint of* A, denoted as $adjA$, is defined as the n square matrix below.

$$adjA = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \cdots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \cdots & \alpha_{n2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \cdots & \alpha_{nn} \end{bmatrix}$$

- The cofactors of the elements of the ith row (column) of A, are the elements of the ith column (row) of $adjA$.

- An n square matrix A is called *singular* if $detA = 0$; if $detA \neq 0$, it is called *non-singular.*
- If A and B are n square matrices such that $AB = BA = I$ where I is the identity matrix, B is called the *inverse* of A, denoted as $B = A^{-1}$, and likewise, A is called the *inverse* of B, that is, $A = B^{-1}$.
- If a matrix A is non-singular, we can compute its inverse from the relation

$$A^{-1} = \frac{1}{detA} adjA$$

- The relation

$$\boxed{X = A^{-1}B}$$

  where A and B are matrices whose elements are known, X is a matrix whose elements are the unknowns, and A and X are conformable for multiplication, can be used to solve any set of simultaneous equations that have solutions. We call this method, the *inverse matrix method of solution.*
- We can also use Microsoft Excel's MINVERSE (Matrix Inversion) and MMULT (Matrix Multiplication) functions to obtain the values of the unknowns in a system of equations.
- The *matrix left division* operation $X = A \backslash B$ is MATLAB's solution of $A^{-1}B$ for the matrix equation $A \cdot X = B$, where matrix X is the same size as matrix B.

3.5 Exercises

For Exercises *1* through *3* below, the matrices A, B, C and D are defined as:

$$A = \begin{bmatrix} 1 & -1 & -4 \\ 5 & 7 & -2 \\ 3 & -5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 9 & -3 \\ -2 & 8 & 2 \\ 7 & -4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 6 \\ -3 & 8 \\ 5 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -4 \end{bmatrix}$$

1. Perform the following computations, if possible. Verify your answers with Excel or MATLAB.

   a. $A+B$ b. $A+C$ c. $B+D$ d. $C+D$ e. $A-B$ f. $A-C$ g. $B-D$ h. $C-D$

2. Perform the following computations, if possible. Verify your answers with Excel or MATLAB.

   a. $A \cdot B$ b. $A \cdot C$ c. $B \cdot D$ d. $C \cdot D$ e. $B \cdot A$ f. $C \cdot A$ g. $D \cdot A$ h. $\dot{D} \cdot C$

3. Perform the following computations, if possible. Verify your answers with Excel or MATLAB.

   a. $detA$ b. $detB$ c. $detC$ d. $detD$ e. $det(A \cdot B)$ f. $det(A \cdot C)$

4. Solve the following system of equations using Cramer's rule. Verify your answers with Excel or MATLAB.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -4 \\ -2x_1 + 3x_2 + x_3 &= 9 \\ 3x_1 + 4x_2 - 5x_3 &= 0 \end{aligned}$$

5. Repeat Exercise 4 using the Gaussian elimination method.

6. Use the MATLAB **det(A)** function to find the unknowns of the system of equations below.

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 + 5x_4 &= 14 \\ x_1 + 3x_2 + 2x_3 - x_4 &= 9 \\ 3x_1 - 3x_2 + 2x_3 + 4x_4 &= 19 \\ 4x_1 + 2x_2 + 5x_3 + x_4 &= 27 \end{aligned}$$

7. Solve the following system of equations using the inverse matrix method. Verify your answers with Excel or MATLAB.

$$\begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & -2 \\ 2 & 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$$

8. Use Excel to find the unknowns for the system

$$\begin{bmatrix} 2 & 4 & 3 & -2 \\ 2 & -4 & 1 & 3 \\ -1 & 3 & -4 & 2 \\ 2 & -2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ -14 \\ 7 \end{bmatrix}$$

Verify your answers with the MATLAB left division operation.

3.6 Solutions to Exercises

1.

a. $A+B = \begin{bmatrix} 1+5 & -1+9 & -4-3 \\ 5-2 & 7+8 & -2+2 \\ 3+7 & -5-4 & 6+6 \end{bmatrix} = \begin{bmatrix} 6 & 8 & -7 \\ 3 & 15 & 0 \\ 10 & -9 & 12 \end{bmatrix}$ b. $A+C$ not conformable for addition

c. $B+D$ not conformable for addition d. $C+D$ not conformable for addition

e. $A-B = \begin{bmatrix} 1-5 & -1-9 & -4+3 \\ 5+2 & 7-8 & -2-2 \\ 3-7 & -5+4 & 6-6 \end{bmatrix} = \begin{bmatrix} -4 & -10 & -1 \\ 7 & -1 & -4 \\ -4 & -1 & 0 \end{bmatrix}$ f. $A-C$ not conformable for subtraction

g. $B-D$ not conformable for subtraction h. $C-D$ not conformable for subtraction

2.

a.

$$A \cdot B = \begin{bmatrix} 1\times5+(-1)\times(-2)+(-4)\times7 & 1\times9+(-1)\times8+(-4)\times(-4) & 1\times(-3)+(-1)\times2+(-4)\times6 \\ 5\times5+7\times(-2)+(-2)\times7 & 5\times9+7\times8+(-2)\times(-4) & 5\times(-3)+7\times2+(-2)\times6 \\ 3\times5+(-5)\times(-2)+6\times7 & 3\times9+(-5)\times8+6\times(-4) & 3\times(-3)+(-5)\times2+6\times6 \end{bmatrix}$$

$$= \begin{bmatrix} -21 & 17 & -29 \\ -3 & 109 & -13 \\ 67 & -37 & 17 \end{bmatrix}$$

Check with MATLAB:

```
A=[1 –1 –4; 5 7 –2; 3 –5 6]; B=[5 9 –3; –2 8 2; 7 –4 6]; A*B
ans =
   -21    17   -29
    -3   109   -13
    67   -37    17
```

b. $$A \cdot C = \begin{bmatrix} 1\times4+(-1)\times(-3)+(-4)\times5 & 1\times6+(-1)\times8+(-4)\times(-2) \\ 5\times4+7\times(-3)+(-2)\times5 & 5\times6+7\times8+(-2)\times(-2) \\ 3\times4+(-5)\times(-3)+6\times5 & 3\times6+(-5)\times8+6\times(-2) \end{bmatrix} = \begin{bmatrix} -13 & 6 \\ -11 & 90 \\ 57 & -34 \end{bmatrix}$$

c. $B \cdot D$ not conformable for multiplication

d. $$C \cdot D = \begin{bmatrix} 4\times1+6\times(-3) & 4\times(-2)+6\times6 & 4\times3+6\times(-4) \\ (-3)\times1+8\times(-3) & (-3)\times(-2)+8\times6 & (-3)\times3+8\times(-4) \\ 5\times1+(-2)\times(-3) & 5\times(-2)+(-2)\times6 & 5\times3+(-2)\times(-4) \end{bmatrix} = \begin{bmatrix} -14 & 28 & -12 \\ -27 & 54 & -41 \\ 11 & -22 & 23 \end{bmatrix}$$

e.
$$B \cdot A = \begin{bmatrix} 5\times 1+9\times 5+(-3)\times 3 & (-2)\times 1+8\times 5+2\times 3 & 7\times 1+(-4)\times 5+6\times 3 \\ 5\times(-1)+9\times 7+(-3)\times(-5) & (-2)\times(-1)+8\times 7+2\times(-5) & 7\times(-1)+(-4)\times 7+6\times(-5) \\ 5\times(-4)+9\times(-2)+(-3)\times 6 & (-2)\times(-4)+8\times(-2)+2\times 6 & 7\times(-4)+(-4)\times(-2)+6\times 6 \end{bmatrix}$$
$$= \begin{bmatrix} 41 & 73 & -56 \\ 44 & 48 & 4 \\ 5 & -65 & 16 \end{bmatrix}$$

f. $C \cdot A$ not conformable for multiplication

g.
$$D \cdot A = \begin{bmatrix} 1\times 1+(-2)\times 5+3\times 3 & 1\times(-1)+(-2)\times 7+3\times(-5) & 1\times(-4)+(-2)\times(-2)+3\times 6 \\ (-3)\times 1+6\times 5+(-4)\times 3 & (-3)\times(-1)+6\times 7+(-4)\times(-5) & (-3)\times(-4)+6\times(-2)+(-4)\times 6 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -30 & 18 \\ 15 & 65 & -24 \end{bmatrix}$$

h. $$D \cdot C = \begin{bmatrix} 1\times 4+(-2)\times(-3)+3\times 5 & 1\times 6+(-2)\times 8+3\times(-2) \\ (-3)\times 4+6\times(-3)+(-4)\times 5 & (-3)\times 6+6\times 8+(-4)\times(-2) \end{bmatrix} = \begin{bmatrix} 25 & -16 \\ -50 & 38 \end{bmatrix}$$

3.

a.
$$detA = \begin{matrix} 1 & -1 & -4 & 1 & -1 \\ 5 & 7 & -2 & 5 & 7 \\ 3 & -5 & 6 & 3 & -5 \end{matrix}$$
$$= 1\times 7\times 6+(-1)\times(-2)\times 3+(-4)\times 5\times(-5)-[3\times 7\times(-4)+(-5)\times(-2)\times 1+6\times 5\times(-1)]$$
$$= 42+6+100-(-84)-10-(-30) = 252$$

b.
$$detB = \begin{matrix} 5 & 9 & -3 & 5 & 9 \\ -2 & 8 & 2 & -2 & 8 \\ 7 & -4 & 6 & 7 & -4 \end{matrix}$$
$$= 5\times 8\times 6+9\times 2\times 7+(-3)\times(-2)\times(-4)-[7\times 8\times(-3)+(-4)\times 2\times 5+6\times(-2)\times 9]$$
$$= 240+126-24-(-168)+40-(-108) = 658$$

c. $detC$ does not exist; matrix must be square

d. $detD$ does not exist; matrix must be square

e. $det(A \cdot B) = detA \cdot detB$ and from parts (a) and (b), $det(A \cdot B) = 252 \times 658 = 165816$

f. $det(A \cdot C)$ does not exist because $detC$ does not exist

4.

$$\Delta = \begin{array}{ccccc} 1 & -2 & 1 & 1 & -2 \\ -2 & 3 & 1 & -2 & 3 \\ 3 & 4 & -5 & 3 & 4 \end{array}$$

$$= 1 \times 3 \times (-5) + (-2) \times 1 \times 3 + 1 \times (-2) \times 4 - [3 \times 3 \times 1 + 4 \times 1 \times 1 + (-5) \times (-2) \times (-2)]$$

$$= -15 - 6 - 8 - 9 - 4 + 20 = -22$$

$$D_1 = \begin{array}{ccccc} -4 & -2 & 1 & 4 & -2 \\ 9 & 3 & 1 & 9 & 3 \\ 0 & 4 & -5 & 0 & 4 \end{array}$$

$$= -4 \times 3 \times (-5) + (-2) \times 1 \times 0 + 1 \times 9 \times 4 - [0 \times 3 \times 1 + 4 \times 1 \times 4 + (-5) \times 9 \times (-2)]$$

$$= 60 + 0 + 36 - 0 + 16 - 90 = 22$$

$$D_2 = \begin{array}{ccccc} 1 & -4 & 1 & 1 & -4 \\ -2 & 9 & 1 & -2 & 9 \\ 3 & 0 & -5 & 3 & 0 \end{array}$$

$$= 1 \times 9 \times (-5) + (-4) \times 1 \times 3 + 1 \times (-2) \times 0 - [3 \times 9 \times 1 + 0 \times 1 \times 1 + (-5) \times (-2) \times (-4)]$$

$$= -45 - 12 - 0 - 27 - 0 + 40 = -44$$

$$D_3 = \begin{array}{ccccc} 1 & -2 & -4 & 1 & -2 \\ -2 & 3 & 9 & -2 & 3 \\ 3 & 4 & 0 & 3 & 4 \end{array}$$

$$= 1 \times 3 \times 0 + (-2) \times 9 \times 3 + (-4) \times (-2) \times 4 - [3 \times 3 \times (-4) + 4 \times 9 \times 1 + 0 \times (-2) \times (-2)]$$

$$= 0 - 54 + 32 + 36 - 36 - 0 = -22$$

$$x_1 = \frac{D_1}{\Delta} = \frac{22}{-22} = -1 \qquad x_2 = \frac{D_2}{\Delta} = \frac{-44}{-22} = 2 \qquad x_3 = \frac{D_3}{\Delta} = \frac{-22}{-22} = 1$$

5.

$$x_1 - 2x_2 + x_3 = -4 \quad (1)$$

$$-2x_1 + 3x_2 + x_3 = 9 \quad (2)$$

$$3x_1 + 4x_2 - 5x_3 = 0 \quad (3)$$

Multiplication of (1) by 2 yields

$$2x_1 - 4x_2 + 2x_3 = -8 \quad (4)$$

Addition of (2) and (4) yields

$$-x_2 + 3x_3 = 1 \quad (5)$$

Multiplication of (1) by -3 yields

$$-3x_1 + 6x_2 - 3x_3 = 12 \quad (6)$$

Addition of (3) and (6) yields

$$10x_2 - 8x_3 = 12 \quad (7)$$

Multiplication of (5) by 10 yields

$$-10x_2 + 30x_3 = 10 \quad (8)$$

Addition of (7) and (8) yields

$$22x_3 = 22 \quad (9)$$

or

$$x_3 = 1 \quad (10)$$

Substitution of (10) into (7) yields

$$10x_2 - 8 = 12 \quad (11)$$

or

$$x_2 = 2 \quad (12)$$

and substitution of (10) and (12) into (1) yields

$$x_1 - 4 + 1 = -4 \quad (13)$$

or

$$x_1 = -1 \quad (14)$$

6.

```
Delta=[–1  2  –3  5; 1  3  2  –1; 3  –3  2  4; 4  2  5  1];
D1=[14  2  –3  5; 9  3  2  –1; 19  –3  2  4; 27  2  5  1];
D2=[–1  14  –3  5; 1  9  2  –1; 3  19  2  4; 4  27  5  1];
D3=[–1  2  14  5; 1  3  9  –1; 3  –3  19  4; 4  2  27  1];
D4=[–1  2  –3  14; 1  3  2  9; 3  –3  2  19; 4  2  5  27];
x1=det(D1)/det(Delta), x2=det(D2)/det(Delta),...
x3=det(D3)/det(Delta), x4=det(D4)/det(Delta)
x1=1         x2=2         x3=3         x4=4
```

7.

$$detA = \begin{array}{ccccc} 1 & 3 & 4 & 1 & 3 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & 3 & 5 & 2 & 3 \end{array}$$

$$= 1 \times 1 \times 5 + 3 \times (-2) \times 2 + 4 \times 3 \times 3 - [2 \times 1 \times 4 + 3 \times (-2) \times 1 + 5 \times 3 \times 3]$$

$$= 5 - 12 + 36 - 8 + 6 - 45 = -18$$

$$adjA = \begin{bmatrix} 11 & -3 & -10 \\ -19 & -3 & 14 \\ 7 & 3 & -8 \end{bmatrix}$$

$$A^{-1} = \frac{1}{detA} \cdot adjA = \frac{1}{-18} \cdot \begin{bmatrix} 11 & -3 & -10 \\ -19 & -3 & 14 \\ 7 & 3 & -8 \end{bmatrix} = \begin{bmatrix} -11/18 & 3/18 & 10/18 \\ 19/18 & 3/18 & -14/18 \\ -7/18 & -3/18 & 8/18 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11/18 & 3/18 & 10/18 \\ 19/18 & 3/18 & -14/18 \\ -7/18 & -3/18 & 8/18 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 33/18 - 6/18 + 0 \\ -57/18 - 6/18 + 0 \\ 21/18 + 6/18 + 0 \end{bmatrix} = \begin{bmatrix} 27/18 \\ -63/18 \\ 27/18 \end{bmatrix} = \begin{bmatrix} 1.50 \\ -3.50 \\ 1.50 \end{bmatrix}$$

1	Spreadsheet for Matrix Inversion and						
2	Matrix Multiplication - Exercise 7						
3							
4		1.00	3.00	4.00			-3.00
5	A=	3.00	1.00	-2.00		B=	-2.00
6		2.00	3.00	5.00			0.00
7							
8		-0.61	0.17	0.56			1.50
9	A^{-1}	1.06	0.17	-0.78		X=	-3.50
10		-0.39	-0.17	0.44			1.50

8.

	A	B	C	D	E	F	G	H
1	Spreadsheet for Matrix Inversion and							
2	Matrix Multiplication - Exercise 8							
3								
4		2.00	4.00	3.00	-2.00			1.00
5	A=	2.00	-4.00	1.00	3.00		B=	10.00
6		-1.00	3.00	-4.00	2.00			-14.00
7		2.00	-2.00	2.00	1.00			7.00
8								
9		-1.58	-4.08	1.17	6.75			-11.50
10	A^{-1}	0.58	1.08	-0.17	-1.75		X=	1.50
11		1.50	3.50	-1.00	-5.50			12.00
12		1.33	3.33	-0.67	-5.00			9.00

```
A=[2  4  3  –2;  2  –4  1  3; –1  3  –4  2;  2  –2  2  1];
B=[1  10  –14  7]'; A\B
 ans =
   -11.5000
     1.5000
    12.0000
     9.0000
```

Chapter 4

Fundamentals of Geometry

This chapter discusses the basic geometric figures. It is intended for readers who need to learn the basics of geometry. Readers with a strong mathematical background may skip this chapter. Others will find it useful, as well as a convenient source for review.

4.1 Introduction

Geometry is the mathematics of the properties, measurement, and relationships of points, lines, angles, surfaces, and solids. The science of geometry also includes *analytic geometry, descriptive geometry, fractal geometry, non-Euclidean geometry**, and spaces with four or more dimensions. It is beyond the scope of this text to discuss the different types of geometry in detail; we will only present the most common figures and their properties such as the number of sides, angles, perimeters, and areas. In this text, we will only be concerned with the so-called *Euclidean Geometry*†.

4.2 Plane Geometry Figures

A *triangle* is a plane figure that is formed by connecting three points, not all in a straight line, by straight line segments. It is a special case of a *polygon* which is defined as a closed plane figure

* *In **analytic geometry** straight lines, curves, and geometric figures are represented by algebraic expressions. Any point in a plane may be located by specifying the distance of the point from each of a pair of perpendicular axes. Points in three-dimensional space can be similarly located with respect to three axes. A straight line can always be represented by an equation in the form* $ax + by + c = 0$.

*The science of making accurate, two-dimensional drawings of three-dimensional geometrical forms is called **descriptive geometry**. The usual technique is by means of orthographic projection, in which the object to be drawn is referred to one or more imaginary planes that are at right angles to one another.*

*A **fractal** is a geometric pattern that is repeated at ever smaller scales to produce irregular shapes and surfaces that cannot be represented by classical geometry. Fractals are used especially in computer modeling of irregular patterns and structures in nature.*

† *Euclidean Geometry is based on one of Euclid's postulates which states that through a point outside a line it is possible to draw only one line parallel to that line. For many centuries mathematicians believed that this postulate could be proved on the basis of Euclid's other postulates, but all efforts to discover such a proof were unsuccessful. In the first part of the 19th century the German mathematician Carl Friedrich Gauss, the Russian mathematician Nikolay Ivanovich Lobachevsky, and the Hungarian mathematician János Bolyai independently demonstrated a consistent system of geometry in which Euclid's postulate was replaced by one stating that an infinite number of parallels to a particular line could be drawn. About 1860 the German mathematician Georg Friedrich Bernhard Riemann showed that a geometry with no parallel lines was equally possible. For comparatively small distances, Euclidean geometry and the non-Euclidean geometries are about the same. However, in dealing with astronomical distances and relativity, non-Euclidean geometries give a more precise description of the observed facts than Euclidean geometry.*

bounded by three or more line segments. The general form of a triangle is shown in Figure 4.1 where the capital letters A, B, and C denote the angles, and the lower case letters denote the three sides of the triangle.

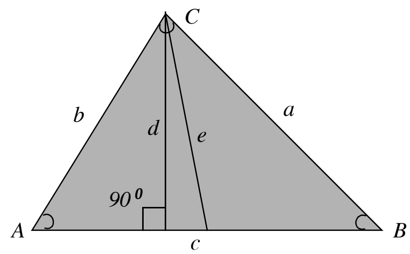


Figure 4.1. General Form of a Triangle

It is customary to show the sides of a triangle opposite to the angles. Thus, side a is opposite to angle A, side b is opposite to angle B and so on.

We will denote the angles with the symbol $\angle$. Thus, $\angle A$ will mean angle A.

The *perimeter* of a polygon is defined as the closed curve that bounds the plane area of the polygon. In other words, it is the entire length of the polygon's boundary.

The *area* of a polygon is the space inside the boundary of the polygon, and it is expressed in square units. In Figure 4.1, the shaded portion represents the area of this triangle.

Note 4.1

The area of some basic polygons can be computed with simple formulas. However, the formulas for the area of most polygons involve trigonometric functions. We will discuss the most common on the next chapter. It is also possible to compute the areas of irregular polygons with a spreadsheet, as we will see later in this chapter.

A *perpendicular* line is one that intersects another line to form a $90°$ angle. Thus, a vertical line that intersects a horizontal line, is a perpendicular to the horizontal since it forms a $90°$ angle with it.

For the triangle of Figure 4.1, line d is a perpendicular line drawn from $\angle C$ down to line c forming $90°$ angle as shown.

In geometry, the line that joins a vertex of a triangle to the midpoint of the opposite side is called the *median*. Thus in Figure 4.1, line e is a median from $\angle C$ to the midpoint of line c.

Property 1

The sum of the three angles of any triangle is $180°$.

Example 4.1

In Figure 4.1, $\angle A = 53°$ and $\angle B = 48°$. Find $\angle C$.

Solution:

From Property 1,

$$\angle A + \angle B + \angle C = 180°$$

or

$$\angle C = 180° - (\angle A + \angle B) \tag{4.1}$$

and since

$$\angle A + \angle B = 53° + 48° = 101°$$

by substitution into (4.1),

$$\angle C = 180° - 101° = 79°$$

Example 4.2

In Figure 4.1, $a = 45\ cm$, $b = 37\ cm$, and $c = 57\ cm$. Find the perimeter of the triangle.

Solution:

The perimeter is the sum of the lengths of the three sides of the triangle. Therefore,

$$perimeter = a + b + c = 45 + 37 + 57 = 139\ cm$$

The right, the isosceles, and the equilateral triangles are special cases of the general triangle of Figure 4.1. We will discuss each separately. Formulas involving trigonometric functions will be given on the next chapter.

A triangle containing an angle of $90°$ is a *right triangle*. Thus, Figure 4.2 is a right triangle.

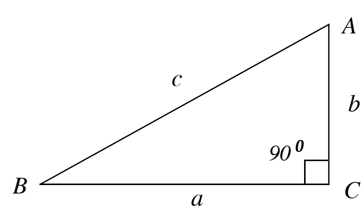


Figure 4.2. Right Triangle

Property 2

If, in a right triangle, $\angle C = 90°$, then $\angle A + \angle B = 90°$. This is a consequence of the fact that in a triangle, the sum of the three angles is $180°$.

In a right triangle, the side opposite to the right angle is called the *hypotenuse* of the right triangle. Thus, side c in Figure 4.2, is the hypotenuse of that triangle.

Property 3

If c is the hypotenuse of a right triangle, then

$$\boxed{c^2 = a^2 + b^2} \tag{4.2}$$

Pythagorean Theorem

This relation is known as the *Pythagorean theorem.*

Example 4.3

In Figure 4.2, $c = 5\ cm$ and $b = 3\ cm$. Find the length of the line segment a.

Solution:

We express (4.2) as

$$a^2 = c^2 - b^2$$

or

$$a = \sqrt{c^2 - b^2}$$

Then, with the given values

$$a = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4\ in.^*$$

Property 4

If c is the hypotenuse of a right triangle, the area of this triangle is found from the relation

$$\boxed{Area = \frac{1}{2} \cdot a \cdot b} \tag{4.3}$$

Area of Right Triangle

Example 4.4

In Figure 4.2, $a = 4\ cm.$ and $b = 3\ cm$. Find the area of this triangle.

Solution:

From (4.3),

$$Area = \frac{1}{2}ab = \frac{1}{2} \times 4 \times 3 = 6\ cm^2.^\dagger$$

* *As we know, $\sqrt{16} = \pm 4$ but we reject the negative value since it is unrealistic for this example.*

† *In area calculations, besides multiplication of the numbers, we must multiply the units also. Thus, for this example, inches × inches = square inches. Otherwise, the result is meaningless.*

A triangle that has two equal sides is called *isosceles* triangle. Figure 4.3 shows an isosceles triangle where sides a and b are equal. Consequently, $\angle A = \angle B$.

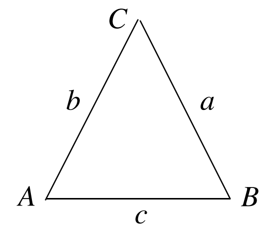


Figure 4.3. Isosceles Triangle

Example 4.5

In Figure 4.3, $a = 5\ cm$, $c = 4\ cm$ and $\angle A = 70°$. Compute:

a. the perimeter

b. $\angle C$

c. the area

Solution:

a.

$$Perimeter = a + b + c$$

and since this is an isosceles triangle,

$$a = b$$

Therefore,

$$perimeter = 2a + c = 2 \times 5 + 4 = 14\ cm$$

b. By Property 1,

$$\angle A + \angle B + \angle C = 180°$$

or

$$\angle C = 180° - (\angle A + \angle B)$$

Since this is an isosceles triangle,

$$\angle B = \angle A = 70°$$

Therefore,

$$\angle C = 180° - (70° + 70°) = 180° - 140° = 40°$$

c. No formula was given to compute the area of an isosceles triangle; however, we can use the formula of the right triangle after we split the isosceles triangle into two equal right triangles. We do this by drawing a perpendicular line from $\angle C$ down to line c forming a $90°$ angle as shown in Figure 4.4. We denote this line as h (for height).

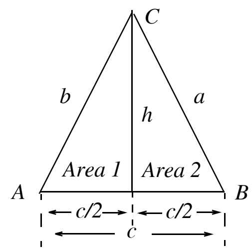


Figure 4.4. Isosceles Triangle for Example 4.5

Since the perpendicular line h divides the isosceles triangle into two equal right triangles, it follows that $Area\ 1 = Area\ 2$. Therefore, we only need to compute $Area\ 1$ and multiply it by 2 to obtain the total area. Thus,

$$Area\ 1 = \frac{1}{2} \times \frac{c}{2} \times h \tag{4.4}$$

where $c/2 = 4/2 = 2\ cm$ Next, we need to find the height h. This is found from the Pythagorean theorem stated in Property 3. Then, for the right triangle of the left side of Figure 4.4,

$$b^2 = (c/2)^2 + h^2$$

or

$$h^2 = b^2 - (c/2)^2$$

By substitution of the given values,

$$h^2 = 5^2 - (4/2)^2 = 25 - 4 = 21$$

or

$$h = \sqrt{21} = 4.5826$$

Finally, by substitution into (4.5), and multiplying by 2 to obtain the total area, we get

$$Total\ Area = 2 \times Area\ 1 = 2 \times \frac{1}{2} \times \frac{c}{2} \times h = 2 \times 4.5826 = 9.1652\ in^2$$

A triangle with all sides equal is called *equilateral triangle*. Figure 4.5 shows an equilateral triangle where sides a, b, and c are equal. Consequently, $\angle A$, $\angle B$, and $\angle C$ are equal.

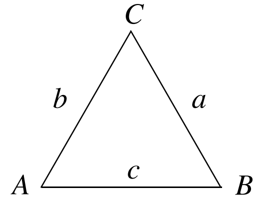


Figure 4.5. Equilateral Triangle

For an equilateral triangle,

$$\angle A = \angle B = \angle C = 60° \tag{4.5}$$

$$Area = \frac{\sqrt{3}}{4}a^2 \tag{4.6}$$

C*ongruent triangles* are identical in shape and size. If two triangles are congruent, the following statements are true:

I When two triangles are congruent, the six corresponding pairs of angles and sides are also congruent.

II If all three pairs of corresponding sides in two triangles are the same as shown in Figure 4.6, the triangles are congruent.

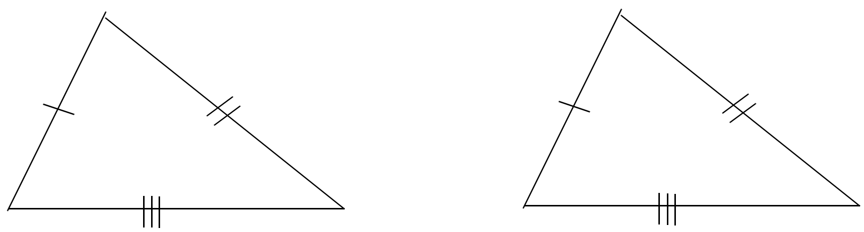


Figure 4.6. The three sides of one triangle are congruent to the three sides of the other triangle

III If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle as shown in Figure 4.7, the triangles are congruent.

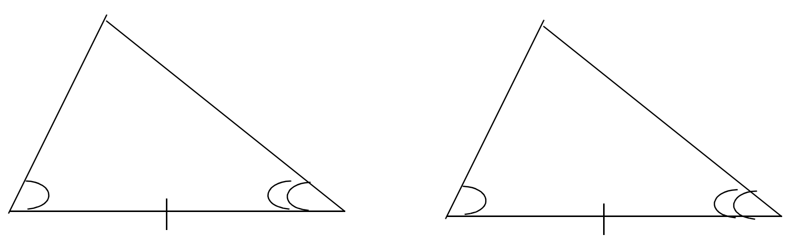


Figure 4.7. Two angles and the included side of one triangle are congruent to two angles and the included side of the other triangle

IV If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle as shown in Figure 4.8, the triangles are congruent.

V If two angles and a non-included side of one triangle are congruent to the two angles and the corresponding non-included angle of another triangle as shown in Figure 4.9, the triangles are congruent.

VI If the hypotenuse and the leg of one right triangle are congruent to the corresponding parts of another triangle as shown in Figure 4.10, the two triangles are congruent.

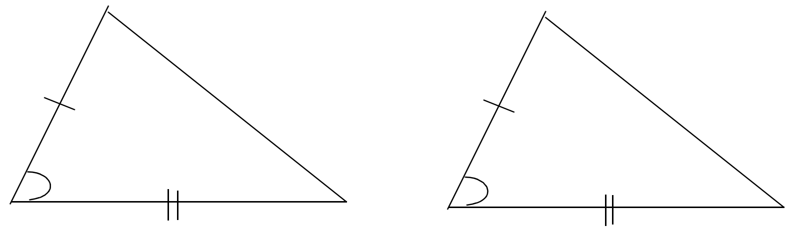


Figure 4.8. Two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle

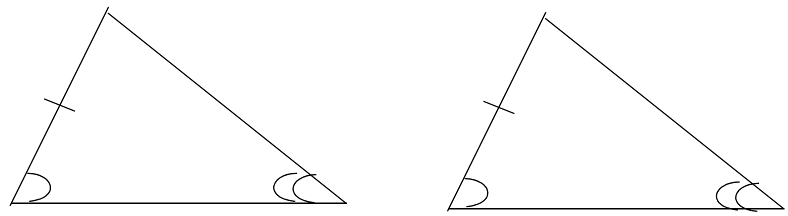


Figure 4.9. Two angles and a non-included side of one triangle are congruent to the two angles and the non-included side of the other triangle

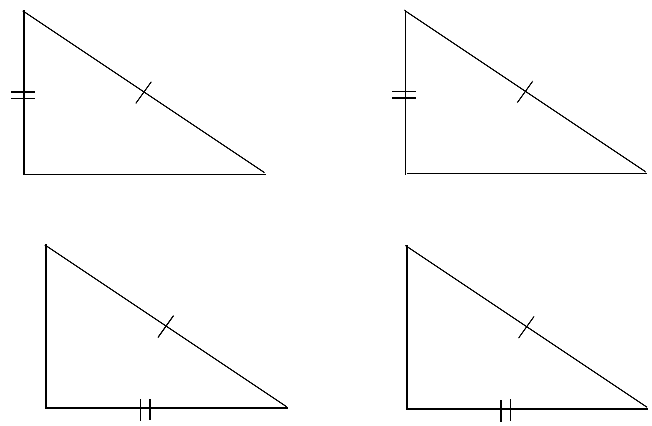


Figure 4.10. The hypotenuse and the leg of one right triangle are congruent to the corresponding parts of another right triangle

VII Two triangles with two sides and a non-included angle may or may not be congruent.

VIII Every triangle can be partitioned into four congruent triangles by connecting the middle points of its three sides as shown in Figure 4.11.

IX A triangle that can be partitioned into n congruent parts, it can also partitioned into $4 \times n, 4^2 \times n, 4^3 \times n, \ldots$ congruent parts.

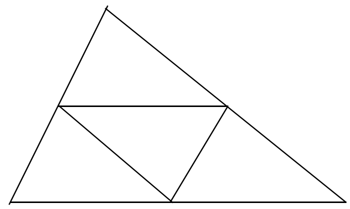


Figure 4.11. Triangle partitioned into four congruent triangles

X Any isosceles triangle can be partitioned into two congruent triangles by one of its heights.

XI Any equilateral triangle can be partitioned into three congruent triangles by segments connecting its center to its vertices.

XII A right triangle whose sides forming the right angle have a ratio 2 to 1 can be partitioned into five congruent triangles as shown on Figure 4.12.

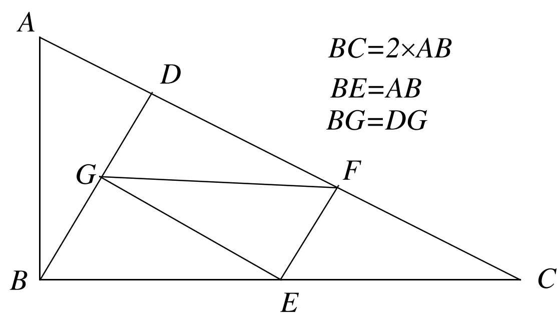


Figure 4.12. Right triangle partitioned into five congruent triangles

In Figure 4.12, line BD is drawn to form right angles with line AC and line EF is drawn to be parallel to line BD. From points E and F lines are drawn to point G which is the middle of line BD.

Figure 4.13 shows two parallel lines x and y intersected by a *transversal line* z.

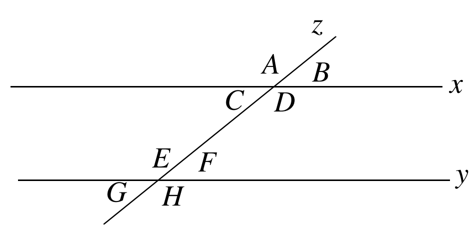


Figure 4.13. Two parallel lines intersected by a transversal

With reference to the lines shown on Figure 4.13:

1. When the transversal line z crosses a line x or line y or both, two sets of opposite angles are formed. Then, using the symbol $\angle$ to denote an angle, and the symbol $\cong$ to indicate congruence, the following relations are true:

$$\angle A \cong \angle D \qquad \angle B \cong \angle C$$
$$\angle E \cong \angle H \qquad \angle F \cong \angle G \tag{4.7}$$

2. Corresponding angles on the same side of the transversal are congruent. Thus, on the left side of the transversal we have

$$\angle A \cong \angle E \qquad \angle C \cong \angle G \tag{4.8}$$

and on the right side of the transversal we have

$$\angle B \cong \angle F \qquad \angle D \cong \angle H \tag{4.9}$$

3. The angles between the two parallel lines x and y are referred to as the *interior* (or inside) angles. Thus, in Figure 4.1 $\angle C$, $\angle D$, $\angle E$, and $\angle F$ are interior angles. Then,

$$\angle C \cong \angle F \qquad \angle D \cong \angle E \tag{4.10}$$

4. The angles above and below the two parallel lines x and y are referred to as the *exterior* (or outside) angles. Thus, in Figure 4.1 $\angle A$, $\angle B$, $\angle G$, and $\angle H$ are exterior angles. Then,

$$\angle A \cong \angle G \qquad \angle B \cong \angle H \tag{4.11}$$

Similar triangles have corresponding angles that are congruent (equal) while the lengths of the corresponding sides are in proportion. Thus, the triangles shown on Figure 4.14 are similar if

$$\angle A \cong \angle D \quad and \quad \angle C \cong \angle E \tag{4.12}$$

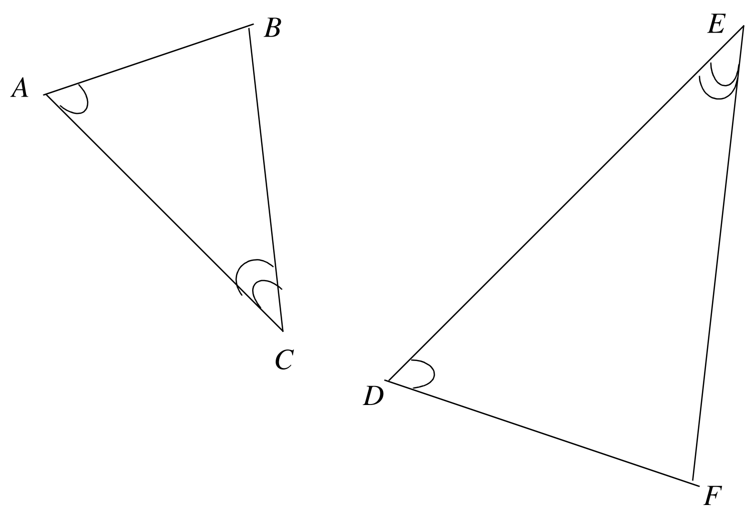


Figure 4.14. Similar triangles

If (4.11) holds, then

$$\frac{AC}{DE} = \frac{BC}{EF} = \frac{AB}{DF} \tag{4.13}$$

We should also remember that:

1. All congruent triangles are similar, but not all similar triangles are congruent.
2. If the angles in two triangles are equal and the corresponding sides are the same size, the triangles are congruent.
3. If the angles in two triangles are equal the triangles are similar.
4. If the angles in two triangles are not equal the triangles are neither similar nor congruent.

A plane figure with four sides and four angles is called *quadrilateral*. Figure 4.15 shows a general quadrilateral.

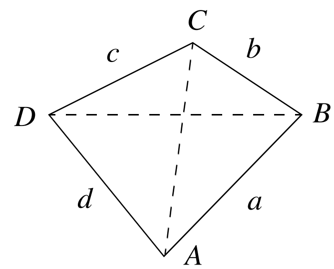


Figure 4.15. General Quadrilateral

The lines joining two non-adjacent vertices of a quadrilateral are called *diagonals*. Thus, in Figure 4.15, the lines AC and BD are the diagonals of that quadrilateral.

Property 5

In any quadrilateral, the sum of the four angles is $360°$

Property 6

The diagonals of a quadrilateral with consecutive sides a, b, c, and d are perpendicular if and only if

$$a^2 + c^2 = b^2 + d^2 \tag{4.14}$$

The *square, rectangle, parallelogram, rhombus,* and *trapezoid* are special cases of quadrilaterals. We will discuss each separately.

Property 7

Two lines are said to be in parallel, when the distance between them is the same everywhere. In other words, two lines on the same plane that never intersect, are said to be parallel lines. *

* *This statement is consistent with the principles of Euclidean geometry.*

A four-sided plane figure with opposite sides in parallel, is called *parallelogram.* Figure 4.16 shows a parallelogram.

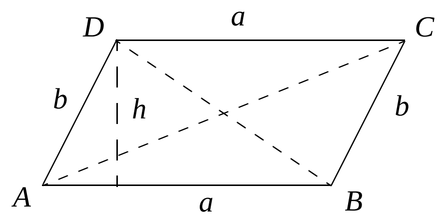


Figure 4.16. Parallelogram

For the parallelogram of Figure 4.16,

$$\begin{aligned} \angle A &= \angle C \\ \angle B &= \angle D \\ \angle A + \angle B &= \angle C + \angle D = 180° \end{aligned} \tag{4.15}$$

$$\boxed{Area = ah} \tag{4.16}$$

Area of Parallelogram

Note: The perpendicular distance from the center of a regular polygon to any of its sides is called *apothem.*

An equilateral parallelogram is called *rhombus*. It is a special case of a parallelogram where all sides are equal. Figure 4.17 shows a rhombus.

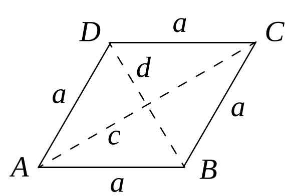


Figure 4.17. Rhombus

For the rhombus of Figure 4.17,

$$c^2 + d^2 = 4a^2 \tag{4.17}$$

$$\boxed{Area = \frac{1}{2}cd} \tag{4.18}$$

Area of Rhombus

A four-sided plane figure with four right angles, is a *rectangle*. It is a special case of a parallelogram where all angles are equal, that is, $90°$ each. Figure 4.18 shows a rectangle.

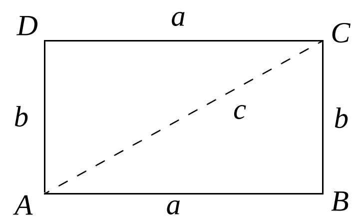


Figure 4.18. Rectangle

For the rectangle of Figure 4.18,

$$\angle A = \angle B = \angle C = \angle D = 90° \tag{4.19}$$

$$c = \sqrt{a^2 + b^2} \tag{4.20}$$

$$\boxed{Area = ab} \tag{4.21}$$

Area of Rectangle

A plane figure with four equal sides and four equal angles, is a *square*. It is a special case of a rectangle where all sides are equal. Figure 4.19 shows a square.

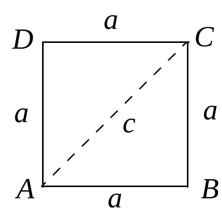


Figure 4.19. Square

For the square of Figure 4.19,

$$\angle A = \angle B = \angle C = \angle D = 90° \tag{4.22}$$

$$c = a\sqrt{2} \tag{4.23}$$

$$\boxed{Area = a^2} \tag{4.24}$$

Area of Square

A quadrilateral with two unequal parallel sides is a *trapezoid*. Figure 4.20 shows a *trapezoid.*

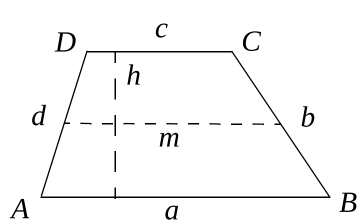


Figure 4.20. Trapezoid

For the trapezoid of Figure 4.20,

$$m = \frac{1}{2}(a + c) \tag{4.25}$$

$$\boxed{Area = \frac{1}{2}(a + c)h = mh} \tag{4.26}$$

Area of Trapezoid

A plane curve that is everywhere equidistant from a given fixed point, referred to as the *center*, is called a *circle*. Figure 4.21 shows a circle where point *o* is the *center* of the circle. The line segment that joins the center of a circle with any point on its circumference is called *radius*, and it is denoted with the letter r. A straight line segment passing through the center of a circle, and terminating at the circumference is called *diameter*, and is denoted with the letter d. Obviously, $d = 2r$.

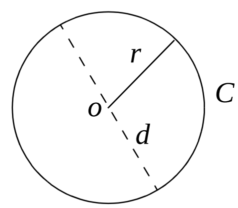


Figure 4.21. Circle

The perimeter of a circle is called *circumference* and it is denoted with the letter C.

Note 4.2

The Greek letter π represents the ratio of the circumference of a circle to its diameter. This ratio is an irrational (endless) number, so the decimal places go on infinitely without repeating. It is approximately equal to $22/7$; to five decimal places, is equal to 3.14159. With computers, this value has been figured to more than 100 million decimal places.

For the circle of Figure 4.21,

$$C = 2\pi r = \pi d \tag{4.27}$$

$$\boxed{Area = \pi r^2 = \frac{1}{4}\pi d^2} \tag{4.28}$$

Area of Circle

Note 4.3

Locus is defined as the set of all points whose coordinates satisfy a single equation or one or more algebraic conditions.

The locus of points for which the sum of the distances from each point to two fixed points is equal forms an *ellipse**. The two fixed points are called *foci* (plural for focus). Figure 4.22 shows an ellipse.

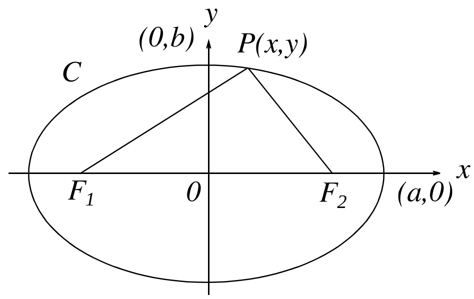


Figure 4.22. Ellipse

For the ellipse of Figure 4.22, F_1 and F_2 are the *foci* and C is the circumference.

In an ellipse,

$$PF_1 + PF_1 = 2a = constant \tag{4.29}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.30}$$

$$C \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}} \tag{4.31}$$

$$Area = \pi ab \tag{4.32}$$

The line segments $2a$ and $2b$, are referred to as the *major* and *minor* axes respectively. The foci are always on the major axis; they are located symmetrically from the center 0 that is the intersection of the major and minor axes. The foci locations can be determined with the aid of Figure 4.14, where the distance $0F_2$, denoted as c, is found from the Pythagorean theorem, that is,

$$c = \sqrt{a^2 - b^2} \tag{4.33}$$

* *The ellipse, parabola and hyperbola are topics of analytic geometry and may be skipped. Here, we give a brief description of each because they find wide applications in science and technology. For instance, we hear phrases such as the"elliptic" orbit of a satellite or planet, a"parabolic" reflector antenna of a radar system, that comets move in "hyperbolic" orbits, etc.*

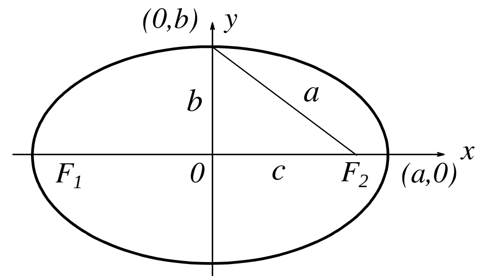


Figure 4.23. Computation of the foci locations

We observe that when the major and minor axes are equal, the ellipse becomes a circle.

Note 4.4

In Figure 4.23, the ratio $e = c/a$ is called the *eccentricity* of the ellipse; it indicates the degree of departure from circularity. In other words, if we keep point a fixed and we vary the line segment c over the range $0 \le c \le a$, we will obtain ellipses which will vary in shape. An ellipse becomes a circle when $c = 0$, and a straight line when $c = a$.

A plane curve formed by the locus of points equidistant from a fixed line, and a fixed point not on the line, is called *parabola*. Figure 4.24 shows a parabola. The fixed line is called *directrix*, and the fixed point *focus*. The lines PF and PQ at any point on the parabola, are equal.

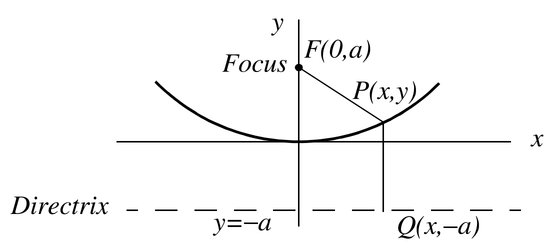


Figure 4.24. Parabola

For the parabola of Figure 4.24,

$$x^2 = 4ay \tag{4.34}$$

The locus of points for which the difference of the distances from two given points, called foci, is a constant, is called *hyperbola*. Figure 4.25 shows a hyperbola.

For the hyperbola of Figure 4.25,

$$PF_2 - PF_2 = 2a \tag{4.35}$$

$$e = c/a \tag{4.36}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{4.37}$$

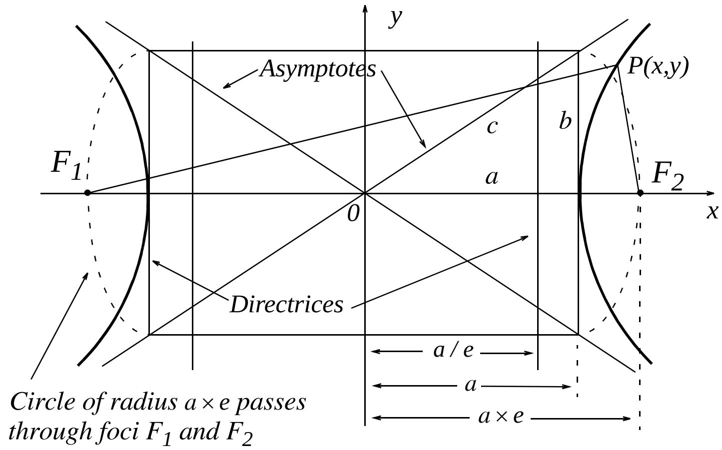


Figure 4.25. Hyperbola

In a hyperbola, the *eccentricity* e, can never be less than 1.

Note 4.5

An *asymptote* is a line considered a limit to a curve in the sense that the perpendicular distance from a moving point on the curve to the line approaches zero as the point moves an infinite distance from the origin. Two such asymptotes are shown in Figure 4.25.

4.3 Solid Geometry Figures

A solid with six faces, each a parallelogram, and each being parallel to the opposite face, is called *parallelepiped*.

A solid with six faces, each a rectangle and each being parallel to the opposite face, is called *rectangular parallelepiped*. Figure 4.26 shows a rectangular parallelepiped.

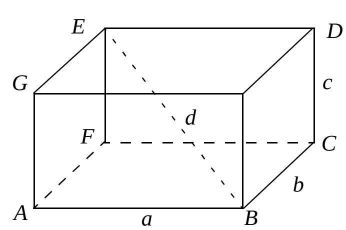


Figure 4.26. Rectangular Parallelepiped

For the rectangular parallelepiped of Figure 4.26, let $d = diagonal$, $T = total\ surface\ area$, and $V = volume$. Then,

$$d = \sqrt{a^2 + b^2 + c^2} \tag{4.38}$$

$$T = 2(ab + bc + ca) \tag{4.39}$$

$$V = abc \tag{4.40}$$

A regular solid that has six *congruent** square faces, is called a *cube*. Alternately, a cube is a special case of a rectangular parallelepiped where all faces are squares. Figure 4.27 shows a cube.

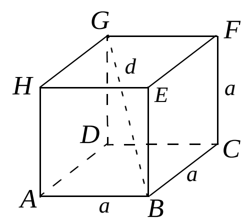


Figure 4.27. Cube

For the cube of Figure 4.27, if we let $d = diagonal$, $T = total\ surface\ area$, and $V = volume$, then,

$$d = a\sqrt{3} \tag{4.41}$$

$$T = 6a^2 \tag{4.42}$$

$$V = a^2 \tag{4.43}$$

A solid figure, whose bases or ends have the same size and shape, are parallel to one another, and each of whose sides is a parallelogram, is called *prism*. Figure 4.28 shows triangular, quadrilateral and pentagonal prisms.

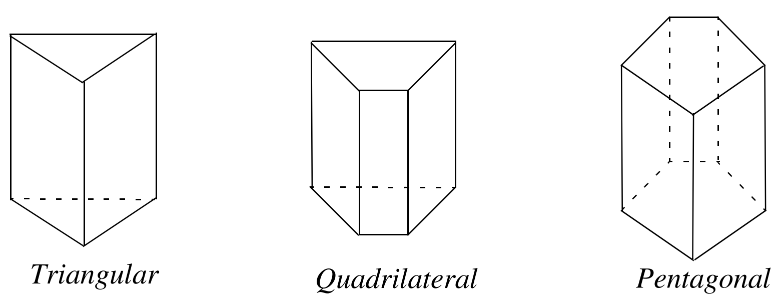


Figure 4.28. Different shapes of prisms

Prisms can assume different shapes; therefore, no general formulas for the total surface and volume are given here. The interested reader may find this information in mathematical tables.

* *Two or more coplanar (on the same plane) figures, such as triangles, rectangles, etc., are said to be congruent when they coincide exactly when superimposed (placed one over the other).*

A solid figure with a polygonal base, and triangular faces that meet at a common point, is called *pyramid*. Figure 4.29 shows a pyramid whose base is a square.

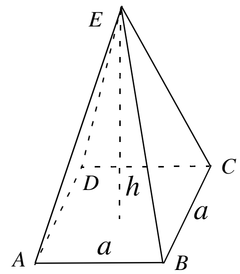


Figure 4.29. Pyramid with square base

For the pyramid of Figure 4.29, the volume V is given by

$$V = \frac{1}{3}(Area\ of\ base) \times height = \frac{1}{3}a^2h \tag{4.44}$$

The part of a pyramid between two parallel planes cutting the pyramid, especially the section between the base and a plane parallel to the base of a pyramid, is called *frustum of a pyramid*. Figure 4.30 shows a frustum of a pyramid.

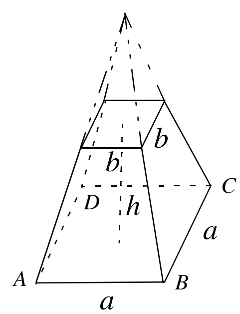


Figure 4.30. Frustrum of a pyramid.

For the frustum of a pyramid of Figure 4.30, the volume V is

$$V = \frac{1}{3}h(a^2 + b^2 + ab) \tag{4.45}$$

A surface generated by a straight line that moves along a closed curve while always passing through a fixed point, is called a *cone*. The straight line is called the *generatrix*, the fixed point is called the *vertex*, and the closed curve is called the *directrix*. If the generatrix is of infinite length, it generates two conical surfaces on opposite sides of the vertex. If the directrix of the cone is a circle, the cone is referred to as a *circular cone*. Figure 4.31 shows a right circular cone.

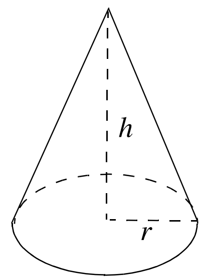


Figure 4.31. Right circular cone

For the cone of Figure 4.31, the volume V is

$$V = \frac{1}{3}\pi r^2 h \tag{4.46}$$

The part of a cone between two parallel planes cutting the cone, especially the section between the base and a plane parallel to the base, is called *frustum of a cone*. Figure 4.32 shows a frustum of a cone.

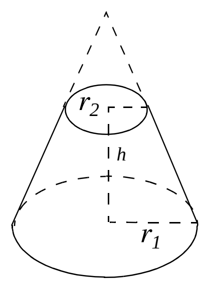


Figure 4.32. Frustum of right circular cone

For the frustum of the cone in Figure 4.32, the volume V is

$$V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2) \tag{4.47}$$

The surface generated by a straight line intersecting and moving along a closed plane curve, the directrix, while remaining parallel to a fixed straight line that is not on or parallel to the plane of the directrix, is called *cylinder*. Figure 4.33 shows a *right circular cylinder.*

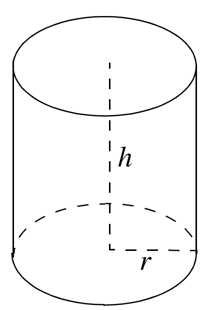


Figure 4.33. Right circular cylinder

For the cylinder of Figure 4.33, the volume V is

$$V = \pi r^2 h \tag{4.48}$$

A three-dimensional surface, all points of which are equidistant from a fixed point is called *sphere*. Figure 4.34 shows a sphere.

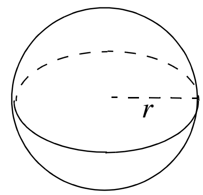


Figure 4.34. Sphere

For the sphere of Figure 4.34, the volume V is

$$V = \frac{4}{3}\pi r^3 \tag{4.49}$$

4.4 Using Spreadsheets to Find Areas of Irregular Polygons

The area of any polygon can be found from the relation

$$\begin{aligned} Area = \frac{1}{2}[&(x_0 y_1 + x_1 y_2 + x_2 y_3 + \ldots + x_{n-1} y_n + x_n y_0) \\ &-(x_1 y_0 + x_2 y_1 + x_3 y_2 + \ldots + x_n y_{n-1} + x_0 y_n)] \end{aligned} \tag{4.50}$$

where x_i and y_i are the coordinates of the vertices (corners) of the polygon whose area is to be found. *The formula of 4.50 dictates that we must go around the polygon in a counterclockwise direction.* The computations can be made easy with a spreadsheet, such as Excel. The procecure is illustrated with the following example.

Example 4.6

A land developer bought a parcel from a seller who did not know its area. Instead, the seller gave the buyer the sketch of Figure 4.26. The distances are in miles. Compute the area of this parcel.

Solution:

We arbitrarily choose the origin $(0, 0)$ as our starting point, and we go around the polygon in a counterclockwise direction as indicated in Figure 4.35.

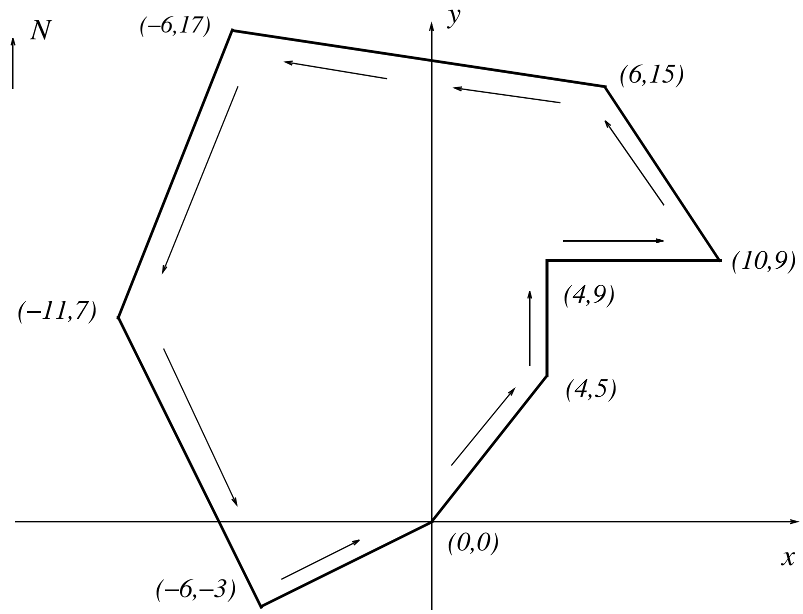


Figure 4.35. Figure for Example 4.6

The spreadsheet of Figure 4.36 below, computes the area of any polygon with up to 10 vertices. The formula in F3 is

=0.5*((B3*D4+B4*D5+B5*D6+B6*D7+B7*D8+B8*D9+B9*D10+B10*D11

+B11*D12+B12*D13+B13*D3)

-(B4*D3+B5*D4+B6*D5+B7*D6+B8*D7+B9*D8+B10*D9+B11*D10+B12*D11

+B13*D12+B3*D13))

and this represents the general formula of (4.50).

	A	B	C	D	E	F
1	Area Calculation for Example 4.6					
2					Area =	235
3	x_0=	0	y_0=	0		
4	x_1=	4	y_1=	5		
5	x_2=	4	y_2=	9		
6	x_3=	10	y_3=	9		
7	x_4=	6	y_4=	15		
8	x_5=	-6	y_5=	17		
9	x_6=	-11	y_6=	7		
10	x_7=	-6	y_7=	-3		
11	x_8=		y_8=			
12	x_9=		y_9=			
13	x_{10}=		y_{10}=			
14						

Figure 4.36. Spreadsheet to compute areas of irregular polygons.

The answer is displayed in F2. Thus, the total area is *235* square miles. To verify that the spreadsheet displays the correct result, we can divide the given polygon into triangles and rectangles, as shown in Figure 4.37, find the area of each using the formulas of (4.3) and (4.16), and add these areas. The area of each triangle or rectangle, is computed and the values are entered in Table 4.1.

TABLE 4.1 *Computation of the total area of Figure 4.27*

Triangle/ Rectangle	***Area***	***Triangle/ Rectangle***	***Area***
1 (abc)	10	**7** (lmp)	1
2 (efg)	12	**8** (mnop)	6
3 (degh)	12	**9** (bhko)	110
4 (gij)	12	**10** (pqs)	16
5 (ijk)	1	**11** (acqr)	30
6 (kln)	16	**12** (ars)	9
Total Area (**1** through **12**) in square miles			***235***

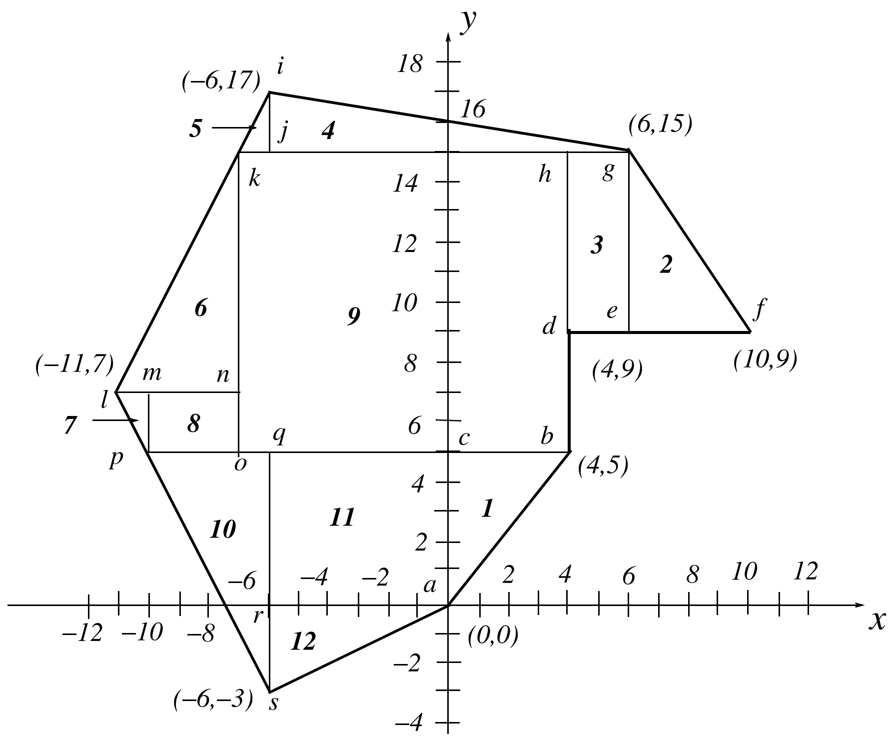


Figure 4.37. Division of Figure 4.26 to rectangles and triangles

4.5 Summary

- *Geometry* is the mathematics of the properties, measurement, and relationships of points, lines, angles, surfaces, and solids.
- In analytic geometry straight lines, curves, and geometric figures are represented by algebraic expressions.
- Descriptive geometry is the science of making accurate, two-dimensional drawings of three-dimensional geometrical forms.
- A fractal is a geometric pattern that is repeated at ever smaller scales to produce irregular shapes and surfaces that cannot be represented by classical geometry. Fractals are used especially in computer modeling of irregular patterns and structures in nature.
- Euclidean Geometry is based on one of Euclid's postulates which states that through a point outside a line it is possible to draw only one line parallel to that line.
- In dealing with astronomical distances and relativity, non-Euclidean geometries give a more precise description of the observed facts than Euclidean geometry.

- A *triangle* is a plane figure that is formed by connecting three points, not all in a straight line, by straight line segments.
- A *polygon* is defined as a closed plane figure bounded by three or more line segments.
- The *perimeter* of a polygon is defined as the closed curve that bounds the plane area of the polygon. It is the entire length of the polygon's boundary.
- The *area* of a polygon is the space inside the boundary of the polygon, and it is expressed in square units.
- A *perpendicular* line is one that intersects another line to form a $90°$ angle.
- The line that joins a vertex of a triangle to the midpoint of the opposite side is called the *median*.
- The sum of the three angles of any triangle is $180°$.
- A triangle containing an angle of $90°$ is a *right triangle*.
- In a right triangle, the side opposite to the right angle is called the *hypotenuse* of the right triangle.
- If c is the hypotenuse of a right triangle, the Pythagorean Theorem states that

$$c^2 = a^2 + b^2$$

- If c is the hypotenuse of a right triangle, the area of this triangle is found from the relation

$$Area = \frac{1}{2} \cdot a \cdot b$$

- A triangle that has two equal sides is called *isosceles* triangle.
- A triangle with all sides equal is called *equilateral triangle*.
- *Congruent triangles* are identical in shape and size. If two triangles are congruent, the following statements are true:

  I When two triangles are congruent, the six corresponding pairs of angles and sides are also congruent.

  II If all three pairs of corresponding sides in two triangles are the same, the triangles are congruent.

  III If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the triangles are congruent.

  IV If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the triangles are congruent.

V If two angles and a non-included side of one triangle are congruent to the two angles and the corresponding non-included angle of another triangle, the triangles are congruent.

VI If the hypotenuse and the leg of one right triangle are congruent to the corresponding parts of another triangle, the two triangles are congruent.

VII Two triangles with two sides and a non-included angle may or may not be congruent.

VIII Every triangle can be partitioned into four congruent triangles by connecting the middle points of its three sides.

IX A triangle that can be partitioned into n congruent parts, it can also partitioned into $4 \times n, 4^2 \times n, 4^3 \times n, \ldots$ congruent parts.

X Any isosceles triangle can be partitioned into two congruent triangles by one of its heights.

XI Any equilateral triangle can be partitioned into three congruent triangles by segments connecting its center to its vertices.

XII A right triangle whose sides forming the right angle have a ratio 2 to 1 can be partitioned into five congruent triangles.

- A *transversal* is a line that intersects two parallel lines.
- *Similar triangles* have corresponding angles that are congruent (equal) while the lengths of the corresponding sides are in proportion.
- All congruent triangles are similar, but not all similar triangles are congruent.
- If the angles in two triangles are equal and the corresponding sides are the same size, the triangles are congruent.
- If the angles in two triangles are equal the triangles are similar.
- If the angles in two triangles are not equal the triangles are neither similar nor congruent.
- A plane figure with four sides and four angles is called *quadrilateral.*
- The lines joining two non-adjacent vertices of a quadrilateral are called *diagonals*.
- In any quadrilateral, the sum of the four angles is $360°$
- The diagonals of a quadrilateral with consecutive sides a, b, c, and d are perpendicular if and only if

$$a^2 + c^2 = b^2 + d^2$$

- Two lines are said to be in parallel, when the distance between them is the same everywhere. In other words, two lines on the same plane that never intersect, are said to be parallel lines.
- A four-sided plane figure with opposite sides in parallel, is called *parallelogram.*

- The perpendicular distance from the center of a regular polygon to any of its sides is called *apothem.*
- An equilateral parallelogram is called *rhombus*. It is a special case of a parallelogram where all sides are equal.
- A four-sided plane figure with four right angles, is a *rectangle*. It is a special case of a parallelogram where all angles are equal, that is, $90°$ each.
- A plane figure with four equal sides and four equal angles, is a *square*. It is a special case of a rectangle where all sides are equal.
- A quadrilateral with two unequal parallel sides is a *trapezoid*.
- A plane curve that is everywhere equidistant from a given fixed point, referred to as the *center*, is called a *circle*.
- The line segment that joins the center of a circle with any point on its circumference is called *radius*.
- A straight line segment passing through the center of a circle, and terminating at the circumference is called *diameter*.
- The perimeter of a circle is called *circumference*.
- In any circle, the ratio of the circumference of a circle to its diameter, denoted as π, is an endless number. To five decimal places is equal to 3.14159
- *Locus* is defined as the set of all points whose coordinates satisfy a single equation or one or more algebraic conditions.
- The locus of points for which the sum of the distances from each point to two fixed points is equal forms an *ellipse*. The two fixed points are called *foci* (plural for focus). The long and short line segments are referred to as the *major* and *minor* axes respectively. The foci are always on the major axis; they are located symmetrically from the center 0 that is the intersection of the major and minor axes.
- The *eccentricity* of the ellipse indicates the degree of departure from circularity.
- A plane curve formed by the locus of points equidistant from a fixed line, and a fixed point not on the line, is called *parabola*. The fixed line is called *directrix*, and the fixed point *focus*.
- The locus of points for which the difference of the distances from two given points, called foci, is a constant, is called *hyperbola*. In a hyperbola, the *eccentricity* e, can never be less than 1.
- An *asymptote* is a line considered a limit to a curve in the sense that the perpendicular distance from a moving point on the curve to the line approaches zero as the point moves an infinite distance from the origin.

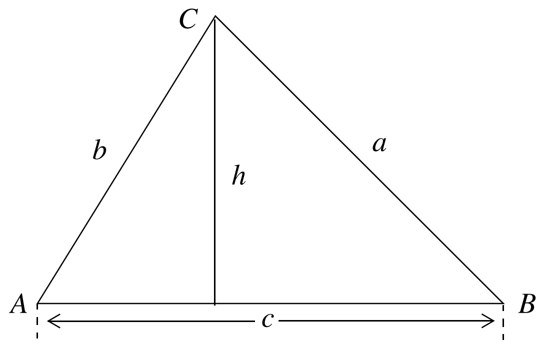
- A solid with six faces, each a parallelogram, and each being parallel to the opposite face, is called *parallelepiped*.
- A solid with six faces, each a rectangle and each being parallel to the opposite face, is called *rectangular parallelepiped*.
- A regular solid that has six *congruent* square faces, is called a *cube*. Alternately, a cube is a special case of a rectangular parallelepiped where all faces are squares.
- A solid figure, whose bases or ends have the same size and shape, are parallel to one another, and each of whose sides is a parallelogram, is called *prism*.
- A solid figure with a polygonal base, and triangular faces that meet at a common point, is called *pyramid*.
- The part of a pyramid between two parallel planes cutting the pyramid, especially the section between the base and a plane parallel to the base of a pyramid, is called *frustum of a pyramid*.
- A surface generated by a straight line that moves along a closed curve while always passing through a fixed point, is called a *cone*. The straight line is called the *generatrix*, the fixed point is called the *vertex*, and the closed curve is called the *directrix*. If the generatrix is of infinite length, it generates two conical surfaces on opposite sides of the vertex. If the directrix of the cone is a circle, the cone is referred to as a *circular cone*.
- The part of a cone between two parallel planes cutting the cone, especially the section between the base and a plane parallel to the base, is called *frustum of a cone*.
- The surface generated by a straight line intersecting and moving along a closed plane curve, the directrix, while remaining parallel to a fixed straight line that is not on or parallel to the plane of the directrix, is called *cylinder*.
- A three-dimensional surface, all points of which are equidistant from a fixed point is called *sphere*.
- The area of any polygon can be found from the relation

$$Area = \frac{1}{2}[(x_0y_1 + x_1y_2 + x_2y_3 + \dots + x_{n-1}y_n + x_ny_0)$$
$$-(x_1y_0 + x_2y_1 + x_3y_2 + \dots + x_ny_{n-1} + x_0y_n)]$$

where x_i and y_i are the coordinates of the vertices (corners) of the polygon whose area is to be found. This relation dictates that we must go around the polygon in a counterclockwise direction. The computations can be made easy with a spreadsheet, such as Excel.

4.6 Exercises

1. Compute the area of the general triangle shown in Figure in terms of the side c and height h.

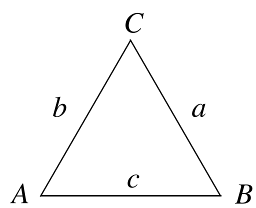


2. For the equilateral triangle below, prove that:

a. $$\angle A = \angle B = \angle C = 60°$$

b. $$Area = \frac{\sqrt{3}}{4}a^2$$

where a is any of the three equal sides.



3. Compute the area of the irregular polygon shown in Figure 4.38 with the relation (4.51)

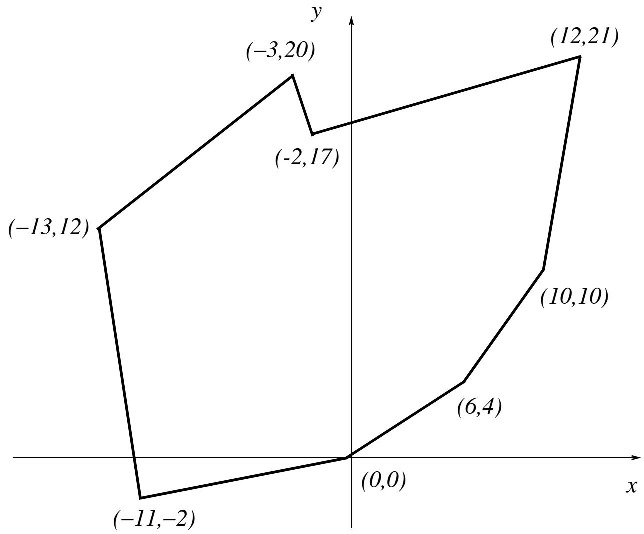


Figure 4.38. Figure for Exercise 3

4. For the triangle shown on Figure 4.39, $BD = DC$, $\angle CDB = 90°$, and $DF = AD$. Prove that the triangles *ADB* and *CDF* are congruent.

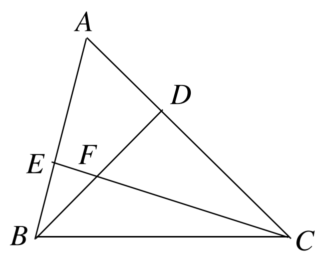
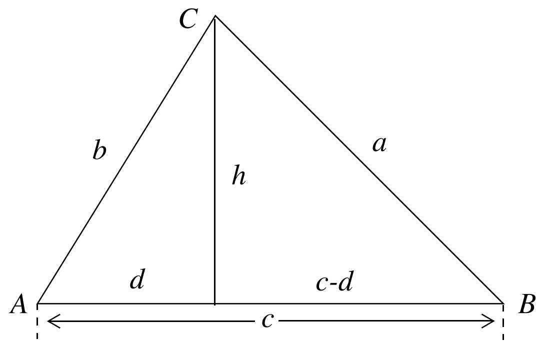


Figure 4.39. Figure for Exercise 4

5. Prove the Phythagorean theorem.

4.7 Solutions to Exercises

1.



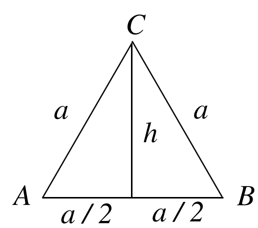
$$Area = \frac{1}{2}\cdot d\cdot h + \frac{1}{2}\cdot(c-d)\cdot h = \frac{1}{2}\cdot d\cdot h + \frac{1}{2}\cdot c\cdot h - \frac{1}{2}\cdot d\cdot h = \frac{1}{2}\cdot c\cdot h$$

2.

a. For any triangle $\angle A + \angle B + \angle C = 180°$, and since the sides a, b, and c are equal, it follows that the angles opposite to the sides are also equal. Therefore,

$$\angle A = \angle B = \angle C = 60°$$

b.



$$Area = \frac{1}{2}\cdot\frac{a}{2}\cdot h + \frac{1}{2}\cdot\frac{a}{2}\cdot h = \frac{a}{2}\cdot h \quad (1)$$

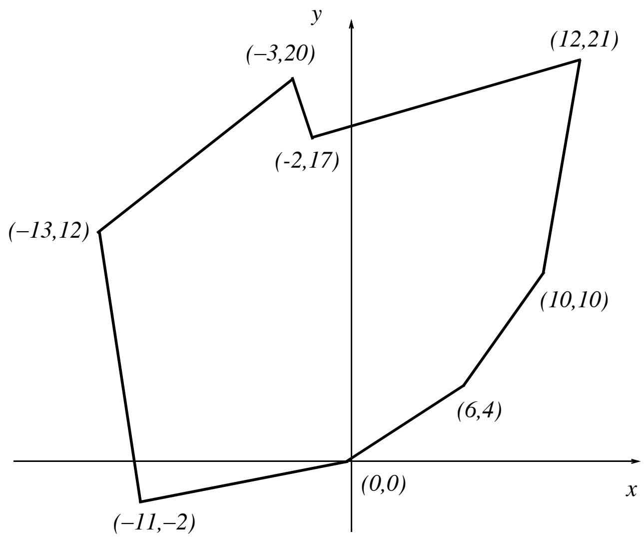
$$h^2 = a^2 - \left(\frac{a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3}{4}\cdot a^2$$

$$h = \sqrt{\frac{3}{4}}\cdot a = \frac{\sqrt{3}}{2}\cdot a$$

and by substitution into (1)

$$Area = \frac{\sqrt{3}}{4}a^2$$

3.



Solution:

$$Area = \frac{1}{2}[(x_0 y_1 + x_1 y_2 + x_2 y_3 + \ldots + x_{n-1} y_n + x_n y_0)$$
$$-(x_1 y_0 + x_2 y_1 + x_3 y_2 + \ldots + x_n y_{n-1} + x_0 y_n)]$$

We arbitrarily choose the origin $(0, 0)$ as our starting point, and we go around the polygon in a counterclockwise direction as in Example 4.6.

The formula in F2 is

=0.5*((B3*D4+B4*D5+B5*D6+B6*D7+B7*D8+B8*D9+B9*D10+B10*D11

+B11*D12+B12*D13+B13*D3)

−(B4*D3+B5*D4+B6*D5+B7*D6+B8*D7+B9*D8+B10*D9+B11*D10+B12*D11

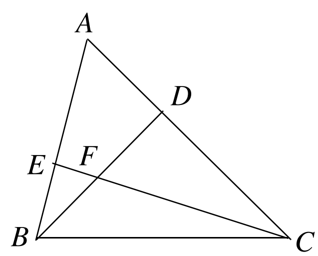
+B13*D12+B3*D13))

and this represents the general formula for the area above.

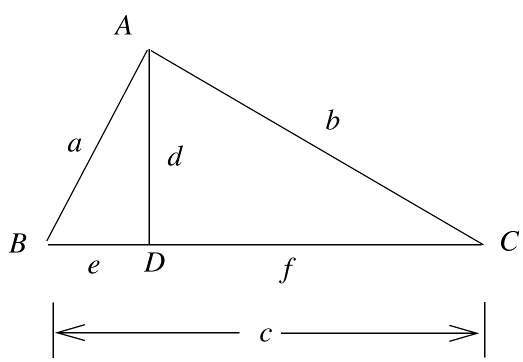
The answer is displayed in F2. Thus, the total area is *374.5* square units.

	A	B	C	D	E	F
1	Area Calculation for Exercise 3					
2					Area =	374.5
3	x_0=	0	y_0=	0		
4	x_1=	6	y_1=	4		
5	x_2=	10	y_2=	10		
6	x_3=	12	y_3=	21		
7	x_4=	-2	y_4=	17		
8	x_5=	-3	y_5=	20		
9	x_6=	-13	y_6=	12		
10	x_7=	-11	y_7=	-2		

4. Since $BD = DC$, $\angle CDB = 90°$, and $DF = AD$. by the Phythagorean theorem the hypotenuses AB and CF of triangles ADB and DFC are equal also. Therefore, these triangles are congruent.



5. Consider the right triangle shown below where $\angle A = 90°$, BC is the hypotenuse, and AD is perpendicular to BC.



We observe that triangles ABD and ABC are similar since both are right triangles and $\angle B$ is common in both triangles. Likewise, triangles ADC and ABC are similar. Since triangles ABD and ABC are similar, it follows that

$$\frac{c}{a} = \frac{a}{e} \quad or \quad ce = a^2 \qquad (1)$$

Also since triangles ADC and ABC are similar, it follows that

$$\frac{c}{b} = \frac{b}{f} \quad or \quad cf = b^2 \qquad (2)$$

Addition of (1) with (2) yields

$$ce + cf = a^2 + b^2$$

or

$$c(e + f) = a^2 + b^2$$

and since

$$e + f = c$$

we obtain the Phythagorean theorem relation

$$c^2 = a^2 + b^2$$