Chapter 7

Mathematics of Finance and Economics

This chapter is an introduction to the mathematics used in the financial community. The material presented in this appendix is indispensable to all business and technology students as well as those enrolled in continuing and adult education. Also, an executive lacking a background in finance is seriously considered handicapped in this vital respect.

7.1 Common Terms

Bond

A *bond* is a debt investment. That is, we loan money to an entity (company or government) that needs funds for a defined period of time at a specified interest rate. In exchange for our money, the entity will issue us a certificate, or bond, that states the interest rate that we are to be paid and when our loaned funds are to be returned (maturity date). Interest on bonds is usually paid every six months, i.e., semiannually.

Corporate bond

A *corporate bond* is a bond issued by a corporation.

Municipal bond

A *municipal bond* is a bond issued by a municipality and that generally is tax-free. That is, we pay no taxes on the interest that we earn. Because it its tax-free, the interest rate is usually lower than for a taxable bond.

Treasury bond

A treasury bond is a bond issued by the US Government. These are considered safe investments because they are backed by taxing authority of the US government. The interest on Treasury bonds is not subject to state income tax. Treasury bonds, or T-bonds for short, have maturities greater than 10 years, while notes and bills have lower maturities.

Perpetuity

A perpetuity is a constant stream of identical cash flows with no maturity date.

Perpetual bond

A *perpetual bond* is a bond with no maturity date. Perpetual bonds are not redeemable and pay a steady stream of interest forever. Such bonds have been issued by the British government.

Convertible Bond

A *convertible bond* is a bond that can be converted into a predetermined amount of a company's equity at certain times during it life. Usually, convertible bonds offer a lower rate of return in exchange for the option to trade the bond into stock.

The conversion ratio can vary from bond to bond. We can find the terms of the convertible, such as the exact number of shares or the method of determining how many shares the bond is converted into, in the *indenture**. For example a conversion ratio of $45:1$ means that every bond we hold (with a $\$1000$ par value) can be exchanged for 45 shares of stock. Occasionally, the indenture might have a provision that states the conversion ratio will change over the years, but this is uncommon.

Treasury note

A *treasury note* is essentially a treasury bond. The only difference is that a treasury note is issued for a shorter time (e.g., two to five years) than a treasury bond.

Treasury bill

A *treasury bill* is a treasury note that is held for a shorter time (e.g., three, six, or nine months to two years) than either a treasury bond or a treasury note. Interest on T-bills are paid at the time the bill matures, and the bills are priced accordingly.

Face value

Face value is the dollar amount assigned to a bond when first issued.

Par value

The *par value* of a bond is the face value of a bond, generally $\$1,000$ for corporate issues, and as much as $\$10,000$ for some government issues. The price at which a bond is purchased is generally not the same as the face value of the bond. When the purchase price of a bond as its par value, it is said to be purchased at par; when the purchase price exceeds the par value, it is said to be purchased above par, or at a premium. When the purchase price is below the par value, it is said to be purchased below par, or at a discount. Thus, the difference between the par value of a bond and its purchase price is termed the premium or discount whichever applies.

For example, if the face value of a bond is $\$5,000$ and it sells for $\$4,600$, it sells at a discount of $\$400$; if the same bond sells for $\$5,300$, then it sells at a premium of $\$300$.

* *Indenture is a contract between an issuer of bonds and the bondholder stating the time period for repayment, amount of interest paid, if the bond is convertible and if so at what price or what ratio, and the amount of money that is to be re-paid.*

Book value

Book value is the value of a bond at any date intermediate between its issue and its redemption.

Coupon bond

A *coupon bond* is a debt obligation with coupons representing semiannual interest payments attached. No record of the purchaser is kept by the issuer, and the purchaser's name is not printed on the certificate.

Zero coupon bond

A *zero coupon bond* is a bond that generates no periodic interest payments and is issued at a discount from face value. The Return is realized at maturity. Because it pays no interest, it is traded at a large discount.

Junk bond

A *junk bond* is a bond purchased for speculative purposes. They have a low rating and a higher default risk. Typically, junk bonds offer interest rates three to four percentage points higher than safer government bonds. Generally, a junk bond is issued by a corporation or municipality with a bad credit rating. In exchange for the risk of lending money to a bond issuer with bad credit, the issuer pays the investor a higher interest rate. "High-yield bond" is a name used by the for junk bond issuer.

Bond rating systems

Standard and Poor's and Moody's are the bond-rating systems used most often by investors. Their ratings are as shown on Table 7.1.

Table 7.1 Bond-Ratings

Moody's	Standard & Poor's	Grade	Risk
Aaa	AAA	Investment	Lowest risk/highest quality
Aa	AA	Investment	Low risk/high quality
A	A	Investment	Low risk/upper medium quality
Baa	BBB	Investment	Medium risk/medium quality
Ba, B	BB, B	Junk	High risk/low quality
Caa, Ca, C	CCC, CC, C	Junk	Highest risk/highly speculative
C	D	Junk	In default

Promissory note

A *promissory note* is an unconditional written promise made by one party to another, to pay a stipulated sum of money either on demand or at a definite future date. The sum of money stated in the note is its face value and if the note is payable at a definite date, this date is termed its “due date” or “date of maturity”. If no mention of interest appears, the maturity value of the note is its face value. But if payment of interest is specified, the maturity value is the face value of the note plus the accrued interest.

For example, let us assume that a promissory noted dated January 10, 2004, reads as follows: "Six months after date I promise to pay to the order of John Doe the sum of $\$10,000$ with interest at 8% per annum. Here, the face value of the note is $\$10,000$, its maturity date is July 10, 2004, and the maturity value is computed as

$$Interest = \$10,000 \times 0.08 \times \frac{1}{2} = \$400$$

Therefore, the maturity value is

$$Maturity\ value = \$10,000 + \$400 = \$10,400$$

Discount Rate

Discount rate is the interest rate charged by Reserve Banks when they extend credit to depository institutions either through advances or through the discount of certain types of paper, including ninety-day commercial paper. Discount rate also refers to the fee a merchant pays its merchant bank for the privilege to deposit the value of each's day's credit purchases. This fee is usually a small percentage of the purchase value.

Prime Rate

Prime rate is the interest rate charged by banks to their most creditworthy customers (usually the most prominent and stable business customers). The rate is almost always the same amongst major banks. Adjustments to the prime rate are made by banks at the same time; although, the prime rate does not adjust on any regular basis.

Mortgage Loan

In the case of many types of loans, regardless of the duration, the creditor will require some form of security for the repayment of the loan, such as property owned by the debtor. The legal instrument under which the property is pledged is termed a "mortgage," or "trust deed." In general, title and possession of the property remain with the mortgagor, or borrower. However, since the mortgagee, or lender, has a vested interest in maintaining the value of this property unimpaired, at least to the extent of the unpaid balance of the loan, the terms of the mortgage generally require that the mortgagor keep the property adequately insured, make necessary repairs, and pay all taxes as they become due.

Should it occur that there is more than one lien applying to a particular piece of property, then the mortgages are referred to as the first mortgage, second mortgage, home-equity, etc., corresponding to the priority of the claim. In the event of default in repaying the loan, or in the event of a violation of the terms of the mortgage on the part of the owner, the mortgagee can request a court of law to undertake the sale of the property in order to recover the balance of his loan.

Property is often purchased with borrowed funds, and the property thus acquired constitutes the security behind the loan. This is frequently true, for example, in the purchase of homes. Mortgage loans of this type generally possess an amortization feature. Although the term "amortization" means simply the liquidation of a debt, it is generally used to denote the process of gradually extinguishing a debt through a series of equal periodic payments extending over a stipulated period of time. Each payment can be regarded as consisting of a payment of the interest accrued for that particular period on the unpaid balance of the loan and a partial payment of principal. With each successive payment, the interest charge diminishes, thereby accelerating the reduction of the debt.

Predatory Lending Practices

Several types of predatory lending have been identified by federal investigators. The most common are:

Equity stripping: Occurs when a loan is based on the equity of a home rather than the borrower's ability to repay. These loans often have high fees, prepayment penalties, and more strict terms than a regular loan.

Packing. The practice of adding credit insurance and other extras to the loan. The supplements to the loan are very profitable to lenders and are generally financed in a single up-front of balloon payment.

Flipping: A form of equity stripping, this occurs when a lender persuades a borrower to repeatedly refinance a loan within a short period of time. The lender typically charges high fees each time.

Annuity

An *annuity* is series of equal payments made at equal intervals or periods of time. When paid into a fund which is invested at compound interest for a specified number of years, the annuity is referred to as a sinking fund.

Ordinary Annuity

An *ordinary annuity* is a series of equal payments or receipts occurring over a specified number of periods with the payments or receipts occurring at the end of each period.

Sinking Fund

A *sinking fund* is an interest-earning fund in which equal deposits are made at equal intervals of time. Thus, if the sum of *$1,000* is placed in a fund every three months, a sinking fund has been established. See also annuity.

As an example, let us assume that a company plans to purchase a new asset five years from now, and the purchase price at that time will be *$10,000*. To accumulate this sum, it will make annual deposits in a sinking fund for the next five years and the fund will be earning interest at *4%* per annum. If the fund did not earn interest, the required deposit would be *$2,000* per annum. However, since the principle in the fund will be continuously augmented through the accrual of interest, the actual deposit required will be less than $\$2,000$.

7.2 Interest

A *paid interest* is charge for a loan, usually a percentage of the amount loaned. If we have borrowed money, from a bank or credit union, we have to pay them interest. An *earned interest* is a percentage of an amount that we receive from a bank or credit union for the amount of money that we have deposited. In other words, if we put money into a bank or credit union they will pay us interest on this money. In this section we will discuss simple interest, simple interest over multiple years, and simple interest over a fraction of a year.

Simple Interest

Simple interest is calculated on a yearly basis (annually) and depends on the interest rate. The rate is often given per annum (p.a.) which means per year.

Let

P_0 = *principal of the loan, i.e., present value*

i = *interest rate per period (simple interest) expressed as a decimal*

n = *number of periods constituting the life of the loan*

P_n = *maturity value of the loan, i.e., future value*

Then, the maturity value of a loan is computed as

$$P_n = P_0 + P_0 ni = P_0(1 + ni) \tag{7.1}$$

If we know the maturity (future) value P_n, we can compute the present value P_0 using the formula

$$P_0 = \frac{P_n}{1 + ni} \tag{7.2}$$

Example 7.1

We deposit $\$500.00$ into a bank account with an interest rate of 2% per annum. We want to find how much money we have in the account after one year.

Solution:

For this example, $P_0 = 500$, $n = 1$, and $i = 0.02$. Therefore, after one year the maturity value P_n will be

$$P_n = P_0(1 + ni) = 500(1 + 1 \times 0.02) = \$510$$

That is, we have earned $\$10.00$ interest.

Example 7.2

We deposit $\$350.00$ in a simple interest account for 3 years. The account pays interest at a rate of 3% per annum. How much do we have in this account after three years?

Solution:

$$P_n = P_0(1 + ni) = 350(1 + 3 \times 0.03) = \$381.50$$

That is, after 3 years we will have $\$381.50$ in this account.

If money is not left in a bank account for a whole year then only a fraction of the interest is paid.

Example 7.3

We deposit $\$50,000$ in a simple interest account for 6 months. This account pays interest at a rate of 8.5% per annum. How much do we have in this account after 6 months?

Solution:

Since the annual rate is 8.5%, the semiannual rate is

$$\frac{1}{2} \times 0.085 = 0.0425$$

Then,

$$A = P(1 + ni) = 50000(1 + 0.0425) = \$52125$$

That is, after 6 months we will have $\$52125$ in this account.

Compound Interest

Compound interest includes interest earned on interest. In other words, with compound interest, the interest rate is applied to the original principle and any accumulated interest.

Let

P_0 = *principal of the loan, i.e., present value*

i = *interest rate per period (simple interest) expressed as a decimal*

n = *number of periods constituting the life of the loan*

P_n = *maturity value of the loan, i.e., future value*

Then, for

Period 1

$$\begin{aligned} \textit{Principal at start of period} &= P_0 \\ \textit{Interest earned} &= P_0 i \\ \textit{Principal at end of Period 1} &= P_0 + P_0 i \\ P_1 &= P_0 (1+i) \end{aligned}$$

Period 2

$$\begin{aligned} \textit{Principal at start of period} &= P_0 (1+i) \\ \textit{Interest earned} &= P_0 (1+i) i \\ \textit{Principal at end of Period 2} &= P_0 (1+i) + P_0 (1+i) i = P_0 (1+i)(1+i) \\ P_2 &= P_0 (1+i)^2 \end{aligned}$$

Period 3

$$\begin{aligned} \textit{Principal at start of period} &= P_0 (1+i)^2 \\ \textit{Interest earned} &= P_0 (1+i)^2 i \\ \textit{Principal at end of Period 3} &= P_0 (1+i)^2 + P_0 (1+i)^2 i = P_0 (1+i)^2 (1+i) \\ P_3 &= P_0 (1+i)^3 \end{aligned}$$

and so on. Thus, for Period n

$$P_n = P_0 (1+i)^n \tag{7.3}$$

The formula of (7.3) assumes that the interest is compounded annually. But if the interest is compounded q times per year, the maturity value is computed using the formula

$$P_n = P_0 \left(1 + \frac{i}{q}\right)^{nq} \tag{7.4}$$

Example 7.4

We deposit *$350.00* in a compound interest account for 3 years. The account pays interest at a rate of 3% per annum compounded annually. How much do we have in this account after three years?

Solution:

Since the interest rate is compounded annually, the formula of (7.3) applies for this example. Thus,

$$P_n = P_0(1+i)^n = 350(1+0.03)^3 = \$382.45$$

That is, after 3 years we will have $\$382.45$ in this account.

We will discuss the Microsoft Excel financial functions at the end of this chapter. For our present discussion, we will use the appropriate function to verify our answers.

The function

```
=FV(0.03,3,0,–350,0)
```

returns $\$382.45$. In the formula above 0.03 represents the annual interest rate expressed in decimal form, 3 represents the number of periods (3 years for our example), 0 represents the payments made in each period. For this example it is zero since there are no periodic payments made. The present value of $\$350.00$ is shown as a negative number since it is an outgoing cash flow. Cash we receive, such as dividends, is represented by positive numbers. The last 0 in the formula indicates that maturity of this investment occurs at the end of the 3-year period. Cash we receive, such as dividends, is represented by positive numbers.

Example 7.5

We deposit $\$350.00$ in a compound interest account for 3 years. The account pays interest at a rate of 3% per annum compounded monthly. How much do we have in this account after three years?

Solution:

Since the interest rate is compounded monthly, the formula of (7.4) applies for this example. Thus,

$$P_n = P_0\left(1+\frac{i}{q}\right)^{nq} = 350\left(1+\frac{0.03}{12}\right)^{3\times 12} = \$382.92$$

Check with Microsoft Excel: The function

```
=FV(0.03/12,3*12,0,–350,0)
```

returns $\$382.92$. In the above formula we divided the interest rate by 12 to represent the compounded interest on a monthly basis, and we multiplied the period (3 years) by 12 to convert to 36 monthly periods.

Example 7.6

This example computes interest on a principal sum to illustrate the differences between simple interest and compound interest. We have used a $\$100.00$ principal and 7% interest.

Simple Interest

The interest rate is applied only to the original principal amount in computing the amount of interest as shown on Table 7.2.

Table 7.2 Simple Interest

Year	**Principal($)**	**Interest($)**	**Ending Balance($)**
1	100.00	7.00	107.00
2	100.00	7.00	114.00
3	100.00	7.00	121.00
4	100.00	7.00	128.00
5	100.00	7.00	135.00
	Total interest	**35.00**	

Compound Interest

The interest rate is applied to the original principle and any accumulated interest as shown on Table 7.3.

Table 7.3 Compound Interest

Year	**Principal($)**	**Interest($)**	**Ending Balance($)**
1	100.00	7.00	107.00
2	107.00	7.49	114.49
3	114.49	8.01	122.50
4	122.50	8.58	131.08
5	131.08	9.18	140.26
	Total interest	**40.26**	

Quite often, both the present value P_0 and maturity (future) value P_n are known and we are interested in finding the interest rate i or the period n. Since equation (7.3) contains the interest rate and the period, we can express it in logarithmic form by taking the common log of both sides of that equation. Thus,

$$logP_n = log[P_0(1+i)^n] = logP_0 + log[(1+i)^n] = logP_0 + nlog(1+i) \tag{7.5}$$

or

$$log\frac{P_n}{P_0} = nlog(1+i) \tag{7.6}$$

Example 7.7

On January 1, 1999, we borrowed $\$5000.00$ from a bank and the debt was discharged on December 31, 2003, with a payment of $\$6100.00$. Assuming that the interest was compounded annually, was the interest rate that we paid?

Solution:

For the example, $P_0 = 5000$, $P_5 = 6100$, and $n = 5$. By substitution in (7.6) we get

$$log\frac{6100}{5000} = 5log(1+i)$$

or

$$log\frac{6.1}{5} = 5log(1+i)$$

Using Microsoft Excel, we find that =LOG10(6.1/5) returns 0.0864

Then,

$$log(1+i) = \frac{0.0864}{5} = 0.0173$$

In Chapter 1 we learned that $log_{10}M = x$ implies that $M = 10^x$. Using Microsoft Excel we find that =10^0.0173 returns 1.041. Thus,

$$1+i = 10^{0.0173} = 1.041$$

and solving for i we get

$$i = 1.041-1 = 0.041$$

Therefore, the interest rate that we paid is 4.1% to the nearest tenth of one per cent.

Check with Microsoft Excel:

The function =RATE(5,0,5000,–6100,0.05) returns 4.06%. In this formula 5 represents the number of the periods (5 years), 0 represents the number of periodic payments made during the 5-year period which is zero for this example, 5000 is the amount of the loan we received, -6100- is the amount we paid, and 0.05 is our guess of what the interest rate might be.

Example 7.8

On January 1, 2004, we deposited $\$5000.00$ to an account paying 4.5% annual interest. How long will it take for this amount to grow to $\$7500.00$?

Solution:

For this example, $P_0 = 5000$, $P_n = 7500$, and $i = 0.045$. By substitution in (7.6) we get

$$log\frac{7500}{5000} = nlog(1 + 0.045)$$

or

$$log1.5 = nlog(1.045)$$

or

$$n = \frac{log1.5}{log(1.045)}$$

Using Excel we find that =LOG10(1.5) returns 0.1761 and =LOG10(1.045) returns 0.0191. Then,

$$n = \frac{0.1761}{0.0191} = 9.22$$

That is, it will take 9.22 years for our $\$5000.00$ investment to grow to $\$7500.00$.

Check with Microsoft Excel: The function =NPER(0.045,0,–5000,7500,0) returns 9.21 years.

Often, we want to translate a given sum of money to its equivalent value at a prior valuation date. To adhere to our previous notation, we shall let P_0 denote the value of a given sum of money at a specified valuation or zero date, and let P_{-n} denote the value of P_0 at a valuation date n periods prior to that of P_0. We will assume that the sum of money P_{-n} is deposited in a fund at interest rate i per period. Thus, its value at the end of n periods will be P_0. Applying equation (7.3) but substituting P_{-n} for P_0, and P_0 for P_n, we obtain

$$P_0 = P_{-n}(1 + i)^n$$

or

$$P_{-n} = \frac{P_0}{(1 + i)^n}$$

or

$$P_{-n} = P_0(1 + i)^{-n} \tag{7.7}$$

Example 7.9

We possess two promissory notes each having a maturity value of $\$1000.00$. The first note is due 2 years hence, and the second 3 years hence. At an annual interest rate of 6% what proceeds will we obtain by discounting the notes at the present date?

Solution:

Equation (7.7) applies to this example. Thus, the value of the first note is

$$P_{-2} = P_0(1+i)^{-2} = 1000(1+0.06)^{-2} = 1000 \times 0.89000 = 890.00$$

and the value second note is

$$P_{-3} = P_0(1+i)^{-3} = 1000(1+0.06)^{-3} = 1000 \times 0.83962 = 839.62$$

and the total discount value is

$$890.00 + 839.62 = 1729.62$$

Check with Microsoft Excel: The function =PV(0.06,2,0,–1000) returns $\$890.00$, and the function =PV(0.06,3,0,–1000) returns $\$839.62$.

The discount applicable to the sum P_0 for n periods at interest rate i is

$$P_0 - P_{-n} = P_0 - P_0(1+i)^{-n} = P_0[1-(1+i)^{-n}] \tag{7.8}$$

Example 7.10

A note whose maturity value is $\$5000.00$ is to be discounted 1 year before maturity at an interest rate of 6%. Compute the discount value using

a. Equation (7.7)

b. Equation (7.8)

Solution:

a.

$$P_{-1} = P_0(1+i)^{-1} = 5000(1+0.06)^{-1} = 5000(0.94340) = 4716.98$$

$$Discount\ value = 5000.00 - 4716.98 = 283.02$$

b.

$$P_0 - P_{-1} = P_0[1-(1+i)^{-1}] = 5000[1-(1+0.06)^{-1}] = 5000(1-0.94340) = 28302$$

Check with Microsoft Excel: We will subtract the function =PV(0.06,1,1,–5000) from $\$5000.00$. Thus, =5000-PV(0.06,1,1,–5000) returns $\$283.96$.

Example 7.11

Let us assume that the following transactions occurred in a fund whose annual interest rate is 8%.

1. An initial deposit of *$1000.00* was made on January 1, 1992
2. A withdrawal of *$600.00* was made on January 1, 1996
3. A deposit of *$800.00* was made on January 1, 2001

If the account was closed on December 31, 2002, what was the final principal?

Solution:

For convenience, the history of the principal in the fund is shown in Figure 7.1.

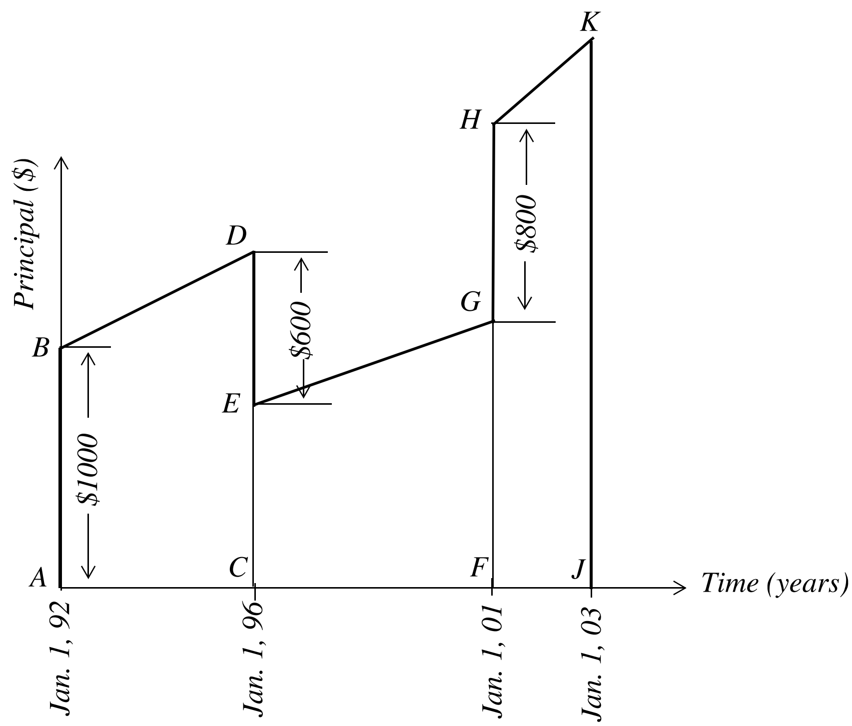


Figure 7.1. Plot for Example 7.11.

The four significant dates and the principal at these dates are computed as follows:

$$AB = 1000$$

$$CD = 1000(1.08)^4 = 1000(1.36049) = 1360.49$$

$$CE = CD - ED = 1360.49 - 600 = 760.49$$

$$FG = 760.49(1.08)^5 = 760.49(1.46933) = 1117.41$$

$$FH = FG + GH = 1117.41 + 800 = 1917.41$$

$$JK = 1917.41(1.08)^2 = 1917.41(1.1664) = 2236.47$$

Therefore, the principal on December 31, 2002 was *$2236.47*.

The principal at any nonsignificant date can be found by equation (7.3).

Quite often, it is desirable to translate a given group of money values to a second group equivalent to the first at another date. The procedure is illustrated with the following example.

Example 7.12

Johnson owes Taylor *$3000* due December 31, 2003, and *$2000* due on December 31, 2005. By mutual consent, the terms of payment are altered to allow Johnson to discharge the debt by making a payment of *$4000* on December 31, 2006, and a payment for the balance on December 31, 2007. If the annual interest rate is 5%, what will be the amount of Johnson's final payment?

Solution:

The given data are tabulated in Table 7.4, where the final payment whose value is to be computed, is denoted as x.

Table 7.4 Data for Example 7.12

Payments-Original Plan	*Payments-Revised Plan*
$3,000 on 12/31/03	$4,000 on 12/31/06
$2,000 on 12/31/05	x on 12/31/07

Since the two groups of payments are equivalent to one another, based on an interest rate of 5%, the total value of one group must equal the total value of the other group at every instant of time. To determine x, we select some standard date for evaluating all sums of money involved. For convenience, we shall choose Dec. 31, 2007, as the valuation date. Then,

$$\begin{aligned} 4000(1.05)+x &= 3000(1.05)^4+2000(1.05)^2 \\ 4200+x &= 3000(1.21551)+2000(1.10250) \\ 4200+x &= 3646.53+2205 \\ x &= 3646.53+2205-4200 = 1651.53 \end{aligned}$$

That is, with the revised payment plan Johnson must pay Taylor *$4000* on December 31, 2006, and *$1651.53* on December 31, 2007.

Had we selected December 31, 2003 as our valuation date, our solution would be

$$\begin{aligned} 4000(1.05)^{-3}+x(1.05)^{-4} &= 3000+2000(1.05)^{-2} \\ 3455.36+x(1.05)^{-4} &= 3000+1814.06 \\ x(1.05)^{-4} &= 4814.06-3455.36 = 1358.70 \\ x &= 1358.70(1.05)^4 = 1358.70(1.21551) = 1651.51 \end{aligned}$$

We can also verify the result by application of a chronological sequence as shown in Table 7.5.

Table 7.5 Chronological sequence in table form for Example 7.12

Value of first debt on 12/31/05 = $3000(1.05)^2$	=	\$3,307.50
Value of second debt on 12/31/05	=	2,000.00
Total principal on loan on 12/31/05	=	5,307.50
Interest earned in 2006 = $5307.50(0.05)$	=	265.38
Principal on loan on 12/31/06	=	5,572.88
Payment on 12/31/06	=	4,000.00
Principal on loan on 01/01/07	=	1,572.88
Interest earned in 2007 = $1572.88(0.05)$	=	78.64
Principal on loan on 12/31/07	=	1,651.52
Payment required on 12/31/07	=	**1,651.52**

We should emphasize that these two groups of payments are equivalent to one another only for an interest rate of 5%. Implicit in our reasoning is the assumption that the creditor can reinvest each sum of money he receives in a manner that continues to yield 5%. Should there occur any variation of the interest rate, then the equivalence of the two groups of payments becomes invalid. For example, assume that the creditor is able to reinvest his capital at a rate of only 4% per cent. For this situation, the revised terms of payment are more advantageous to him since, by deferring the collection of the money due him, he enables his capital to earn the higher rate of interest for a longer period of time.

In the preceding example, the replacement of one group of money values with an equivalent one was accomplished with a known interest rate. Many problems arise in practice, however, in which the equivalent groups of money values are known, and it is necessary to determine the interest rate on which their equivalence is predicated. Problems of this nature can only be solved by a trial-and-error method, and if the exact interest rate is not one of those listed in the interest tables, it can be approximated by means of straight-line interpolation.

Example 7.13

Smith owed Jones the sum of $\$1000$ due Dec. 31, 2000, and \$4,000, due Dec. 31, 2002. Because Smith was unable to meet these obligations as they became due, the debt was discharged by means of a payment of $\$2000$ on Dec. 31, 2003, and a second payment of $\$3850$ on Dec. 31, 2004. What annual interest rate was intrinsic in these payments?

Solution:

Let i denote the interest rate. If the expression $(1+i)^n$ is expanded by the binomial theorem* and all terms beyond the second are discarded, we obtain

$$(1+i)^n \approx 1+ni \tag{7.9}$$

We will apply this approximate value to obtain a first approximation of the value of i. This procedure is tantamount to basing our first calculation on the use of simple rather than compound interest. This approximation understates the value of $(1+i)^n$, and the degree of error varies in proportion to n.

Since the two groups of payments in this problem are equivalent, the value of one group equals the value of the other at any valuation date that we select. Equating the two groups at the end of 2004 on the basis of simple interest, we obtain

$$\begin{aligned}
[1,000+4i(1000)]+[4,000+2i(4000)] &= [2000+i(2000)]+3850 \\
4000i+8000i-2000i &= 2000+3850-1000-4000 \\
10000i &= 850 \\
i &= 0.085
\end{aligned}$$

or $i = 8.5\%$ as a first approximation.

But because application of equation (7.9) yields an approximation, our computations have produced an understatement of the value of each sum of money except the last. Moreover, since the amount of error increases as n increases, it is evident that we have reduced the importance of the money values having early valuation dates and inflated the importance of those having late valuation dates. The true rate, therefore, will be less than 8.5%. Let us assume an 8% rate. If this were the actual rate, then the payment x required at the end of 2004 is determined as follows:

$$\begin{aligned}
1000(1.08)^4+4000(1.08)^2 &= 2000(1.08)+x \\
x &= 1000(1.08)^4+4000(1.08)^2-2000(1.08) = 3866.09
\end{aligned}$$

Since the actual payment was $\$3850.00$, the true interest rate is less than 8% per cent. Let us try a 7.5% rate.

$$\begin{aligned}
1000(1.075)^4+4000(1.075)^2 &= 2000(1.075)+x \\
x &= 1000(1.075)^4+4000(1.075)^2-2000(1.075) = 3807.97
\end{aligned}$$

Since this value is less than $\$3850.00$, we expect the interest rate to lie between 7.5% and 8% as shown on the graph of Figure 7.2.

* *The binomial theorem is discussed in Chapter 11.*

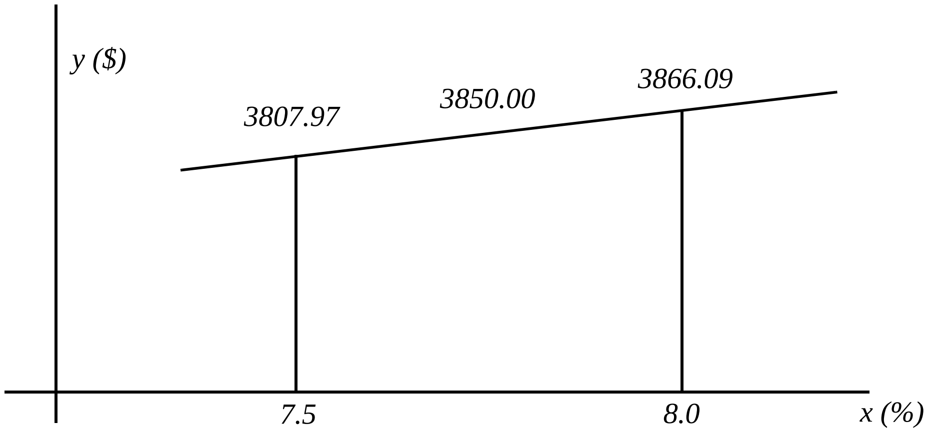


Figure 7.2. Graph for Example 7.13

The interest rate between 7.5% and 8% can be found by applying straight-line interpolation as we discussed on Chapter 2, Relation (2.48), that is,

$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1$$

For our example,

$$x_{y = 3850} = \frac{1}{slope} \times (3850.00 - 3807.97) + 0.075 = \frac{1}{\dfrac{3866.09 - 3807.97}{0.080 - 0.075}} \times 42.03 + 0.075$$

$$= \frac{0.005}{58.12} \times 42.03 + 0.075 = 0.0786$$

Therefore, the actual interest rate is 7.86%.

The problem of calculating an interest rate is often encountered where an investment is made that produces certain known returns at future dates and we wish to determine the rate of return earned by the investment. The following example illustrates the procedure.

Example 7.14

An undeveloped lot was purchased in January, 1995, for $\$2000$. Taxes and assessments were charged at the end of each year are as shown in Table 7.6.

The owner paid the charges for 1997 and 1998, but not for subsequent years. At the end of 2002, the lot was sold for $\$3000$, the seller paying the back charges at 7% interest compounded annually and also paying a commission of 5% to a real estate agent. What rate of return was realized on the investment?

Table 7.6 Data for Example 7.14

Year	Taxes Paid
1997	30
1998	30
1999	200
2000	30
2001	30
2002	30

Solution:

The payment made by the owner at the end of 2002 for the back charges consisted of the following:

Table 7.7 Computations for Example 7.14

Year	Computations	$ Amount
1999	$200\times(1.07)^3$	$245.01
2000	$30\times(1.07)^2$	34.35
2001	$30\times(1.07)$	32.10
	Total back charges	$311.46
2002	Taxes	$30.00
2002	Real Estate Commission $3,000×5%	$150.00
	Total payment at end of 2002	$491.46
2002	Selling Price	$3,000.00
2002	Net profit ($3,000.00 – $491.46)	$2,508.54

Next, we form the following two equivalent groups of money shown in Table 7.8.

Table 7.8

Group 1 – Disbursements		Group 2 – Receipts	
Date	Amount	Date	Amount
1/1/1997	$2,000.00	12/31/2002	$2508,54
12/31/1997	30.00		
12/31/1998	30.00		

Let i denote the investment rate. Using simple interest to obtain a first approximation and selecting Dec. 31, 2002, as our valuation date, we obtain

$$[2000 + 2000 \times 6i] + [30 + 30 \times 5i] + [30 + 30 \times 4i] = 2508.54$$

or

$$12270i = 2508.54 - 2060 = 448.54$$

from which

$$i = \frac{448.54}{12270} = 0.03655583 \approx 3.66\%$$

That is, as a first approximation, the interest rate is 3.66%. But our approximation has given insufficient weight to the disbursements and therefore, the true investment rate is less than this. To understand this, let us consider the factor $(1 + i)^n$. Using the binomial theorem and discarding all terms beyond the second, we obtain

$$(1 + i)^n \approx 1 + ni \tag{7.10}$$

It is obvious that this approximation understates the value of $(1 + i)^n$ and the error varies in proportion to n.

Let us try a 3.5% approximation. The income y required to yield this rate is found as follows:

$$2000 \times (1.035)^6 + 30 \times (1.035)^5 + 30 \times (1.035)^4 = y$$

or

$$y_{x\,=\,0.035} = 2458.11 + 35.63 + 34.43 = 2528.57$$

Since the actual income was $\$2508.54$, the investment rate is less than 3.5%. Let us try a 3.0% rate. Then,

$$2000 \times (1.03)^6 + 30 \times (1.03)^5 + 30 \times (1.03)^4 = y$$

or

$$y_{x\,=\,0.030} = 2388.10 + 34.78 + 33.77 = 2456.65$$

Since this value is less than $\$2508.54$, we expect the interest rate to lie between 3.0% and 3.5% as shown on the graph of Figure 7.3.

The interest rate between 3.0% and 3.5% can be found by applying straight-line interpolation as we discussed on Chapter 2, Relation (2.48), that is,

$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1$$

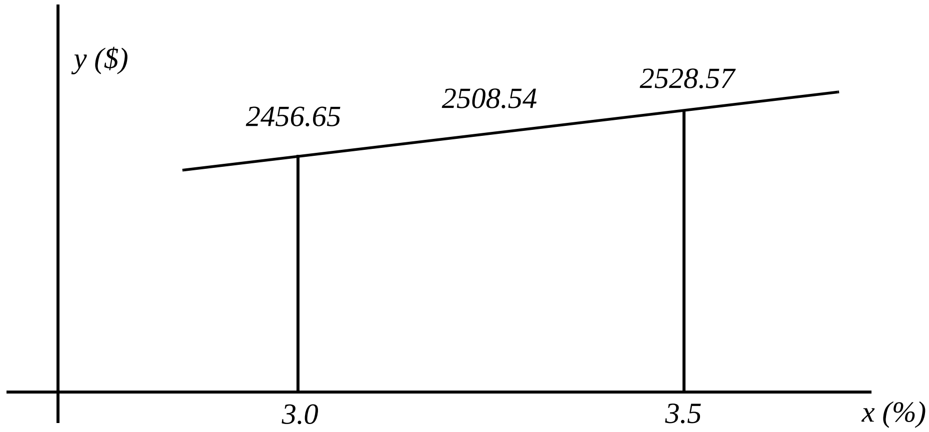


Figure 7.3. Graph for Example 7.14

For our example,

$$x_{y\,=\,2508.54} = \frac{1}{slope} \times (2508.54 - 2456.65) + 0.03 = \frac{1}{\frac{2528.57 - 2456.65}{0.035 - 0.030}} \times 51.89 + 0.03$$

$$= \frac{0.005}{71.92} \times 51.89 + 0.03 = 0.0336$$

Therefore, the actual interest rate is 3.36%.

Effective Interest Rate

Consider the sum of $\$1000$ to be deposited in a fund earning interest at the rate of 8% per annum compounded quarterly. Since the interest is compounded quarterly, 8% is merely a nominal interest rate; the true interest rate is 2% per quarterly period. At the end of 6 months, or two interest periods, this sum has expanded to

$$P_2 = 1000(1 + 0.02)^2 = 1040.40$$

If this fund, instead of earning 2% interest per quarterly period, were earning interest at the rate of 4.04% per cent per semiannual period, then the principal at the end of any 6-month period would be the same. Hence, assuming that the principal is not withdrawn from the fund except at the end of a 6-month period, we may regard the two interest rates as being equivalent to one another.

At the expiration of 1 year, or four interest periods, the original sum of $\$1000$ has expanded to

$$P_4 = 1000(1 + 0.02)^4 = 1082.43$$

Similarly, if this fund earned interest at the rate of 8.243% per cent compounded annually, the principal at the end of each year would be the same. This interest rate, then, is also equivalent to the given rate.

We have thus illustrated the equivalence of these three interest rates:

2% per quarterly period

4.04% per semiannual period

8.243% per annual period

If a given interest rate applies to a period less than 1 year, then its equivalent rate for an annual period is referred to as its *effective rate*. Thus, the effective rate corresponding to a rate of 2% per quarterly period is 8.243% per cent. The effective rate is numerically equal to the interest earned by a principal of $\$1.00$ for 1 year.

In general, let

j = nominal interest rate

r = effective interest rate

m = number of compoundings per year

Then,

$$Effective\ rate = r = \left(1 + \frac{j}{m}\right)^m - 1 \tag{7.11}$$

Example 7.15

We wish to make a ratio comparison of the following two interest rates:

First rate, 6% per annum compounded monthly = $1/2\%$ per month

Second rate, 4% per annum compounded semiannually = 2% per semiannual period

Since the two rates apply to unequal periods of time, a direct comparison is not possible. However, we can establish a basis of comparison by determining the effective rate corresponding to each:

Effective rate for 6% compounded monthly

$$r_1 = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 6.168\%$$

Effective rate for 4% compounded semiannually

$$r_2 = \left(1 + \frac{0.04}{2}\right)^2 - 1 = 4.04\%$$

Therefore,

$$\frac{r_1}{r_2} = \frac{6.168\%}{4.04\%} = 1.527$$

The calculation of effective interest rates is therefore highly useful for comparative purposes.

7.3 Sinking Funds

We defined a sinking fund in Section 7.1. As a review, a sinking fund is an interest-earning fund in which equal deposits are made at equal intervals of time. Thus, if the sum of $\$100$ is placed in a fund every 3 months, a sinking fund has been established. A sinking fund is generally created for the purpose of gradually accumulating a specific sum of money required at some future date. For example, when a corporation has floated an issue of bonds it will often set aside a portion of its annual earnings to assure itself sufficient funds to retire the bonds at their maturity. The money reserve need not remain idle but can be invested in an interest-earning fund.

Assume that a business firm plans to purchase a new machine 5 years hence, the purchase price being $\$10000$. To accumulate this sum, it will make annual deposits in a sinking fund for the next 5 years, the fund earning interest at the rate of 4% per annum. How much should each deposit be? (In reality, it would be pointless to deposit the fifth sum of money, since the fund will be closed at the same date. However, it will simplify our discussion if we consider that the deposit is actually made.)

Now, if the fund did not earn interest, the required annual deposit would be simply $\$10000/5$ or $\$2000$. However, since the principal in the fund will be continuously augmented through the accrual of interest, the actual deposit required will be somewhat less than $\$2000$. We shall soon derive a formula for calculating the required deposit.

In the discussion that follows, we shall assume, if nothing is stated to the contrary, that the following conditions prevail:

1. The sinking fund is established at the beginning of a specific interest period. We shall identify the date on which the fund is established as the *origin date* of the fund.
2. The interval between successive deposits, known as the *deposit period*, is equal in length to an interest period.
3. Each deposit is made at the end of an interest period. Consequently, there is a time interval of one interest period between the date the fund is established and the date the first deposit is made.

4. The date at which a sinking fund is to terminate is always the last day of a specific interest period. We shall call this date the *terminal date* of the fund. The principal in the fund at its termination will, of course, include the interest earned during the last period and the final deposit in the fund, made on the terminal date.

A sinking fund satisfying the above requirements is referred to as an *ordinary sinking fund*. The duration of the fund is referred to as the *term of the fund*.

To illustrate the above definitions, assume that a sinking fund is created on Jan. 1, 2003, to consist of 10 deposits of $\$1000$ each. The interest (and deposit) period of the fund is 1 year. Then

Origin date = Jan. 1, 2003

Date of first deposit = Dec. 31, 2003

Terminal date (and date of last deposit) = Dec. 31, 2012

Term of fund = 10 years

At the end of each interest period, the principal in the fund increases abruptly as a result of the compounding of the interest earned during that period and the receipt of the periodic deposit. Where the principal in a sinking fund is to be calculated at a date intermediate between its origin and termination, we shall in all cases determine the principal on the last day of a particular interest period, immediately after these two events have transpired.

Example 7.16

The sum of $\$500$ is deposited in a sinking fund at the end of each year for 4 years. If the interest rate is 6% compounded annually, what is the principal in the fund at the end of the fourth year?

Solution:

The development of the principal in the fund is recorded graphically in Figure 7.4 by the method of chronological sequence. This graph is intended solely as an aid in visualizing the growth of the principal, not as a means of achieving a graphical solution of the problem.

At the end of the first year, the sum of $\$500$, represented by AB, is placed in the fund. During the second year, the principal earns interest at the rate of 6% and amounts to $\$530$ at the end of the second year (CD). A deposit of $\$500$ at this time (DE) enhances the principal to the amount of $\$1030$ (CE). This two-cycle process is repeated each year until the final deposit KL is made. The growth of the principal is recorded in Table 7.9.

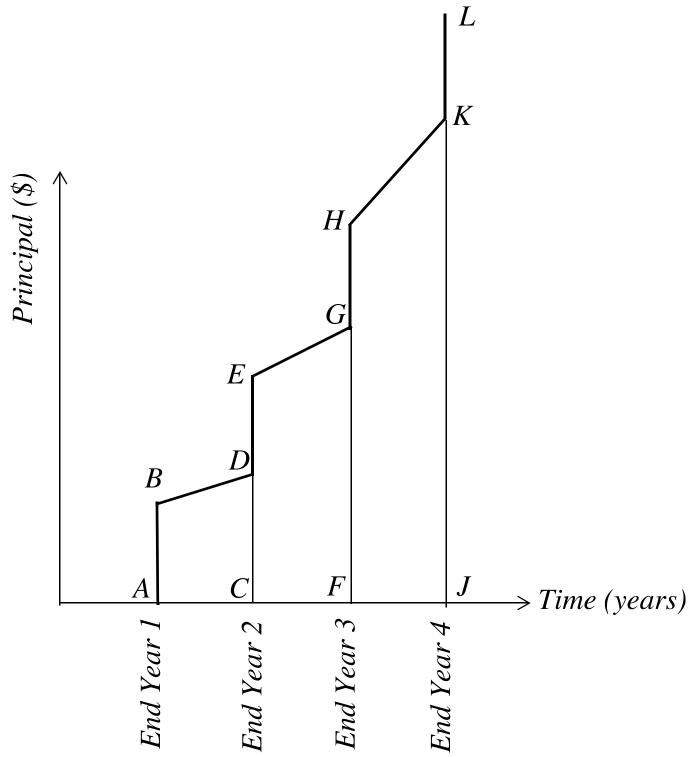


Figure 7.4. Graph for Example 7.16

Table 7.9 Computations for Example 7.16

Year	Principal in fund at beginning of period	Interest earned	Deposit at end of period	Principal in fund at end of period
1	0.00	0.00	500.00	500.00
2	500.00	30.00	500.00	1030.00
3	1030.00	61.80	500.00	1591.80
4	1591.80	95.51	500.00	2187.31

Thus, the principal in the sinking fund at the end of the fourth year is *$2187.31*.

Of course, we could have computed the principal at the end of the fourth year as shown in Table 7.10.

We can derive an equation that computes the principal at the end of a period in a sinking fund as follows:

Let

R = periodic deposit

Table 7.10 Alternate Computations for Example 7.16

Deposit Number	Amount	Number of years in fund	Value at the end of the fourth year		
1	\$500.00	3	$500(1.06)^3$	=	\$595.51
2	500.00	2	$500(1.06)^2$	=	561.80
3	500.00	1	$500(1.06)$	=	530.00
4	500.00	0	500	=	500.00
			Total	=	2187.31

i = interest rate

n = number of deposits made (number of interest periods contained in the term of the fund)

S_n = principal in the sinking fund at the end of the nth period

To apply the method of chronological sequence, we must observe that the principal at the end of any period equals the principal at the end of the preceding period multiplied by the factor $(1+i)$ and augmented by the periodic deposit R. That is,

$$S_r = S_{r-1}(1+i)+R \tag{7.12}$$

where r has any integral value less than n. Hence, we obtain the following relations:

$$\begin{aligned} S_1 &= R \\ S_2 &= R(1+i)+R \\ S_3 &= R(1+i)^2+R(1+i)+R \\ &\dots \\ S_n &= R(1+i)^{n-1}+R(1+i)^{n-2}+\dots+R(1+i)+R \end{aligned}$$

For convenience, we shall reverse the sequence of the terms, thereby obtaining

$$S_n = R+R(1+i)+R(1+i)^2+R(1+i)^{n-2}+R(1+i)^{n-1} \tag{7.13}$$

We recognize the right side of this equation as a geometric series where R is the first term and the ratio of each term to the preceding term is $(1+i)$. In Chapter 2, we derived Equation (2.61) which is repeated here for convenience.

$$s_n = a_1\frac{1-r^n}{1-r} = a_1\frac{r^n-1}{r-1} \tag{7.14}$$

Applying this equation to the sinking-fund principal, but substituting R for a_1 and $(1 + i)$ for r, we obtain

$$S_n = R\frac{(1+i)^n - 1}{(1+i) - 1}$$

or

$$S_n = R\frac{(1+i)^n - 1}{i} \tag{7.15}$$

For the special case where the periodic deposit R is $\$1$, we shall use the notation $s_{\bar{n}}$ to denote the principal in the fund at the end of the nth period. Hence,

$$s_{\bar{n}} = \frac{(1+i)^n - 1}{i} \tag{7.16}$$

For the general case where the periodic deposit R has any value other than $\$1$, the principal S_n can be expressed in terms of $s_{\bar{n}}$ as

$$S_n = Rs_{\bar{n}} \tag{7.17}$$

Equation (7.17) is useful when a math tables book such as CRC *Standard Mathematical Tables* that contains financial tables is available.

Example 7.17

A sinking fund consists of 15 annual deposits of $\$1000$ each, with interest earned at the rate of 4% compounded annually. What is the principal in the fund at its terminal date?

Solution:

For this example, $R = 1000$, $i = 0.04$, and $n = 15$. Using Equation (7.15) we get

$$S_n = R\frac{(1+i)^n - 1}{i} = 1000 \times \frac{(1+0.04)^{15} - 1}{0.04} = 1000 \times 20.05359 = 20,023.59$$

Check with Microsoft Excel: The function =FV(0.04,15,–1000) returns $20,023.59.

In many problems, the principal in the sinking fund at its terminal date is the known quantity, and we must determine the periodic deposit required to accumulate this principal. Solving Equation (7.15) for R we get

$$R = S_n\frac{i}{(1+i)^n - 1} \tag{7.18}$$

Example 7.18

A corporation is establishing a sinking fund for the purpose of accumulating a sufficient capital to retire its outstanding bonds at maturity. The bonds are redeemable in 10 years, and their maturity value is $\$150,000$. How much should be deposited each year if the fund pays interest at the rate of 3%?

Solution:

For this example, $S_n = 150000$, $i = 0.03$, and $n = 10$. Using Equation (7.18) we get

$$R = S_n \frac{i}{(1+i)^n - 1} = 150000 \times \frac{0.03}{(1+0.03)^{10} - 1} = 150000 \times 0.08723 = \$13,084.58$$

Check with Microsoft Excel: The function =PMT(0.03,10,0,–150000) returns $13,084.58.

7.4 Annuities

Annuity: A series of equal payments or receipts occurring over a specified number of periods.

Ordinary annuity: A series of equal payments or receipts occurring over a specified number of periods with the payments or receipts occurring at the end of each period.

Annuity due: A series of equal payments or receipts occurring over a specified number of periods with the payments or receipts occurring at the beginning of each period.

NOTE: While technically correct, the last two definitions shown above can be a bit confusing. Whether a cash flow appears to occur at the end or the beginning of a period often depends on our perspective. For example, the end of year 2 is also the beginning of year 3.

Therefore, the real key to distinguishing between an ordinary annuity and an annuity due is the point at which either a future or present value is to be calculated. Remembering the following characteristics should help us identify the type of annuity that we are dealing with:

1. For an ordinary annuity, future value is calculated as of the last cash flow, while present value is calculated as of one period before the first cash flow.
2. For an annuity due, future value is calculated as of one period after the last cash flow, while present value is calculated as of the first cash flow.

The equations used in the computations of sinking funds apply also to annuities.

Example 7.19

Suppose that on January 1, 2003 our bank account was $\$6000$, and withdrawals of $\$500$ each were made at the end of each year for 4 years. If the account earned 4% annual interest, what would be the balance in the account at the end of 2006 immediately after the last withdrawal is made?

Solution:

The computations are shown in Table 7.11.

Table 7.11 Computations for Example 7.19

Year	Principal at beginning	Interest Earned	End-of-year withdrawal	Principal at end of period
2003	$6000.00	6000.00 × 0.04 = $240.00	$500.00	6000.00 + 240.00 – 500.00 = $5740.00
2004	5740.00	5740.00 × 0.04 = $229.60	$500.00	5740.00 + 229.60 – 500.00 = $5469.60
2005	5469.60	5469.60 × 0.04 = $218.78	$500.00	5469.60 + 218.78 – 500.00 = $5188.38
2006	5188.38	5188.38 × 0.04 = $207.54	$500.00	5188.38 + 207.54 – 500.00 = $4895.92

As shown in the table above, the principal at the end of 2006 would be $4895.92.

In many instances, it is desirable to find the value of these n sums of money at the origin date of the annuity. The equation that will compute the amount at the origin date is derived as follows:

We recall from (7.7) that

$$P_{-n} = P_0(1+i)^{-n} \tag{7.19}$$

and from (7.15) that

$$S_n = R\frac{(1+i)^n - 1}{i} \tag{7.20}$$

As stated earlier, these equations apply to both sinking funds and annuities. For convenience, we will denote (7.19) as

$$PV_{annuity} = FV_{annuity}(1+i)^{-n} \tag{7.21}$$

and (7.20) as

$$FV_{annuity} = Payment\frac{(1+i)^n - 1}{i} \tag{7.22}$$

By substitution of (7.22) into (7.21) we get

$$\begin{aligned} PV_{annuity} &= Payment\frac{(1+i)^n - 1}{i}(1+i)^{-n} \\ &= Payment\frac{1-(1+i)^{-n}}{i} \end{aligned} \tag{7.23}$$

Example 7.20

Suppose that an annuity has been established that will enable us to receive $\$300$ at the end of each year for 5 years starting with December 31, 2003. If the interest rate is 6%, what amount can we receive on January 1, 2003 should we decide to close this annuity?

Solution:

Equation (7.23) is applicable for this example. Thus,

$$PV_{annuity} = 300\frac{1-(1+0.06)^{-5}}{0.06} = 300 \times 4.21236 = 1263.71$$

Example 7.21

A business firm must make a disbursement of $\$800$ at the end of each year from 2004 to 2007 inclusive. In order to meet the obligations as they fall due, it must deposit in a fund at the beginning of 2004 an amount of money that will be just sufficient to provide the necessary payments. If the interest rate is 3% compounded annually, what sum should be deposited?

Solution:

Let x denote the sum to be deposited in the fund on Jan. 1, 2004. The controlling relationship is

$$x = Value\ of\ deposit = Value\ of\ annuity$$

This relationship holds for any valuation date we select; the most convenient date to use is obviously the beginning of 2004, which is both the date of the deposit and the origin date of the annuity. Then,

$$PV_{annuity} = 800\frac{1-(1+0.03)^{-3}}{0.03} = 800 \times 3.7171 = 2973.68$$

Example 7.22

A father wishes to establish a fund for his newborn son's college education. The fund is to pay $\$2,000$ on the 18th, 19th, 20th, and 21st birthdays of the son. The fund will be built up by the deposit of a fixed sum on the son's 1st to 17th birthdays, inclusive. If the fund earns 2%, what should the yearly deposit into the fund be?

Solution:

Let R denote the periodic deposit in the fund. Selecting the end of the 17th year as our valuation date, we have

$$PV_{annuity} = Payment\frac{1-(1+i)^{-n}}{i} = 2000\frac{1-(1+0.02)^{-4}}{0.02} = 2000 \times 3.80773 = \$7615.46$$

That is, at the end of the 17th year, the fund must have accumulated $\$7615.46$.

Now, we must find the annual payments that must be deposited in order to accumulate this amount in 17 years. Solving (7.22) for $Payment$, we get

$$Payment = FV_{annuity}\frac{i}{(1+i)^n - 1} = 7615.46\frac{0.02}{(1+0.02)^{17} - 1} = 7615.46 \times 0.04997 = \$380.54$$

Determining the Interest Rate of an Annuity

In many problems pertaining to annuities that arise in practice, the known quantities are the value of the annuity at either the origin or terminal date and the amount of the periodic payment, while the interest rate by which the given quantities are related is unknown. This is often true, for example, where an asset is purchased on the installment plan. The purchaser knows the purchase price of the asset and the periodic payment he is obligated to make, but he is not directly aware of the interest rate implicit in the loan. Only an approximate solution of this type of problem is possible, and a solution by straight-line interpolation yields results of a sufficient degree of accuracy.

Example 7.23

An asset costing $\$5,000$ was purchased on the installment plan, the terms of sale requiring that the buyer make a down payment of $\$2,000$ and five annual payments of $\$720$ each. The first of these periodic payments is to be made 1 year subsequent to the date of purchase. What is the interest rate pertaining to this loan?

Solution:

From (7.23),

$$PV_{annuity} = Payment\frac{1-(1+i)^{-n}}{i} \tag{7.24}$$

Assuming that $PV_{annuity}$ is the value at the origin date, this value is $5000 - 2000 = 3000$ and the $Payment$ is $\$720$. Substitution of these values into (7.24) and rearranging yields

$$\frac{1-(1+i)^{-n}}{i} = \frac{3000}{720}$$

Obviously, it is a formidable task to solve this equation for the interest rate i. Instead, we construct the following spreadsheet where we observe that the interest rate corresponding to the value at the origin is between 6% and 7%.

i	R	n	Value at origin date
1%	720	5	3494.47
2%	720	5	3393.69
3%	720	5	3297.39
4%	720	5	3205.31
5%	720	5	3117.22
6%	720	5	**3032.90**
7%	720	5	**2952.14**
8%	720	5	2874.75
9%	720	5	2800.55
10%	720	5	2729.37

We will use the graph of Figure 7.5 below to apply straight-line interpolation

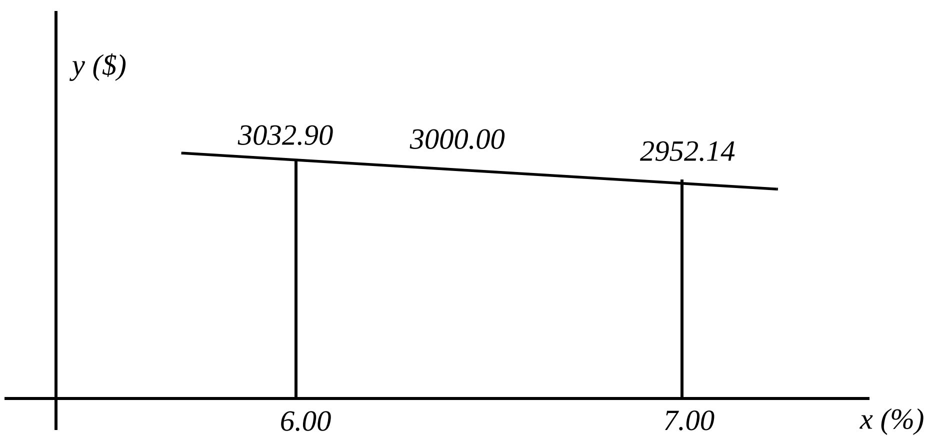


Figure 7.5. Graph for Example 7.23

$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1$$

For our example,

$$x_{y\,=\,3000.00} = \frac{1}{slope} \times (3000.00 - 3032.90) + 0.06 = \frac{1}{\dfrac{2952.14 - 3032.90}{0.07 - 0.06}} \times (-32.90) + 0.06$$

$$= \frac{0.01}{-80.76} \times (-32.90) + 0.06 = 0.06407$$

Therefore, the interest rate is 6.4%.

Check with the Microsoft Excel IRR function: In a blank spreadsheet we enter the values 3000 (5000–2000), –720, –720, –720, –720, and –720 in A1 through A6 respectively, and we use the formula =IRR(A1:A6) and Excel returns 6.402%.

7.5 Amortization

Amortization is the liquidation (a debt, such as a mortgage) by installment payments or payment into a sinking fund. In the course of doing business, we will likely acquire what are known as intangible assets. These assets can contribute to the revenue growth of your business and, as such, they can be expensed against these future revenues. An example of an intangible asset is when you buy a patent for an invention.

Calculating amortization

The formula for calculating the amortization on an intangible asset is similar to the one used for calculating straight-line depreciation*. We divide the initial cost of the intangible asset by the estimated useful life of the intangible asset. The amortization per year is computed with the formula

$$Amortization\ per\ year = \frac{Initial\ cost}{Useful\ life} \tag{7.25}$$

Example 7.24

Suppose that it costs $\$10,000$ to acquire a patent and it has an estimated useful life of ten years. The amortization per year is

$$Amortization\ per\ year = \frac{Initial\ cost}{Useful\ life} = \frac{\$10000}{10} = \$1000$$

The amount of amortization accumulated since the asset was acquired appears on the balance sheet as a deduction under the amortized asset.

The formula of (7.25) assumes that the interest rate is zero. However, when a debt such as a mortgage is amortized over a number of years, there is always an interest rate associated with that debt. In this case, the following formula is used.

$$Payment = Principal \times \frac{Interest}{1-(1+Interest)^{-n}} \tag{7.26}$$

Amortization tables can be easily constructed with spreadsheets such as that shown on Figure 7.6. For this spreadsheet we have assumed an annual interest rate of 7.5% and thus the monthly interest rate is computed and shown in cell C8 as =0.075/12.

* *We will discuss depreciation in Chapter 8*

	A	B	C	D	E	F
1	30 YEAR MORTGAGE LOAN					
2						
3	Inputs					
4		**Terms of Loan**				
5						
6		***Principal***	$500,000			
7		***Term***	360			
8		***Interest (Month)***	0.625%			
9		***Payment***	$3,496.07			
10						
11	Loan Payment Schedule					
12						
13	Payment	Regular Payment	Interest	Principle	Extra	Principal
14	Period	Year 2002	Expense	Payment	Principal Paid	Balance
15						
16	1	$3,496.07	$3,125.00	$371.07	$0.00	$499,628.93
17	2	$3,496.07	$3,122.68	$373.39	$0.00	$499,255.54
18	3	$3,496.07	$3,120.35	$375.73	$0.00	$498,879.81

Figure 7.6. 30-Year Amortization Table

With Microsoft Excel, in all arguments, cash we pay out such as deposits to savings, is represented by negative numbers; cash we receive, such as dividend checks, is represented by positive numbers.

Other formulas in other cells are as follows:

C9: =PMT(C8,C7,-C6)

B16: =C9

C16: =C6*C8

D16: =B16–C16

E16: =IF(AND(F7<=A16,F8>=A16),MIN(F6,C6–D16),0)

F16: =C6–D16–E16

B17: =MIN(C9,F16+C17)

C17: =F16*C8

D17: =B17–C17

E17:=IF(AND(F7<=A17,F8>=A17),MIN(F6,F16–D17),
IF(AND(F10<=A17,F11>=A17),MIN(F9,F16-D17),0))

F17: =F16–D17–E17

The remaining entries in columns B though F are obtained by copying B17:F17 to B18:F549

Example 7.25

A *$10,000* second mortgage is being amortized by means of *20* equal yearly installments at an interest rate of *6%*. The agreement provides for paying off the mortgage in a lump sum at any time with an amount equal to the unpaid balance plus a charge of *1%* of the unpaid balance. What would have to be paid to discharge the mortgage after *10* payments have been made?

Solution:

Each yearly installment is found by

$$Payment = Principal \times \frac{Interest}{1-(1+Interest)^{-n}}$$

For our example,

$$Payment = 10000 \times \frac{0.06}{1-(1+0.06)^{-20}} = 10000 \times \frac{0.06}{1-0.3118} = 871.85$$

To find the unpaid balance at the end of *10* years, we construct the spreadsheet of Figure 7.7.

20 YEAR MORTGAGE LOAN PAID IN 10 YEARS

Inputs

Terms of Loan		Extra Payments	
Principal	$10,000	Extra Amount	$0.00
Term	20	Start Period	$0.00
Interest (Month)	6.00%	End Period	$0.00
Payment	$871.85		

Loan Payment Schedule

Payment Period	Regular Payment	Interest Expense	Principle Payment	Extra Principal Paid	Principal Balance
1	$871.85	$600.00	$271.85	$0.00	$9,728.15
2	$871.85	$583.69	$288.16	$0.00	$9,440.00
3	$871.85	$566.40	$305.45	$0.00	$9,134.55
4	$871.85	$548.07	$323.77	$0.00	$8,810.78
5	$871.85	$528.65	$343.20	$0.00	$8,467.58
6	$871.85	$508.05	$363.79	$0.00	$8,103.79
7	$871.85	$486.23	$385.62	$0.00	$7,718.17
8	$871.85	$463.09	$408.76	$0.00	$7,309.42
9	$871.85	$438.57	$433.28	$0.00	$6,876.14
10	$871.85	$412.57	$459.28	$0.00	$6,416.86

Figure 7.7. Spreadsheet for Example 7.25

We can also use the equation

$$Unpaid\ balance = Payment \times \frac{1 - (1 + i)^{-(n - r)}}{i}$$

where

n = specified period of the loan

r = actual period when the loan is paid in full

For this example,

$$Unpaid\ balance = 871.85 \times \frac{1 - (1 + 0.06)^{-(20 - 10)}}{0.06} = 6486.89$$

7.6 Perpetuities

As stated earlier, a perpetuity is a constant stream of identical cash flows with no maturity date. In other words, a perpetuity is an annuity that continues indefinitely. To provide for a perpetuity, it is necessary to deposit in a fund a sum of money of such magnitude that the interest earned between successive payments exactly equals the amount of the periodic payment, thereby maintaining the original principal permanently intact. Thus,

$$Amount\ deposited = \frac{Earned\ interest}{Interest\ rate} \tag{7.27}$$

Example 7.26

An individual wishes to establish a scholarship for a university of $\$5,000$ annually and that the fund in which the money is to be held earns 4% interest per annum. To earn $\$5,000$ interest annually, the sum to be deposited is

$$Amount\ deposited = \frac{5000}{0.04} = \$125000$$

Of course, this sum must be deposited 1 year prior to the date of the first payment.

In the above example, the payment period coincided with the interest period, but this may not be always the case. In general, let

R = periodic payment of a perpetuity

i = interest rate

m = number of interest periods contained in one payment period

M = principal to be deposited in the fund one payment period prior to the date of the first payment

Equating the periodic payment R to the interest earned by M in one payment period (= m interest periods), we obtain

$$M(1+i)^m - M = R$$

$$M[(1+i)^m - 1] = R$$

or

$$M = \frac{R}{(1+i)^m - 1} \tag{7.28}$$

Example 7.27

What amount must be donated to build an institution having an initial cost of $\$500,000$, provide an annual upkeep of $\$50,000$, and have $\$500,000$ at the end of each 50-year period to rebuild the institution. Assume that invested funds return 4%.

Solution:

The annual upkeep of $\$50,000$ can be regarded as a single disbursement made at the end of each year. The required payments can be grouped in the following manner:

1. An initial payment of $\$500,000$ to construct the building.

2. A perpetuity consisting of payments of $\$500,000$ each made at 50-year intervals.

3. A perpetuity consisting of annual payments of $\$50,000$ each for maintenance.

$$Payments = Initial\ payment + Perpetuity\ at\ 50\text{-}year\ intervals + Perpetuity\ for\ maintenance$$

or

$$M = 500000 + \frac{500000}{(1+0.04)^{50} - 1} + \frac{50000}{0.04} = 500000 + \frac{500000}{6.107} + \frac{50000}{0.04}$$

$$= 500000 + 81873.26 + 1250000 = 1,831,873.26$$

7.7 Valuation of Bonds

Generally, the price at which a bond is purchased is not identical with the face value of the bond. When the purchase price of a bond does coincide with its par value, it is said to be purchased at par; when the purchase price exceeds the par value it is said to be purchased above par, or at a premium; finally, when the purchase price is below the par value, it is said to be purchased below par, or at a discount.

The difference between the par value of a bond and its purchase price is termed the premium or discount, whichever applies. For example, if the face value of a bond is $\$1000$ and it sells for $\$940$, it has been sold at a discount of $\$60.00$; if the same bond sells for $\$1020$, then it has been sold at a premium of $\$20$. We shall hereafter refer to the periodic interest payments received by the bondholder as the *dividends* and to the interval between successive dividends as the *dividend period.*

We will assume that the economic conditions remain unchanged. Accordingly, we will assume that both the price and the prevailing interest rate remain unaltered during the entire period. Therefore, if an investor has purchased a bond at its date of issue at a price to yield a return of 6% on his investment, it follows that at any date prior to the maturity of the bond he can transfer the bond to a new purchaser at such price as to maintain a 6% return on his original investment and to yield the new investor the same rate of return. The value of a bond at any date intermediate between its issue and its redemption is termed its *book value*.

Example 7.28

Assume that a bond of $\$1000$ face value, redeemable at par, and earning interest at the rate of 6% annually is purchased by an investor for $\$980$ at a date 1 year prior to its maturity. What is the rate of return at the end of the year?

Solution:

At the end of the year the investor receives the following sums of money:

$$\begin{aligned} \text{Dividend } 6\% \text{ of } \$1000 &= \$60.00 \\ \text{Redemption price (face value)} &= 1000.00 \\ \text{Total income at end of year} &= 1060.00 \\ \text{Investment at beginning of year} &= \;\;980.00 \\ \text{Earning for year} &= \;\;\;80.00 \\ \text{Rate of return on investment} &= \;\;80/980 = 8.16\% \end{aligned}$$

Calculating the Purchase Price of a Bond

We will now present an example to find out the price we should pay for a bond to yield a stipulated investment rate. We will assume that the date of purchase coincides with the beginning of a dividend period.

Example 7.29

A $\$1000$ bond, redeemable at par in 5 years, pays dividends at the rate of 5% per year. Compute the purchase price of the bond and the book value at the beginning of each dividend period if the bond is purchased to yield a rate of return of

a. 5%

b. 8%

c. 3%

Solution:

a. Since the investment rate equals the dividend rate, the purchase price of the bond at its date of issue equals its par value of $\$1000$. At date of issue the bond is worth $\$1000$ and during the first year, the bond earns interest at the rate of 5% and therefore attains the value of $\$1050$ at the end of that date. At this date, however, a dividend of $\$50$ is received by the bondholder restoring the bond to its original value of $\$1000$. This process is repeated every period. The result is that at the beginning of each dividend period the value of the bond has reverted to its par value.

b. The sums of money which the bondholder is to receive are the following:

1. The series of five annual dividends, each of which is 5% of the face value, or $\$50$. These dividends constitute an annuity whose value at date of issue of the bond, based on a rate of 8%, is given by Equation (7.24) as

$$PV_{annuity} = Payment\frac{1-(1+i)^{-n}}{i} = 50 \times \frac{1-(1+0.08)^{-5}}{0.08} = 50 \times 3.99271 = 199.64$$

2. The redemption value of the bond, $\$1000$, which will be received 5 years after the date of purchase. The value of this sum at date of purchase is by Equation (7.19)

$$P_{-n} = P_0(1+i)^{-n} = 1000(1+0.08)^{-5} = 1000 \times 0.68058 = 680.58$$

Then, the purchase price of the bond is

$$199.64 + 680.58 = 880.22$$

Hence, the bond is purchased at a discount of

$$1000.00 - 880.22 = 119.78$$

The book value of the bond at the commencement of each subsequent year is computed by repeating the above steps. The results are shown in Table 7.12.

c. We will follow the same method of solution as in Part (b) with all computations based on an interest rate of 3%. The computations are shown Table 7.13.

The first line in Table 7.13 indicates that the purchase price is

$$Purchase\ price = 50 \times \frac{1-(1+0.03)^{-5}}{0.08} + 1000(1+0.03)^{-5} = 228.99 + 862.61 = \$1091.60$$

The bond is, therefore, purchased at a premium of $1091.60 - 1000.00 = \$91.60$.

Table 7.12 Computations for Part (b), Example 7.29

Beginning of year	Value of future dividends	Redemption price	Book value of bond
1	$50 \times \frac{1-(1.08)^{-5}}{0.08} = \$199.64$	$1000(1.08)^{-5} = \$680.58$	$199.64 + 680.58 = \$880.22$
2	$50 \times \frac{1-(1.08)^{-4}}{0.08} = 165.61$	$1000(1.08)^{-4} = 735.03$	$165.61 + 735.03 = 900.64$
3	$50 \times \frac{1-(1.08)^{-3}}{0.08} = 128.85$	$1000(1.08)^{-3} = 793.83$	$128.85 + 793.83 = 922.68$
4	$50 \times \frac{1-(1.08)^{-2}}{0.08} = 89.16$	$1000(1.08)^{-2} = 857.34$	$89.16 + 857.34 = 946.50$
5	$50 \times \frac{1-(1.08)^{-1}}{0.08} = 46.30$	$1000(1.08)^{-1} = 925.93$	$46.30 + 925.93 = 972.23$
End of 5th year	0.00	1000.00	1000.00

Table 7.13 Computations for Part (c), Example 7.29

Beginning of year	Value of future dividends	Redemption price	Book value of bond
1	$50 \times \frac{1-(1.03)^{-5}}{0.03} = \$228.99$	$1000(1.03)^{-5} = \$862.61$	$228.99 + 862.61 = \$1091.60$
2	$50 \times \frac{1-(1.03)^{-4}}{0.03} = 185.85$	$1000(1.03)^{-4} = 888.49$	$185.85 + 888.49 = 1074.34$
3	$50 \times \frac{1-(1.03)^{-3}}{0.03} = 141.43$	$1000(1.03)^{-3} = 915.14$	$141.43 + 915.14 = 1056.57$
4	$50 \times \frac{1-(1.03)^{-2}}{0.03} = 95.67$	$1000(1.03)^{-2} = 942.60$	$95.67 + 942.60 = 1038.27$
5	$50 \times \frac{1-(1.03)^{-1}}{0.03} = 48.54$	$1000(1.03)^{-1} = 970.87$	$48.54 + 970.87 = 1019.41$
End of 5th year	0.00	1000.00	1000.00

Total Periodic Bond Disbursement

To accumulate the funds necessary to retire an issue of bonds at its maturity, the seller must make periodic deposits in a sinking fund. Let us assume that the deposit period of the fund coinciding with the dividend period of the bonds. Also let

F = total face value of the bonds

M = total redemption value of the bonds

d = dividend rate of the bonds

i = interest rate of sinking fund

The total disbursement pertaining to the bonds which the seller must make at the end of each period consists of the following:

1. The periodic dividend D payable to the bondholders. This amount is computed as

$$D = Fd \tag{7.29}$$

2. The periodic contribution R to the sinking fund

$$R = M\frac{i}{(1+i)^n - 1} \tag{7.30}$$

Therefore, the total periodic disbursements (annual costs) are

$$Periodic\ disbursements = D + R = Fd + M\frac{i}{(1+i)^n - 1} \tag{7.31}$$

Example 7.30

A community wishes to purchase an existing utility valued at $\$5,000,000$ by selling 5% bonds that will mature in 30 years. The money to retire the bonds will be raised by paying equal annual amounts into a sinking fund that will earn 4% per cent. The bonds will be sold at par. What will be the total annual cost of the bonds until they mature?

Solution:

For this example $F = M = 5,000,000$, $d = 0.05$, $i = 0.04$, and $n = 30\ years$. By substitution into (7.31) we get

$$Annual\ disbursements = 5,000,000 \times 0.05 + 5,000,000 \times \frac{0.04}{(1+0.04)^{30} - 1}$$
$$= 250,000 + 89,150.50 = 339,150.50$$

Example 7.31

A municipality wishes to raise funds for improvements by issuing 5.5% bonds. There is $\$200,000$ available per year for interest payments and retirement of the bonds at 10% above face value. The interest rate of the sinking fund is 3%. What should be the amount of the bond issue if all the bonds are to be retired in 20 years?

Solution:

For this example $M = 1.10F$, $d = 0.055$, $i = 0.03$, and $n = 20\ years$. By substitution into (7.31) we get

$$\begin{aligned} Annual\ disbursements = 200,000 &= 0.055F + 1.10F\frac{0.03}{(1+0.03)^{20} - 1} \\ &= 0.055F + 1.10F(0.03722) \end{aligned}$$

or

$$\begin{aligned} 0.055F + 0.041F &= 200000 \\ 0.096F &= 200000 \end{aligned}$$

and solving for F we get $F = \$2,083,333.33$.

Calculation of Interest Rate of Bond

In the previous examples we calculated the purchase price of a bond corresponding to a particular interest rate. In many cases, however, the purchase price of a bond is the known quantity, and it is necessary to determine the investment rate secured by the purchaser. A calculation of this nature is also required to enable the issuing corporation to determine the true rate of interest it is paying for borrowed money. The basis for calculating this interest rate is not the actual selling price of the bonds but rather the sum of money remaining after the payment of administrative and legal expenses.

Assuming there are no bond tables available for obtaining the interest rate corresponding to a given purchase price, we shall calculate the approximate rate using straight line approximation.

Example 7.32

Williams paid $\$11,000$ for a $\$10,000$ bond that pays $\$400$ per year. In 20 years the bond will be redeemed for $\$10,500$. What net rate of interest will Williams get on his investment?

Solution:

Let i denote the interest rate. Evaluating all sums at the maturity date, we get

$$11000(1+i)^{20} = 400\frac{(1+i)^{20} - 1}{i} + 10500$$

To obtain a first approximation, we substitute $1 + ni$ for the expression $(1 + i)^n$. Then

$$11000(1 + 20i) = 400 + 10500 = 400 \times 20 + 10500$$

Solving for i we get as a first approximation

$$220000i = 18500 - 11000$$

$$i = \frac{7500}{220000} = 0.0341 = 3.41\%$$

Since we have understated the value both of the $\$11,000$ investment and of the periodic dividends, it is reasonable to presume that this value does not deviate markedly from the true value. Therefore, let us compute at 3.0% and at 3.5% and perform straight line approximation as before.

At 3.0% we have

$$11000(1.03)^{20} = 11000 \times 1.80611 = 19,867.20$$

$$400\frac{(1 + 0.03)^{20} - 1}{0.03} = 400 \times 26.87037 = 10,748.20$$

$$Redemption\ price\ for\ 3.0\%\ rate = 19,867.20 - 10,748.20 = \$9,119.00$$

At 3.5% we have

$$11000(1.035)^{20} = 11000 \times 1.98979 = 21,887.70$$

$$400\frac{(1 + 0.035)^{20} - 1}{0.035} = 400 \times 28.27968 = 11,319.90$$

$$Redemption\ price\ for\ 3.5\%\ rate = 21,887.70 - 11,319.90 = \$10,578.80$$

Applying straight-line interpolation, we obtain

$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1$$

For our example,

$$x_{y\,=\,10,500} = \frac{1}{slope} \times (10,500.00 - 9,119.00) + 0.03 = \frac{1}{\dfrac{10,578.80 - 9,119.00}{0.035 - 0.030}} \times 1,381.00 + 0.03$$

$$= \frac{0.005}{1,456.80} \times 1,381.00 + 0.03 = 0.03474$$

or 3.47% approximately.

7.8 Spreadsheet Financial Functions

Spreadsheets such as Microsoft Excel, Lotus 1-2-3, and Quattro Pro have built-in financial functions that we can use to obtain quick answers. To appreciate the usefulness of these functions, we present the following example which we will solve by classical methods.

Microsoft Excel solves for one financial argument in terms of the others. If the argument rate is not zero, then,

$$pv \times (1 + rate)^{nper} + pmt(1 + rate \times type) \times \left(\frac{(1 + rate)^{nper} - 1}{rate}\right) + fv = 0 \tag{7.32}$$

$$provided\ that\ rate \neq 0$$

If $rate = 0$, the following formula is used.

$$(pmt \times nper) + pv + fv = 0 \tag{7.33}$$

In equations (7.32) and (7.33) the arguments are defined as follows:

pv = present value or the lump sum amount that a series of future payments is worth at the present.

rate = the interest rate per period. For example, if we obtain an automobile loan at a 10% annual interest rate and make monthly payments, our interest rate per month is $10\%/12$, or 0.83%. Accordingly, we must enter $10\%/12$, or 0.83%, or 0.0083, into the formula as the rate.

nper = the total number of payment periods in an annuity. For example, if we get a four-year car loan and make monthly payments, our loan has $4 \times 12 = 48$ periods. We must enter 48 into the formula for nper.

pmt = the payment made each period and cannot change over the life of the annuity. Typically, pmt includes principal and interest but no other fees or taxes. For example, the monthly payments on a $\$10,000$, 4-year car loan at 12% are $\$263.33$. We must enter -263.33 into the formula as the pmt.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period.

fv = the future value, or a cash balance we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (the future value of a loan, for example, is 0). For example, if we want to save $\$50,000$ to pay for a special project in 18 years, then $\$50,000$ is the future value. We could then make a conservative guess at an interest rate and determine how much we must save each month.

The Microsoft Excel financial functions are listed below.

PV Function

The PV function returns the present value of an investment. The present value is the total amount that a series of future payments is worth now. For example, when we borrow money, the loan amount is the present value to the lender.

This function also computes the present value of an ordinary annuity. An *ordinary annuity* is a series of payments made at equally spaced intervals.

The present value pv of an annuity allows us to compare different investment alternatives or potential obligations.

The formula for computing the present value of an ordinary annuity is

$$pv = Payment \times \frac{1-(1+interest)^{-n}}{interest} \tag{7.34}$$

The syntax for Excel is

=PV(rate,nper,pmt,fv,type) where*

rate = the interest rate per period. For example, if we obtain an automobile loan at a 10% annual interest rate and make monthly payments, our interest rate per month is $10\%/12$, or 0.83%. Accordingly, we must enter $10\%/12$, or 0.83%, or 0.0083, into the formula as the rate.

nper = the total number of payment periods in an annuity. For example, if we get a four-year car loan and make monthly payments, our loan has $4 \times 12 = 48$ periods. We must enter 48 into the formula for nper.

pmt = the payment made each period and cannot change over the life of the annuity. Typically, pmt includes principal and interest but no other fees or taxes. For example, the monthly payments on a $\$10,000$, 4-year car loan at 12% are $\$263.33$. We must enter -263.33 into the formula as the pmt. If pmt is omitted, we must include the fv argument.

fv = the future value, or a cash balance we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (the future value of a loan, for example, is 0). For example, if we want to save $\$50,000$ to pay for a special project in 18 years, then $\$50,000$ is the future value. We could then make a conservative guess at an interest rate and determine how much we must save each month. If fv is omitted, we must include the pmt argument.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

* *These terms were defined on the previous page but ar repeated here for convenience.*

Remarks:

1. We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.
2. As stated earlier, for all the arguments, cash we pay out, such as deposits to savings, is represented by negative numbers; cash we receive, such as dividend checks, is represented by positive numbers.

Example 7.33

We wish to receive $\$12,000$ at the end of each year for the next 20 years. If the interest rate is 7%, what amount must we deposit now to achieve this goal?

Solution:

We use the Excel PV function =PV(0.07,20,12000,0,0) and this returns $-\$127, 128.17$. This result is negative because it represents the amount that we must pay. In other words, this is an outgoing cash flow.

FV Function

The FV function computes the future value of an ordinary annuity. An ordinary annuity is a series of payments made at equally spaced intervals.

The future value of an annuity allows us to compare different investment alternatives or potential obligations.

The formula for computing the future value of an ordinary annuity is

$$FV = Payment \times \frac{(1 + interest)^n - 1}{interest} \tag{7.35}$$

The syntax for Excel is

=FV(rate,nper,pmt,pv,type) where

rate = interest rate per period

nper = the total number of payment periods in an annuity

pmt = payment made each period; it cannot change over the life of the annuity. Typically, pmt contains principle and interest but no other fees or taxes. If pmt is omitted, we must include the pv argument.

pv = present value or the lump sum amount that a series of future payments is worth at the present. If pv is omitted, it is assumed to be 0 (zero) and we must include the pmt argument.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

Remarks:

1. The payment returned by PMT includes principal and interest but no taxes, reserve payments, or fees sometimes associated with loans.
2. We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.

Example 7.34

Suppose we deposit $\$2,000$ each year for the next 20 years at 7.5% interest rate and we make each year's contribution on the last day of the year. We want to find out the amount accumulated at the end of the 20-year period.

Solution:

We use the Excel FV function =FV(0.075,20,–2000,0,0) and this returns $\$86,609.36$. This is the amount that we have accumulated over the 20-year period.

PMT Function

The PMT function calculates the payment for a loan based on constant payments and a constant interest rate.

The formula for computing the payment is

$$PMT = Principal \times \frac{interest}{1 - (1 + interest)^{-n}} \tag{7.36}$$

The syntax for Excel is

=PMT(rate,nper,pv,fv,type) where

rate = interest rate for the loan per period

nper = the total number of payments for the loan

pv = present value or the total amount that a series of future payments is worth at the present. It is also referred to as the principal.

fv = future value or the cash balance that we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (zero), that is, the future value of a loan is zero.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

Remarks:

1. The payment returned by PMT includes principal and interest but no taxes, reserve payments, or fees sometimes associated with loans.
2. We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.

Example 7.35

Suppose we buy an automobile and we finance a loan of $\$15,000$ for the next 5 years at 8% interest rate. We want to find out the monthly payment for this loan.

Solution:

We use the Excel PMT function =PMT(0.08/12,5*12,15000,0,0) and this returns -304.15. This represents our monthly payment. This amount is shown as a negative number since it represents an outgoing cash flow. The interest rate is divided by 12 to obtain the monthly payment, and the years that the money is paid out is multiplied by 12 to obtain the number of payments.

RATE Function

The RATE function returns the interest rate per period of an annuity. We can use RATE to determine the yield of a zero-coupon bond that is sold at a discount of its face value. This function is also useful in forecasting applications to calculate the compound growth rate between current and projected revenues and earnings.

RATE is calculated by iteration and can have zero or more solutions. If the successive results of RATE do not converge to within 0.0000001 after 20 iterations, RATE returns the #NUM! error value.

The syntax for Excel is

=RATE(nper,pmt,pv,fv,type,guess) where

nper = the total number of payment periods in an annuity.

pmt = the payment made each period and cannot change over the life of the annuity. Typically, pmt includes principal and interest but no other fees or taxes. If pmt is omitted, we must include the fv argument.

pv = present value or the total amount that a series of future payments is worth at the present.

fv = future value or the cash balance that we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (zero), that is, the future value of a loan is zero.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

guess = our guess for what the rate will be. If we omit guess, it is assumed to be 10%. If RATE does not converge, we can try different values for guess. RATE usually converges if guess is between 0 and 1.

Remark:

We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.

Example 7.36

Suppose that for $\$3,500$ we can purchase a zero-coupon bond with $\$10,000$ face value maturing in 10 years. We want to determine the annual interest rate that this bond will yield.

Solution:

We use the Excel RATE function =RATE(10,0,–3500,10000,0,0.1) and this returns 11.07%. This represents the annual interest rate that this bond will yield.

NPER* Function

The NPER function returns the number of periods for an investment based on periodic, constant payments, and a constant interest rate.

The syntax for Excel is

=NPER(rate,pmt,pv,fv,type) where

rate = the interest rate per period.

pmt = the payment made each period and cannot change over the life of the annuity. Typically, pmt includes principal and interest but no other fees or taxes.

pv = present value or the total amount that a series of future payments is worth at the present.

* *We discussed this function in Chapter 2 also.*

fv = future value or the cash balance that we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (zero), that is, the future value of a loan is zero.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

Example 7.37

Suppose that we deposit *$2,000* at the end of each year into a savings account. This account earns *7.5%* per year compounded annually. How long will it take to accumulate *$100,000*?

Solution:

We use the Excel NPER function =NPER(0.075,-2000,0,100000) and this returns *21.54* years. This represents the time it will take to accumulate *$100,000*.

NPV Function

The NPV function calculates the net present value of an investment by using a discount rate and a series of future payments (negative values) and income (positive values).

The syntax for Excel is

=NPV(rate,value1,value2,...) where

rate = the rate of discount over the length of one period.

value1, value2,... = *1* to *29* arguments representing the payments and income and must be equally spaced in time and occur at the end of each period.

Remarks:

1. NPV uses the order of value1, value2,... to interpret the order of cash flows. We must enter our payment and income values in the correct sequence.
2. Arguments that are numbers, empty cells, logical values, or text representations of numbers are counted; arguments that are error values or text that cannot be translated into numbers are ignored.
3. If an argument is an array or reference, only numbers in that array or reference are counted. Empty cells, logical values, text, or error values in the array or reference are ignored.
4. The NPV investment begins one period before the date of the valued cash flow and ends with the last cash flow in the list. The NPV calculation is based on future cash flows. If our first cash flow occurs at the beginning of the first period, the first value must be added to the NPV result, not included in the values arguments. For more information, see the examples below.
5. The formula for NPV is

$$NPV = \sum_{i=1}^{n} \frac{values_i}{(1+rate)^i} \tag{7.37}$$

6. NPV is similar to the PV function (present value). The primary difference between PV and NPV is that PV allows cash flows to begin either at the end or at the beginning of the period whereas in NPV the arguments value1, value2,... must occur at the end of each period as stated above. Also, unlike the variable NPV cash flow values, PV cash flows must be constant throughout the investment. For information about annuities and financial functions, see PV.

7. NPV is also related to the IRR function (Internal Rate of Return) which is discussed below. IRR is the rate for which NPV equals zero, that is,

$$NPV(IRR(\ldots), \ldots) = 0 \tag{7.38}$$

Example 7.38

Suppose that in a year we make 12 irregular distributions shown as value1, value2,...,value12 on the spreadsheet of Figure 7.8 at 11.5% annual interest rate, and we want to compute the net present value of those distributions.

	A	B
1	**Distributions**	**Amount**
2	value1	0
3	value2	0
4	value3	2500
5	value4	2500
6	value5	3000
7	value6	5000
8	value7	6000
9	value8	9000
10	value9	3000
11	value10	2500
12	value11	0
13	value12	7500
14	Annual Interest rate	11.5%
15	**=NPV(B15/12,B2:B13)**	**$38,084.13**

Figure 7.8. Spreadsheet for Example 7.38

Solution:

We use the formula =NPV(B15/12,B2:B13) which returns $\$38,084.13$. This value represents the net present value of these distributions. As expected, this amount is less than the sum of $\$41,000$ of the 12 monthly distributions.

IRR Function

The IIR function returns the internal rate of return for a series of cash flows represented by the numbers in values. These cash flows need not be even as they would be for an annuity. However, the cash flows must occur at regular intervals, such as monthly or annually. The internal rate of return is the interest rate received for an investment consisting of payments (negative values) and income (positive values) that occur at regular periods.

The syntax for Excel is

=IRR(values,guess) where

values = an array or a reference to cells that contain numbers for which we want to calculate the internal rate of return. This argument must contain at least one positive value and one negative value to calculate the internal rate of return. IRR uses the order of values to interpret the order of cash flows. We must enter our payment and income values in the sequence we want. Normally, the first cash flow value in the range is a negative number that represents an outgoing cash flow. If an array or reference argument contains text, logical values, or empty cells, those values are ignored.

guess = a number that we guess is close to the result of IRR.

Excel uses an iterative method for calculating IRR. Starting with guess, IRR cycles through the calculation until the result is accurate within *0.00001%*. If IRR cannot find a result that works after *20* tries, the #NUM! error value is returned.

In most cases we need not provide guess for the IRR calculation. If guess is omitted, it is assumed to be *0.1* (*10%*).

If IRR gives the #NUM! error value, or if the result is not close to what we expected, we should try again with a different value for guess.

Example 7.39

Suppose that we make an initial investment of *$10,000* and we receive *12* unequal payments as shown in Column A of the spreadsheet of Figure 7.9. What is the internal rate of return?

Solution:

In Cell C2 of the spreadsheet we enter =IRR(A2:A14) and this returns *10.10%*.

Remark:

IRR is closely related to NPV, the net present value function. The rate of return calculated by IRR is the interest rate corresponding to a 0 (zero) net present value. The following formula illustrates how NPV and IRR are related:

NPV(IRR(A2:A14),A2:A14) returns 9.29×10^{-13}.

	A	B	C
1	**Cash Flows**		**IRR**
2	-10000		10.10%
3	1800		
4	1500		
5	1250		
6	1400		
7	1375		
8	1280		
9	1625		
10	1560		
11	1230		
12	1425		
13	1580		
14	1620		

Figure 7.9. Spreadsheet for Example 7.39

Example 7.40

Suppose we started our business with an initial investment of *$60,000* and our net income for the first five years is as shown on the spreadsheet of Figure 7.10. We want to compute:

a. the internal rate of return after *4* years

b. the internal rate of return after *5* years

c. the internal rate of return after *2* years

	A	B
1	**Description**	**Cash Flows**
2	Initial Cost of Business	-60000
3	Net Income for 1st year	10000
4	Net Income for 2nd year	13000
5	Net Income for 3rd year	15000
6	Net Income for 4th year	18000
7	Net Income for 5th year	23000
8		
9	**Formula**	**Result**
10	=IRR(B2:B6)	-2%
11	=IRR(B2:B7)	9%
12	=IRR(B2:B4)	#NUM!
13	=IRR(B2:B4, -0.10)	-44%

Figure 7.10. Spreadsheet for Example 7.40

Solution:

The formulas that we used for this example are shown in Cells A10:A13. The internal rate of return for the first 4, first 5, and first 2 years are shown in Cells B10, B11, and B13. We observe that for the last computation it was necessary to enter a guess of -10%.

MIRR Function

The MIRR function returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.

The syntax for Excel is

=MIRR(values,finance_rate,reinvest_rate) where

values = an array or a reference to cells that contain numbers for which we want to calculate the internal rate of return. This argument must contain at least one positive value and one negative value to calculate the internal rate of return; otherwise, MIRR returns the #DIV/0! error value. MIRR uses the order of values to interpret the order of cash flows. We must enter our payment and income values in the sequence we want. Normally, the first cash flow value in the range is a negative number that represents an outgoing cash flow. If an array or reference argument contains text, logical values, or empty cells, those values are ignored; however, cells with zero value are included.

finance_rate = the interest rate that we pay on the money used in the cash flows.

reinvest_rate = the interest rate that we receive on the cash flows as we reinvest them.

MIRR uses the following formula:

$$MIRR = \left(\frac{-NPV(reinvest\_rate, values[positive]) \times (1 + reinvest\_rate)^n}{NPV(finance\_rate, values[negative]) \times (1 + finance\_rate)}\right)^{\frac{1}{n-1}} - 1 \qquad (7.39)$$

where n is the number of cash flows in values.

Example 7.41

Suppose we started our business with a loan of $\$120,000$ and our net income for the first five years including interest, is as shown on the spreadsheet of Figure 7.11. We want to compute:

a. the modified rate of return after 5 years based on a reinvest rate of 12%

b. the modified rate of return after 3 years based on a reinvest rate of 12%

c. the 5-year modified rate of return based on a reinvest rate of 14% instead of 12%

	A	B
1	**Description**	**Cash Flows & Interest**
2	Initial Cost of Business (Loan)	-120000
3	Return for 1st year	39000
4	Return for 2nd year	30000
5	Return for 3rd year	21000
6	Return for 4th year	37000
7	Return for 5th year	46000
8	Annual interest rate for the $120,000 loan	10%
9	Annual interest rate for the reinvested profits	12%
10		
11	**Formula**	**Result**
12	=MIRR(B2:B7,B8,B9)	13%
13	=MIRR(B2:B5,B8,B9)	-5%
14	=MIRR(B2:B7,B8,0.14)	13%

Figure 7.11. Spreadsheet for Example 7.41

Solution:

The formulas that we used for this example are shown in Cells A12:A14. The modified rate of return after *5* years and *3* years based on a reinvest rate of *12%* is shown on B12 and B13 respectively. The modified rate of return after *5* years based on a reinvest rate of *14%* is shown on B14.

IPMT Function

The IPMT function returns the interest payment for a given period for an investment based on periodic, constant payments, and a constant interest rate.

The syntax for Excel is

=IPMT(rate,per,nper,pv,fv,type) where

rate = interest rate per period

per = the period for which we want to find the interest. It must be in the range 1 to nper.

nper = the total number of payment periods

pv = present value or the total amount that a series of future payments is worth at the present.

fv = future value or the cash balance that we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (zero), that is, the future value of a loan is zero.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

Remarks:

1. For all arguments, cash that we pay out, such as deposits to savings, is represented by negative numbers. Cash we receive, such as dividends, is represented by positive numbers.
2. We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.

Example 7.42

We took out an *$8,000* loan for 3 years at an annual interest rate of 10%. We want to compute:

a. the interest due in the first month of the first year

b. the interest due for the last year of the loan assuming that the payments are made annually.

Solution:

a. We use the formula =IPMT(0.1/12,1,3,8000) where the interest rate is divided by 12 to obtain the monthly rate. This formula returns *–$66.67* and this amount represents the interest paid in the first month of the first period.

b. We use the formula =IPMT(0.1,3,3,8000) where the interest rate is not divided by 12 since we are interested in the annual interest payment for the last (third) period. This formula returns *$292.45* and this amount represents the interest paid for the entire third year.

PPMT Function

The PPMT function returns the payment on the principal for a given period for an investment based on periodic, constant payments, and a constant interest rate.

The syntax for Excel is

=PPMT(rate,per,nper,pv,fv,type) where

rate = interest rate per period

per = specifies the period. It must be in the range 1 to nper.

nper = the total number of payment periods

pv = present value or the total amount that a series of future payments is worth at the present.

fv = future value or the cash balance that we want to attain after the last payment is made. If fv is omitted, it is assumed to be 0 (zero), that is, the future value of a loan is zero.

type = the number 0 or 1 which indicates when payments are due. If 0, the payments are due at the end of the period. If 1, the payments are due at the beginning of the period. If omitted, it is assumed to be 0.

Remark:

We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at *11%* annual interest rate, we must use *11%/12* for rate and 4×12 for nper. If we make annual payments on the same loan, we must use *11%* for rate and *4* for nper.

Example 7.43

We took out an *$6,000* loan for *5* years at an annual interest rate of *11%*. We want to compute the principal payment for the first month of the first year.

Solution:

We use the formula =PPMT(0.11/12,1,5*12,6000) where the interest rate is divided by *12* to obtain the monthly interest rate, and the number of years is multiplied by *12* to obtain the total number of monthly payments. The formula above returns *–$75.45* and this amount represents the interest paid for the first month of the first year.

Example 7.44

We took out an *$300,000* loan for *10* years at an annual interest rate of *9%*. We want to compute the principal payment for the last year of the *10*-year period.

Solution:

We use the formula =PPMT(0.09,10,10,300000). This formula above returns *–$42,886* and this amount represents the principal payment for the last year of the *10*-year period.

ISPMT Function

The ISPMT function calculates the interest paid during a specific period of an investment.

The syntax for Excel is

=ISPMT(rate,per,nper,pv) where

rate = interest rate for the investment

per = specifies the period for which we want to find the interest and must be in the range 1 to nper.

nper = the total number of payment periods

pv = present value of the investment. For a loan, pv is the loan amount.

Remarks:

1. For all arguments, cash that we pay out, such as deposits to savings, is represented by negative numbers. Cash we receive, such as dividends, is represented by positive numbers.
2. We must make sure that we are consistent with the units that we use for specifying rate and nper. For example, if we make monthly payments on a four-year loan at 11% annual interest rate, we must use $11\%/12$ for rate and 4×12 for nper. If we make annual payments on the same loan, we must use 11% for rate and 4 for nper.

Example 7.45

We have decided to take out a *$6,000,000* loan for 8 years at an annual interest rate of 8.5%. We want to compute

a. the interest that we must pay the first month of the first year

b. the interest that we must pay for first year of the 8-year period

Solution:

a. We use the formula =ISPMT(0.085/12,1,8*12,6000000) where the interest rate is divided by 12 to obtain the monthly interest rate, and the number of years is multiplied by 12 to obtain the total number of monthly payments. The formula above returns *–$42,057.29* and this amount represents the interest that we must pay the first month of the first year.

b. We use the formula =ISPMT(0.085,1,8,6000000). The formula above returns *–$446,250.00* and this amount represents the interest that we must pay the entire first year.

7.9 The MATLAB Financial Toolbox

The MATLAB Student Version, Release 13, includes the Financial Toolbox that contains several cash-flow functions to compute interest rates, rates of return, present or future values, depreciation, and annuities. These are very similar to the financial functions provided by Microsoft Excel. It also includes the Financial Derivatives Toolbox and the Financial Time Series Toolbox. However, in this section, we will restrict our discussion on the most common functions of the Financial Toolbox.

Like the Microsoft Excel functions, with all of the MATLAB financial functions investments are considered negative cash flows, and return payments are considered positive cash flows.

irr function

The **irr** function computes the internal rate of return of periodic cash flows. Its syntax is

r=irr(cf) where **cf** is a cash flow vector.

The first entry in **cf** is the initial investment and it is entered as a negative number. If **cf** is entered as a matrix, each column is treated as a separate cash flow.

Example 7.46

Suppose that an initial investment of *$150,000* is made and the annual income for this investment is as shown in Table 7.14.

Table 7.14 Income for Example 7.46

Year	Income
1	15,000
2	25,000
3	35,000
4	50,000
5	60,000

We can compute the internal rate of return with the MATLAB statement

internal_rate=irr([–150000 15000 25000 35000 45000 60000]) and this returns

```
internal_rate =

          0.05
```

effrr function

The **effrr** function computes the effective rate of return given an annual interest rate, also known as Annual Percentage Rate (APR), and number of compounding periods per year. Its syntax is

r=effrr(apr,per) where

apr is the annual interest rate and **per** is the number of compounding periods per year.

Example 7.47

Suppose that the annual percentage interest rate is *7%* and it is compounded monthly. We can compute the effective annual rate with the MATLAB statement

effective_rate=effrr(0.07,12) and this returns

```
effective_rate =

    0.0723
```

That is, the effective rate is *7.23%*.

pvfix function

The **pvfix** function computes the present value of cash flows at regular time intervals with equal or unequal payments. Its syntax is

p=pvfix(rate,nper,p,fv,due) where

rate is the periodic interest rate, **nper** is the number of periods, **p** is the periodic payment, **fv** is a payment received other than **p** in the last period, and **due** specifies whether the payments are made at the beginning (**due=1**) or end (**due=0**) of the period. The default arguments are **fv=0** and **due=0**.

Example 7.48

A *$400* payment is made monthly into a savings account earning *4%*. The payments are made at the end of the month for *6* years. To compute the present value of these payments in a fixed format with two decimal places, we use the statements

format bank

present_value_equal_fixed_payments=pvfix(0.04/12,6*12,400,0,0) and these return

```
present_value_equal_fixed_payments =
       25566.97
```

Example 7.49

Payments of *$300*, *$325*, *$350*, *$375*, *$400*, *$425*, *$450*, and *$475* are made semiannually into a savings account earning *5%* per year. The payments are made at the end of June and at the end of December for *4* years. To compute the present value of these payments in a fixed format with two decimal places, we use the statements

format bank

payments=[300 325 350 375 400 425 450 475];

present_value_unequal_fixed_payments=pvfix(0.05/2,4*2,payments,0,0) and these return

```
present_value_unequal_fixed_pay =
  2151.04  2330.29  2509.55  2688.80  2868.05  3047.31  3226.56  3405.82
```

These values represent the present value at the end of each semiannual period.

pvvar function

The **pvvar** function computes the present value of cash flows at irregular time intervals with equal or unequal payments. Its syntax is

pv=pvvar(cf,rate,df)

where **cf** is a vector representing the cash flows, **rate** is the periodic interest rate, and **df** represents the dates that the cash flows occur. If **df** omitted, it is understood that the cash flows occur at the end of the period.

Example 7.50

Suppose we make an initial investment of *$15,000* at an annual interest rate of *7%* and the yearly income realized by this investment is as shown in Table 7.15.

Table 7.15 Cash flows for Example 7.50

Year	*Return*
1	$3,500
2	4,000
3	3,750
4	4,200
5	4,100

To compute the net present value of the periodic cash flows, we use the following statements:

```
format bank
```

```
CF=[–15000  3500  4000  3750  4200  4100]; present_value=pvvar(CF,0.07)
```

and these return

```
present_value =
        953.30
```

Example 7.51

An investment of *$15,000* returns the cash flows shown in Table 7.16 where the first cash flow value is shown as the first cash flow value. The annual interest rate is *8%*.

Table 7.16 Cash flows for Example 7.51

Cash Flow	*Date*
$–15,000	January 23, 2001
3,500	February 27, 2002
4,000	March 18, 2002
3,750	July 15, 2002
4,200	September 9, 2002
4,100	November 20, 2002

To compute the present value of these cash flows we use the following statements:

```
format bank
```

```
CF=[–15000 3500 4000 3750 4200 4100];
DF=['01/23/2001'; '02/27/2002'; '03/18/2002'; '07/15/2002'; '09/09/2002'; '11/20/2002'];
present_value=pvvar(CF,0.08, DF)
```

and these return

```
present_value =

       2494.95
```

fvfix function

The **fvfix** function computes the future value of cash flows at regular time intervals with equal or unequal payments. Its syntax is

f=fvfix(rate,nper,p,pv,due)

where **rate** is the periodic interest rate, **nper** is the number of periods, **p** is the periodic payment, **pv** is the initial value, and **due** specifies whether the payments are made at the beginning (**due=1**) or end (**due=0**) of the period. The default arguments are **pv=0** and **due=0**.

Example 7.52

Suppose that we already have *$2,000* in a savings account and we add *$250* at the end of each month for *10* years. This account pays *6%* interest compounded monthly. To compute the future value of our investment at the end of *10* years, we use the following statements:

```
format bank
future_value=fvfix(0.06/12,10*12,250,2000,0)
```

and these return

```
future_value =

      44608.63
```

fvvar function

The **fvvar** function computes the future value of cash flows at irregular time intervals with equal or unequal payments. Its syntax is

fv=fvvar(cf,rate,df)

where **cf** is a vector representing the cash flows, **rate** is the periodic interest rate, and **df** represents the dates that the cash flows occur. If **df** omitted, it is understood that the cash flows occur at the end of the period. The initial investment is included as the initial cash flow value.

Example 7.53

Suppose we make an initial investment of *$15,000* at an annual interest rate of *7%* and the yearly income realized by this investment is as shown in Table 7.17.

Table 7.17 Cash flows for Example 7.53

Year	Return
1	$3,500
2	4,000
3	3,750
4	4,200
5	4,100

To compute the future value of the periodic cash flows, we use the following statements:

format bank

CF=[–15000 3500 4000 3750 4200 4100];

future_value=fvvar(CF,0.07) and these return

```
future_value =
       1337.06
```

Example 7.54

An investment of *$15,000* returns the cash flows shown in Table 7.18 where the first cash flow value is shown as the first cash flow value. The annual interest rate is *8%*.

Table 7.18 Cash flows for Example 7.54

Cash Flow	Date
$–15,000	January 23, 2001
3,500	February 27, 2002
4,000	March 18, 2002
3,750	July 15, 2002
4,200	September 9, 2002
4,100	November 20, 2002

To compute the present value of these cash flows we use the following statements:

format bank

CF=[–15000 3500 4000 3750 4200 4100];

DF=['01/23/2001'; '02/27/2002'; '03/18/2002'; '07/15/2002'; '09/09/2002'; '11/20/2002'];

future_value=fvvar(CF,0.08, DF) and these return

```
future_value =

       2871.10
```

annurate function

The **annurate** function computes the periodic interest rate paid on a loan or annuity. Its syntax is

r=annurate(nper,pv,fv,due)

where **rate** is the periodic interest rate, **nper** is the number of periods, **p** is the periodic payment, **pv** is the present value of the loan or annuity, **fv** is the future value of the loan or annuity, and **due** specifies whether the payments are made at the beginning (**due=1**) or end (**due=0**) of the period. The default arguments are **fv=0** and **due=0**.

Example 7.55

Suppose that we have taken a loan of *$8,000* for *5* years and we make monthly payments of *$225* at the end of each month. To compute the interest rate that we pay for this loan, we use the following statements:

format bank

monthly_rate=annurate(5*12,225,8000,0,0) and these return

```
monthly_rate =

          0.02
```

Therefore, the annual interest rate is $0.02 \times 12 = 0.24$ or 24%.

amortize function

The **amortize** function computes the principal and interest portions of a loan paid by a periodic payment and returns the remaining balance of the original loan amount and the periodic payment. Its syntax is

[principal, interest, balance, periodic_payment]=amortize(rate,nper,pv,fv,due)

where **rate** is the periodic interest rate that we pay, **nper** is the number of periods, **pv** is the present value of the loan or annuity, **fv** is the future value of the loan or annuity, and **due** specifies whether the payments are made at the beginning (**due=1**) or end (**due=0**) of the period. The default arguments are **fv=0** and **due=0**. On the left side of above statement **principal** is a vector of the principal paid in each period, **interest** is a vector of the interest paid in each period, **balance** is the remaining balance of the loan in each payment period, and **periodic_payment** is the periodic payment.

Example 7.56

Suppose we took a loan of *$2,000* to be paid in *6* bimonthly installments at an annual interest rate of 7.5%. The following statements will compute the principal paid in each period, the interest paid in each period, the remaining balance in each period, and the periodic payment.

```
format bank
```

`[principal, interest, balance, periodic_payment]=amortize(0.075/6, 6, 2000)` and these return

```
principal =
  323.07  327.11  331.19  335.33  339.53  343.77
interest =
  25.00 20.96 16.87 12.73 8.54 4.30
balance =
  1676.93  1349.83  1018.63  683.30  343.77  -0.00
periodic_payment =
        348.07
```

The MATLAB Financial Toolbox includes the Securities Industry Association (SIA)* compliant functions to compute accrued interest, determine prices and yields, and calculate duration of fixed-income securities. It also includes a set of functions to generate and analyze term structure of interest rates.

7.10 Comparison of Alternate Proposals

Quite often, one must perform economic analysis to select equipment or property from two or more proposals. In this section we will discuss the annual cost comparison method by comparing the total annual costs for each proposal.

Let

$$PP = Purchase\ Price\ (initial\ cost)\ of\ asset$$

$$SV = Salvage\ Value\ of\ asset$$

$$n = life\ of\ asset\ in\ years$$

$$i = Annual\ interest\ rate$$

$$AE = Annual\ Expenditures$$

The annual expenditures include all costs except depreciation and interest. These expenditures include taxes, insurance, materials, labor, repairs and maintenance, supplies, and utilities.

* *SIA-compliant functions are normally used with U.S. Treasury bills, bonds (corporate and municipal), and notes.*

The effective purchase price at the retirement date of an asset is computed as

$$PP(1+i)^n - SV$$

Converting this to an equivalent annuity, we get

$$Total\ Annual\ Cost = [PP(1+i)^n - SV]\frac{i}{(1+i)^n - 1} + AE$$

$$= \frac{PPi(1+i)^n - SVi + PPi - PPi}{(1+i)^n - 1} + AE = \frac{(PP - SV)i + PPi[((1+i)^n) - 1]}{(1+i)^n - 1} + AE$$

or

$$Total\ Annual\ Cost = (PP - SV)\frac{i}{(1+i)^n - 1} + PPi + AE \tag{7.40}$$

The total annual cost of an asset is a highly useful tool in arriving at a business decision where an economic choice is to be made among several alternative courses of action. The following examples will illustrate the applicability of annual cost for comparison purposes.

Example 7.57

A manufacturer has bought a piece of property on which he will build a storage warehouse. Two types of construction are considered, brick and galvanized iron. The data are available are shown in Table 7.19.

Table 7.19 Data for Example 7.57

	Brick	Galvanized Iron
Initial cost (purchase price)	$24,000	$10,000
Salvage Value	5,000	2,000
Estimated life	50 years	15 years
Annual maintenance	200	300
Annual insurance premium	48	50
Annual taxes	300	125
Interest	4%	4%

Which type of construction is the most economical?

Solution:

The equation of (7.40) applies to this example. For convenience, we construct Table 7.20 to consolidate the data shown in Table 7.19.

Table 7.20 Consolidation of the data of Table 7.19

	Brick	Galvanized Iron
PP (Purchase Price)	\$24,000	\$10,000
SV (Salvage Value)	5,000	2,000
n (Estimated life)	50 years	15 years
AE (Maintenance, Insurance, Taxes)	200+48+300=548	300+50+125=475
Interest	4%	4%

Then,

$$\begin{aligned}\textit{Total Annual Cost}_{brick} &= (24000-5000)\frac{0.04}{(1+0.04)^{50}-1}+24000\times 0.04+548\\ &= 19000\times 0.00655+960+548 = \$1,632.45\end{aligned}$$

Likewise,

$$\begin{aligned}\textit{Total Annual Cost}_{galvanized\ iron} &= (10000-2000)\frac{0.04}{(1+0.04)^{15}-1}+10000\times 0.04+475\\ &= 8000\times 0.04994+400+475 = 1,274.53\end{aligned}$$

Therefore, the galvanized construction is the most economical choice.

Example 7.58

Two possible routes for a power line are under consideration. Route A is around a lake, 15 miles in length. The first cost will be $\$6,000$ per mile, the yearly maintenance will be $\$2,000$ per mile, and there will be a salvage value at the end of 15 years of $\$3,000$ per mile. Route B is a submarine cable across the lake, 5 miles long. The first cost will be $\$31,000$ per mile, the annual maintenance $\$400$ per mile, and the salvage value at the end of 15 years will be $\$6,000$ per mile. The yearly power loss will be $\$550$ per mile for both routes. Interest rate is 4.5%; taxes are 3% of the first cost. Compare the two routes on the basis of annual costs.

Solution:

For convenience, we tabulate the given data on Table 7.21.

Then, for Route A

$$\begin{aligned}\textit{Total Annual Cost}_{Route\ A} &= (90000-45000)\frac{0.045}{(1+0.045)^{15}-1}+90000\times 0.045+40950\\ &= 45000\times 0.04811+4050+40950 = \$47,165.12\end{aligned}$$

Likewise, for Route B

Table 7.21 Given data for Example 7.58

	Route A	Route B
PP (Purchase Price, i.e., Initial Cost)	$6,000×15=$90,000	$31,000×5=$155,000
SV (Salvage Value)	3,000×15=45,000	6,000×5=30,000
Interest	4.5%	4.5%
Annual Expenditures		
Maintenance	2,000×15=30,000	400×5=2,000
Power loss ($550/mile)	550×15=8,250	550×5=2,750
Taxes at 3% of initial cost	90,000×0.03=2,700	155,000×0.03=4,650
Total Annual Expenditures	30,000+8,250+2,700=$40,950	2,000+2,750+4,650=$9,400

$$Total\ Annual\ Cost_{Route\ B} = (155000 - 30000)\frac{0.045}{(1+0.045)^{15}-1} + 155000 \times 0.045 + 9400$$
$$= 125000 \times 0.04811 + 4050 + 9400 = \$22,389.23$$

Therefore, Route B is the most economical choice.

7.11 Kelvin's Law

On the previous section, we discussed how we can make an economic choice by computing the annual costs of two or more alternatives. In this section, we will discuss a class of problems in which the total cost can be found by the following formula:

$$y = ax + \frac{b}{x} + c \tag{7.41}$$

where

$y = total\ cost$

$a, b, and\ c\ are\ constants$

$x = a\ variable$

The total cost y becomes a minimum when

$$\frac{dy}{dx} = 0 = a - \frac{b}{x^2}$$

or when

$$x = \sqrt{\frac{b}{a}} \tag{7.42}$$

The pair of the equations of (7.41) and (7.42) are known as *Kelvin's law*. This law can be applied to special types of problems such as the most economical span length of a bridge, most economical lot size, and most economical diameter of an oil pipeline.

Example 7.59

A railroad company proposes to build a double-track half-through plate-girder bridge 980 feet long, with spans of equal length. The cost of steel in place is estimated at $\$0.10$ cents per pound. Disregarding the cost of the track as constant, find the economical length of span. For superstructure use the equation

$$Weight_{lb/linear\ foot} = 25s + 2050$$

where s is the span length in feet.

For the center piers use the equation

$$Cost_{center\ piers/pier} = 50s + 30000$$

and for the end piers use the equation

$$Cost_{end\ piers/pier} = 50s + 20000$$

Solution:

The total weight of steel in the superstructure is

$$Steel_{total\ weight} = 980\ feet \times Weight/linearfoot = 980(25s + 2050) = 24500s + 2009000$$

Cost of steel at $\$0.10/lb$ is

$$Cost_{steel} = 0.10(24500s + 2009000) = 2450s + 200900$$

As stated, s is the span length in feet. Then, the number of spans is

$$Number\ of\ spans = \frac{Bridge\ length}{Span\ length} = \frac{980}{s}$$

There are 2 end piers and the number of center piers is one less than the number of spans. Thus,

$$Cost_{end\ piers} = 2(50s + 20000) = 100s + 40000$$

and

$$Cost_{center\ piers} = \left(\frac{980}{s} - 1\right)(50s + 30000) = 49000 - 50s + \frac{29400000}{s} - 30000$$
$$= \frac{29400000}{s} - 50s + 19000$$

Therefore, denoting the total cost of bridge as y, we get

$$y = Cost_{steel} + Cost_{end\ piers} + Cost_{center\ piers}$$
$$= 2450s + 200900 + 100s + 40000 + \frac{29400000}{s} - 50s + 19000$$
$$= 2500s + \frac{29400000}{s} + 259900$$

By comparison of the above expression with the equation of (7.41) we see that

$$a = 2500 \qquad b = 29400000 \qquad x = s$$

Therefore, the most economical length of each bridge span is when y is minimum, and by equation (7.42) it occurs when

$$x = s = \sqrt{\frac{b}{a}} = \sqrt{\frac{29400000}{2500}} = 108.44353 \approx 108.5\ feet$$

and the number of spans is

$$Number\ of\ spans = \frac{Bridge\ length}{Span\ length} = \frac{980}{108.5} = 9.03 \approx 9\ spans$$

Example 7.60

A manufacturer must to produce $50,000$ parts a year and wishes to select the most economical lot size to be equally spaced during the year. The annual cost of storage and interest on investment varies directly with the lot size and is $\$1,000$ when the entire annual requirement is manufactured in one lot. The cost of setting up and dismantling the machine for each run in $\$30$. What is the most economical lot size?

Solution:

Let x be the most economical lot size. Then, the number of machine setups will be $50000/x$.

Annual cost of setups will be

$$30 \times \frac{50000}{x} = \frac{1500000}{x}$$

and since the entire annual cost of storage and interest varies directly with the lot size and is $\$1,000$ when manufactured in one lot, by proportion we have

$$\frac{\$1000}{50000\ parts} \times x = 0.02x$$

Now, denoting the total cost to produce $50,000$ units as y, we have

$$y = 0.02x + \frac{1500000}{x}$$

Therefore, $a = 0.02$, $b = 1500000$, and per equation (7.42), y will be a minimum cost when

$$x = \sqrt{\frac{b}{a}} = \sqrt{\frac{1500000}{0.02}} = 8660\ parts\ per\ lot$$

and the number of lots required will be

$$\frac{50000}{8660} = 5.8$$

If we round this number to 6, we find that the most economical lot size is

$$\frac{50000}{6} = 8333\ parts$$

7.12 Summary

- A bond is a debt investment. That is, we loan money to an entity (company or government) that needs funds for a defined period of time at a specified interest rate.
- A corporate bond is a bond issued by a corporation.
- A municipal bond is a bond issued by a municipality and that generally is tax-free.
- A treasury bond is a bond issued by the US Government.
- A perpetuity is a constant stream of identical cash flows with no maturity date.
- A perpetual bond is a bond with no maturity date. Perpetual bonds are not redeemable and pay a steady stream of interest forever. Such bonds have been issued by the British government.
- A convertible bond is a bond that can be converted into a predetermined amount of a company's equity at certain times during it life.
- A treasury note is treasury bond that is issued for a shorter time.
- A treasury bill is a treasury bond that is held for a shorter time than either a treasury bond or a treasury note.
- Face value is the dollar amount assigned to a bond when first issued.
- Par value is the face value of a bond.
- Book value is the value of a bond at any date intermediate between its issue and its redemption.
- A coupon bond is a debt obligation with coupons representing semiannual interest payments attached.
- A zero coupon bond is a bond that generates no periodic interest payments and is issued at a discount from face value. The return is realized at maturity.
- A junk bond is a bond purchased for speculative purposes. It has a low rating and a high default risk.
- Bonds are assigned a rating by Standard and Poor's and Moody's
- A promissory note is an unconditional written promise made by one party to another, to pay a stipulated sum of money either on demand or at a definite future date.
- The discount rate the rate charged by Reserve Banks when they extend credit to depository institutions either through advances or through the discount of certain types of paper, including ninety-day commercial paper.
- The prime rate is the interest rate charged by banks to their most creditworthy customers.

- An annuity is a series of equal payments made at equal intervals or periods of time. When paid into a fund which is invested at compound interest for a specified number of years, the annuity is referred to as a sinking fund.
- An ordinary annuity is a series of equal payments or receipts occurring over a specified number of periods with the payments or receipts occurring at the end of each period.
- A sinking fund is an interest-earning fund in which equal deposits are made at equal intervals of time. It is also referred to as an annuity.
- Interest is charge for a loan.
- Simple interest is calculated on a yearly basis (annually) and depends on the interest rate.
- Compound interest includes interest earned on interest.
- If a given interest rate applies to a period less than 1 year, then its equivalent rate for an annual period is referred to as its effective rate. Thus, the effective rate corresponding to a rate of 2% per quarterly period is 8.243% per cent. The effective rate is numerically equal to the interest earned by a principal of $\$1.00$ for 1 year.
- The Excel PV function returns the present value of an investment. The present value is the total amount that a series of future payments is worth now. This function also computes the present value of an ordinary annuity.
- The Excel FV function computes the future value of an ordinary annuity.
- The Excel PMT function calculates the payment for a loan based on constant payments and a constant interest rate.
- The Excel RATE function returns the interest rate per period of an annuity. We can use RATE to determine the yield of a zero-coupon bond that is sold at a discount of its face value.
- The Excel NPER function returns the number of periods for an investment based on periodic, constant payments, and a constant interest rate.
- The Excel NPV function calculates the net present value of an investment by using a discount rate and a series of future payments (negative values) and income (positive values).
- The Excel IIR function returns the internal rate of return for a series of cash flows represented by the numbers in values. These cash flows need not be even as they would be for an annuity.
- The Excel MIRR function returns the modified internal rate of return for a series of periodic cash flows. MIRR considers both the cost of the investment and the interest received on reinvestment of cash.
- The Excel IPMT function returns the interest payment for a given period for an investment based on periodic, constant payments, and a constant interest rate.

- The Excel PPMT function returns the payment on the principal for a given period for an investment based on periodic, constant payments, and a constant interest rate.
- The Excel ISPMT function calculates the interest paid during a specific period of an investment.
- The MATLAB **irr** function computes the internal rate of return of periodic cash flows.
- The MATLAB **effrr** function computes the effective rate of return given an annual interest rate, also known as Annual Percentage Rate (APR), and number of compounding periods per year.
- The MATLAB **pvfix** function computes the present value of cash flows at regular time intervals with equal or unequal payments.
- The MATLAB **pvvar** function computes the present value of cash flows at irregular time intervals with equal or unequal payments.
- The MATLAB **fvfix** function computes the future value of cash flows at regular time intervals with equal or unequal payments.
- The MATLAB **fvvar** function computes the future value of cash flows at irregular time intervals with equal or unequal payments.
- The MATLAB **annurate** function computes the periodic interest rate paid on a loan or annuity.
- The MATLAB **amortize** function computes the principal and interest portions of a loan paid by a periodic payment and returns the remaining balance of the original loan amount and the periodic payment.
- The material presented in this chapter allows us to perform economic analysis to select equipment or property from two or more proposals.
- Kelvin's law can be applied to special types of problems such as the most economical span length of a bridge, most economical lot size, and most economical diameter of an oil pipeline.

7.13 Exercises

1. Jack's bank account pays interest at a rate of 4.3% per year. If he puts $\$850$ into his account, how much will Jack have after a year?
2. Mary earns 4.5% interest per year on the money she has saved in her bank account. Her initial investment was $\$200$. How much is on her account after 5 years?
3. Lisa has $\$560$. She deposits it into a bank account that pays 3.7% interest per annum. She leaves it in the bank for 2 months. How much money does Lisa have now?
4. If we deposit $\$1,500$ into an account earning interest rate of 5% compounded quarterly, what will our principal be at the end of 7 years?
5. An individual possesses a promissory note, due 2 years hence, whose maturity value is $\$3,200$. What is the discount value of this note based on an interest rate of 7%?
6. On July 1, 1992, a bank account had a balance of $\$8000$. A deposit of $\$1000$ was made on Jan. 1, 1993, and a deposit of $\$750$ on Jan. 1, 1994. On Apr. 1, 1995, the sum of $\$1,200$ was withdrawn. What was the balance in the account on Jan. 1, 1998, if interest is earned at the rate of 4% per cent compounded quarterly?
7. A manufacturing firm contemplates retiring an existing machine at the end of 2012. The new machine to replace the existing one will have an estimated cost of $\$10,000$. This expense will be partially defrayed by sale of the old machine as scrap for $\$750$. To accumulate the balance of the required capital, the firm will deposit the following sums in an account earning interest at 5% compounded quarterly:

   $\$1,500$ at the end of 2009 $\$1,500$ at the end of 2010 $\$2,000$ at the end of 2011

   What cash disbursement will be necessary at the end of 2012 to purchase the new machine?
8. A father wishes to have $\$50,000$ available at his son's eighteenth birthday. What sum should be set aside at the son's fifth birthday if it will earn interest at the rate of 3% per cent compounded semiannually?
9. In Exercise 8, if the father will set aside equal sums of money at the son's fifth, sixth, and seventh birthdays, what should each sum be?
10. Brown owes Smith the following sums:

    $\$1,000$ due 2 years hence $\$1,500$ due 3 years hence $\$1,800$ due 4 years hence

    Having received an inheritance, he has decided to liquidate the debts at the present date. If the two parties agree on a 5% interest rate, what sum must Brown pay? In order to verify the solution obtained, evaluate all sums of money, including the required payment, at any date other than the present.

11. How much interest will $\$1,000$ earn if it is invested at 6% for 5 years?

12. A business firm contemplating the installation of labor-saving machinery has a choice between two different models. Machine A will cost $\$36,500$, while machine B will cost $\$36,300$. The repairs required for each machine are as follows:

    Machine A: $\$1,500$ at end of 5th year and $\$2,000$ at end of 10th year

    Machine B: $\$3,800$ at end of 9th year

    The machines are alike in all other respects. If this firm is earning a 7% per cent return on its capital, which machine should be purchased?

13. A fund was established in 1990, whose history is recorded below:

    Deposit of $\$1,000$ on 1/1/90

    Deposit of $\$2,000$ on 1/1/92

    Withdrawal of $\$300$ on 7/1/92

    Deposit of $\$1,600$ on 7/1/93

    Withdrawal of $\$1,200$ on 1/1/94

    The fund earned interest at the rate of 3% compounded semiannually until the end of 1992. At that date, the interest rate was augmented to 4% compounded semiannually. What was the principal in the fund at the end of 1996?

14. If the sum of $\$2,000$ is invested in a fund earning 7.5% interest compounded semiannually, what will be its final value at the end of 5 years?

15. In Exercise 14, how long will it take the original sum of $\$2,000$ to expand to $\$3,000$?

16. On Apr. 1, 2001, an investor purchased stock of the XYZ Corp. at a total cost of $\$3,600$. He then received the following semiannual dividends:

    $\$105$ on Oct. 1, 2001 $\quad$ $\$110$ on Apr. 1, 2002

    $\$110$ on Oct. 1, 2002 $\quad$ $\$100$ on Apr. 1, 2003

    After receipt of the last dividend, the investor sold his stock, receiving $\$3,800$ after deduction of brokerage fees. What semiannual rate did this individual realize on his investment?

17. An investor is considering alternate plans for the investment of $\$200,000$.

Plan A involves the purchase of non appreciating tax-free securities paying 1.5% per annum. Dividends will be invested at a net return of 2%, and the securities will be sold at the end of 20 years.

Plan B calls for the purchase of income property consisting of land worth $\$40,000$ and apartments worth $\$160,000$. It is estimated that the units will have an average occupancy of 90%, a useful life of 20 years and a salvage value at the end of that time of 10%. The land is expected to triple in value during this period. Taxes are $\$60$ per thousand on a 50% valuation. Insurance is $\$600$ per year, repairs will be $\$1,000$ annually, and income at full occupancy will be $\$1,200$ per month. A real estate agency must be paid 3% of gross income. Income taxes are estimated at 20% of net income. The annual net profit is invested at a net return of 2%. Depreciation as a credit against income taxes is assumed to be straight line.

Calculate the value of each plan at the end of 20 years, state which plan would be more profitable and show the difference in sums.

18. Fourteen years ago a 1200-Kw steam electric plant was constructed at a cost of $\$220$ per Kw. Annual operating expenses have been $\$31,000$ to produce the annual demand of $5,400,000$ Kw-hours. It is estimated that the annual operating expenses and demand for current will continue at the current level. The original estimate of a 20-year life with a 5% salvage value at that time is still expected to be correct.

    The company is contemplating the replacement of the old steam plant with a new 1200-Kw diesel plant. The old plant can be sold now for $\$75,000$, while the new diesel plant will cost $\$245$ per Kw to construct. The diesel plant will have a life of 25 years with a salvage value of 10% at the end of that time and will cost $\$23,000$ annually to operate. Annual taxes and insurance will be 2.3% of the first cost of either plant. Using an interest rate of 5%, determine whether the company is financially justified in replacing the old steam plant now.

19. The yearly requirement of a manufacturer is 1000 units of a part that is used at a uniform rate throughout the year. The machine set-up cost per lot is $\$40$. Production cost is $\$5.20$ per unit. Interest, insurance and taxes are estimated at 12% on average inventory. The cost of storing the parts is estimated at $\$0.80$ per unit per year. Calculate the economic lot size.

7.14 Solutions to Exercises

1. $850 \times (1 + 0.043) = \$886.55$ or with Excel =FV(0.043,1,0,-850,0)= 886.55

2. $200 \times (1 + 0.045)^5 = \$249.24$ or with Excel =FV(0.045,5,0,-200,0) = 249.24

3. $560 \times (1 + (0.037/12) \times 2) = \$563.45$ or with Excel =FV(0.037/12,1*2,0,-560,0)=563.46

4. $1500 \times \left(1 + \frac{0.05}{4}\right)^{7 \times 4} = 1500 \times (1.0125)^{28} = \$2,123.99$

   or with Excel =FV(0.05/4,7*4,0,-1500)=2123.99

5. $3200 \times (1 + 0.07)^{-2} = \$2,795$ or with Excel =PV(0.07,2,0,3200) = –2,795

6. $8000(1.01)^{22} + 1000(1.01)^{20} + 750(1.01)^{16} - 1200(1.01)^{11} = x$

   $8000(1.24472) + 1000(1.22019) + 750(1.17258) - 1200(1.11567) = x$

   $9957.76 + 1220.19 + 879.43 - 1338.80 = x$ from which $x = 10718.58$

   and with Excel

   =FV(0.04/4,(5*4+2),0,-8000,0)+FV(0.04/4,5*4,0,-1000,0)+FV(0.04/4,4*4,0,-750,0)
   +FV(0.04/4,(2*4+3),0,1200,0)=10,718.55

7. $1500(1.0125)^{12} + 1500(1.0125)^8 + 2000(1.0125)^4$

   $1500(1.16075) + 1500(1.10449) + 2000(1.05095)$

   $1741.125 + 1656.735 + 2101.90 = 5499.76$

Cost of new machine	10,000
Trade-in	750
Net cost	9250

   $9250 - 5499.76 = 3750.24$

8. $P_n = P_0(1 + i)^n$

   $$P_0 = \frac{P_n}{(1 + i)^n} = P_n(1 + i)^{-n}$$

   $$P_0 = 50000(1 + 0.15)^{-26} = 50000(0.67902) = \$33,951$$

9. $x(1.015)^{26} + x(1.015)^{24} + x(1.015)^{20} = 50000$

$x(1.47271) + x(1.42950) + x(1.34686) = 50000$

$4.25x = 50000$ and thus $x = 11, 765$

10. $P_{-n} = P_0(1+i)^{-n}$

$P_{-n1} = 1000(1+1.05)^{-2} = 1000(0.90703) = 907.03$

$P_{-n2} = 1500(1+1.05)^{-3} = 1500(0.86384) = 1295.22$

$P_{-n3} = 1800(1+1.05)^{-4} = 1800(0.82270) = 1480.86$

$907.03 + 1295.22 + 1480.86 = 3683.11$

11. $P_n = P_0(1+i)^n = 1000(1.06)^5 = 1000(1.41852) = 1418.52$

Therefore, interest earned is $1418.52 - 1000 = 418.52$

12. Assumptions:

1. Purchase date is selected as our valuation date

2. Interest of 7% is compounded annually.

Machine A:

$$\begin{aligned} 36500 + 1500(1.07)^{-5} + 2000(1.07)^{-10} &= 36500 + 1500(0.71299) + 2000(0.50835) \\ &= 36500 + 1069.50 + 1016.70 = 38586.20 \end{aligned}$$

Machine B:

$$36300 + 3800(1.07)^{-9} = 36300 + 3800 \times 0.54393 = 38367$$

Savings by choosing Machine B, $38586 - 38367 = 219$

13. $1000(1.0175)^6 + 2000(1.0175)^4 - 300(1.0175)$

$1000(1.10970) + 2000(1.07186) - 300(1.0175) = 2948.17$

$2948.17(1.02)^8 + 1600(1.02)^7 - 1200(1.02)^6$

$2948.17(1.17166) + 1600(1.14869) - 1200(1.12616)$

$3454.25 + 1837.90 - 1351.39 = 3940.76$

14. $P_n = P_0(1+i)^n = 2000(1.0375)^{10} = 2890.08$

15. $P_n = P_0(1+i)^n$

$$logP_n = logP_0 + nlog(1+i)$$

Solving for n we get:

$$n = \frac{logP_n - logP_0}{log(1+i)} = \frac{log3000 - log2000}{log(1.0375)} = \frac{3.477 - 3.301}{0.016} = \frac{0.176}{0.016} = 11$$

Therefore, it will take 11 semiannual periods or 5.5 years.

16. $3600 + 4i(3600) = [105 + 3i(105)] + [110 + 2i(110)] + [110 + i(110)] + 100 + 3800$

This yields $i = 4.15\%$. Since our approximation has given insufficient weight to receipts, the true investment rate is higher than this. Let us try an interest rate of 4.3%.

The amount y required to yield this rate is

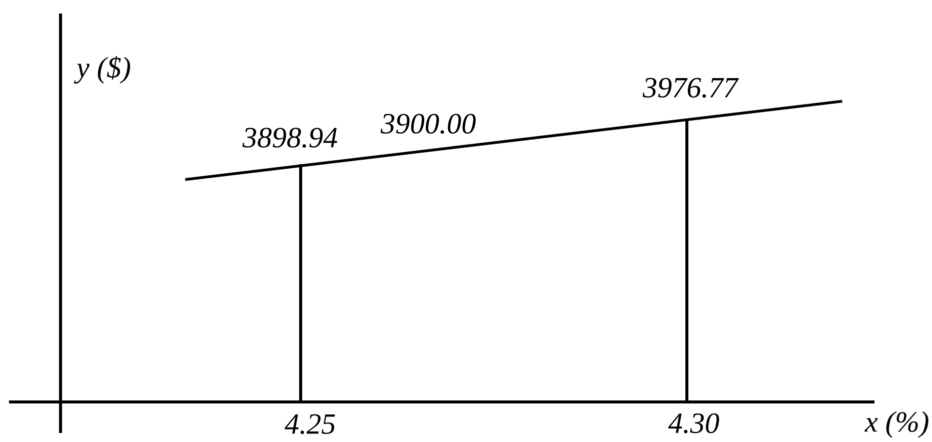
$$3600(1.043)^4 - 105(1.043)^3 + 110(1.043)^2 + 110(1.043) = y$$

or $y = 3906.77$. But this number is higher than $100 + 3800 = 3900$ which is the amount the investor received at the end of the period. Therefore, the interest rate must be slightly lower. Let us try an interest rate of 4.25%.

$$3600(1.0425)^4 - 105(1.0425)^3 + 110(1.0425)^2 + 110(1.0425) = y$$

or $y = 3898.94$

We will use the graph below to apply straight-line interpolation



$$x_i = \frac{1}{slope} \times (y_i - y_1) + x_1 = \frac{1}{m} \times (y_i - y_1) + x_1$$

$$x_{y\,=\,3900.00} = \frac{1}{slope} \times (3900.00 - 2898.94) + 0.0425 = \frac{1}{\dfrac{3906.77 - 3898.94}{0.0430 - 0.0425}} \times 1.06 + 0.0425$$

$$= \frac{0.0005}{7.83} \times 1.06 + 0.0425 = 0.0426$$

Therefore, the actual interest rate is 4.26%. This is the semiannual rate. The annual rate is 8.52%.

17. Plan A:

$$FV = Payment \times \frac{(1 + interest)^n - 1}{interest}$$

The lump sum at the end of 20 years will be

$$Lump\ sum = 200000 + (200000 \times 0.015) \times \frac{(1 + 0.02)^{20} - 1}{0.02}$$

$$= 200000 + 3000 \times 24.2974 = \$272,892.10$$

Plan B:

	PP (Purchase Price, i.e., Initial Investment)	SV (Salvage Value)	PP-SV
Apartment Units	$160,000	10%×160,000=$16,000	160,000–16,000=$144,000
Land	40,000	120,000	–80,000
Total	$200,000	$136,000	$64,000

$$Taxable\ Income = Gross\ Revunue - (Operating\ Expenses + Depreciation)$$

where

$$Gross\ Revenue = 12\ months \times 1,200/month \times 0.9\ occupancy = \$12,960$$

$$Operating\ Expenses$$
$$= Real\ EstateFee\ at\ 3\%\ of\ \$12{,}960 + Property\ Taxes + Insurance + Repairs$$
$$= 0.03 \times 12960 + \frac{60}{1000} \times (200,000 \times 0.50) + 600 + 1000 = \$7,988.80$$

Depreciation is allowable on the apartment units only and it will subject the investor to a capital gains tax at the end of 20 years. Thus,

$$Straight\ Line\ Depreciation = \frac{160,000 - 16,000}{20} = \$7,200$$

and

$$\begin{aligned} Total\ credits\ against\ income\ taxes &= Total\ Operating\ Expenses + Depreciation \\ &= \$7,988.80 + \$7,200 = \$15,188.80 \end{aligned}$$

Since total credits of $\$15,188.80$ exceed the gross revenue of $\$12,960$, there would be no annual income tax and the amount available for investment would be

$$\begin{aligned} Amount\ Available\ for\ Investment &= Gross\ Revenue - Credits\ against\ Income\ Taxes \\ &= \$12,960.00 - \$7,988.80 = \$4,971.20 \end{aligned}$$

and if this amount is invested for a return of 2%, the lump sum available at the end of 20 years would be

$$\begin{aligned} LumpSum &= Amount\ Available\ for\ Investment + SalvageValue \\ &= 4,971.20 \times \frac{(1+0.02)^{20}}{0.02} + 136,000 = 4,971.20 \times 24.2974 + 136,000 \\ &= \$256,787.10 \end{aligned}$$

The lump sum in Plan A was found to be $\$272,892.10$. Therefore, before capital gains tax, Plan A is more profitable by $\$272,892.10 - \$256,787.10 = \$16,105$.

18. For convenience, we tabulate the given data as shown below.

	Electric Plant	Diesel Plant
Plant Capacity	1200 Kw	1200 Kw
Investment per Kw	$220	$245
Original Investment (PP)	264,000	294,000
Estimated Life	20 years	25 years
End of Life Salvage Value (ESV)	5%PP=13,200	10%PP=29,400
Present Age of Investment	14 years	N/A
Remaining Life (RL)	6 years	N/A
Present Salvage Value (PSV)	75,000	N/A
Annual Expenditures (AE)		
Operating Expenses	31,000	23,000
Taxes and Insurance	2.3%PP=6072	2.3%PP=6762
Total Annual Expenditures	37,072	29,762

We will use (7.40) to compute the annual costs for each plant, i.e.,

$$Total\ Annual\ Cost = (PP - SV)\frac{i}{(1+i)^n - 1} + PPi + AE$$

where for the electric plant $n = 20 - 14 = 6$, and the diesel plant $n = 25$

For the existing (electric) plant, let $SA = Annual\ Savings\ if\ existing\ plant\ is\ replaced$

Then,

$$SA = (PSV - ESV)\frac{i}{(1+i)^n - 1} + PSVi + AE$$

$$= (75000 - 13200)\frac{0.05}{(1+0.05)^6} + 75000 \times 0.05 + 37072$$

$$= 61800 \times 0.14702 + 3750 + 37072 = \$49,907.68$$

For the diesel plant

$$AC = (PP - SV)\frac{i}{(1+i)^n - 1} + PPi + AE$$

$$= (294000 - 29400)\frac{0.05}{(1+0.05)^{25}} + 294000 \times 0.05 + 29762$$

$$= 264600 \times 0.020952 + 14700 + 29762 = \$50,006.02$$

From a practical point of view, the economics indicate a standoff due to the small differential between values of SA and AC.

19. Let us denote the economic lot size as x where lot size is the number of units manufactured with one machine set-up. Then

$$Number\ of\ lots\ per\ year = Number\ of\ machine\ setups = \frac{1000}{x}$$

and

$$Annual\ cost\ for\ all\ setups = \$40 \times \frac{1000}{x} = \frac{40,000}{x}$$

Next,

$$Average\ number\ of\ units\ in\ storage = Average\ inventory = \frac{x}{2}$$

and

$$\text{Inventory cost per year} = \text{Production cost per unit} \times \text{Average inventory}$$
$$= \$5.20 \times \frac{x}{2} = \$2.60x$$

Also,

$$\text{Interest, Insurance, and Taxes on Average Inventory} = 12\% \times \$2.60x = \$0.312x$$

and

$$\text{Storage costs} = \$0.80 \times \frac{x}{2} = \$0.40x$$

The annual production costs are

$$AC = \$5.20 \times 1000 = \$5,200$$

Therefore, total annual costs

$$y = (0.312 + 0.40)x + \frac{40,000}{x} + 5,200 = 0.712x + \frac{40,000}{x} + 5,200$$

By inspection,

$$a = 0.712 \qquad b = 40,000$$

and the economic lot size is

$$x = \sqrt{\frac{b}{a}} = \sqrt{\frac{40,000}{0.712}} = 237 \text{units}$$

This lot size would require

$$\frac{1000}{237} = 4.22 \text{ lots}$$

and rounding this number to 4 lots, we find that the 1000 units should be manufactured in 4 lots of 250 units each lot.