Chapter 2

Sequences and Series

2.1 Sequences and Sequence Limits

Exercises

1. Find the first five elements of the following sequences

   (a) $a_n = 2n - 5$;

   (b) $a_n = -\frac{1}{2}n + 3$;

   (c) $a_n = \frac{2n+3}{n+1}$;

   (d) $a_n = \left(1 + \frac{1}{n}\right)^n$;

   (e) $a_n = \frac{n+(-1)^n}{2}$.

2. Suggest the general term a_n for the following sequences

   (a) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

   (b) $\frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \ldots$

   (c) $1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$

3. Find $\lim\limits_{n\to\infty} a_n$ for the following sequences

(a) $a_n = \frac{n}{n+1}$;

(b) $a_n = (-1)^n \frac{1}{2n}$;

(c) $a_n = \frac{3n-1}{2n+3}$;

(d) $a_n = \frac{5n-7}{1-3n}$;

(e) $a_n = \frac{n^3-2n-1}{n-3}$;

(f) $a_n = \frac{n-1}{-n^3+2n+1}$;

(g) $a_n = \frac{3-n}{-n^2+5n-2}$.

4. Find

   (a) $\lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2-5}}$;

   (b) $\lim\limits_{n\to\infty} \frac{3n^3-2n+4}{5n^2\sqrt{n^2+2}}$;

   (c) $\lim\limits_{n\to\infty} (\sqrt{n^2+n} - \sqrt{n^2-n})$.

5. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *monotonically increasing* if each its term is smaller than the term after it; i.e., for every n holds $a_n < a_{n+1}$. Which of the following sequences are monotonically increasing?

   (a) $a_n = \frac{1}{n}$

   (b) $a_n = 1 - \frac{1}{n}$

   (c) $a_n = 1 + \frac{1}{n}$

   (d) $a_n = \frac{1}{2^n}$

   (e) $a_n = (-1)^n \frac{1}{2n}$

6. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *monotonically decreasing* if each its term is greater than the term after it; i.e., for every n holds $a_n > a_{n+1}$. Which of the following sequences are monotonically decreasing?

   (a) $a_n = \frac{1}{n}$

   (b) $a_n = 1 - \frac{1}{n}$

   (c) $a_n = 1 + \frac{1}{n}$

(d) $a_n = 1 - \frac{1}{2^n}$

(e) $a_n = (-1)^n \frac{1}{2n}$

7. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be *bounded* if there exist numbers m and M such that every term of the sequence is between them; i.e., for every n holds $m \leq a_n \leq M$. The numbers m and M are called a *lower bound* and an *upper bound* of the sequence a_n. Are the following sequences bounded? If yes, find an upper bound and a lower bound.

   (a) $a_n = \frac{1}{n}$

   (b) $a_n = 1 - \frac{1}{n}$

   (c) $a_n = 1 + \frac{1}{n}$

   (d) $a_n = 1 - \frac{1}{2^n}$

   (e) $a_n = (-1)^n \frac{1}{2n}$

8. Find $\lim\limits_{n\to\infty} a_n$ for the following sequences

   (a) $a_n = \left(1 + \frac{1}{2n}\right)^n$;

   (b) $a_n = \left(1 - \frac{1}{n}\right)^n$;

   (c) $a_n = \left(1 + \frac{3}{n}\right)^n$;

   (d) $a_n = \left(1 + \frac{1}{n}\right)^{5n}$;

   (e) $a_n = \left(1 + \frac{4}{n}\right)^{5n}$;

   (f) $a_n = \left(1 - \frac{2}{3n}\right)^{-4n}$.

9. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be $\mathrm{O}(b_n)$ (we write $a_n = \mathrm{O}(b_n)$) if there exist constants C and n_0 such that $a_n \leq C b_n$ for all $n > n_0$. Prove the following transformations in an expression that uses the O-notation.

   (a) $\mathrm{O}(1) = \mathrm{O}(2)$

   (b) $a_n = \mathrm{O}(a_n)$

   (c) $c\,\mathrm{O}(a_n) = \mathrm{O}(a_n)$

(d) $\mathrm{O}(ca_n) = \mathrm{O}(a_n)$

(e) $\mathrm{O}(a_n)\,\mathrm{O}(b_n) = \mathrm{O}(a_n b_n)$

(f) If $a_n = \mathrm{O}(b_n)$, then $\mathrm{O}(a_n) + \mathrm{O}(b_n) = \mathrm{O}(b_n)$.

10. Using the facts that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

and that for every k and every $\alpha > 1$

$$\lim_{n\to\infty} \frac{n^k}{\alpha^n} = 0,$$

show that

(a) $\ln n = \mathrm{O}(n)$, but $n \neq \mathrm{O}(\ln n)$;

(b) $n \ln n = \mathrm{O}(n^{3/2})$, but $n^{3/2} \neq \mathrm{O}(n \ln n)$;

(c) $n^m = \mathrm{O}(\alpha^n)$, but $\alpha^n \neq \mathrm{O}(n^m)$.