1.3 Introduction to Differential Equations

Exercises

1. Find the general solutions of the given differential equations.
   (a) $\frac{dy}{dx} = 3x^2 + 5x - 6$;
   (b) $\frac{dP}{dt} = \sqrt{t} + e^{-t}$;
   (c) $\frac{dy}{dx} = 3y$;
   (d) $\frac{dy}{dx} = y^2$;
   (e) $\frac{dy}{dx} = e^y$;
   (f) $\frac{dy}{dx} = e^{x+y}$;
   (g) $\frac{dy}{dx} = \frac{x}{y}$;
   (h) $\frac{dy}{dx} = \frac{y}{x}$;
   (i) $\frac{dy}{dx} = \sqrt{xy}$;
   (j) $\frac{dy}{dx} = \frac{y^2+4}{xy}$;
   (k) $\frac{dy}{dx} = \frac{y}{x-1}$;
   (l) $\frac{dy}{dx} = e^y\sqrt{x+1}$;
   (m) $\frac{dy}{dx} = \frac{y+3}{(2x-5)^6}$;
   (n) $\frac{dy}{dx} = (e^y + 1)(x-2)^9$;
   (o) $\frac{dx}{dt} = \frac{xt}{2t+1}$;
   (p) $\frac{dy}{dt} = \frac{te^y}{2t-1}$;

2. An investment grows at a rate equal to 7% of its size. Write a differential equation describing the given situation. Find the general solution of the differential equation.

3. The Mitscherlich model, a useful model of agricultural production, specifies that the size $Q(t)$ of a crop changes in such a way that the rate of change is proportional to $B - Q(t)$, where B is the maximum possible size of the crop.

(a) Write the relationship as a differential equation and find its general solution.

(b) Note that this model is similar to the learning model. Is this just a coincidence or is there some meaningful analogy linking the two situations? Explain.

4. The residents of a certain community have voted to discontinue the fluoridation of their water supply. The local reservoir currently holds 800 million liters of fluoridated water that contains 800 kilograms of fluoride. The fluoridated water is flowing out of the reservoir at the rate of 16 million liters per day and is being replaced at the same rate by unfluoridated water. At all times, the remaining fluoride is evenly distributed in the reservoir. Express the amount of fluoride in the reservoir as a function of time.

   (a) Let $Q(t)$ be the amount of fluoride in the reservoir at time t. Explain why $Q(t)$ satisfies the differential equation

   $$\frac{dQ}{dt} = -\frac{Q}{50}$$

   (b) Exress the amount of fluoride in the reservoir as a function of time.

5. Suppose that a particular commodity has linear demand and supply functions, $D(p) = a - bp$ and $S(p) = r + sp$, for price p and positive constants a, b, r and s. Further assume that price is a function of time t and that the time rate of change of price is proportional to the shortage $D - S$, so that

   $$\frac{dp}{dt} = k(D - S).$$

   Solve this differential equation and sketch the graph of $p(t)$. What happens to $p(t)$ "in the long run" (as $t \to \infty$)?

6. Let D and I denote the national debt and national income, and assume that both are functions of time t. One of several Domar debt models assumes that the time rates of change of D and I are both proportional to I, so that

   $$\frac{dD}{dt} = aI \quad \text{and} \quad \frac{dI}{dt} = bI.$$

Suppose $I(0) = I_0$ and $D(0) = D_0$.

(a) Solve both of these differential equations and express $D(t)$ and $I(t)$ in terms of a, b, I_0 and D_0.

(b) The economist, Evsey Domar, who first studied this model, was interested in the ratio of national debt to national income. What happens to the ratio $\frac{D(t)}{I(t)}$ as $t \to +\infty$?