Partial Derivatives

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Aims and Objectives

- Differentiating a function of several variables with respect to one of its varaibles
- Geometric representation and applications of partial derivatives
- Second order partial derivatives

Contents

Partial Derivatives

- Find the rate of change of a function of two variables with respect to one of its variables when the other is held constant.
- Differentiate the function with respect to the particular variable while keeping the other variable fixed.

Definition and Notation

Partial Derivatives

Suppose $z = f(x, y)$.

- The partial derivative of f with respect to x is denoted by
$$f_x(x, y) = \frac{\partial z}{\partial x}$$
  and is the function obtained by differentiating f with respect to x, treating y as a constant.
- The partial derivative of f with respect to y is denoted by
$$f_y(x, y) = \frac{\partial z}{\partial y}$$
  and is the function obtained by differentiating f with respect to y, treating x as a constant.

Computation of Partial Derivatives

Example

Find the partial derivatives f_x and f_y if $f(x, y) = xe^{-2xy}$.

Solution.

Use the product rule:

$$f_x(x, y) = e^{-2xy} + x(-2ye^{-2xy}) = (1 - 2xy)e^{-2xy}.$$

Use the constant multiple rule:

$$f_y(x, y) = x(-2xe^{-2xy}) = -2x^2 e^{-2xy}.$$

□

Geometric Interpretation of Partial Derivatives

- The value $z = f(x, y)$: assigning a "height" to the point (x,y,0).
- If y is kept fixed at $y = y_0$, the points $(x, y_0, f(x, y_0))$ form a curve: the intersection of the surface $z = f(x, y)$ with the plane $y = y_0$.
- At each point of the curve, the partial derivative $\frac{\partial z}{\partial x}$ is the slope of the curve in the plane $y = y_0$.

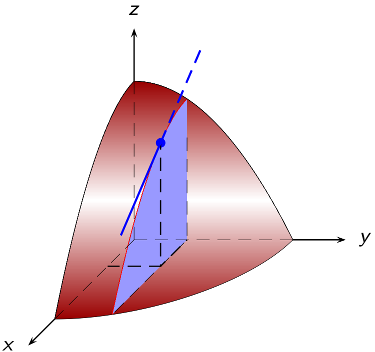

Figure: $\frac{\partial z}{\partial x}$ = slope in x direction.

Marginal Analysis

Example

It is estimated that the weekly output of a certain plant is given by the function $Q(x, y) = 1{,}200x + 500y + x^2y - x^3 - y^2$ units, where x is the number of skilled workers
and y is the number of unskilled workers employed at the plant.
Currently the workforce consists of 30 skilled workers
and 60 unskilled workers.
Use marginal analysis to estimate the change in the weekly output that will result from the addition of 1 more skilled worker
if the number of unskilled workers in unchanged.

Marginal Analysis. (Continued)

Solution.

$$Q_x(x, y) = 1{,}200 + 2xy - 3x^2$$

The resulting change in output is approximately:

$$\begin{aligned} Q(31, 60) - Q(30, 60) &\approx Q_x(30, 60) \\ &= 1{,}200 + 2 \cdot 30 \cdot 60 - 3 \cdot 30^2 = 2{,}100 \end{aligned}$$

units. □

Substitute and Complementary Commodities

Partial Derivatives

- Two commodities are said to be *substitute commodities* if an increase in the demand of either results in a decrease in demand of the other (e.g. butter, margarine).
- Two commodities are said to be *complementary commodities* if an decrease in the demand of either results in a decrease in demand of the other (e.g. camera, film).

Substitute and Complementary Commodities. (Continued)

Partial Derivatives

- Suppose $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$ units of the two commodities are demanded when the unit prices of the commodities are p_1 and p_2, respectively.
- We have

$$\frac{\partial D_1}{\partial p_1} < 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_2} < 0.$$

Substitute and Complementary Commodities. (Continued)

Partial Derivatives

- For substitute commodities:

$$\frac{\partial D_1}{\partial p_2} > 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_1} > 0.$$

- For complementary commodities:

$$\frac{\partial D_1}{\partial p_2} < 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_1} < 0.$$

Substitute and Complementary Commodities. (Continued)

Example

Suppose the demand function for flour in a certain community is given by

$$D_1(p_1, p_2) = 500 + \frac{10}{p_1 + 2} - 5p_2,$$

while the corresponding demand for bread is given by

$$D_2(p_1, p_2) = 400 - 2p_1 + \frac{7}{p_2 + 3},$$

where p_1 is the price in euros of a kilogram of flour and p_2 is the price of a loaf of bread.
Determine whether flour and bread are substitute or complementary or neither.

Substitute and Complementary Commodities. (Continued)

Solution.

$$\frac{\partial D_1}{\partial p_2} = -5 < 0 \quad \text{and} \quad \frac{\partial D_2}{\partial p_1} = -2 < 0.$$

Since both partial derivatives are negative,
it follows that flour and bread are complemetary commodities. □

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- L. D. Hofmann, G. L. Bradley, *Calculus – for business, economics and life sciences,* pp. 519–533.

Summary

- Partial derivatives of $z = f(x, y)$:

$$f_x = \frac{\partial z}{\partial x} \quad f_y = \frac{\partial z}{\partial y}$$