Sequences and Series

Sequences and Sequence Limits

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Aims and Objectives

- Learning the notions of an infinite sequence and its limit
- Calculating the limit of a sequence by using algebraic rules
- Introducing the number e: natural exponential base

Contents

Finite Sequences

- A *finite sequence* is a row matrix $[a_1, a_2, a_3, \ldots, a_p]$ of order $1 \times p$.
- When speaking about sequences, we usually avoid brackets:

$$a_1, a_2, a_3, \ldots, a_p$$

  or, shorter, $\{a_n\}_{n=1}^{p}$.
- If we have a finite sequence $\{a_n\}_{n=1}^{p}$, then each integer n from the interval $1 \leq n \leq p$ is associated a real number a_n:

n	1	2	3	$\ldots$	p
a_n	a_1	a_2	a_3	$\ldots$	a_p

Infinite Sequences

- If the integer n has no upper bounded p;
  i.e., every integer $n \geq 1$ (natural number)
  is associated a real number a_n,
  then an *infinite sequence* (*sequence*, for short) is obtained.
- The association:

n	1	2	3	...	n	...
a_n	a_1	a_2	a_3	...	a_n	...

- Notation:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

or, shorter, $\{a_n\}_{n=1}^{\infty}$

Examples of Sequences

Example (...)

1. If the rule of association of n with a_n is $a_n = \frac{1}{n}$, then $a_1 = \frac{1}{1} = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, ..., hence the obtained sequence is
$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$
2. If $a_n = \frac{n}{n+1}$, then the obtained sequence is
$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$$

Examples of Sequences. (Continued)

Example (...)

3. If $a_n = (-1)^n \frac{1}{2n}$, then the obtained sequence is

$$-\frac{1}{2}, \frac{1}{4}, -\frac{1}{6}, \dots, (-1)^n \frac{1}{2n}, \dots$$

The Limit of a Sequence

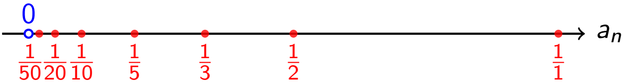
- "Behaviour" of the sequence $a_n = \frac{1}{n}$ when n increases without bound:

n	10	100	200	500	1000	10000
a_n	0.1	0.01	0.005	0.002	0.001	0.0001

- The element a_n gets closer and closer to 0 as the number n increases without bound:

$$\lim_{n \longrightarrow \infty} a_n = 0.$$

- Geometrical interpretation of the expression $\lim_{n \longrightarrow \infty} a_n = 0$



The Limit of a Sequence. (Continued)

The Limit of a Sequence

If a_n gets closer and closer to a number L
when n increases without bound,
then L is the *limit of a_n when n approaches ∞:*

$$\lim_{n \to \infty} a_n = L$$

Properties of Limits

Algebraic Properties of Limits

If $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ exist, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty}(ka_n) = k \lim_{n\to\infty} a_n \quad \text{for any constant } k,$$

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0,$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } \lim_{n\to\infty} a_n^p \text{ exists.}$$

Properties of Limits. (Continued)

Limits of Two Elementary Sequences

- The limit of a constant is the constant itself:

$$\lim_{n \to \infty} k = k.$$

- The limit of $a_n = n$ when $n \to \infty$ is:

$$\lim_{n \to \infty} n = \infty.$$

Properties of Limits. (Continued)

Example

Let $k > 0$ be a constant.

- The limit of a power n^k:

$$\lim_{n \to \infty} n^k = \infty.$$

- The limit of the reciprocal of a power:

$$\lim_{n \to \infty} \frac{1}{n^k} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^k = \left(\lim_{n \to \infty} \frac{1}{n}\right)^k = 0^k = 0.$$

Properties of Limits. (Continued)

The Limit of a Polynomial Expression

If $a_n \neq 0$, then

$$\lim_{n \to \infty} (a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0) = \lim_{n \to \infty} a_k n^k.$$

That is, to find the limit, take the limit of the term of highest degree.

Properties of Limits. (Continued)

Example

Find

$$\lim_{n\to\infty}(1 - n^2 + 4n^3 - 3n^4).$$

Solution.

$$\begin{aligned}\lim_{n\to\infty}(1 - n^2 + 4n^3 - 3n^4) &= \lim_{n\to\infty}(-3n^4)\\ &= -3(\lim_{n\to\infty} n^4) = -\infty.\end{aligned}$$

□

Properties of Limits. (Continued)

The Limit of a Rational Expression

To find the limit when $n \to \infty$ of a rational expression over n:

1. Compare the degrees of the numerator and the denominator and divide numerator and denominator by n raised to the smaller of these degrees.
2. Take the limits of the new numerator and denominator.

Properties of Limits. (Continued)

Example

Find

$$\lim_{n\to\infty} \frac{3n^2 - 2n + 1}{2n^2 + 3n - 1}.$$

Solution.

Divide the numerator and denominator by n^2 to get

$$\lim_{n\to\infty} \frac{3n^2 - 2n + 1}{2n^2 + 3n - 1} = \lim_{n\to\infty} \frac{\frac{3n^2-2n+1}{n^2}}{\frac{2n^2+3n-1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{3 - \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{3}{n} - \frac{1}{n^2}} = \frac{3 - 0 + 0}{2 + 0 - 0} = \frac{3}{2}.$$

$\square$

Properties of Limits. (Continued)

Example

Find

$$\lim_{n \to \infty} \frac{-n^2 + 2n + 1}{5n - 2}.$$

Properties of Limits. (Continued)

Solution.

Divide the numerator and denominator by n to get

$$\lim_{n\to\infty} \frac{-n^2+2n+1}{5n-2} = \lim_{n\to\infty} \frac{-n+2+\frac{1}{n}}{5-\frac{2}{n}}.$$

Since

$$\lim_{n\to\infty} \left(-n+2+\frac{1}{n}\right) = -\infty \quad \text{and} \quad \lim_{n\to\infty} \left(5-\frac{2}{n}\right) = 5,$$

it follows that

$$\lim_{n\to\infty} \frac{-n^2+2n+1}{5n-2} = -\infty.$$

□

The Number e

- Consider the limit of the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n .$$

- The "behaviour" of the sequence when $n \to \infty$:

n	1	10	100	1,000	10,000	100,000
$\left(1 + \frac{1}{n}\right)^n$	2	2.5937	2.7048	2.7169	2.7182	2.7183

- The value of the number e:

$$e = 2.71828...$$

The Number e. (Continued)

The Natural Exponential Base e

The number

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828...$$

is called the *natural exponential base.*

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- S. T. Karris, *Mathematics for business, science and technology,* pp. 2-18–2-32.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* pp. 41-50.

Summary

- An infinite sequence $\{a_n\}_{n=1}^{\infty}$ and its limit $\lim_{n\to\infty} a_n$
- Properties of limits:
  - Algebraic properties of limits
  - Limits of elementary sequences $\lim_{n\to\infty} k$ and $\lim_{n\to\infty} n$
  - The limit of a polynomial expression
  - The limit of a rational expression
- The natural exponential base e