Arithmetic and Geometric Progressions

F. M. Berisha

South East European University, Tetovo

Aims and Objectives

- Learning the notions of two important kinds of sequences: arithmetic sequence and geometric sequence
- Calculating an arbitrary term and the sum of terms of an arithmetic or geometric sequence
- Applications in business and economics.

Contents

Arithmetic Sequence (Progression)

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A sequence a_1, a_2, a_3, . . . , a_n, . . . is an *arithmetic sequence* (or *progression*) if each term can be found by adding a constant d (called *difference*) to the preceding term; i.e., if for every $n > 1$ holds

$$a_n - a_{n-1} = d.$$

Arithmetic Sequence (Progression). (Continued)

Example

A business of producing detergent had an annual production of 147 tons in 2007.
The owner plans to increase its production each year by 4.5 ton.
What is the planned production for 2009?

Arithmetic Sequence (Progression). (Continued)

Solution.

The sequence of planned annual productions
is an arithmetic progression:

Year	2007	2008	2009	...
n	1	2	3	...
a_n	147	151.5	156	...

Thus, the annual production in 2009 is planned to be
$a_3 = 156$ tons of detergent. □

The General Term of an Arithmetic Sequence

The n-th term of an arithmetic sequence:

$$\begin{aligned} a_n = a_{n-1} + d = (a_{n-2} + d) + d &= a_{n-2} + 2d \\ &= a_{n-3} + 3d = \cdots = a_1 + (n-1)d. \end{aligned}$$

The n-th Term of an Arithmetic Sequence

The n-th term of an arithmetic sequence with the first term a_1 and the difference d is

$$\boxed{a_n = a_1 + (n-1)d.}$$

The General Term of an Arithmetic Sequence. (Continued)

Example

Find the amount of the annual production planned for 2007 in the application from the previous example.

Solution.

As we saw, the planned annual productions form an arithmetic sequence with $a_1 = 147$, $d = 4.5$. The term a_{11} of the sequence is required.

$$a_{11} = a_1 + (11 - 1)d = 147 + 10 \cdot 4.5 = 192$$

tons. □

The Sum of Terms of an Arithmetic Sequence

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The sum of first n terms of an arithmetic sequence with the first term a_1 and the difference d is

$$\boxed{S_n = \frac{n}{2}(a_1 + a_n),}$$

or

$$S_n = \frac{n}{2}[2a_1 + (n-1)d].$$

The Sum of Terms of an Arithmetic Sequence. (Cont.)

Example

It has been estimated that contributions in response to a fund-raising campaign will be 5,000 € in the first week and will decrease during the following weeks by 600 € each week. Calculate the total amount of money raised during the first 8 weeks.

The Sum of Terms of an Arithmetic Sequence. (Cont.)

Solution.

The sequence of the weekly amounts raised by the campaign is an arithmetic progression with the first term $a_1 = 5000$ and the difference $d = -600$.

$$S_8 = \frac{8}{2}[2 \cdot 5{,}000 + (8-1) \cdot (-600)] = 23{,}200$$

euros. □

Geometric Sequence (Progression)

Geometric Sequence (Progression)

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is an *geometric sequence* (or *progression*) if each term can be found by multiplying the preceding term by a constant r (called *ratio*); i.e., if for every $n > 1$ holds

$$\frac{a_n}{a_{n-1}} = r.$$

The General Term of a Geometric Sequence

The n-th term of a geometric sequence:

$$a_n = a_{n-1}r = (a_{n-2}r)r = a_{n-2}r^2$$
$$= a_{n-3}r^3 = \cdots = a_1 r^{n-1}.$$

The n-th Term of a Geometric Sequence

The n-th term of a geometric sequence with the first term a_1 and the ratio r is

$$\boxed{a_n = a_1 r^{n-1}.}$$

The General Term of a Geometric Sequence. (Continued)

Example

A producer estimates that the annual revenue from producing and selling a product will increase each year by 15%.
What is the estimated annual revenue for 2015
if by the end of 2005 the annual revenue was 100,000 €?

The General Term of a Geometric Sequence. (Continued)

Solution...

For the annual revenue during the first period we put
$R_1 = 100{,}000$.
Since the annual revenue increases each year by 15%, we have

$$R_2 = R_1 + R_1 \cdot \frac{15}{100} = R_1 \left(1 + \frac{15}{100}\right).$$

If we denote by R_n the annual revenue for the n-th period and by R_{n-1} that for the previous period, then

$$R_n = R_{n-1} + R_{n-1} \cdot \frac{15}{100} = R_{n-1} \left(1 + \frac{15}{100}\right),$$

which means that the sequence of annual revenues is a geometric progression with the ratio $r = 1 + \frac{15}{100} = 1.15$. $\square$

The General Term of a Geometric Sequence. (Continued)

. . . Solution.

The estimated annual revenue for 2015 is

$$R_{11} = 100{,}000 \cdot 1.15^{10} \approx 404{,}556$$

euros. □

The Sum of Terms of a Geometric Sequence

The Sum of Terms of a Geometric Sequence

The sum of first n terms of a geometric sequence with the first term a_1
and the ratio r is

$$\boxed{S_n = a_1 \frac{q^n - 1}{q - 1}.}$$

The Sum of Terms of a Geometric Sequence. (Continued)

Example

Suppose that during the first year of its use
a certain industrial machinery generates a 3,000 € profit
and that each of the following years the profit decreases by 13%.
What is the total profit generated by the use of the machinery
during 15 years?

The Sum of Terms of a Geometric Sequence. (Continued)

Solution.

Denote by P_n the profit for the n-th year. Then

$$P_n = P_{n-1} - P_{n-1} \cdot \frac{13}{100} = P_{n-1}\left(1 - \frac{13}{100}\right);$$

i.e., the sequence of annual profits is a geometric progression with the ratio $r = 1 - \frac{13}{100} = 0.87$ and the first term $P_1 = 3{,}000$.

$$S_{15} = 3{,}000 \cdot \frac{0.87^{15} - 1}{0.87 - 1} \approx 20{,}219.6$$

euros. □

For Further Reading

- http://fberisha.netfirms.com
- Homework: Exercises from teaching materials
- S. T. Karris, *Mathematics for business, science and technology,* pp. 2-18–2-32.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* fq. 50–56.

Summary

- The notions of arithmetic and geometric progressions
- Finding and using in applications the general term and the sum of terms of an arithmetic or a geometric sequence
- Understanding the relationship between an increase or a decrease by a fixed percentage and the geometric sequence.