Series

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Aims and Objectives

- Learning the notion of a series and its convergence
- Calculating the sum of a geometric series
- Determining the nature of a simple series

Contents

Notion of a Series

Let a_1, a_2, a_3, ..., a_n, ... be a sequence.
Form the sequence of sums $\{S_n\}_{n=1}^{\infty}$ of terms of the sequence $\{a_n\}_{n=1}^{\infty}$:

$$
\begin{aligned}
S_1 &= a_1, \\
S_2 &= a_1 + a_2, \\
S_3 &= a_1 + a_2 + a_3, \\
&\vdots \\
S_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k, \\
&\vdots
\end{aligned}
$$

The sequence $\{S_n\}_{n=1}^{\infty}$ is called a *series.*
If it exists, the limit $S = \lim_{n \to \infty} S_n$ is called the *sum of the series.*

Notion of a Series. (Continued)

Series

- If $\{a_n\}_{n=1}^{\infty}$ is a number sequence,
  then the sequence of its *partial sums* $S_n = \sum\limits_{k=1}^{n} a_k$
  is called a *series.*
- If $S = \lim\limits_{n \to \infty} S_n$, then S is the *sum of the series:*

$$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k.$$

- If there exists the sum of a series,
  then the series is said to be *convergent;*
  if not, the series is *divergent.*

The Series $\sum_{n=1}^{\infty} 1/n$

Shembull

Determine the nature (convergent or divergent) of the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

The Series $\sum_{n=1}^{\infty} 1/n$. (Continued)

Zgjidhje.

The following table gives an intuitive idea about the nature of the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

n	1	2	5	10	100	1000	10000
$\sum_{k=1}^{n} \frac{1}{k}$	1	1.5	2.2833	2.9290	5.1874	7.4855	9.7876

So, the impression is that the partial sums increase without bound:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

$\square$

The Series $\sum_{n=1}^{\infty} 1/n^2$

Shembull

What is the nature of the following series?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Zgjidhje.

n	10	100	10^3	10^4	10^5	10^6
$\sum_{k=1}^{n} \frac{1}{k^2}$	1.5498	1.6350	1.6439	1.6448	1.6449	1.6449

From the table, we intuitively see that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Its sum is not greater than, say, 2. $\square$

The Series $\sum_{n=1}^{\infty} 1/n^k$

The nature of the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$

- If $k > 1$, then the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ converges.
- If $k \leq 1$, then the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ diverges.

Geometric Series

- If a sequence $\{a_n\}_{n=1}^{\infty}$ is a geometric sequence, then the series $\sum_{n=1}^{\infty} a_n$ is called a *geometric series.*
- Its sum is (recollect the previous section)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a_1 \frac{q^n - 1}{q - 1} = a_1 \frac{\left(\lim_{n\to\infty} q^n\right) - 1}{q - 1}.$$

- What can be said about $\lim_{n\to\infty} q^n$ for given values of the constant q?

Geometric Series. (Continued)

The table illustrates the behaviour of the sequence $\{q^n\}_{n=1}^{\infty}$ for the values $q = \frac{2}{3}$, $q = -\frac{3}{4}$, $q = 2$ and $q = -1.5$.

n	1	2	5	10	50	100
$\left(\frac{2}{3}\right)^n$	0.6667	0.4444	0.1317	0.017	$1.6 \cdot 10^{-9}$	$2.5 \cdot 10^{-18}$
$\left(-\frac{3}{4}\right)^n$	-0.75	0.5625	-0.2373	0.0563	$5.7 \cdot 10^{-7}$	$3.2 \cdot 10^{-13}$
2^n	2	4	32	1024	$1.1 \cdot 10^{15}$	$1.3 \cdot 10^{30}$
$(-1.5)^n$	-1.5	2.25	-7.594	57.67	$6.4 \cdot 10^{8}$	$4.1 \cdot 10^{17}$

Geometric Series. (Continued)

The nature of a sequence $\{q^n\}_{n=1}^{\infty}$

- If $-1 < q < 1$, then
$$\lim_{n\to\infty} q^n = 0.$$
- If $q > 1$, then
$$\lim_{n\to\infty} q^n = \infty.$$
- If $q < -1$, then the sequence $\{q^n\}_{n=1}^{\infty}$ has no limit.

Geometric Series. (Continued)

The Nature of a Geometric Series

- If $-1 < q < 1$, then
$$\sum_{k=0}^{\infty} aq^k = \frac{a}{1-q}.$$
- If $q > 1$, then
$$\sum_{k=0}^{\infty} aq^k = \infty.$$
- If $q < -1$, then the series $\sum_{k=0}^{\infty} aq^k$ diverges.

A Criterion for Divergence of a Series

A Criterion for Divergence of a Series

- If a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.
- Hence, if it is **not** $\lim_{n \to \infty} a_n = 0$,

  then the series $\sum_{n=1}^{\infty} a_n$ diverges.

D'Alembert's Criterion

D'Alembert's Criterion for Convergence of a Series

- For each n let $a_n > 0$ and
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
- If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

For Further Reading

- `http://fberisha.netfirms.com`
- Homework: Exercises from teaching materials
- S. T. Karris, *Mathematics for business, science and technology,* pp. 2-18–2-32.
- F. M. Berisha, M. Q. Berisha, *Matematikë – për biznes dhe ekonomiks,* fq. 57–63.

Summary

- Notions of a series, its convergence and sum
- The nature of a sequence $\{q^n\}_{n=1}^{\infty}$ and its limit
- Calculating the sum of a geometric series
- Criteria for determining the nature of a series