I Counting

Counting things is a central problem in Discrete Mathematics. Once we can count, we can determine the likelihood of a particular even and we can estimate how long a computer algorithm takes to complete a task.

1 Sets and Lists

Sets and lists are fundamental concepts that arise in various contexts, including computer algorithms. We study basic counting problems in terms of these concepts.

Sorting. A common computational task is to rearrange elements in order. Given a linear array $A[1..n]$ of integers, rearrange them such that $A[i] \leq A[i+1]$ for $1 \leq i < n$.

```
for i = 1 to n - 1 do
  for j = i + 1 downto 2 do
    if A[j] > A[j - 1] then
      aux = A[j]; A[j] = A[j - 1]; A[j] = aux
    endif
  endfor
endfor.
```

We wish to count the number of comparisons made in this algorithm. For example, sorting an array of five elements uses 15 comparisons. In general, we make $1 + 2 + \cdots + (n-1) = \sum_{i=1}^{n-1} i$ comparisons.

Sums. We now derive a closed form for the above sum by adding it to itself. Arranging the second sum in reverse order and adding the terms in pairs, we get

$$[1 + (n-1)] + \ldots + [(n-1) + 1] \quad = \quad n(n-1).$$

Since each number of the original sum is added twice, we divide by two to obtain

$$\sum_{i=1}^{n-1} i \quad = \quad \frac{n(n-1)}{2}.$$

As with many mathematical proofs, this is not the only way to derive this sum. We can think of the sum as two sets of stairs that stack together, as in Figure 1. At the base, we have $n-1$ gray blocks and one white block. At each level, one more block changes from gray to white, until we have one gray block and $n-1$ white blocks. Together, the stairs form a rectangle divided into $n-1$ by n squares, with exactly half the squares gray and the other half white. Thus, $\sum_{i=1}^{n} i = \frac{n(n-1)}{2}$, same as before. Notice that this sum can appear in other forms, for example,

$$\begin{aligned} \sum_{i=1}^{n-1} i &= 1 + 2 + \ldots + (n-1) \\ &= (n-1) + (n-2) + \ldots + 1 \\ &= \sum_{i=1}^{n-1} (n-i). \end{aligned}$$

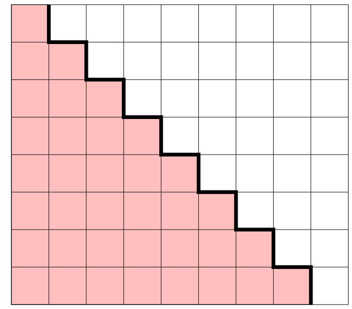

Figure 1: The number of squares in the grid is twice the sum from 1 to 8.

Sets. A *set* is an unordered collection of distinct elements. The *union* of two sets is the set of elements that are in one set or the other, that is, $A \cup B = \{x \mid x \in A$ or $x \in B\}$. The *intersection* of the same two sets is the set of elements that are in both, that is, $A \cap B = \{x \mid x \in A$ and $x \in B\}$. We say that A and B are disjoint if $A \cap B = \emptyset$. The *difference* is the set of elements that belong to the first but not to the second set, that is, $A - B = \{x \mid x \in A$ and $x \notin B\}$. The *symmetric difference* is the set of elements that belong to exactly one of the two sets, that is, $A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. Look at Figure 2 for a visual description of the sets that result from the four types of operations. The number of elements in a set A is denoted as $|A|$. It is referred to as the *size* or the *cardinality* of A. The number of elements in the union of two sets cannot be larger than the sum of the two sizes.

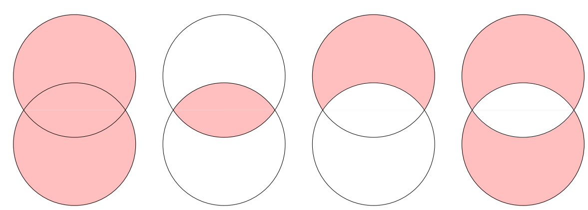

Figure 2: From left to right: the union, the intersection, the difference, and the symmetric difference of two sets represented as disks in the plane.

SUM PRINCIPLE 1. $|A \cup B| \leq |A| + |B|$ with equality if A and B are disjoint.

To generalize this observation to more than two sets, we call the sets $S_1, S_2, \ldots, S_m$ a *covering* of $S = S_1 \cup S_2 \cup \ldots \cup S_m$. If $S_i \cap S_j = \emptyset$ for all $i \neq j$, then the covering is called a *partition*. To simplify the notation, we write $\bigcup_{i=1}^m S_i = S_1 \cup S_2 \cup \cdots \cup S_m$.

SUM PRINCIPLE 2. Let $S_1, S_2, \ldots, S_m$ be a covering of S. Then, $|S| \leq \sum_{i=1}^m |S_i|$, with equality if the covering is a partition.

Matrix multiplication. Another common computational task is the multiplication of two matrices. Assuming the first matrix is stored in a two-dimensional array $A[1..p, 1..q]$ and the second matrix is stored in $B[1..q, 1..r]$, we match up rows of A with the columns of B and form the sum of products of corresponding elements. For example, multiplying

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

with

$$B = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & 5 \\ 4 & 0 & 1 \end{bmatrix}$$

results in

$$C = \begin{bmatrix} 11 & 8 & 20 \\ 18 & 4 & 14 \end{bmatrix}.$$

The algorithm we use to get C from A and B is described in the following pseudo-code.

```
for i = 1 to p do
  for j = 1 to r do
    C[i, j] = 0;
    for k = 1 to q do
      C[i, j] = C[i, j] + A[i, k] · B[k, j]
    endfor
  endfor
endfor.
```

We are interested in counting how many multiplications the algorithm takes. In the example, each entry of the result uses three multiplications. Since there are six entries in C, there are a total of $6 \cdot 3 = 18$ multiplications. In general, there are q multiplications for each of pr entries of the result. Thus, there are pqr multiplications in total. We state this observation in terms of sets.

PRODUCT PRINCIPLE 1. Let $S = \bigcup_{i=1}^m S_i$. If the sets $S_1, S_2, \ldots, S_m$ form a partition and $|S_i| = n$ for each $1 \leq i \leq m$ then $|S| = nm$.

We can also encode each multiplication by a triplet of integers, the row number in A, the column number in A which is also the row number in B, and the column number in B. There are p possibilities for the first number, q for the second, and r for the third number. We generalize this method as follows.

PRODUCT PRINCIPLE 2. If S is a set of lists of length m with i_j possibilities for position j, for $1 \leq j \leq m$, then $|S| = i_1 \cdot i_2 \cdot \ldots \cdot i_m = \prod_{j=1}^m i_j$.

We can use this rule to count the number of cartoon characters that can be created from a book giving choices for head, body, and feet. If there are p choices for the head, q choices for the body, and r choices for the legs, then there are pqr different cartoon characters we can create.

Number of passwords. We apply these principles to count the passwords that satisfy some conditions. Suppose a valid password consists of eight characters, each a digit or a letter, and there must be at least two digits. To count the number of valid passwords, we first count the number of eight character passwords without the digit constraint: $(26+10)^8 = 36^8$. Now, we subtract the number of passwords that fail to meet the digit constraint, namely the passwords with one or no digit. There are 26^8 passwords without any digits. To count the passwords with exactly one digit, we note that there are 26^7 ways to choose an ordered set of 7 letters, 10 ways to choose one digit, and 8 places to put the digit in the list of letters. Therefore, there are $26^7 \cdot 10 \cdot 8$ passwords with only one digit. Thus, there are $36^8 - 26^8 - 26^7 \cdot 10 \cdot 8$ valid passwords.

Lists. A *list* is an ordered collection of elements which are not necessarily different from each other. We note two differences between lists and sets:

(1) a list is ordered, but a set is not;

(2) a list can have repeated elements, but a set can not.

Lists can be expressed in terms of another mathematical concept in which we map elements of one set to elements of another set. A *function* f from a *domain* D to a *range* R, denoted as $f : D \to R$, associates exactly one element in R to each element $x \in D$. A list of k elements is a function $\{1, 2, \ldots, k\} \to R$. For example, the function in Figure 3 corresponds to the list $a, b, c, b, z, 1, 3, 3$. We can use the Product Principle 2 to count the number of different functions from a finite domain, D, to a finite range, R.

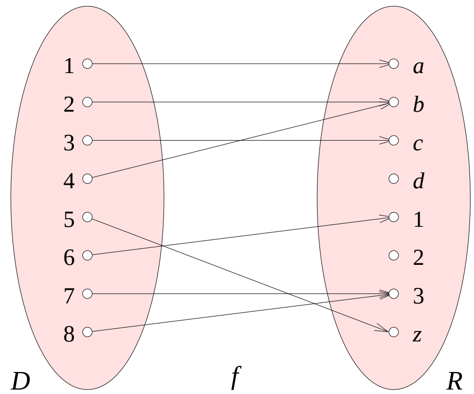

Figure 3: A function representing a list.

Specifically, we have a list of length $|D|$ with $|R|$ possibilities for each position. Hence, the number of different functions from D to R is $|R|^{|D|}$.

Bijections. The function $f : D \to R$ is *injective* or *one-to-one* if $f(x) \neq f(y)$ for all $x \neq y$. It is *surjective* or *onto* if for every $r \in R$, there exists some $x \in D$ with $f(x) = r$. The function is *bijective* or a *one-to-one correspondence* if it is both injective and surjective.

BIJECTION PRINCIPLE. Two sets D and R have the same size if and only if there exists a bijection $f : D \to R$.

Thus, asking how many bijections there are from D to R only makes sense if they have the same size. Suppose this size is finite, that is, $|D| = |R| = n$. Then being injective is the same as being bijective. To count the number of bijections, we assign elements of R to elements of D, in sequence. We have n choices for the first element in the domain, $n-1$ choices for the second, $n-2$ for the third, and so on. Hence the number of different bijections from D to R is $n \cdot (n-1) \cdot \ldots \cdot 1 = n!$.

Summary. Today, we began with the building blocks of counting: sets and lists. We went through some examples using the sum and product principles: counting the number of times a loop is executed, the number of possible passwords, and the number of combinations. Finally, we talked about functions and bijections.

2 Binomial Coefficients

In this section, we focus on counting the number of ways sets and lists can be chosen from a given set.

Permutations. A *permutation* is a bijection from a finite set D to itself, $f : D \to D$. For example, the permutations of $\{1, 2, 3\}$ are: $123, 132, 213, 231, 312$, and 321. Here we list the permutations in lexicographic order, same as they would appear in a dictionary. Assuming $|D| = k$, there are $k!$ permutations or, equivalently, orderings of the set. To see this, we note that there are k choices for the first element, $k - 1$ choices for the second, $k - 2$ for the third, and so on. The total number of choices is therefore $k(k - 1) \cdot \ldots \cdot 1$, which is the definition of $k!$.

Let $N = \{1, 2, \ldots, n\}$. For $k \leq n$, a *k-element permutation* is an injection $\{1, 2, \ldots, k\} \to N$. In other words, a k-element permutation is a list of k distinct elements from N. For example, the 3-element permutations of $\{1, 2, 3, 4\}$ are

123,	124,	132,	134,	142,	143,
213,	214,	231,	234,	241,	243,
312,	314,	321,	324,	341,	342,
412,	413,	421,	423,	431,	432.

There are 24 permutations in this list. There are six orderings of the subset $\{1, 2, 3\}$ in this list. In fact, each 3-element subset occurs six times. In general, we write $n^{\underline{k}}$ for the number of k-element permutations of a set of size n. We have

$$\begin{aligned} n^{\underline{k}} &= \prod_{i=0}^{k-1} (n - i) \\ &= n(n-1)\cdots(n-(k-1)) \\ &= \frac{n!}{(n-k)!}. \end{aligned}$$

Subsets. The *binomial coefficient* $\binom{n}{k}$, pronounced n choose k, is by definition the number of k-element subsets of a size n set. Since there are $k!$ ways to order a set of size k, we know that $n^{\underline{k}} = \binom{n}{k} \cdot k!$ which implies

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

We fill out the following tables with values of $\binom{n}{k}$, where the row index is the values of n and the column index is the value of k. Values of $\binom{n}{k}$ for $k > n$ are all zero and are omitted from the table.

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

By studying this table, we notice several patterns.

- $\binom{n}{0} = 1$. In words, there is exactly one way to choose no item from a list of n items.
- $\binom{n}{n} = 1$. In words, there is exactly one way to choose all n items from a list of n items.
- Each row is symmetric, that is, $\binom{n}{k} = \binom{n}{n-k}$.

This table is also known as Pascal's Triangle. If we draw it symmetric between left and right then we see that each entry in the triangle is the sum of the two entries above it in the previous row.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Pascal's Relation. We express the above recipe of constructing an entry as the sum of two previous entries more formally. For convenience, we define $\binom{n}{k} = 0$ whenever $k < 0$, $n < 0$, or $n < k$.

PASCAL'S RELATION. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

PROOF. We give two arguments for this identity. The first works by algebraic manipulations. We get

$$\begin{aligned} \binom{n}{k} &= \frac{(n-k)(n-1)! + k(n-1)!}{(n-k)!k!} \\ &= \frac{(n-1)!}{(n-k-1)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!} \\ &= \binom{n-1}{k} + \binom{n-1}{k-1}. \end{aligned}$$

For the second argument, we partition the sets. Let $|S| = n$ and let a be an arbitrary but fixed element from S. $\binom{n}{k}$ counts the number of k-element subsets of S. To get the number of subsets that contain a, we count the $(k-1)$-element subsets of $S - \{a\}$, and to get the number of subsets that do not contain a, we count the k-element subsets of $S - \{a\}$. The former is $\binom{n-1}{k-1}$ and the latter is $\binom{n-1}{k}$. Since the subsets that contain a are different from the subsets that do not contain a, we can use the Sum Principle 1 to get the number of k-element subsets of S equal to $\binom{n-1}{k-1} + \binom{n-1}{k}$, as required. $\square$

Binomials. We use binomial coefficients to find a formula for $(x+y)^n$. First, let us look at an example.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + yx + xy + yy\\ &= x^2 + 2xy + y^2.\end{aligned}$$

Notice that the coefficients in the last line are the same as in the second line of Pascal's Triangle. This is more generally the case and known as the

BINOMIAL THEOREM. $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$.

PROOF. If we write each term of the result before combining like terms, we list every possible way to select one x or one y from each factor. Thus, the coefficient of $x^{n-i}y^i$ is equal to $\binom{n}{n-i} = \binom{n}{i}$. In words, it is the number of ways we can select $n - i$ factors to be x and have the remaining i factors to be y. This is equivalent to selecting i factors to be y and have the remaining factors be x. $\square$

Corollaries. The Binomial Theorem can be used to derive a number of other interesting sums. We prove three such consequences.

COROLLARY 1. $\sum_{i=0}^{n} \binom{n}{i} = 2^n$.

PROOF. Let $x = y = 1$. Then, by the Binomial Theorem we have

$$(1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i.$$

This implies the claimed identity. $\square$

COROLLARY 2. $\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$.

PROOF. We use Pascal's Relation to prove this identity. It is instructive to trace our steps graphically, in the triangle above. In a first step, we replace $\binom{n+1}{k+1}$ by $\binom{n}{k}$ and $\binom{n}{k+1}$. Keeping the first term, we replace the second, $\binom{n}{k+1}$, by $\binom{n-1}{k}$ and $\binom{n-1}{k+1}$. Repeating this operation, we finally replace $\binom{k+1}{k+1}$ by $\binom{k}{k} = 1$ and $\binom{k}{k+1} = 0$. In other words, $\binom{n+1}{k+1}$ is equal to the sum of the $\binom{j}{k}$ for j running from n down to k. $\square$

COROLLARY 3. $\sum_{i=1}^{n} i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

PROOF. We first express the summands in terms of binomial coefficients and then use Corollary 2 to get the result.

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= 2\sum_{i=1}^{n} \frac{i^2 - i}{2} + \sum_{i=1}^{n} i\\ &= 2\sum_{i=1}^{n} \binom{i}{2} + \sum_{i=1}^{n} \binom{i}{1}\\ &= 2\binom{n+1}{3} + \binom{n+1}{2}\\ &= \frac{2(n+1)n(n-1)}{1 \cdot 2 \cdot 3} + \frac{(n+1)n}{1 \cdot 2}\\ &= \frac{n^3 - n}{3} + \frac{n^2 + n}{2}.\end{aligned}$$

This implies the claimed identity. $\square$

Summary. The binomial coefficient, $\binom{n}{k}$, counts the different ways we can choose k elements from a set of n. We saw how it can be used to compute $(x+y)^n$. We proved several corollaries and saw that describing the identities as counting problems can lead us to different, sometimes simpler proofs.

3 Equivalence Relations

Equivalence relations are a way to partition a set into subsets of equivalent elements. Being equivalent is then interpreted as being the same, such as different views of the same object or different ordering of the same elements, etc. By counting the equivalence classes, we are able to count the items in the set that are different in an essential way.

Labeling. To begin, we ask how many ways are there to label three of five elements red and the remaining two elements blue? Without loss of generality, we can call our elements A, B, C, D, E. A labeling is an function that associates a color to each element. Suppose we look at a permutation of the five elements and agree to color the first three red and the last two blue. Then the permutation $ABDCE$ would correspond to coloring A, B, D red and C, E blue. However, we get the same labeling with other permutations, namely

$$\begin{array}{lll} ABD;CE & BAD;CE & DAB;CE \\ ABD;EC & BAD;EC & DAB;EC \\ ADB;CE & BDA;CE & DBA;CE \\ ADB;EC & BDA;EC & DBA;EC\,. \end{array}$$

Indeed, we have $3!2! = 12$ permutations that give the same labeling, simply because there are $3!$ ways to order the red elements and $2!$ ways to order the blue elements. Similarly, every other labeling corresponds to 12 permutations. In total, we have $5! = 120$ permutations of five elements. The set of 120 permutations can thus be partitioned into $\frac{120}{12} = 10$ blocks such that any two permutations in the same block give the same labeling. Any two permutations from different blocks give different labelings, which implies that the number of different labelings is 10. More generally, the number of ways we can label k of n elements red and the remaining $n - k$ elements blue is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$. This is also the number of k-element subsets of a set of n elements.

Now suppose we have three labels, red, green, and blue. We count the number of different labelings by dividing the total number of orderings by the orderings within in the color classes. There are $n!$ permutations of the n elements. We want i elements red, j elements blue, and $k = n - i - j$ elements green. We agree that a permutation corresponding to the labeling we get by coloring the first i elements red, the next j elements blue, and the last k elements green. The number of repeated labelings is thus $i!$ times $j!$ times $k!$ and we have $\frac{n!}{i!j!k!}$ different labelings.

Equivalence relations. We now formalize the above method of counting. A *relation* on a set S is a collection R of ordered pairs, (x, y). We write $x \sim y$ if the pair (x, y) is in R. We say that a relation is

- *reflexive* if $x \sim x$ for all $x \in S$;
- *symmetric* if $x \sim y$ implies $y \sim x$;
- *transitive* if $x \sim y$ and $y \sim z$ imply $x \sim z$.

We say that the relation is an *equivalence relation* if R is reflexive, symmetric, and transitive. If S is a set and R an equivalence relation on S, then the *equivalence class* of an element $x \in S$ is

$$[x] \quad = \quad \{y \in S \mid y \sim x\}.$$

We note here that if $x \sim y$ then $[x] = [y]$. In the above labeling example, S is the set of permutations of the elements A, B, C, D, E and two permutations are equivalent if they give the same labeling. Recalling that we color the first three elements red and the last two blue, the equivalence classes are $[ABC; DE]$, $[ABD; CE]$, $[ABE; CD]$, $[ACD; BE]$, $[ACE; BD]$, $[ADE; BC]$, $[BCD; AE]$, $[BCE; AD]$, $[BDE; AC]$, $[CDE; AB]$.

Not all relations are equivalence relations. Indeed, there are relations that have none of the above three properties. There are also relations that satisfy any subset of the three properties but none of the rest.

An example: modular arithmetic. We say an integer a is *congruent* to another integer b modulo a positive integer n, denoted as $a = b \bmod n$, if $b - a$ is an integer multiple of n. To illustrate this definition, let $n = 3$ and let S be the set of integers from 0 to 11. Then $x = y \bmod 3$ if x and y both belong to $S_0 = \{0, 3, 6, 9\}$ or both belong to $S_1 = \{1, 4, 7, 10\}$ or both belong to $S_2 = \{2, 5, 8, 11\}$. This can be easily verified by testing each pair. Congruence modulo 3 is in fact an equivalence relation on S. To see this, we show that congruence modulo 3 satisfies the three required properties.

reflexive. Since $x - x = 0 \cdot 3$, we know that $x = x \bmod 3$.

symmetric. If $x = y \bmod 3$ then x and y belong to the same subset S_i. Hence, $y = x \bmod 3$.

transitive. Let $x = y \bmod 3$ and $y = z \bmod 3$. Hence x and y belong to the same subset S_i and so do y and z. It follows that x and z belong to the same subset.

More generally, congruence modulo n is an equivalence relation on the integers.

Block decomposition. An equivalence class of elements is sometimes called a *block*. The importance of equivalence relations is based on the fact that the blocks partition the set.

THEOREM. Let R be an equivalence relation on some set S. Then the blocks $S_x = \{y \in S \mid x \sim y, y \in S\}$ for all $x \in S$ partition S.

PROOF. In order to prove that $\bigcup_x S_x = S$, we need to show two things, namely $\bigcup_{x \in S} S_x \subseteq S$ and $\bigcup_{x \in S} S_x \supseteq S$. Each S_x is a subset of S which implies the first inclusion. Furthermore, each $x \in S$ belongs to S_x which implies the second inclusion. Additionally, if $S_x \neq S_y$, then $S_x \cap S_y = \emptyset$ since $z \in S_x$ implies $z \sim x$, which means that $S_x = S_z$, which means that $S_z \neq S_y$. Therefore, z is not related to y, and so $z \notin S_y$. □

Symmetrically, a partition of S defines an equivalence relation. If the blocks are all of the same size then it is easy to count them.

QUOTIENT PRINCIPLE. If a set S of size p can be partitioned into q classes of size r each, then $p = qr$ or, equivalently, $q = \frac{p}{r}$.

Multisets. The difference between a set and a *multiset* is that the latter may contain the same element multiple times. In other words, a multiset is an unordered collection of elements, possibly with repetitions. We can list the repetitions,

$$\langle\langle c, o, l, o, r \rangle\rangle$$

or we can specify the multiplicities,

$$m(c) = 1, m(o) = 2, m(r) = 1.$$

The *size* of a multiset is the sum of the multiplicities. We show how to count multisets by considering an example, the ways to distribute k (identical) books among n (different) shelves. The number of ways is equal to

- the number of size-k multisets of the n shelves;
- the number of ways to write k as a sum of n non-negative integers.

We count the ways to write k as a sum of n non-negative integers as follows. Choose the first integer of the sum to be p. Now we have reduced the problem to counting the ways to write $k - p$ as the sum of $n - 1$ non-negative integers. For small values of n, we can do this.

For example, let $n = 3$. Then, we have $p + q + r = k$. The choices for p are from 0 to k. Once p is chosen, the choices for q are fewer, namely from 0 to $k - p$. Finally, if p and q are chosen then r is determined, namely $r = k - p - q$. The number of ways to write k as a sum of three non-negative integers is therefore

$$\begin{aligned}
\sum_{p=0}^{k} \sum_{q=0}^{k-i} 1 &= \sum_{p=0}^{k} (k - p + 1) \\
&= \sum_{p=1}^{k+1} p \\
&= \binom{k+2}{2}.
\end{aligned}$$

There is another (simpler) way of finding this solution. Suppose we line up our n books, then place $k - 1$ dividers between them. The number of books between the i-th and the $(i-1)$-st dividers is equal to the number of books on the i-th shelf; see Figure 4. We thus have $n + k - 1$ objects, k books plus $n - 1$ dividers. The number of ways to

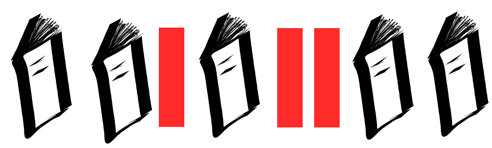

Figure 4: The above arrangement of books and blocks represents two books placed on the first and last shelves, and one book on the second shelf. As a sum, this figure represents $2 + 1 + 0 + 2$.

choose $n - 1$ dividers from $n + k - 1$ objects is $\binom{n+k-1}{n-1}$. We can easily see that this formula agrees with the result we found for $n = 3$.

Summary. We defined relations and equivalence relations, investigating several examples of both. In particular, modular arithmetic creates equivalence classes of the integers. Finally, we looked at multisets, and saw that counting the number of size-k multisets of n elements is equal to the number of ways to write k as a sum of n non-negative integers.

First Homework Assignment

Write the solution to each question on a single page. The deadline for handing in solutions is January 26.

Question 1. ($20 = 10 + 10$ points). If n basketball teams play each other team exactly once, how many games will be played in total? If the teams then compete in a single elimination tournament (similar to March Madness), how many additional games are played?

Question 2. ($20 = 10 + 10$ points).

(a) (Problem 1.2-7 in our textbook). Let $|D| = |R| = n$. Show that the following statement is true: The function $f : D \to R$ is surjective if and only if f is injective.

(b) Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ a bijection? Prove or give a counterexample.

Question 3. ($20 = 6 + 7 + 7$ points).

(a) What is the coefficient of the x^8 term of $(x - 2)^{30}$?

(b) What is the coefficient of the $x^i y^j z^k$ term of $(x + y + z)^n$?

(c) Show that $\binom{n}{k} = \binom{n}{n-k}$.

Question 4. ($20 = 6+7+7$ points). For (a) and (b), prove or disprove that the relations given are equivalence relations. For (c), be sure to justify your answer.

(a) Choose some $k \in \mathbb{Z}$. Let $x, y \in \mathbb{Z}$. We say $x \sim y$ if $x \equiv y \mod k$.

(b) Let x, y be positive integers. We say $x \sim y$ if the greatest common factor of x and y is greater than 1.

(c) How many ways can you distribute k identical cookies to n children?