

## 1. Mathematical Induction and Combinatorics

- (1) Show that for each positive integer  $n$ , we have

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

- (2) Show that the cube of a positive integer can always be written as the difference of two squares.

- (3) Establish a formula for  $\sum_{k=2}^n \frac{1}{k^2 - 1}$  valid for each positive integer  $n$ .

- (4) Establish a formula allowing one to obtain the sum of the first  $n$  positive even integers.

- (5) Show that the formula  $\sum_{j=1}^n (-1)^j j^2 = (-1)^n \sum_{j=1}^n j$  holds for each positive integer  $n$ .

- (6) Show that  $a + b$  is a factor of  $a^{2n-1} + b^{2n-1}$  for each integer  $n \geq 1$ .

- (7) Show that  $a^2 + b^2$  is a factor of  $a^{4n} - b^{4n}$  for each integer  $n \geq 1$ .

- (8) Show that for each positive integer  $n$ ,

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

- (9) Show that  $\sum_{j=1}^n j \cdot j! = (n+1)! - 1$  for each positive integer  $n$ .

- (10) Prove, using induction, that  $(2n)! < 2^{2n}(n!)^2$  for each integer  $n \geq 1$ .

- (11) Use induction in order to prove that  $n^3 < n!$  for each integer  $n \geq 6$ .

- (12) Let  $\theta$  be a real number such that  $\theta \geq -1$ . Prove, using induction, that for each integer  $n \geq 0$ , we have  $(1 + \theta)^n \geq 1 + n\theta$ .

- (13) Let  $\theta$  be a nonnegative real number. Show, using induction, that for each positive integer  $n$ , we have  $(1 + \theta)^n \geq 1 + n\theta + \frac{n(n-1)}{2}\theta^2$ .

- (14) Show that for each positive integer  $n$ ,  $\frac{1}{3}(n^3 + 2n)$  is an integer.

- (15) Show that  $\frac{10^n + 3 \cdot 4^{n+2} + 5}{9}$  is an integer for each positive integer  $n$ .

- (16) Show that if  $n$  is a positive integer, then

$$\binom{n}{k} = \binom{n}{k+1} \iff n = 2k + 1.$$

- (17) Show that if  $n$  is a positive integer, then

$$(a) \quad \binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = 2^{2n-1};$$

$$(b) \quad \binom{2n}{1} + \binom{2n}{3} + \cdots + \binom{2n}{2n-1} = 2^{2n-1}.$$

- (18) Prove that for each integer  $n \geq 1$ , we have

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

- (19) Show that each integer  $n > 7$  can be written as a sum containing only the numbers 3 and 5. For example,  $8 = 3 + 5$ ,  $9 = 3 + 3 + 3$ ,  $10 = 5 + 5$ .  
 (20) Assume that amongst  $n$  points,  $n \geq 2$ , in a given plane, no three points are on the same line. Show that the number of possible lines passing through these points is  $n(n-1)/2$ .  
 (21) Show that for each integer  $n \geq 2$ ,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

- (22) Prove that for each positive integer  $k$ ,

$$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2-1).$$

- (23) We saw in problem 1 that, for each integer  $n \geq 1$ ,

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}; \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}; \\ 1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}. \end{aligned}$$

Hence, letting  $S_k(n) = 1^k + 2^k + \cdots + n^k$  and in light of these three relations, it is normal to conjecture that, for each integer  $k \geq 1$ ,  $S_k(n)$  is a polynomial of degree  $k+1$ . In fact, in 1654, Blaise Pascal (1623–1662) established that indeed it was the case. His proof used induction and the expansion of the expression  $(n+1)^{k+1} - 1$ . Provide the details.

- (24) Find a formula, valid for each integer  $n \geq 2$ , for

$$\prod_{i=2}^n \left(1 - \frac{1}{i}\right), \quad \text{and the same for } \prod_{i=2}^n \left(1 - \frac{1}{i^2}\right).$$

- (25) Show that, whatever the value of the integer  $n \geq 1$ , we always have

$$\sum_{i=1}^n \frac{i}{i^4 + i^2 + 1} < \frac{1}{2}.$$

- (26) Show that if  $m$ ,  $n$  and  $r$  are three positive integers such that

$$S := \frac{1}{m} + \frac{1}{n} + \frac{1}{r} < 1, \quad \text{then} \quad S \leq \frac{41}{42}.$$

- (27) Given a positive integer  $n$ , let  $s(n)$  be the sum of its digits (in basis 10). For each pair of positive integers  $k, \ell$  smaller than 10, let  $A_k(\ell)$  be the number of  $\ell$ -digit positive integers  $n$  whose sum of digits is equal to  $k$ . In other words,

$$A_k(\ell) = \#\{n : 10^{\ell-1} \leq n < 10^\ell, s(n) = k\}.$$

Show that

$$A_k(\ell) = \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1},$$

and conclude in particular that  $A_k(\ell) = A_\ell(k)$ .

- (28) Using induction, prove the formulas due to Mariares (1913):

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \cdots + n^2 &= \binom{n+2}{3}, & \text{if } n \text{ is odd;} \\ 2^2 + 4^2 + 6^2 + \cdots + n^2 &= \binom{n+2}{3}, & \text{if } n \text{ is even.} \end{aligned}$$

- (29) Let  $S$  be a set of 10 distinct integers chosen amongst the numbers  $1, 2, \dots, 99$ . Show that  $S$  must contain two disjoint subsets for which the sum of their respective elements is the same.
- (30) Given 51 arbitrary positive integers, show that one can always find two of them whose difference is 50.
- (31) In order to acquire problem solving skills, a student decides to solve at least one problem per day and at most 11 per week and to do this for a whole year. Show that there exists a period of consecutive days during which he will solve exactly 20 problems.
- (32) On a rectangular table of dimension 120 inches by 150 inches, we set 14 001 marbles. Show that, no matter how these are arranged, one can place a cylindrical glass with a diameter of 5 inches over at least 8 marbles.
- (33) Choose  $n$  points on a circle and join them pairwise by secants. Taking for granted that no more than two secants can meet at the same point, in how many regions is the circle thus divided?
- (34) Say we have three posts and  $n$  disks of different diameters placed on one of the posts, ordered by increasing diameters, the largest at the bottom of the post, the smaller at the top. The problem consists in transferring the tower of disks from the first post to the third post, using if need be the second post, but in such a way that, with each move, we do not place the moving disk on a smaller one. Establish the function of  $n$  which gives the minimum number of moves. (This problem is known as the “Tower of Hanoi Problem”.)
- (35) Let  $\{F_n : n \in \mathbb{N}\}$  be the sequence of Fibonacci numbers defined by  $F_1 = 1, F_2 = 1$  and  $F_n = F_{n-1} + F_{n-2}$  for  $n \geq 3$ . Show that each positive integer can be written as the sum of distinct Fibonacci numbers.
- (36) One easily checks that

$$\begin{aligned} 1 &= 1^2, \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2, \\ 3 &= -1^2 + 2^2, \\ 4 &= -1^2 - 2^2 + 3^2, \\ 5 &= 1^2 + 2^2, \\ 6 &= 1^2 - 2^2 + 3^2. \end{aligned}$$

Hence, we may be tempted to formulate a conjecture, namely that each positive integer  $n$  can be written as

$$n = e_1 1^2 + e_2 2^2 + e_3 3^2 + e_4 4^2 + \cdots + e_k k^2,$$

for a certain positive integer  $k$  (depending on  $n$ ), where the  $e_i \in \{-1, 1\}$ . Prove this conjecture.