2. Divisibility

(37) The mathematician Duro Kurepa defined $!n = 0! + 1! + \cdots + (n-1)!$ for $n \geq 1$ and conjectured that $(!n, n!) = 2$ for all $n \geq 2$. This conjecture has been verified by Ivić and Mijajlović [**20**] for $n < 10^6$. Using computer software, write a program showing that this conjecture is true up to $n = 1000$.

(38) Consider the situation where the positive integer a is divided by the positive integer b using the euclidian division (see Theorem 7) yielding

$$(*) \qquad a = 652b + 8634.$$

By how much can we increase both a and b without changing the quotient $q = 652$?

(39) Consider the number $N = 111\ldots11$, here written in basis 2. Write N^2 in basis 2.

(40) Show that $39|7^{37} + 13^{37} + 19^{37}$.

(41) Show that, for each integer $n \geq 1$, the number $49^n - 2352n - 1$ is divisible by 2304.

(42) Given any integer $n \geq 1$, show that the number $n^4 + 2n^3 + 2n^2 + 2n + 1$ is never a perfect square.

(43) Let N be a two digit number. Let M be the number obtained from N by interchanging its two digits. Show that 9 divides $M - N$ and then find all the integers N such that $|M - N| = 18$.

(44) Is it true that 3 never divides $n^2 + 1$ for every positive integer n? Explain.

(45) Is it true that 5 never divides $n^2 + 2$ for every positive integer n? Explain. Is the result the same if one replaces the number 5 by the number 7?

(46) Given $s + 1$ integers $a_0, a_1, \ldots, a_s$ and a prime number p, show that p divides the integer

$$N(n) := a_0 + a_1 n + \cdots + a_{s-1} n^{s-1} + a_s n^s$$

if and only if p divides $N(r)$, for an integer r, $0 \leq r \leq p - 1$. Use this to find all integers n such that 7 divides $3n^2 + 6n + 5$.

(47) Compute the value of the expression

$$\frac{(10^4 + 324)(22^4 + 324)(34^4 + 324)(46^4 + 324)(58^4 + 324)}{(4^4 + 324)(16^4 + 324)(28^4 + 324)(40^4 + 324)(52^4 + 324)}.$$

(48) Show that, in any basis, the number 10101 is composite.

(49) Show that the product of four consecutive integers is necessarily divisible by 24.

(50) Show that the number

$$1^{47} + 2^{47} + 3^{47} + 4^{47} + 5^{47} + 6^{47}$$

is a multiple of 7.

(51) Show that the product of any five consecutive positive integers cannot be a perfect square.

(52) Show that $30|n^5 - n$ for each positive integer n.

(53) Show that $6|n(n+1)(2n+1)$ for each positive integer n.

(54) Given any integer $n \geq 0$, show that $64^{n+1} - 63n - 64$ is divisible by 3969. More generally, given $a \in \mathbb{N}$, show that for each integer $n \geq 0$, $(a+1)^{n+1} - an - (a+1)$ is divisible by a^2.

(55) Find all positive integers n such that $(n+1)|(n^2+1)$.

(56) Find all positive integers n such that $(n^2+2)|(n^6+206)$.

(57) Identify, if any exist, the positive integers n such that $(n^3+2)|(n^6+216)$.

(58) If a and b are positive integers such that $b|(a^2+1)$, do we necessarily have that $b|(a^4+1)$? Explain.

(59) Let n and k be positive integers.

(a) For $n \geq k$, show that

$$\frac{n}{(n,k)} \;\Big|\; \binom{n}{k}.$$

(b) For $n \geq k$, show that

$$\frac{n+1-k}{(n+1,k)} \;\Big|\; \binom{n}{k}.$$

(c) For $n \geq k-1 \geq 1$, show that

$$\frac{(n+1,k-1)}{n+2-k}\binom{n}{k-1} \quad \text{is an integer.}$$

(60) For each integer $n \geq 1$, let $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n), \qquad \text{for each integer } n \geq 1.$$

(61) Show that, for each positive integer n,

$$49|2^{3n+3} - 7n - 8.$$

(62) Find all positive integers a for which $a^{10}+1$ is divisible by 10.

(63) Is it true that $3|2^{2n} - 1$ for each positive integer n? Explain.

(64) Show that if an integer is of the form $6k+5$, then it is necessarily of the form $3k-1$, while the reverse is false.

(65) Can an integer $n > 1$ be of the form $8k+7$ and also of the form $6\ell+5$? Explain.

(66) Let $M_1 = 2+1$, $M_2 = 2\cdot 3+1$, $M_3 = 2\cdot 3\cdot 5+1$, $M_4 = 2\cdot 3\cdot 5\cdot 7+1$, $M_5 = 2\cdot 3\cdot 5\cdot 7\cdot 11+1, \ldots$. Prove none of the numbers M_k is a perfect square.

(67) Verify that if an integer is a square and a cube, then it must be of the form $7k$ or $7k+1$.

(68) If x and y are odd integers, prove that x^2+y^2 cannot be a perfect square.

(69) Show that, for each positive integer n, we have $n^2|(n+1)^n - 1$.

(70) Let $k, n \in \mathbb{N}$, $n \geq 2$. Show that $(n-1)^2|(n^k-1)$ if and only if $(n-1)|k$. More generally, show the following result: Let $a \in \mathbb{Z}$ and $k, n \in \mathbb{N}$ with $n \neq a$; then $(n-a)^2|(n^k-a^k)$ if and only if $(n-a)|ka^{k-1}$.

(71) Let a, b be integers and let n be a positive integer.

(a) If $a - b \neq 0$, show that

$$\left(\frac{a^n-b^n}{a-b}, a-b\right) = \left(n(a,b)^{n-1}, a-b\right).$$

(b) If $a+b \neq 0$ and if n is odd, show that

$$\left(\frac{a^n+b^n}{a+b}, a+b\right) = \left(n(a,b)^{n-1}, a+b\right).$$

(c) Show that if a and b are relatively prime with $a+b \neq 0$ and if $p > 2$ is a prime number, then

$$\left(\frac{a^p+b^p}{a+b}, a+b\right) = \begin{cases} 1 & \text{if } p \not| (a+b), \\ p & \text{if } p|(a+b). \end{cases}$$

(72) Let k and n be positive integers. Show that the only solutions (k,n) of the equation $(n-1)! = n^k - 1$ are $(1,2)$, $(1,3)$ and $(2,5)$.

(73) According to Euclid's algorithm, assuming that $b \geq a$ are positive integers, we have

$$\begin{array}{rcll} b &=& aq_1 + r_1, & 0 < r_1 < a, \\ a &=& r_1q_2 + r_2, & 0 < r_2 < r_1, \\ r_1 &=& r_2q_3 + r_3, & 0 < r_3 < r_2, \\ &\vdots&&\\ r_{j-2} &=& r_{j-1}q_j + r_j, & 0 < r_j < r_{j-1}, \\ r_{j-1} &=& r_jq_{j+1}, & \end{array}$$

where $r_j = (a,b)$.

(a) Show that $b > 2r_1$, $a > 2r_2$ and for $k \geq 1$, $r_k > 2r_{k+2}$.

(b) Deduce that $b > 2^{j/2}$ and therefore that the maximum number of steps in Euclid's algorithm is $[2(\log b/\log 2)]$.

(74) Show that there exist infinitely many positive integers n such that $n|2^n+1$.

(75) Let a be an integer ≥ 2. Show that for positive integers m and n we have

$$a^n - 1 | a^m - 1 \iff n|m.$$

(76) Let N_n be an integer formed of n consecutive "1"s. For example, $N_3 = 111$, $N_7 = 1\,111\,111$. Show that $N_n|N_m \iff n|m$.

(77) Prove that no member of the sequence $11, 111, 1\,111, 11\,111, \ldots$ is a perfect square.

(78) What is the smallest positive integer divisible both by 2 and 3 which is both a perfect square and a sixth power? More generally, what is the smallest positive integer n divisible by both 2 and 3 which is both an n–th power and an m–th power, where $n, m \geq 2$?

(79) Three of the four integers, found between 100 and 1000, with the property of being equal to the sum of the cubes of their digits are 153, 370 and 407. What is the fourth of these integers?

(80) How many positive integers $n \leq 1000$ are not divisible by 2, nor by 3, nor by 5?

(81) Prove the following result obtained in the seventeenth century by Pierre de Fermat (1601–1665): *"Each odd prime number p can be written as the difference of two perfect squares."*

(82) Prove that the representation mentioned in problem 81 is unique.

(83) Is the result of Fermat stated in problem 81 still true if p is simply an odd positive integer?

(84) Let $n = 999\,980\,317$. Observing that $n = 10^9 - 3^9$ and factoring this last expression, conclude that $7|n$.

(85) Show that if an odd integer can be written as the sum of two squares, then it is of the form $4n+1$.

(86) Let $a,b,c \in \mathbb{Z}$ be such that $abc \neq 0$ and $(a,b,c) = 1$ and such that $a^2+b^2=c^2$. Prove that at least one of the integers a and b is even.

(87) For which integer values of k is the number 10^k-1 the cube of an integer?

(88) Show that if the positive integer a divides both $42n+37$ and $7n+4$ for a certain integer n, then $a=1$ or $a=13$.

(89) If a and b are two positive integers and if $\dfrac{1}{a}+\dfrac{1}{b}$ is an integer, prove that $a=b$. Moreover, show that a is then necessarily equal to 1 or 2.

(90) Let $a,b \in \mathbb{N}$ such that $(a,b)=4$. Find all possible values of (a^2,b^3).

(91) Let $a,b \in \mathbb{N}$ and $d=(a,b)$. Find the value of $(3a+5b, 5a+8b)$ in terms of d and more generally that of $(ma+nb, ra+sb)$ knowing that $ms-nr=1$, where $m,n,r,s \in \mathbb{N}$.

(92) Let $m,n \in \mathbb{N}$. If $d|mn$ where $(m,n)=1$, show that d can be written as $d=rs$ where $r|m$, $s|n$ and $(r,s)=1$.

(93) Let a,b,d be nonzero integers, d odd, such that $d|(a+b)$ and $d|(a-b)$. Show that $d|(a,b)$.

(94) Given eight positive composite integers ≤ 360, show that at least two of them have a common factor larger than 1.

(95) If a and b are positive integers such that $(a,b)=1$ and ab is a perfect square, show that a and b are perfect squares.

(96) Can $n(n+1)$ be a perfect square for a certain positive integer n? Explain.

(97) What are the possible values of the expression $(n, n+14)$ as n runs through the set of positive integers?

(98) Let $n>1$ an integer. Which of the following statements are true:

$$3|(n^3-n), \qquad 3|n(n+1), \qquad 8|(2n+1)^2-1, \qquad 6|n(n+1)(n+2).$$

(99) Is it true that if n is an even integer, then $24|n(n+1)(n+2)$? Explain.

(100) Let n be an integer such that $(n,2)=(n,3)=1$. Show that $24|n^2+47$.

(101) Let $d=(a,b)$, where a and b are positive integers. Show that there are exactly d numbers amongst the integers a, $2a$, $3a, \ldots, ba$ which are divisible by b.

(102) Let a,b be integers such that $(a,b)=d$, and let x_0,y_0 be integers such that $ax_0+by_0=d$. Show that:
(a) $(x_0,y_0)=1$;
(b) x_0 and y_0 are not unique.

(103) Let a,m and n be positive integers. If $(m,n)=1$, show that $(a,mn)=(a,m)(a,n)$.

(104) For all $n \in \mathbb{N}$, show that $(n^2+3n+2, 6n^3+15n^2+3n-7)=1$.

(105) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, show that
(a) $(a+b, a-b) = 1$ or 2; (b) $(2a+b, a+2b)=1$ or 3;
(c) $(a^2+b^2, a+b)=1$ or 2; (d) $(a+b, a^2-3ab+b^2)=1$ or 5.

(106) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, find the possible values of
(a) (a^3+b^3, a^3-b^3); (b) (a^2-b^2, a^3-b^3).

(107) Let a, b and c be integers. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.
(a) If $(a,b)=(a,c)$, then $[a,b]=[a,c]$.

(b) If $(a,b)=(a,c)$, then $(a^2,b^2)=(a^2,c^2)$.
(c) If $(a,b)=(a,c)$, then $(a,b)=(a,b,c)$.

(108) Let $a,b\in\mathbb{Z}$ and let $m,n\in\mathbb{N}$. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.
(a) If $a^n|b^n$, then $a|b$.
(b) If $a^m|b^n$, $m>n$, then $a|b$.
(c) If $a^m|b^n$, $m<n$, then $a|b$.

(109) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,b)=1$ and $c|a$, then $(c,b)=1$.

(110) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,bc)=1$, then $(a,b)=(a,c)=1$.

(111) Let $a,b\in\mathbb{Z}$. Show that $(a,b)=(a+b,[a,b])$. Using this result, find two positive integers whose sum is 186 and whose LCM is 1440.

(112) Let $a,b,c\in\mathbb{Z}$.
(a) Show that $(a,bc)=(a,(a,b)c)$.
(b) Show that $(a,bc)=(a,(a,b)(a,c))$.

(113) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,c)=1$, then $(ab,c)=(b,c)$.

(114) Let a, b, m and n be integers. If $(m,n)=1$, show that $(ma+nb,mn)=(a,n)(b,m)$. Show that this result generalizes the result of problem 103.

(115) Is it possible that $\binom{n}{r}$ is relatively prime with $\binom{n}{s}$, for certain positive integers r,s,n satisfying $0<r<s\le n/2$? Explain.

(116) Find two positive integers for which the difference between their LCM and their GCD is equal to 143.

(117) Let a,b,c be positive integers. Show that $(a,b,c)=((a,b),c)$ and $[a,b,c]=[[a,b],c]$. Generalize this result. Use this result to compute $(132,102,36)$ and find those integers x,y,z for which $132x+102y+36z=(132,102,36)$.

(118) Let n be a positive integer. Evaluate $(n,n+1,n+2)$ and $[n,n+1,n+2]$.

(119) Let a,b,c be positive integers. If $(a,b)=(b,c)=(a,c)=1$, show that $(a,b,c)[a,b,c]=abc$.

(120) Is it true that if a and b are positive integers such that $(a,b)=1$, then $(a^2,ab,b^2)=1$? Explain.

(121) Is it true that if a, b and c are positive integers, then $[a^2,ab,b^2]=[a^2,b^2]$? Explain.

(122) Is it true that if a, b and c are positive integers, then $(a,b,c)=\big((a,b),(a,c)\big)$? Explain.

(123) Is it true that $[a,b,c]\cdot(a,b,c)=|abc|$, $\forall\ a,b,c\in\mathbb{Z}\setminus\{0\}$? Explain.

(124) Let a, b, d, m and n be positive integers such that $a|d^m-1$, $b|d^n-1$ and $(a,b)=1$. Show that $ab|d^{[m,n]}-1$.

(125) Show that if a is an integer >1, then, for each pair of positive integers m and n,

$$(a^m-1,a^n-1)=a^{(m,n)}-1.$$

What do we obtain for (a^m+1,a^n+1), for (a^m+1,a^n-1)? More generally, given $a>1$ and $b>1$, what are the values of

$$(a^m-b^m,a^n-b^n),\quad (a^m+b^m,a^n+b^n)\ \text{ and }\ (a^m+b^m,a^n-b^n)?$$

(126) Show that there exist infinitely many pairs of integers $\{x,y\}$ satisfying $x+y=40$ and $(x,y)=5$.

(127) Find all pairs of positive integers $\{a, b\}$ such that $(a, b) = 15$ and $[a, b] = 90$. More generally, if d and m are positive integers, show that there exists a pair of positive integers $\{a, b\}$ for which $(a, b) = d$ and $[a, b] = m$ if and only if $d|m$. Moreover, in this situation, show that the number of such pairs is 2^r, where r is the number of distinct prime factors of m/d.

(128) Prove that one cannot find integers m and n such that $m + n = 101$ and $(m, n) = 3$.

(129) Let $a, m, n \in \mathbb{N}$ with $m \neq n$.
(a) Show that $(a^{2^n} + 1)|(a^{2^m} - 1)$ if $m > n$.
(b) Show that $(a^{2^n} + 1, a^{2^m} + 1) = \begin{cases} 1 & \text{if } a \text{ is even,} \\ 2 & \text{if } a \text{ is odd.} \end{cases}$

(130) Let n be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \ldots, \binom{2n}{2n-1}.$$

(131) Given $n + 1$ distinct positive integers $a_1, a_2, \ldots, a_{n+1}$ such that $a_i \leq 2n$ for $i = 1, 2, \ldots, n + 1$, show that there exists at least one pair $\{a_j, a_k\}$ with $j \neq k$ such that $a_j | a_k$.

(132) Let $n > 2$. Consider the three n-tuples $(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)})$, $i = 1, 2, 3$, where each $a_j^{(i)} \in \{+1, -1\}$ and assume that these three n-tuples satisfy $\displaystyle\sum_{j=1}^{n} a_j^{(i)} a_j^{(k)} = 0$ for each pair $\{i, k\}$ such that $1 \leq i < k \leq 3$. Show that $4|n$.

(133) Let A be the set of natural numbers which, in their decimal representation, do not have "7" amongst their digits. Prove that
$$\sum_{n \in A} \frac{1}{n} < +\infty.$$

(134) Let $u_1, u_2, \ldots$ be a strictly increasing sequence of positive integers. Denoting by $[a, b]$ the lowest common multiple of a and b, show that the series
$$\sum_{n=1}^{\infty} \frac{1}{[u_n, u_{n+1}]} \qquad \text{converges.}$$