3. Prime Numbers

(135) Using computer software, write a program
(a) to generate all Mersenne primes up to $2^{525} - 1$;
(b) to determine the smallest prime number larger than $10^{100} + 1$.

(136) Write a program that generates prime numbers up to a given number N. One can, of course, use Eratosthenes' sieve.

(137) Use a computer to find four consecutive integers having the same number of prime factors (allowing repetitions).

(138) (a) By reversing the digits of the prime number 1009, we obtain the number 9001, which is also prime. Write a program to find the prime numbers in [1,10000] verifying this property.
(b) By reversing the digits of the prime number 163, we obtain the number 361, which is a perfect square. Using computer software, write a program to find all prime numbers in $[1, 10000]$ with this property.

(139) Using a computer, find all prime numbers $p \leq 10\,000$ with the property that p, $p + 2$ and $p + 6$ are all primes.

(140) Let p_k be the k-th prime number. Show that $p_k < 2^k$ if $k \geq 2$.

(141) If a prime number $p_k > 5$ is equally isolated from the prime numbers appearing before and after it, that is $p_k - p_{k-1} = p_{k+1} - p_k = d$, say, show that d is a multiple of 6. Then, for each of the cases $d = 6$, 12 and 18, find, by using a computer, the smallest prime number p_k with this property.

(142) Prove that none of the numbers

$$12321,\ 1234321,\ 123454321,\ 12345654321,\ 1234567654321,$$
$$123456787654321,\ 12345678987654321$$

is prime.

(143) For each integer $k \geq 1$, let n_k be the k-th composite number, so that for instance $n_1 = 4$ and $n_{10} = 18$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(144) For each integer $k \geq 1$, let n_k be the k-th number of the form p^α, where p is prime, α a positive integer, so that for instance $n_1 = 2$ and $n_{10} = 16$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(145) Find all positive integers $n < 100$ such that $2^n + n^2$ is prime. To which class of congruence modulo 6 do these numbers n belong?

(146) Show that if the integer $n \geq 4$ is not an odd multiple of 9, then the corresponding number $a_n := 4^n + 2^n + 1$ is necessarily composite. Then, use a computer in order to find all positive integers $n < 1000$ for which a_n is prime.

(147) Consider the sequence (a_n) defined by $a_1 = a_2 = 1$ and, for $n \geq 3$, by $a_n = n! - (n-1)! + \cdots + (-1)^n 2! + (-1)^{n+1} 1!$. Use a computer in order to find the smallest number n such that a_n is a composite number.

(148) The mathematicians Minác and Willans have obtained a formula for the n-th prime number p_n which is more of a theoretical interest than of a

practical interest:

$$p_n = 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{1+\sum_{j=2}^{m}\left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j}\right]\right]} \right]^{1/n} \right],$$

where as usual $[x]$ stands for the largest integer $\leq x$. Prove this formula.

(149) Develop an idea used by Paul Erdős (1913–1996) to show that, for each integer $n \geq 1$,

$$\prod_{p \leq n} p \leq 4^n.$$

His idea was to write

$$\prod_{p \leq n} p = \prod_{p \leq \frac{n+1}{2}} p \cdot \prod_{\frac{n+1}{2} < p \leq n} p$$

and to use the fact that each prime number $p > (n+1)/2$ appears in the factorization of the binomial coefficient $\binom{n}{(n+1)/2}$. Provide the details.

(150) Show that if four positive integers a, b, c, d are such that $ab = cd$, then the number $a^2 + b^2 + c^2 + d^2$ is necessarily composite.

(151) Show that, for each integer $n \geq 1$, the number $4n^3 + 6n^2 + 4n + 1$ is composite.

(152) Show that if p and q are two consecutive odd prime numbers, then $p + q$ is the product of at least three prime numbers (not necessarily distinct).

(153) Does there exist a positive integer n such that $n/2$ is a perfect square, $n/3$ a cube and $n/5$ a fifth power?

(154) Given any integer $n \geq 2$, show that $n^{42} - 27$ is never a prime number.

(155) Let $\theta(x) := \sum_{p \leq x} \log p$. Prove that Bertrand's Postulate follows from the fact that

$$c_1 x < \theta(x) < c_2 x,$$

where $c_1 = 0.73$ and $c_2 = 1.12$.

(156) Use Bertrand's Postulate to show that, for each integer $n \geq 4$,

$$p_{n+1}^2 < p_1 p_2 \cdots p_n,$$

where p_n stands for the n-th prime number.

(157) Certain integers $n \geq 3$ can be written in the form $n = p + m^2$, with p prime and $m \in \mathbb{N}$. This is the case for example for the numbers 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21. Let q^r be a prime power, where r is a positive even integer such that $2q^{r/2} - 1$ is composite. Show that q^r cannot be written as $q^r = p + m^2$, with p prime and $m \in \mathbb{N}$.

(158) Show that if p and $8p - 1$ are primes, then $8p + 1$ is composite.

(159) Show that all positive integers of the form $3k + 2$ have a prime factor of the same form, that all positive integers of the form $4k + 3$ have a prime factor of the same form, and finally that all positive integers of the form $6k + 5$ have a prime factor of the same form.

(160) A positive integer n has a *Cantor expansion* if it can be written as

$$n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1!,$$

where the a_j's are integers satisfying $0 \leq a_j \leq j$.

(a) Find the Cantor expansion of 23 and of 57.

(b) Show that all positive integers n have a Cantor expansion and moreover that this expansion is unique.

(161) If $p > 1$ and $d > 0$ are integers, show that p and $p+d$ are both primes if and only if

$$(p-1)!\left(\frac{1}{p} + \frac{(-1)^d d!}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d}$$

is an integer.

(162) Find all prime numbers p such that $p+2$ and p^2+2p-8 are primes.

(163) Is it true that if p and p^2+8 are primes, then p^3+4 is prime? Explain.

(164) Let $n \geq 2$. Show that the integers n and $n+2$ form a pair of twin primes if and only if

$$4\left((n-1)!+1\right)+n \equiv 0 \pmod{n(n+2)}.$$

(165) Identify each prime number p such that $2^p + p^2$ is also prime.

(166) For which prime number(s) p is $17p+1$ a perfect square?

(167) Given two integers a and b such that $(a,b) = p$, where p is prime, find all possible values of:

(a) (a^2, b); (b) (a^2, b^2); (c) (a^3, b); (d) (a^3, b^2).

(168) Given two integers a and b such that $(a, p^2) = p$ and $(b, p^4) = p^2$, where p is prime, find all possible values of:

(a) (ab, p^5); (b) $(a+b, p^4)$; (c) $(a-b, p^5)$; (d) $(pa-b, p^5)$.

(169) Given two integers a and b such that $(a, p^2) = p$ and $(b, p^3) = p^2$, where p is a prime number, evaluate the expressions (a^2b^2, p^4) and (a^2+b^2, p^4).

(170) Let p be a prime number and a, b, c be positive integers. For each of the following statements, say if is true or false. If it is true, give a proof; if it is false, provide a counter-example.

(a) If $p|a$ and $p|(a^2+b^2)$, then $p|b$.

(b) If $p|a^n$, $n \geq 1$, then $p|a$.

(c) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2-c^2)$.

(d) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2+c^2)$.

(171) Let a, b and c be positive integers. Show that $abc = (a,b,c)[ab,bc,ac] = (ab,bc,ac)[a,b,c]$.

(172) Let a, b and c be positive integers and assume that $abc = (a,b,c)[a,b,c]$. Show that this necessarily implies that $(a,b) = (b,c) = (a,c) = 1$.

(173) Let a, b and c be positive integers. Show that $(a,b,c) = \dfrac{(a,b)(b,c)(a,c)}{(ab,bc,ac)}$ and that $[a,b,c] = \dfrac{abc\,(a,b,c)}{(a,b)(b,c)(a,c)}$.

(174) Let a, b and c be positive integers. Show that

$$\frac{[a,b,c]^2}{[a,b][b,c][c,a]} = \frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}.$$

(175) Find three positive integers a, b, c such that

$$[a,b,c] \cdot (a,b,c) = \sqrt{abc}.$$

(176) Let $\#n = [1,2,3,\ldots,n]$ be the lowest common multiple of the numbers $1,2,\ldots,n$. Show that

$$\prod_{p\leq n} p \leq \#n = \prod_{p\leq n} p^{[\log n/\log p]}.$$

(177) Let p be a prime number and r a positive integer. What are the possible values of $(p,p+r)$ and of $[p,p+r]$?

(178) Let $p>2$ be a prime number such that $p|8a-b$ and $p|8c-d$, where $a,b,c,d\in\mathbb{Z}$. Show that $p|(ad-bc)$.

(179) Show that, if $\{p,p+2\}$ is a pair of twin primes with $p>3$, then 12 divides the sum of these two numbers.

(180) Let n be a positive integer. Show that if n is a composite integer, then $n|(n-1)!$ except when $n=4$.

(181) For which positive integers n is it true that

$$\sum_{j=1}^{n} j \,\Big|\, \prod_{j=1}^{n} j?$$

(182) Let $\pi = 3.141592\ldots$ be Archimede's constant, and for each positive real number x, let $\pi_2(x)$ be the function that counts the number of pairs of twin primes $\{p,p+2\}$ such that $p\leq x$. Show that

$$\pi_2(x) = 2 + \sum_{7\leq n\leq x} \sin\left(\frac{\pi}{2}(n+2)\left[\frac{n!}{n+2}\right]\right)\cdot\sin\left(\frac{\pi}{2}n\left[\frac{(n-2)!}{n}\right]\right),$$

where $[y]$ stands for the largest integer $\leq y$.

(183) Given an integer $n\geq 2$, show, without using Bertrand's Postulate, that there exists a prime number p such that $n<p<n!$.

(184) In 1556, Niccòlo Tartaglia (1500–1557) claimed that the sums

$$1+2+4,\ 1+2+4+8,\ 1+2+4+8+16,\ \ldots$$

stood successively for a prime number and a composite number. Was he right?

(185) Show that if a^n-1 is prime for certain integers $a>1$ and $n>1$, then $a=2$ and n is prime.

REMARK: *The integers of the form 2^p-1, where p is prime, are called Mersenne numbers. We denote them by M_p in memory of Marin Mersenne* (1588–1648), *who had stated that M_p is prime for*

$$p = 2,3,5,7,13,17,19,31,67,127,257$$

and composite for all the other primes $p<257$. This assertion of Mersenne can be found in the preface of his book Cogita Physico-mathematica, *published in Paris in 1644. Since then, we have found a few errors in the computations of Mersenne: indeed M_p is not prime for $p=67$ and $p=257$, while M_p is prime for $p=61$, $p=89$ and $p=109$. One can find in the appendix C of the book of J.M. De Koninck and A. Mercier* [**8**] *the list of Mersenne primes M_p corresponding to the prime numbers p satisfying $2\leq p\leq 44\,497$. Note on the other hand that it has recently been discovered that $2^{32\,582\,657}-1$ is prime (in September 2006), which brings to 44 the total number of known Mersenne primes. It is also known that the primes*

M_p are closely related to the PERFECT NUMBERS, *in the sense that, as was shown by Leonhard Euler (1707–1783), n is an even perfect number if and only if $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.*

(186) Show that if there exists a positive integer n and an integer $a \geq 2$ such that $a^n + 1$ is prime, then a is even and $n = 2^r$ for a certain positive integer r.

REMARK: *The prime numbers of the form $2^{2^k} + 1$, $k = 0, 1, 2, \ldots$, are called "Fermat primes". The reason is that Pierre de Fermat claimed in 1640 (although saying he could not prove it) that all the numbers of the form $2^{2^k} + 1$ are prime. One hundred years later, Euler proved that*

$$2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417.$$

*As of today, we still do not know if, besides the cases $k = 0, 1, 2, 3, 4$, primes of the form $2^{2^k} + 1$ exist. Nevertheless, it is known that $2^{2^k} + 1$ is composite for $5 \leq k \leq 32$; see H.C. Williams [**41**] and the site* www.prothsearch.net/fermat.html.

(187) Show that the equation $(2^x - 1)(2^y - 1) = 2^{2^z} + 1$ is impossible for positive integers x, y and z. (This implies in particular that a Fermat number, that is a number of the form $2^{2^k} + 1$, cannot be the product of two Mersenne numbers.)

(188) Prove by induction that, for each integer $n \geq 1$,

$$F_0 F_1 F_2 \cdots F_{n-1} = F_n - 2,$$

where $F_i = 2^{2^i} + 1$, $i = 0, 1, 2, \ldots$.

(189) Use the result of problem 188 in order to prove that if m and n are distinct positive integers, then $(F_m, F_n) = 1$.

(190) A positive integer n is said to be *pseudoprime in basis* $a \geq 2$ if it is composite and if $a^{n-1} \equiv 1 \pmod{n}$. Find the smallest number which is pseudoprime in each of the bases 2, 3, 5 and 7.

(191) Use Problem 189 to prove that there exist infinitely many primes.

(192) Consider the numbers $f_n = 2^{3^n} + 1$, $n = 1, 2, \ldots$, and show they are all composite and in particular that, for each positive integer n,
(a) $3^{n+1} | f_n$; (b) $p | f_n \Rightarrow p | f_{n+1}$.

(193) Show that there exist infinitely many prime numbers p such that the numbers $p - 2$ and $p + 2$ are both composite.

(194) Show that 641 divides $F_5 = 2^{2^5} + 1$ without doing the explicit division.

(195) Use an induction argument in order to prove that each Fermat number $F_n = 2^{2^n} + 1$, where $n \geq 2$, ends with the digit 7.

(196) Let n be a positive integer and consider the set $E = \{1, 2, \ldots, n\}$. Let 2^k be the largest power of 2 which belongs to E. Show that for all $m \in E \setminus \{2^k\}$, we have $2^k \not| m$. Using this result, show that $\sum_{j=1}^{n} 1/j$ is not an integer if $n > 1$.

(197) Show that, for each positive integer n, one can find a prime number $p < 50$ such that $p | (2^{5n} - 1)$.

(198) Show that the integers defined by the sequence of numbers

$$M_k = p_1 p_2 \cdots p_k + 1 \qquad (k = 1, 2, \ldots),$$

where p_j stands for the j–th prime number, are prime numbers for $1 \leq k \leq 5$ and composite numbers for $k = 6, 7$. What about M_8, M_9 and M_{10}?

(199) Use the proof of Euclid's Theorem on the infinitude of primes to show that, if we denote by p_r the r-th prime number, then $p_r \leq 2^{2^{r-1}}$ for each $r \in \mathbb{N}$.

(200) In Problem 199, we obtained an upper bound for p_r, the r-th prime number, namely $p_r \leq 2^{2^{r-1}}$. Use this inequality to obtain a lower bound for $\pi(x)$, the number of prime numbers $\leq x$. More precisely, show that, for $x \geq 3$, $\pi(x) \geq \log\log x$.

(201) Show that there exist infinitely many prime numbers of the form $4n + 3$.

(202) Show that there exist infinitely many prime numbers of the form $6n + 5$.

(203) Let $f\colon \mathbb{N} \to \mathbb{R}$ be the function defined by

$$f(x) = a_r x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0,$$

where $a_r \neq 0$ and where each a_i, $0 \leq i \leq r$, is an integer. Show that, by an appropriate choice of a_i, $0 \leq i \leq r$, the set $\{f(n) : n \in \mathbb{N}\}$ contains at least r prime numbers.

(204) Consider the positive integers which can be written as an alternating sequence of 0's and 1's. The number 101 010 101 is such a number and observe that $101\,010\,101 = 41 \cdot 271 \cdot 9091$. Besides 101, do there exist other prime numbers of this form?

(205) Find all prime numbers of the form $2^{2^n} + 5$, where $n \in \mathbb{N}$. Would the question be more difficult if one replaces the number 5 by another number of the form $3k + 2$? Explain.

(206) The largest gaps between two consecutive prime numbers $p_r < p_{r+1} < 100$ occur successively when

$$\begin{aligned}
p_{r+1} - p_r &= 5 - 3 = 2,\\
p_{r+1} - p_r &= 11 - 7 = 4,\\
p_{r+1} - p_r &= 29 - 23 = 6,\\
p_{r+1} - p_r &= 97 - 89 = 8.
\end{aligned}$$

Is it true that these constantly increasing gaps always occur by jumps of length 2? In other words, does the first gap of length $2k$ always occur before the first gap of length $2k + 2$?

(207) Show that $\displaystyle\sum_{\alpha=2}^{\infty}\sum_{p}\frac{1}{p^\alpha} < 1$, where the inner sum runs over all the prime numbers p.

(208) Let

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \cdots,$$

be a series which is in fact a finite sum for each real number $x \geq 1$ since $\pi(x^{1/n}) = 0$ as soon as $n > \log x/\log 2$. Show that

$$\pi(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}f(x^{1/n}).$$

REMARK: It is possible to show that $f(x)$ is a better approximation of $\pi(x)$ than $Li(x) := \int_2^x \frac{dt}{\log t}$ (see H. Riesel [**31**]).

(209) Let $n \geq 2$ be an integer. Show that the interval $[n, 2n]$ contains at least one perfect square.

(210) If n is a positive integer such that $3n^2 - 3n + 1$ is composite, show that n^3 cannot be written as $n^3 = p + m^3$, with p prime and m a positive integer.

(211) It is conjectured that there exist infinitely many prime numbers p of the form $p = n^2 + 1$. Identify the primes $p < 10\,000$ of this particular form. Why is the last digit of such a prime number p always 1 or 7? Is there any reasonable explanation for the fact that the digit 7 appears essentially twice as often?

(212) Show that, for each integer $n \geq 2$,

$$(n!)^{1/n} \leq \prod_{p \leq n} p^{\frac{1}{p-1}}.$$

(213) For each integer $N \geq 1$, let $S_N = \{n^2 + 2 : 6 \leq n \leq 6N\}$. Show that no more than $\frac{1}{6}$ of the elements of S_N are primes.

(214) Let p be a prime number and consider the integer $N = 2 \cdot 3 \cdot 5 \cdots p$. Show that the $(p-1)$ consecutive integers

$$N + 2, N + 3, N + 4, \ldots, N + p$$

are composite.

(215) Let $n > 1$ be an integer with at least 3 digits. Show that
  (a) $2|n$ if and only if the last digit of n is divisible by 2;
  (b) $2^2|n$ if and only if the number formed with the last two digits of n is divisible by 4;
  (c) $2^3|n$ if and only if the number formed with the last three digits of n is divisible by 8.

Can one generalize?

(216) For each integer $n \geq 2$, let

$$P(n) = \prod_{\substack{p|n \\ p > \log n}} \left(1 - \frac{1}{p}\right).$$

Show that $\lim_{n \to \infty} P(n) = 1$.

(217) Prove that there exists an interval of the form $[n^2, (n+1)^2]$ containing at least 1000 prime numbers.

(218) Use the Prime Number Theorem (see Theorem 17) in order to prove that the set of numbers of the form p/q (where p and q are primes) is dense in the set of positive real numbers.

(219) Show that the sum of the reciprocals of a finite number of distinct prime numbers cannot be an integer.

(220) Use the fact that there exists a positive constant c such that if $x \geq 100$,

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + c + R(x) \quad \text{with } |R(x)| < \frac{1}{\log x} \qquad (1)$$

and moreover that, for $x \geq 2$,

$$\pi(x) := \sum_{p \leq x} 1 < \frac{3}{2} \frac{x}{\log x} \tag{2}$$

in order to prove that if $P(n)$ stands for the largest prime factor of n, then

$$\frac{1}{x} \#\{n \leq x : P(n) > \sqrt{x}\} = \log 2 + T(x) \quad \text{with } |T(x)| < \frac{9}{2} \frac{1}{\log x}. \tag{3}$$

Use this result to show that more than $\frac{2}{3}$ of the integers have their largest prime factor larger than their square root, or in other words that the density of the set of integers n such that $P(n) > \sqrt{n}$ is larger than $\frac{2}{3}$.

(221) Prove the following formula (due to Adrien-Marie Legendre (1752–1833)):

$$\pi(x) = \pi(\sqrt{x}) + \sum_{n | p_1 \cdots p_r} \mu(n) \left[\frac{x}{n}\right] - 1,$$

where $r = \pi(\sqrt{x})$.

(222) Consider the following two conjectures:

A. *(Goldbach Conjecture) Each even integer* ≥ 4 *can be written as the sum of two primes.*

B. *Each integer* > 5 *can be written as the sum of three prime numbers.*

Show that these two conjectures are equivalent.

(223) Show that $\pi(m)$, the number of prime numbers not exceeding the positive integer m, satisfies the relation

$$\pi(m) = \sum_{j=2}^{m} \left[\frac{(j-1)! + 1}{j} - \left[\frac{(j-1)!}{j} \right] \right],$$

where $[y]$ stands for the largest integer $\leq y$.

(224) Given a sequence of natural numbers $\mathcal{A}$, let $A(n) = \#\{m \leq n : m \in \mathcal{A}\}$, and let us denote respectively by

$$\underline{\mathbf{d}}\mathcal{A} = \liminf_{n\to\infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{\mathbf{d}}\mathcal{A} = \limsup_{n\to\infty} \frac{A(n)}{n}$$

the *asymptotic lower density* and *asymptotic upper density* of the sequence $\mathcal{A}$. On the other hand, if both these densities are equal, we say that the sequence $\mathcal{A}$ has density $\mathbf{d}\mathcal{A} = \underline{\mathbf{d}}\mathcal{A} = \overline{\mathbf{d}}\mathcal{A}$. Prove that:

(a) the density of the sequence made up of all the multiples of a natural number a is equal to $1/a$;

(b) the density of the sequence made up of all the multiples of a natural number a which are not divisible by the natural number a_0 is equal to $\dfrac{1}{a} - \dfrac{1}{[a, a_0]}$;

(c) the density of the sequence made up of all natural numbers which are not divisible by any of the prime numbers $q_1, q_2, \ldots, q_r$ is equal to $\displaystyle\prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right)$.

(225) Let $\mathcal{A}$ be the set of natural numbers n such that $2^{2k} \leq n < 2^{2k+1}$ for a certain integer $k \geq 0$, so that

$$\mathcal{A} = \{1, 4, 5, 6, 7, 16, 17, \ldots, 31, 64, 65, \ldots, 127, 256, 257, \ldots\}.$$

Show that

$$\underline{\mathbf{d}}\,\mathcal{A} \neq \overline{\mathbf{d}}\,\mathcal{A}.$$

(226) We say that a sequence of natural numbers $\mathcal{A}$ is *primitive* if no element of $\mathcal{A}$ divides another one. Examples of such sequences are: the sequence of prime numbers, the sequence of natural numbers having exactly k prime factors (k fixed), and finally the sequence of integers n belonging to the interval $]k, 2k]$ (k fixed). Show that if $\mathcal{A}$ is a primitive sequence, then $\overline{\mathbf{d}}\,\mathcal{A} \leq \frac{1}{2}$.

(227) Let $\mathcal{A}$ be a primitive sequence (see Problem 226). Show that

$$\sum_{a \in \mathcal{A}} \frac{1}{a \log a} < +\infty.$$

(228) Let $E = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

(a) Show that the sum and the product of elements of E are in E.

(b) Define the norm of an element $z \in E$ by $\|z\| = \|a+b\sqrt{-5}\| = a^2+5b^2$. We say that an element $p \in E$ is *prime* if it is impossible to write $p = n_1 n_2$, with $n_1, n_2 \in E$, $\|n_1\| > 1$, $\|n_2\| > 1$; we say that it is *composite* if it is not prime. Show that, in E, 3 is a prime number and 29 is a composite number.

(c) Show that the factorization of 9 in E is not unique.

(229) Let A be a set of natural numbers and let $A(x) = \#\{n \leq x : n \in A\}$. Show that, for all $x \geq 1$,

$$\sum_{\substack{n \leq x \\ n \in A}} \frac{1}{n} = \sum_{n \leq x} \frac{A(n)}{n(n+1)} + \frac{A(x)}{[x]+1}.$$