Pythagorean Triples

The Pythagorean Theorem, that "beloved" formula of all high school geometry students, says that the sum of the squares of the sides of a right triangle equals the square of the hypotenuse. In symbols,

$$a^2 + b^2 = c^2$$

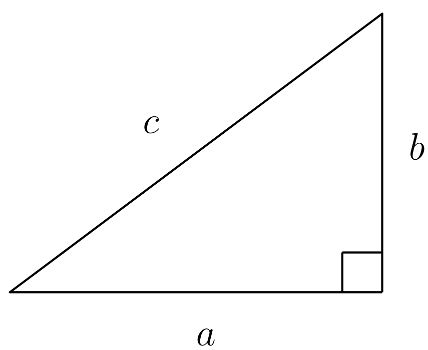

Figure 1: A Pythagorean Triangle

Since we're interested in number theory, that is, the theory of the natural numbers, we will ask whether there are any Pythagorean triangles all of whose sides are natural numbers. There are many such triangles. The most famous has sides 3, 4, and 5. Here are the first few examples:

$$3^2 + 4^2 = 5^2, \quad 5^2 + 12^2 = 13^2, \quad 8^2 + 15^2 = 17^2, \quad 28^2 + 45^2 = 53^2.$$

The study of these *Pythagorean triples* began long before the time of Pythagoras. There are Babylonian tablets that contain lists of parts of such triples, including quite large ones, indicating that the Babylonians probably had a systematic method for producing them. Even more amazing is the fact that the Babylonians may have used their lists of Pythagorean triples as primitive trigonometric tables. Pythagorean triples were also used in ancient Egypt. For example, a rough-and-ready way to produce a right angle is to take a piece of string, mark it into 12 equal segments, tie it into a loop, and hold it taut in the form of a 3-4-5 triangle, as illustrated in Figure 2. This provides an inexpensive right angle tool for use on small construction projects (such as marking property boundaries or building pyramids).

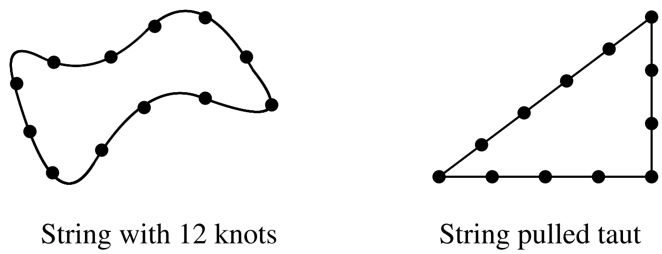

Figure 2: Using a knotted string to create a right triangle

The Babylonians and Egyptians had practical reasons for studying Pythagorean triples. Do such practical reasons still exist? For this particular problem, the answer is "probably not." However, there is at least one good reason to study Pythagorean triples, and it's the same reason why it is worthwhile studying the art of Rembrandt and the music of Beethoven. There is a beauty to the ways in which numbers interact with one another, just as there is a beauty in the composition of a painting or a symphony. To appreciate this beauty, one has to be willing to expend a certain amount of mental energy. But the end result is well worth the effort. Our goal in this book is to understand and appreciate some truly beautiful mathematics, to learn how this mathematics was discovered and proved, and maybe even to make some original contributions of our own.

Enough blathering, you are undoubtedly thinking. Let's get to the real stuff. Our first naive question is whether there are infinitely many *Pythagorean triples*, that is, triples of natural numbers (a, b, c) satisfying the equation $a^2 + b^2 = c^2$. The answer is "YES" for a very silly reason. If we take a Pythagorean triple (a, b, c) and multiply it by some other number d, then we obtain a new Pythagorean triple (da, db, dc). This is true because

$$(da)^2 + (db)^2 = d^2(a^2 + b^2) = d^2c^2 = (dc)^2.$$

Clearly these new Pythagorean triples are not very interesting. So we will concentrate our attention on triples with no common factors. We will even give them a name:

> A *primitive Pythagorean triple* (or PPT for short) is a triple of numbers (a, b, c) such that a, b, and c have no common factors[1] and satisfy
> $$a^2 + b^2 = c^2.$$

The first step is to accumulate some data. I used a computer to substitute in values for a and b and checked if $a^2 + b^2$ is a square. Here are some primitive Pythagorean triples that I found:

$$\begin{array}{cccc} (3,4,5), & (5,12,13), & (8,15,17), & (7,24,25), \\ (20,21,29), & (9,40,41), & (12,35,37), & (11,60,61), \\ (28,45,53), & (33,56,65), & (16,63,65). & \end{array}$$

A few conclusions can easily be drawn even from such a short list. For example, it certainly looks like one of a and b is odd and the other even. It also seems that c is always odd.

It's not hard to prove that these conjectures are correct. First, if a and b are both even, then c would also be even. This means that a, b, and c would have a common factor of 2, so the triple would not be primitive. Next, suppose that a and b are both odd, which means that c would have to be even. This means that there are numbers x, y, and z such that

$$a = 2x + 1, \qquad b = 2y + 1, \qquad \text{and} \qquad c = 2z.$$

We can substitute these into the equation $a^2 + b^2 = c^2$ to get

$$(2x + 1)^2 + (2y + 1)^2 = (2z)^2,$$
$$4x^2 + 4x + 4y^2 + 4y + 2 = 4z^2.$$

Now divide by 2,

$$2x^2 + 2x + 2y^2 + 2y + 1 = 2z^2.$$

This last equation says that an odd number is equal to an even number, which is impossible, so a and b cannot both be odd. Since we've just checked that they cannot both be even and cannot both be odd, it must be true that one is even and

[1] A *common factor* of a, b, and c is a number d such that each of a, b, and c is a multiple of d. For example, 3 is a common factor of 30, 42, and 105, since $30 = 3 \cdot 10$, $42 = 3 \cdot 14$, and $105 = 3 \cdot 35$, and indeed it is their largest common factor. On the other hand, the numbers 10, 12, and 15 have no common factor (other than 1). Since our goal in this chapter is to explore some interesting and beautiful number theory without getting bogged down in formalities, we will use common factors and divisibility informally and trust our intuition.

the other is odd. It's then obvious from the equation $a^2 + b^2 = c^2$ that c is also odd.

We can always switch a and b, so our problem now is to find all solutions in natural numbers to the equation

$$a^2 + b^2 = c^2 \qquad \text{with} \qquad \begin{cases} a \text{ odd,} \\ b \text{ even,} \\ a, b, c \text{ having no common factors.} \end{cases}$$

The tools that we use are *factorization* and *divisibility*.

Our first observation is that if (a, b, c) is a primitive Pythagorean triple, then we can factor

$$a^2 = c^2 - b^2 = (c - b)(c + b).$$

Here are a few examples from the list given earlier, where note that we always take a to be odd and b to be even:

$$\begin{aligned} 3^2 &= 5^2 - 4^2 = (5-4)(5+4) = 1 \cdot 9, \\ 15^2 &= 17^2 - 8^2 = (17-8)(17+8) = 9 \cdot 25, \\ 35^2 &= 37^2 - 12^2 = (37-12)(37+12) = 25 \cdot 49, \\ 33^2 &= 65^2 - 56^2 = (65-56)(65+56) = 9 \cdot 121. \end{aligned}$$

It looks like $c - b$ and $c + b$ are themselves always squares. We check this observation with a couple more examples:

$$\begin{aligned} 21^2 &= 29^2 - 20^2 = (29-20)(29+20) = 9 \cdot 49, \\ 63^2 &= 65^2 - 16^2 = (65-16)(65+16) = 49 \cdot 81. \end{aligned}$$

How can we prove that $c - b$ and $c + b$ are squares? Another observation apparent from our list of examples is that $c - b$ and $c + b$ seem to have no common factors. We can prove this last assertion as follows. Suppose that d is a common factor of $c - b$ and $c + b$; that is, d divides both $c - b$ and $c + b$. Then d also divides

$$(c + b) + (c - b) = 2c \qquad \text{and} \qquad (c + b) - (c - b) = 2b.$$

Thus, d divides $2b$ and $2c$. But b and c have no common factor because we are assuming that (a, b, c) is a primitive Pythagorean triple. So d must equal 1 or 2. But d also divides $(c - b)(c + b) = a^2$, and a is odd, so d must be 1. In other words, the only number dividing both $c - b$ and $c + b$ is 1, so $c - b$ and $c + b$ have no common factor.

We now know that $c-b$ and $c+b$ are positive integers having no common factor, that their product is a square since $(c-b)(c+b)=a^2$. The only way that this can happen is if $c-b$ and $c+b$ are themselves squares.[2] So we can write

$$c+b=s^2 \qquad \text{and} \qquad c-b=t^2,$$

where $s>t\geq 1$ are odd integers with no common factors. Solving these two equations for b and c yields

$$c=\frac{s^2+t^2}{2} \quad \text{and} \quad b=\frac{s^2-t^2}{2},$$

and then

$$a=\sqrt{(c-b)(c+b)}=st.$$

We have (almost) finished our first proof! The following theorem records our accomplishment.

Theorem 1 (Pythagorean Triples Theorem)**.** *We will get every primitive Pythagorean triple (a,b,c) with a odd and b even by using the formulas*

$$a=st, \qquad b=\frac{s^2-t^2}{2}, \qquad c=\frac{s^2+t^2}{2},$$

where $s>t\geq 1$ are chosen to be any odd integers with no common factors.

Why did we say that we have "almost" finished the proof? We have shown that if (a,b,c) is a PPT with a odd, then there are odd integers $s>t\geq 1$ with no common factors so that a, b, and c are given by the stated formulas. But we still need to check that these formulas always give a PPT. We first use a little bit of algebra to show that the formulas give a Pythagorean triple. Thus

$$(st)^2+\left(\frac{s^2-t^2}{2}\right)^2=s^2t^2+\frac{s^4-2s^2t^2+t^4}{4}=\frac{s^4+2s^2t^2+t^4}{4}=\left(\frac{s^2+t^2}{2}\right)^2.$$

We also need to check that st, $\frac{s^2-t^2}{2}$, and $\frac{s^2+t^2}{2}$ have no common factors. This is most easily accomplished using an important property of prime numbers.

[2] This is intuitively clear if you consider the factorization of $c-b$ and $c+b$ into primes, since the primes in the factorization of $c-b$ will be distinct from the primes in the factorization of $c+b$. However, the existence and uniqueness of the factorization into primes is by no means as obvious as it appears.

For example, taking $t = 1$ in Theorem 1 gives a triple $\left(s, \frac{s^2-1}{2}, \frac{s^2+1}{2}\right)$ whose b and c entries differ by 1. This explains many of the examples that we listed. The following table gives all possible triples with $s \le 9$.

s	t	$a = st$	$b = \dfrac{s^2 - t^2}{2}$	$c = \dfrac{s^2 + t^2}{2}$
3	1	3	4	5
5	1	5	12	13
7	1	7	24	25
9	1	9	40	41
5	3	15	8	17
7	3	21	20	29
7	5	35	12	37
9	5	45	28	53
9	7	63	16	65

A Notational Interlude

Mathematicians have created certain standard notations as a shorthand for various quantities. We will keep our use of such notation to a minimum, but there are a few symbols that are so commonly used and are so useful that it is worthwhile to introduce them here. They are

$$\begin{aligned}
\mathbb{N} &= \text{the set of natural numbers} = 1, 2, 3, 4, \dots\,, \\
\mathbb{Z} &= \text{the set of integers} = \dots -3, -2, -1, 0, 1, 2, 3, \dots\,, \\
\mathbb{Q} &= \text{the set of rational numbers (i.e., fractions).}
\end{aligned}$$

In addition, mathematicians often use $\mathbb{R}$ to denote the real numbers and $\mathbb{C}$ for the complex numbers, but we will not need these. Why were these letters chosen? The choice of $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ needs no explanation. The letter $\mathbb{Z}$ for the set of integers comes from the German word "Zahlen," which means numbers. Similarly, $\mathbb{Q}$ comes from the German "Quotient" (which is the same as the English word). We will also use the standard mathematical symbol $\in$ to mean "is an element of the set." So, for example, $a \in \mathbb{N}$ means that a is a natural number, and $x \in \mathbb{Q}$ means that x is a rational number.

Exercises

1. **(a)** We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.

(b) By examining the above list of primitive Pythagorean triples, make a guess about when a, b, or c is a multiple of 5. Try to show that your guess is correct.

2. A nonzero integer d is said to *divide* an integer m if $m = dk$ for some number k. Show that if d divides both m and n, then d also divides $m - n$ and $m + n$.

3. For each of the following questions, begin by compiling some data; next examine the data and formulate a conjecture; and finally try to prove that your conjecture is correct. (But don't worry if you can't solve every part of this problem; some parts are quite difficult.)

(a) Which odd numbers a can appear in a primitive Pythagorean triple (a, b, c)?

(b) Which even numbers b can appear in a primitive Pythagorean triple (a, b, c)?

(c) Which numbers c can appear in a primitive Pythagorean triple (a, b, c)?

4. In our list of examples are the two primitive Pythagorean triples

$$33^2 + 56^2 = 65^2 \qquad \text{and} \qquad 16^2 + 63^2 = 65^2.$$

Find at least one more example of two primitive Pythagorean triples with the same value of c. Can you find three primitive Pythagorean triples with the same c? Can you find more than three?

5. We have seen that the n^{th} triangular number T_n is given by the formula

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

The first few triangular numbers are 1, 3, 6, and 10. In the list of the first few Pythagorean triples (a, b, c), we find $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(9, 40, 41)$. Notice that in each case, the value of b is four times a triangular number.

(a) Find a primitive Pythagorean triple (a, b, c) with $b = 4T_5$. Do the same for $b = 4T_6$ and for $b = 4T_7$.

(b) Do you think that for every triangular number T_n, there is a primitive Pythagorean triple (a, b, c) with $b = 4T_n$? If you believe that this is true, then prove it. Otherwise, find some triangular number for which it is not true.

6. If you look at the table of primitive Pythagorean triples in this chapter, you will see many triples in which c is 2 greater than a. For example, the triples $(3, 4, 5)$, $(15, 8, 17)$, $(35, 12, 37)$, and $(63, 16, 65)$ all have this property.

(a) Find two more primitive Pythagorean triples (a, b, c) having $c = a + 2$.

(b) Find a primitive Pythagorean triple (a, b, c) having $c = a + 2$ and $c > 1000$.

(c) Try to find a formula that describes all primitive Pythagorean triples (a, b, c) having $c = a + 2$.

7. For each primitive Pythagorean triple (a, b, c) in the table in this chapter, compute the quantity $2c - 2a$. Do these values seem to have some special form? Try to prove that your observation is true for all primitive Pythagorean triples.

8. Let m and n be numbers that differ by 2, and write the sum $\frac{1}{m} + \frac{1}{n}$ as a fraction in lowest terms. For example, $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$.

(a) Compute the next three examples.

(b) Examine the numerators and denominators of the fractions in (a) and compare them with the table of Pythagorean triples. Formulate a conjecture about such fractions.

(c) Prove that your conjecture is correct.

9. (a) Read about the Babylonian number system and write a short description, including the symbols for the numbers 1 to 10 and the multiples of 10 from 20 to 50.

(b) Read about the Babylonian tablet called Plimpton 322 and write a brief report, including its approximate date of origin.

(c) The second and third columns of Plimpton 322 give pairs of integers (a, c) having the property that $c^2 - a^2$ is a perfect square. Convert some of these pairs from Babylonian numbers to decimal numbers and compute the value of b so that (a, b, c) is a Pythagorean triple.