

THIRD EDITION

A CONCISE
INTRODUCTION TO
PURE
MATHEMATICS

MARTIN LIEBECK



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Chapter 6

Complex Numbers

We all know that there are simple quadratic equations, such as $x^2 + 1 = 0$, that have no real solutions. In order to provide a notation with which to discuss such equations, we introduce a symbol i , and define

$$i^2 = -1 .$$

A *complex number* is defined to be a symbol $a + bi$, where a, b are real numbers. If $z = a + bi$, we call a the *real part* of z and b the *imaginary part*, and write

$$a = \operatorname{Re}(z), \quad b = \operatorname{Im}(z) .$$

We define addition and multiplication of complex numbers by the rules

$$\text{addition: } (a + bi) + (c + di) = a + c + (b + d)i$$

$$\text{multiplication: } (a + bi)(c + di) = ac - bd + (ad + bc)i .$$

Notice that in the multiplication rule, we multiply out the brackets in the usual way, and replace the i^2 by -1 . For example, $(1 + 2i)(3 - i) = 5 + 5i$ and $(a + bi)(a - bi) = a^2 + b^2$.

It is also possible to subtract complex numbers:

$$(a + bi) - (c + di) = a - c + (b - d)i$$

and, less obviously, to divide them: provided c, d are not both 0,

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2} \right) i .$$

For example, $\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{-2i}{2} = -i$.

We write $\mathbb{C}$ for the set of all complex numbers. If we identify the complex number $a + 0i$ with the real number a , we see that $\mathbb{R} \subseteq \mathbb{C}$.

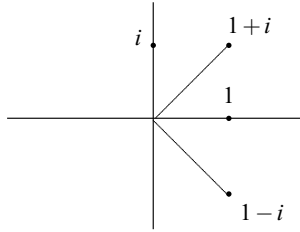
Notice that every quadratic equation $ax^2 + bx + c = 0$ (where $a, b, c \in \mathbb{R}$) has roots in $\mathbb{C}$. For by the famous formula you will be familiar with, the roots are

$$\frac{1}{2a} \left(-b \pm \sqrt{b^2 - 4ac} \right) .$$

If $b^2 \geq 4ac$ these roots lie in $\mathbb{R}$, while if $b^2 < 4ac$ they are the complex numbers $\frac{-b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a}i$.

Geometrical Representation of Complex Numbers

It turns out to be a very fruitful idea to represent complex numbers by points in the xy -plane. This is done in a natural way — the complex number $a + bi$ is represented by the point in the plane with coordinates (a, b) . For example, i is represented by $(0, 1)$; $1 - i$ by $(1, -1)$; and so on:



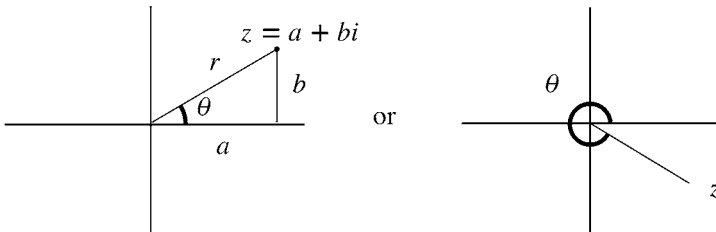
If $z = a + bi$, we define $\bar{z} = a - bi$ and call this the *complex conjugate* of z . Also, the *modulus* of $z = a + bi$ is the distance from the origin to the point (a, b) representing z . It is written as $|z|$. Thus,

$$|z| = \sqrt{a^2 + b^2}.$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2.$$

The *argument* of z is the angle θ between the x -axis and the line joining 0 to z , measured in the counterclockwise direction:



If $z = a + bi$ and $|z| = r$, then we see that $a = r \cos \theta$, $b = r \sin \theta$, so

$$z = r(\cos \theta + i \sin \theta).$$

This is known as the *polar form* of the complex number z .

Example 6.1

The polar forms of i , -1 , $1+i$ and $1-i$ are

$$\begin{aligned} i &= 1 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right), \quad -1 = 1 (\cos \pi + i \sin \pi), \\ 1+i &= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right), \quad 1-i = \sqrt{2} \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right). \end{aligned}$$

Let $z = r(\cos \theta + i \sin \theta)$. Notice that

$$\begin{aligned} \cos \theta + i \sin \theta &= \cos(\theta + 2\pi) + i \sin(\theta + 2\pi) \\ &= \cos(\theta + 4\pi) + i \sin(\theta + 4\pi) = \dots, \end{aligned}$$

so multiples of 2π can be added to θ (or subtracted from θ) without changing z . Thus, z has many different arguments. There is, however, a unique value of the argument of z in the range $-\pi < \theta \leq \pi$, and this is called the *principal argument* of z , written $\arg(z)$. For example, $\arg(1-i) = -\frac{\pi}{4}$.

De Moivre's Theorem

The xy -plane, representing the set of complex numbers as just described, is known as the *Argand diagram*; it is also sometimes called simply the *complex plane*.

The significance of the geometrical representation of complex numbers begins to become apparent in the next result, which shows that complex multiplication has a simple and natural geometric interpretation.

THEOREM 6.1 (De Moivre's Theorem)

Let z_1, z_2 be complex numbers with polar forms

$$z_1 = r_1 (\cos \theta_1 + i \sin \theta_1), \quad z_2 = r_2 (\cos \theta_2 + i \sin \theta_2).$$

Then the product

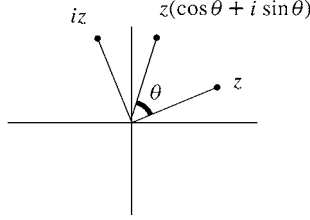
$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

In other words, $z_1 z_2$ has modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$.

PROOF We have

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\ &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \quad \blacksquare \end{aligned}$$

De Moivre's Theorem says that multiplying a complex number z by $\cos \theta + i \sin \theta$ rotates z counterclockwise through the angle θ ; for example, multiplication by i rotates z through $\frac{\pi}{2}$:



We now deduce a significant consequence of De Moivre's Theorem.

PROPOSITION 6.1

Let $z = r(\cos \theta + i \sin \theta)$, and let n be a positive integer. Then

- (i) $z^n = r^n(\cos n\theta + i \sin n\theta)$, and
(ii) $z^{-n} = r^{-n}(\cos n\theta - i \sin n\theta)$.

PROOF (i) Applying Theorem 6.1 with $z_1 = z_2 = z$ gives

$$z^2 = zz = rr(\cos(\theta + \theta) + i \sin(\theta + \theta)) = r^2(\cos 2\theta + i \sin 2\theta) .$$

Repeating, we get

$$z^n = r \dots r(\cos(\theta + \dots + \theta) + i \sin(\theta + \dots + \theta)) = r^n(\cos n\theta + i \sin n\theta) .$$

(ii) First observe that

$$\begin{aligned} z^{-1} &= \frac{1}{z} = \frac{1}{r(\cos \theta + i \sin \theta)} = \frac{1}{r} \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta)} \\ &= \frac{1}{r}(\cos \theta - i \sin \theta) . \end{aligned}$$

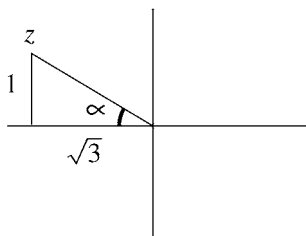
Hence $z^{-1} = r^{-1}(\cos(-\theta) + i \sin(-\theta))$, which proves the result for $n = 1$. And, for general n , we simply note that $z^{-n} = (z^{-1})^n$, which by part (i) is equal to $(r^{-1})^n(\cos(-n\theta) + i \sin(-n\theta))$, hence to $(r^{-n})(\cos(n\theta) - i \sin(n\theta))$. ■

We now give a few examples illustrating the power of De Moivre's Theorem.

Example 6.2

Calculate $(-\sqrt{3} + i)^7$.

Answer We first find the polar form of $z = -\sqrt{3} + i$.



In the diagram, $\sin \alpha = \frac{1}{2}$, so $\alpha = \frac{\pi}{6}$. Hence $\arg(z) = \frac{5\pi}{6}$. Also $|z| = 2$, so the polar form of z is

$$z = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right).$$

Hence, by Proposition 6.1,

$$\begin{aligned} (-\sqrt{3} + i)^7 &= 2^7 \left(\cos \frac{35\pi}{6} + i \sin \frac{35\pi}{6} \right) \\ &= 2^7 \left(\cos \frac{-\pi}{6} + i \sin \frac{-\pi}{6} \right) \end{aligned}$$

(subtracting 6π from the argument $\frac{35\pi}{6}$). Since $\cos \frac{-\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin \frac{-\pi}{6} = -\frac{1}{2}$, this gives

$$(-\sqrt{3} + i)^7 = 2^6 (\sqrt{3} - i).$$

Example 6.3

Find a complex number w such that $w^2 = -\sqrt{3} + i$ (i.e., find a complex square root of $-\sqrt{3} + i$).

Answer From the previous solution, $-\sqrt{3} + i = 2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$. So if we define

$$w = \sqrt{2} \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right),$$

then by Proposition 6.1, $w^2 = -\sqrt{3} + i$. Note that $\sqrt{2}(\cos(\frac{5\pi}{12} + \pi) + i \sin(\frac{5\pi}{12} + \pi))$ works equally well; by Theorem 6.1 this is equal to $w(\cos \pi + i \sin \pi) = -w$.

Example 6.4

In this example we find a formula for $\cos 3\theta$ in terms of $\cos \theta$.

We begin with the equation

$$\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3.$$

Writing $c = \cos \theta, s = \sin \theta$, and expanding the cube, we get

$$\cos 3\theta + i \sin 3\theta = c^3 + 3c^2si + 3cs^2i^2 + s^3i^3 = c^3 - 3cs^2 + i(3c^2s - s^3) .$$

Equating real parts, we have $\cos 3\theta = c^3 - 3cs^2$. Also $c^2 + s^2 = \cos^2 \theta + \sin^2 \theta = 1$, so $s^2 = 1 - c^2$, and therefore

$$\cos 3\theta = c^3 - 3c(1 - c^2) = 4c^3 - 3c .$$

That is,

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta .$$

Example 6.5

We now use the previous example to find a cubic equation having $\cos \frac{\pi}{9}$ as a root.

Putting $\theta = \frac{\pi}{9}$ and $c = \cos \frac{\pi}{9}$, Example 6.4 gives

$$\cos 3\theta = 4c^3 - 3c .$$

However, $\cos 3\theta = \cos \frac{\pi}{3} = \frac{1}{2}$. Hence $\frac{1}{2} = 4c^3 - 3c$. In other words, $c = \cos \frac{\pi}{9}$ is a root of the cubic equation

$$8x^3 - 6x - 1 = 0 .$$

Note that if $\phi = \frac{\pi}{9} + \frac{2\pi}{3}$ or $\frac{\pi}{9} + \frac{4\pi}{3}$, then $\cos 3\phi = \frac{1}{2}$, and hence the above argument shows $\cos \phi$ is also a root of this cubic equation. The roots of $8x^3 - 6x - 1 = 0$ are therefore $\cos \frac{\pi}{9}, \cos \frac{7\pi}{9}$ and $\cos \frac{13\pi}{9}$.

The $e^{i\theta}$ Notation

It is somewhat cumbersome to keep writing $\cos \theta + i \sin \theta$ in our notation for complex numbers. We therefore introduce a rather more compact notation by defining

$$e^{i\theta} = \cos \theta + i \sin \theta$$

for any real number θ . (This equation turns out to be very significant when $e^{i\theta}$ is regarded as an exponential function, but for now it is simply the definition of the symbol $e^{i\theta}$.)

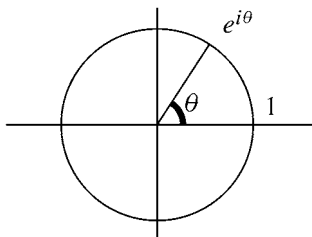
For example,

$$e^{2\pi i} = 1, \quad e^{\pi i} = -1, \quad e^{\frac{\pi}{2}i} = i, \quad e^{\frac{\pi}{4}i} = \frac{1}{\sqrt{2}}(1 + i) .$$

Also, for any integer k ,

$$e^{i\theta} = e^{i(\theta+2k\pi)}.$$

Each of the complex numbers $e^{i\theta}$ has modulus 1, and the set consisting of all of them is the *unit circle* in the Argand diagram – that is, the circle of radius 1 centered at the origin:



The polar form of a complex number z can now be written as

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg(z)$. For example,

$$-\sqrt{3} + i = 2e^{\frac{5\pi i}{6}}.$$

De Moivre's Theorem 6.1 implies that

$$e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)},$$

and Proposition 6.1 says that for any integer n ,

$$(e^{i\theta})^n = e^{in\theta}.$$

From these facts we begin to see some of the significance emerging behind the definition of $e^{i\theta}$.

PROPOSITION 6.2

- (i) If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$.
- (ii) Let $z = re^{i\theta}, w = se^{i\phi}$ in polar form. Then $z = w$ if and only if both $r = s$ and $\theta - \phi = 2k\pi$ with $k \in \mathbb{Z}$.

PROOF (i) We have $z = r(\cos \theta + i \sin \theta)$, so $\bar{z} = r(\cos \theta - i \sin \theta) = r(\cos(-\theta) + i \sin(-\theta)) = re^{-i\theta}$.

(ii) If $r = s$ and $\theta - \phi = 2k\pi$ with $k \in \mathbb{Z}$, then

$$z = re^{i\theta} = se^{i(\phi+2k\pi)} = se^{i\phi} = w.$$

This does the “right to left” implication.

For the “left to right” implication, suppose $z = w$. Then $|z| = |w|$, so $r = s$ and also $e^{i\theta} = e^{i\phi}$. Now

$$\begin{aligned} e^{i\theta} = e^{i\phi} &\Rightarrow e^{i\theta} e^{-i\phi} = e^{i\phi} e^{-i\phi} \Rightarrow e^{i(\theta-\phi)} = 1 \\ &\Rightarrow \cos(\theta - \phi) = 1, \sin(\theta - \phi) = 0 \Rightarrow \theta - \phi = 2k\pi \text{ with } k \in \mathbb{Z}. \quad \blacksquare \end{aligned}$$

Roots of Unity

Consider the equation

$$z^3 = 1.$$

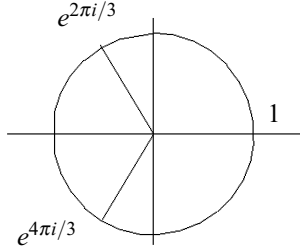
This is easy enough to solve: rewriting it as $z^3 - 1 = 0$, and factorizing this as $(z - 1)(z^2 + z + 1) = 0$, we see that the roots are

$$1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

These complex numbers have polar forms

$$1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}.$$

In other words, they are evenly spaced on the unit circle like this:



These three complex numbers are called the *cube roots of unity*.

More generally, if n is a positive integer, then the complex numbers that satisfy the equation

$$z^n = 1$$

are called the n^{th} roots of unity.

PROPOSITION 6.3

Let n be a positive integer and define $w = e^{\frac{2\pi i}{n}}$. Then the n^{th} roots of unity are the n complex numbers

$$1, w, w^2, \dots, w^{n-1}$$

(i.e., $1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \dots, e^{\frac{2(n-1)\pi i}{n}}$). They are evenly spaced around the unit circle.

PROOF Let $z = re^{i\theta}$ be an n^{th} root of unity. Then

$$1 = z^n = r^n e^{ni\theta}.$$

From Proposition 6.2(ii) it follows that $r = 1$ and $n\theta = 2k\pi$ with $k \in \mathbb{Z}$. Therefore, $\theta = \frac{2k\pi}{n}$, and so $z = e^{\frac{2k\pi i}{n}} = w^k$.

Thus every n^{th} root of unity is a power of w . On the other hand, any power w^k is an n^{th} root of unity, since

$$(w^k)^n = w^{nk} = \left(e^{\frac{2\pi i}{n}}\right)^{nk} = (e^{2\pi i})^k = 1.$$

The complex numbers

$$1, w, w^2, \dots, w^{n-1}$$

are all the distinct powers of w (since $w^n = 1, w^{n+1} = w$, etc.). Hence, these are the n^{th} roots of unity. ■

Example 6.6

The fourth roots of unity are $1, e^{\frac{i\pi}{2}}, e^{i\pi}, e^{\frac{3i\pi}{2}}$, which are just

$$1, i, -1, -i.$$

The sixth roots of unity are $1, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, -1, e^{\frac{4i\pi}{3}}$ and $e^{\frac{5i\pi}{3}}$; these are the corners of a regular hexagon drawn inside the unit circle.

We can use the n^{th} roots of unity to find the n^{th} roots of any complex number. Here is an example.

Example 6.7

Find all solutions of the equation

$$z^5 = -\sqrt{3} + i.$$

(In other words, find all the fifth roots of $-\sqrt{3} + i$.)

Answer Let $p = -\sqrt{3} + i$. Recall that $p = 2e^{\frac{5\pi i}{6}}$. One of the fifth roots of this is clearly

$$\alpha = 2^{\frac{1}{5}} e^{\frac{\pi i}{6}}$$

(where of course $2^{\frac{1}{5}}$ is the real fifth root of 2). If w is a fifth root of unity, then $(\alpha w)^5 = \alpha^5 w^5 = \alpha^5 = z$, so αw is also a fifth root of p . Thus we have found the following 5 fifth roots of $-\sqrt{3} + i$:

$$\alpha, \alpha e^{\frac{2\pi i}{5}}, \alpha e^{\frac{4\pi i}{5}}, \alpha e^{\frac{6\pi i}{5}}, \alpha e^{\frac{8\pi i}{5}}.$$

These are in fact all the fifth roots of p : for if β is any fifth root of p , then $\beta^5 = \alpha^5 = p$, so $(\frac{\beta}{\alpha})^5 = 1$, which means that $\frac{\beta}{\alpha} = w$ is a fifth root of unity, and hence $\beta = \alpha w$ is in the above list.

We conclude that the fifth roots of $-\sqrt{3} + i$ are

$$2^{\frac{1}{5}} e^{\frac{\pi i}{6}}, 2^{\frac{1}{5}} e^{\frac{17\pi i}{30}}, 2^{\frac{1}{5}} e^{\frac{29\pi i}{30}}, 2^{\frac{1}{5}} e^{\frac{41\pi i}{30}}, 2^{\frac{1}{5}} e^{\frac{53\pi i}{30}}.$$

In general, the above method shows that if one of the n^{th} roots of a complex number is β , then the others are $\beta w, \beta w^2, \dots, \beta w^{n-1}$ where $w = e^{\frac{2\pi i}{n}}$.

Exercises for Chapter 6

- Prove the following facts about complex numbers:
 - $u + v = v + u$ and $uv = vu$ for all $u, v \in \mathbb{C}$.
 - $\overline{u + v} = \bar{u} + \bar{v}$ and $\overline{uv} = \bar{u}\bar{v}$ for all $u, v \in \mathbb{C}$.
 - $|u|^2 = u\bar{u}$ for all $u \in \mathbb{C}$.
 - $|uv| = |u||v|$ for all $u, v \in \mathbb{C}$.
 - $u(vw) = (uv)w$ for all $u, v, w \in \mathbb{C}$.

(Hint: For (e), use the polar forms of u, v, w .)
- Prove the "Triangle Inequality" for complex numbers: $|u + v| \leq |u| + |v|$ for all $u, v \in \mathbb{C}$.
- Find the real and imaginary parts of $(\sqrt{3} - i)^{10}$ and $(\sqrt{3} - i)^{-7}$. For which values of n is $(\sqrt{3} - i)^n$ real?
- What is $\sqrt{i}$?
 - Find all the tenth roots of i . Which one is nearest to i in the Argand diagram?
 - Find the seven roots of the equation $z^7 - \sqrt{3} + i = 0$. Which one of these roots is closest to the imaginary axis?
- Let z be a non-zero complex number. Prove that the three cube roots of z are the corners of an equilateral triangle in the Argand diagram.

6. Express $\frac{1+i}{\sqrt{3}+i}$ in the form $x+iy$, where $x, y \in \mathbb{R}$. By writing each of $1+i$ and $\sqrt{3}+i$ in polar form, deduce that

$$\cos \frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad \sin \frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

7. (a) Show that $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$. Deduce that if $\omega = e^{2\pi i/5}$ then $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0$.

(b) Let $\alpha = 2\cos \frac{2\pi}{5}$ and $\beta = 2\cos \frac{4\pi}{5}$. Show that $\alpha = \omega + \omega^4$ and $\beta = \omega^2 + \omega^3$. Find a quadratic equation with roots α, β . Hence show that

$$\cos \frac{2\pi}{5} = \frac{1}{4}(\sqrt{5}-1).$$

8. Find a formula for $\cos 4\theta$ in terms of $\cos \theta$. Hence write down a quartic equation (i.e., an equation of degree 4) that has $\cos \frac{\pi}{12}$ as a root. What are the other roots of your equation?
9. Find all complex numbers z such that $|z| = |\sqrt{2}+z| = 1$. Prove that each of these satisfies $z^8 = 1$.
10. Prove that there is no complex number z such that $|z| = |z+i\sqrt{5}| = 1$.
11. Show that if w is an n^{th} root of unity, then $\bar{w} = \frac{1}{w}$. Deduce that

$$\overline{(1-w)^n} = (w-1)^n.$$

Hence show that $(1-w)^{2n}$ is real.

12. Let n be a positive integer, and let $z \in \mathbb{C}$ satisfy the equation

$$(z-1)^n + (z+1)^n = 0.$$

- (a) Show that $z = \frac{1+w}{1-w}$ for some $w \in \mathbb{C}$ such that $w^n = -1$.
- (b) Show that $w\bar{w} = 1$.
- (c) Deduce that z lies on the imaginary axis.
13. Critic Ivor Smallbrain is discussing the film *Sets, Lines and Videotape* with his two chief editors, Sir Giles Tantrum and Lord Overthetop. They are sitting at a circular table of radius 1. Ivor is bored and notices in a daydream that he can draw real and imaginary axes, with origin at the center of the table, in such a way that Tantrum is represented by a certain complex number z and Overthetop is represented by the complex number $z+1$. Breaking out of his daydream, Ivor suddenly exclaims “you are both sixth roots of 1!”

Prove that Ivor is correct, despite the incredulous editorial glares.

Chapter 7

Polynomial Equations

Expressions like $x^2 - 3x$ or $-7x^{102} + (3 - i)x^{17} - 7$, or more generally,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 ,$$

where the coefficients $a_0, a_1, \dots, a_n$ are complex numbers, are called *polynomials* in x . A *poly-nomial equation* is an equation of the form

$$p(x) = 0$$

where $p(x)$ is a polynomial. The *degree* of such an equation is the highest power of x that appears with a non-zero coefficient.

For example, equations of degree 1 take the form $ax + b = 0$ and are also known as *linear* equations; degree 2 equations $ax^2 + bx + c = 0$ are *quadratic* equations; degree 3 equations are *cubic* equations; degree 4 are *quartic* equations; degree 5 are *quintic* equations; and so on.

A complex number α is said to be a *root* of the polynomial equation $p(x) = 0$ if $p(\alpha) = 0$: in other words, when α is substituted for x , $p(x)$ becomes equal to 0. For example, 1 is a root of the cubic equation $x^3 - 3x + 2 = 0$.

The search for formulae for the roots of polynomial equations was one of the driving forces in mathematics from the time of the Greeks until the 19th century. Let us now taste a tiny flavor of this huge subject, in the hope that appetites are whetted for more.

It is obvious that any linear equation $ax + b = 0$ has exactly one root, namely $-\frac{b}{a}$. We are also familiar with the fact that any quadratic equation $ax^2 + bx + c = 0$ has roots in $\mathbb{C}$, given by the formula $\frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$.

Things are less clear for cubic equations. Indeed, while the formula for the roots of a quadratic was known to the Greeks, it was not until the 16th century that a method for finding the roots of a cubic was found by the Italian mathematicians Scipio Ferreo, Tartaglia and Cardan. Here is their method.

Solution of Cubic Equations

Consider the cubic equation

$$x^3 + ax^2 + bx + c = 0 \quad (7.1)$$

The first step is to get rid of the x^2 term. This is easily done: put $y = x + \frac{a}{3}$. Then $y^3 = (x + \frac{a}{3})^3 = x^3 + ax^2 + \frac{a^2}{3}x + \frac{a^3}{27}$, so Equation (7.1) becomes $y^3 + b'y + c' = 0$ for some b', c' . Write this equation as

$$y^3 + 3hy + k = 0 \quad (7.2)$$

(The coefficients $3h$ and k can easily be worked out, given a, b, c .)

Here comes the clever part. Write $y = u + v$. Then

$$\begin{aligned} y^3 &= (u + v)^3 = u^3 + v^3 + 3u^2v + 3uv^2 = u^3 + v^3 + 3uv(u + v) \\ &= u^3 + v^3 + 3uvy. \end{aligned}$$

Hence, the cubic equation

$$y^3 + 3uvy - (u^3 + v^3) = 0 \quad (7.3)$$

has $u + v$ as a root.

Our aim now is to find u and v so that the coefficients in Equations (7.2) and (7.3) are matched up. To match the coefficients, we require

$$h = -uv, \quad k = -(u^3 + v^3). \quad (7.4)$$

From the first of these equations we have $v^3 = \frac{-h^3}{u^3}$, hence the second equation gives $u^3 - \frac{h^3}{u^3} = -k$, so

$$u^6 + ku^3 - h^3 = 0. \quad (7.5)$$

This is just a quadratic equation for u^3 , and a solution is

$$u^3 = \frac{1}{2} \left(-k + \sqrt{k^2 + 4h^3} \right).$$

Then from (7.4),

$$v^3 = -k - u^3 = \frac{1}{2} \left(-k - \sqrt{k^2 + 4h^3} \right).$$

As $y = u + v$, we have obtained the following formula for the roots of the cubic (7.2):

$$\sqrt[3]{\frac{1}{2} \left(-k + \sqrt{k^2 + 4h^3} \right)} + \sqrt[3]{\frac{1}{2} \left(-k - \sqrt{k^2 + 4h^3} \right)}.$$

Since a complex number has three cube roots, and there are two cube roots to be chosen, it seems that there are nine possible values for this formula. However, the equation $uv = -h$ implies that $v = \frac{-h}{u}$, and hence there are only three roots of (7.2), these being $u - \frac{h}{u}$ for each of the three choices for u .

Specifically, if u is one of the cube roots of $\frac{1}{2}(-k + \sqrt{k^2 + 4h^3})$, the other cube roots are $u\omega, u\omega^2$ where $\omega = e^{\frac{2\pi i}{3}}$, and so the roots of the cubic equation (7.2) are

$$u - \frac{h}{u}, \quad u\omega - \frac{h\omega^2}{u}, \quad u\omega^2 - \frac{h\omega}{u}.$$

Once we know the roots of (7.2), we can of course write down the roots of the general cubic (7.1), since $x = y - \frac{a}{3}$.

Let us illustrate this method with a couple of examples.

Example 7.1

(1) Consider the cubic equation $x^3 - 6x - 9 = 0$. This is (7.2) with $h = -2, k = -9$, so $\frac{1}{2}(-k + \sqrt{k^2 + 4h^3}) = \frac{1}{2}(9 + \sqrt{49}) = 8$. Hence, taking $u = 2$, we see that the roots are $3, 2\omega + \omega^2, 2\omega^2 + \omega$. As $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ and $\omega^2 = \frac{1}{2}(-1 - i\sqrt{3})$, these roots are

$$3, \quad \frac{1}{2}(-3 + i\sqrt{3}), \quad \frac{1}{2}(-3 - i\sqrt{3}).$$

(Of course these could easily have been worked out by cleverly spotting that 3 is a root and factorizing the equation as $(x-3)(x^2 + 3x + 3) = 0$.)

(2) Consider the equation $x^3 - 6x - 40 = 0$. The above formula gives roots of the form

$$\sqrt[3]{20 + 14\sqrt{2}} + \sqrt[3]{20 - 14\sqrt{2}}.$$

However, we cleverly also spot that 4 is a root. What is going on?

In fact, nothing very mysterious is going on. The real cube root of $20 \pm 14\sqrt{2}$ is $2 \pm \sqrt{2}$, as can be seen by cubing the latter. Hence, the roots of the cubic are

$$\begin{aligned} (2 + \sqrt{2}) + (2 - \sqrt{2}) &= 4, \\ (2 + \sqrt{2})\omega + (2 - \sqrt{2})\omega^2 &= -2 + i\sqrt{6}, \\ (2 + \sqrt{2})\omega^2 + (2 - \sqrt{2})\omega &= -2 - i\sqrt{6}. \end{aligned}$$

Higher Degrees

Not long after the solution of the cubic, Ferrari, a pupil of Cardan, showed how to obtain a formula for the roots of a general quartic (degree 4) equation. The next step, naturally enough, was the quintic. However, several hundred years passed without anyone finding a formula for the roots of a general quintic equation.

There was a good reason for this. There is no such formula. Nor is there a formula for equations of degree greater than 5. This amazing fact was first established in the early 19th century by the Danish mathematician Abel (who died at age 26), after which the Frenchman Galois (who died at age 21) built an entirely new theory of equations, linking them to the then recent subject of group theory, which not only explained the non-existence of formulae, but laid the foundations of a whole edifice of algebra and number theory known as *Galois theory*, a major area of modern-day research. If you get a chance, take a course in Galois theory during the rest of your studies in mathematics — you won't regret it!

The Fundamental Theorem of Algebra

So, there is no formula for the roots of a polynomial equation of degree 5 or more. We are therefore led to the troubling question: Can we be sure that such an equation actually has a root in the complex numbers?

The answer to this is yes, we can be sure. This is a famous theorem of another great mathematician — perhaps the greatest of all — Gauss:

THEOREM 7.1 Fundamental Theorem of Algebra

Every polynomial equation of degree at least 1 has a root in $\mathbb{C}$.

This is really a rather amazing result. After all, we introduced complex numbers just to be able to talk about roots of quadratics like $x^2 + 1 = 0$, and we find ourselves with a system that contains roots of *all* polynomial equations.

There are many different proofs of the Fundamental Theorem of Algebra available — Gauss himself found five during his lifetime. Probably the proofs that are easiest to understand are those using various basic results in the subject of complex analysis, and most undergraduate courses on this topic would include a proof of the Fundamental Theorem of Algebra. In Chapter 24 we shall give a proof of the theorem for real polynomials of odd degree.

The Fundamental Theorem of Algebra has many consequences. We shall give just a couple here. Let $p(x)$ be a polynomial of degree n . By the Fundamental Theorem 7.1, $p(x)$ has a root in $\mathbb{C}$, say α_1 . It is rather easy to see from this (but we won't prove it here) that there is another polynomial $q(x)$, of degree $n - 1$, such that

$$p(x) = (x - \alpha_1)q(x) .$$

By Theorem 7.1, $q(x)$ also has a root in $\mathbb{C}$, say α_2 , so as above there is a polynomial $r(x)$ of degree $n - 2$ such that

$$p(x) = (x - \alpha_1)(x - \alpha_2)r(x) .$$

Repeat this argument until we get down to a polynomial of degree 1. We thus obtain a factorization

$$p(x) = a(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_n) ,$$

where a is the coefficient of x^n , and $\alpha_1, \dots, \alpha_n$ are the roots of $p(x)$. These may of course not all be different, but there are precisely n of them if we count repeats.

Summarizing:

THEOREM 7.2

Every polynomial of degree n factorizes as a product of linear polynomials and has exactly n roots in $\mathbb{C}$ (counting repeats).

Next, we briefly consider *real* polynomial equations — that is, equations of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0 ,$$

where all the coefficients $a_0, a_1, \dots, a_n$ are real numbers.

Of course not all the roots of such an equation need be real (think of $x^2 + 1 = 0$). However, we can say something interesting about the roots.

Let $\alpha \in \mathbb{C}$ be a root of the real polynomial equation $p(x) = 0$. Thus,

$$p(\alpha) = a_n \alpha^n + a_{n-1} \alpha^{n-1} + \dots + a_1 \alpha + a_0 = 0 .$$

Consider the complex conjugate $\bar{\alpha}$. We shall show this is also a root. To see this, observe first that $\overline{\alpha^2} = \bar{\alpha}^2$ (just apply Exercise 1(b) of Chapter 6 with $u = v = \alpha$); likewise, $\overline{\alpha^3} = \bar{\alpha}^3, \dots, \overline{\alpha^n} = \bar{\alpha}^n$. Also $\bar{a_i} = a_i$ for all i , since the a_i are all real. Consequently,

$$\begin{aligned} p(\bar{\alpha}) &= a_n \bar{\alpha}^n + a_{n-1} \bar{\alpha}^{n-1} + \dots + a_1 \bar{\alpha} + a_0 \\ &= a_n \overline{\alpha^n} + a_{n-1} \overline{\alpha^{n-1}} + \dots + a_1 \bar{\alpha} + a_0 \\ &= \overline{a_n \alpha^n + a_{n-1} \alpha^{n-1} + \dots + a_1 \alpha + a_0} \\ &= \overline{p(\alpha)} = 0 , \end{aligned}$$

showing that $\bar{\alpha}$ is indeed a root.

Thus for a real polynomial equation $p(x) = 0$, the non-real roots appear in complex conjugate pairs $\alpha, \bar{\alpha}$. Say the real roots are $\beta_1, \dots, \beta_k$ and the non-real roots are $\alpha_1, \bar{\alpha}_1, \dots, \alpha_l, \bar{\alpha}_l$ (where $k + 2l = n$). Then, as discussed above,

$$p(x) = (x - \beta_1) \dots (x - \beta_k) (x - \alpha_1) (x - \bar{\alpha}_1) \dots (x - \alpha_l) (x - \bar{\alpha}_l) .$$

Notice that $(x - \alpha_i)(x - \bar{\alpha}_i) = x^2 - (\alpha_i + \bar{\alpha}_i)x + \alpha_i \bar{\alpha}_i$, which is a quadratic with real coefficients. Thus, $p(x)$ factorizes as a product of real linear and real quadratic polynomials.

Summarizing:

THEOREM 7.3

Every real polynomial factorizes as a product of real linear and real quadratic polynomials and has its non-real roots appearing in complex conjugate pairs.

Relationships Between Roots

Despite the fact that there is no general formula for the roots of a polynomial equation of degree 5 or more, there are still some interesting and useful relationships between the roots and the coefficients.

First consider a quadratic equation $x^2 + ax + b = 0$. If α, β are the roots, then $x^2 + ax + b = (x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta$, and hence, equating coefficients, we have

$$\alpha + \beta = -a, \quad \alpha\beta = b .$$

Likewise, for a cubic equation $x^3 + ax^2 + bx + c = 0$ with roots α, β, γ , we have $x^3 + ax^2 + bx + c = (x - \alpha)(x - \beta)(x - \gamma)$, hence

$$\alpha + \beta + \gamma = -a, \quad \alpha\beta + \alpha\gamma + \beta\gamma = b, \quad \alpha\beta\gamma = -c .$$

Applying this argument to an equation of degree n , we have

PROPOSITION 7.1

Let the roots of the equation

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$$

be $\alpha_1, \alpha_2, \dots, \alpha_n$. If s_1 denotes the sum of the roots, s_2 denotes the sum of all products of pairs of roots, s_3 denotes the sum of all products of

triples of roots, and so on, then

$$\begin{aligned}s_1 &= \alpha_1 + \cdots + \alpha_n = -a_{n-1} , \\s_2 &= a_{n-2} , \\s_3 &= -a_{n-3} , \\&\dots \quad \dots \\s_n &= \alpha_1 \alpha_2 \dots \alpha_n = (-1)^n a_0 .\end{aligned}$$

PROOF We have

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n) .$$

If we multiply out the right-hand side, the coefficient of x^{n-1} is $-(\alpha_1 + \cdots + \alpha_n) = -s_1$, the coefficient of x^{n-2} is s_2 , and so on. The result follows. ■

Here are some examples of what can be done with this result.

Example 7.2

(1) Write down a cubic equation with roots $1+i, 1-i, 2$.

Answer If we call these three roots $\alpha_1, \alpha_2, \alpha_3$, then $s_1 = \alpha_1 + \alpha_2 + \alpha_3 = 4$, $s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = (1+i)(1-i) + 2(1+i) + 2(1-i) = 6$ and $s_3 = \alpha_1\alpha_2\alpha_3 = 4$. Hence, a cubic with these roots is

$$x^3 - 4x^2 + 6x - 4 = 0 .$$

(2) If α, β are the roots of the equation $x^2 - 5x + 9 = 0$, find a quadratic equation with roots α^2, β^2 .

Answer Of course this could be done by using the formula to write down α and β , then squaring them, and so on; this would be rather tedious, and a much more elegant solution is as follows. From the above, we have

$$\alpha + \beta = 5, \quad \alpha\beta = 9 .$$

Therefore, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 5^2 - 18 = 7$ and $\alpha^2\beta^2 = 9^2 = 81$. We therefore know the sum and product of α^2 and β^2 , so a quadratic having these as roots is $x^2 - 7x + 81 = 0$.

(3) Find the value of k if the roots of the cubic equation $x^3 + x^2 + 2x + k = 0$ are in geometric progression.

Answer Saying that the roots are in geometric progression means that they are of the form $\alpha, \alpha r, \alpha r^2$ for some r . We then have

$$\alpha(1+r+r^2) = -1, \quad \alpha^2(r+r^2+r^3) = 2, \quad \alpha^3r^3 = -k .$$

Dividing the first two of these gives $\alpha r = -2$. Hence, the third gives $k = 8$.

Exercises for Chapter 7

- Find the (complex) roots of the quadratic equation $x^2 - 5x + 7 - i = 0$.
 - Find the roots of the quartic equation $x^4 + x^2 + 1 = 0$.
 - Find the roots of the equation $2x^4 - 4x^3 + 3x^2 + 2x - 2 = 0$, given that one of them is $1 + i$.
- Use the method given in this chapter for solving cubics to find the roots of the equation $x^3 - 6x^2 + 13x - 12 = 0$.

Now notice that 3 is one of the roots. Reconcile this with the roots you have found.

- Solve $x^3 - 15x - 4 = 0$ using the method for solving cubics.

Now cleverly spot an integer root. Deduce that

$$\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{11}{2}\right)\right) = \frac{2}{\sqrt{5}}.$$

- Factorize $x^5 + 1$ as a product of real linear and quadratic polynomials.
- Show that $\cos \frac{2\pi}{9}$ is a root of the cubic equation $8x^3 - 6x + 1 = 0$.

Find the other two roots and deduce that

$$\cos \frac{2\pi}{9} + \cos \frac{4\pi}{9} + \cos \frac{8\pi}{9} = 0$$

and

$$\cos \frac{2\pi}{9} \cdot \cos \frac{4\pi}{9} \cdot \cos \frac{8\pi}{9} = -\frac{1}{8}.$$

- Factorize $x^{2n+1} - 1$ as a product of real linear and quadratic polynomials.
 - Write $x^{2n} + x^{2n-1} + \cdots + x + 1$ as a product of real quadratic polynomials.
 - Let $\omega = e^{2\pi i/2n+1}$. Show that $\sum \omega^{i+j} = 0$, where the sum is over all i and j from 1 to $2n+1$ such that $i < j$.

7. (a) Find the value of k such that the roots of $x^3 + 6x^2 + kx - 10 = 0$ are in arithmetical progression (i.e., are $\alpha, \alpha + d, \alpha + 2d$ for some d). Solve the equation for this value of k .
- (b) If the roots of the equation $x^3 - x - 1 = 0$ are α, β, γ , find a cubic equation having roots $\alpha^2, \beta^2, \gamma^2$ and also a cubic equation having roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.
- (c) Given that the sum of two of the roots of the equation $x^3 + px^2 + p^2x + r = 0$ is 1, prove that $r = (p+1)(p^2 + p + 1)$.
- (d) Solve the equation $x^4 - 3x^3 - 5x^2 + 17x - 6 = 0$, given that the sum of two of the roots is 5.
8. Critic Ivor Smallbrain and his friend Polly Gnomialle have entered a competition to try to qualify to join the UK team for the annual Film Critics Mathematical Olympiad. The qualification competition consists of one question. Here it is:

Find all real or complex solutions of the simultaneous equations

$$\begin{aligned}x + y + z &= 3 \\x^2 + y^2 + z^2 &= 3 \\x^3 + y^3 + z^3 &= 3\end{aligned}$$

The first pair to correctly answer the question gets into the team. Ivor overhears his arch-rival Greta Picture, who has also entered the competition, whispering to her partner, "Let's look for a cubic equation which has roots x, y and z ."

Can you help Ivor and Polly win?

