

Elementary Linear Algebra

8e

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Elementary Linear Algebra
Eighth Edition**Ron Larson**

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WCN: 02-200-203

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Library of Congress Control Number: 2015944033

Student Edition

ISBN: 978-1-305-65800-4

Loose-leaf Edition

ISBN: 978-1-305-95320-8

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Technology Guide*

*Available online at **CengageBrain.com**.

1 Systems of Linear Equations

- 1.1 Introduction to Systems of Linear Equations
- 1.2 Gaussian Elimination and Gauss-Jordan Elimination
- 1.3 Applications of Systems of Linear Equations



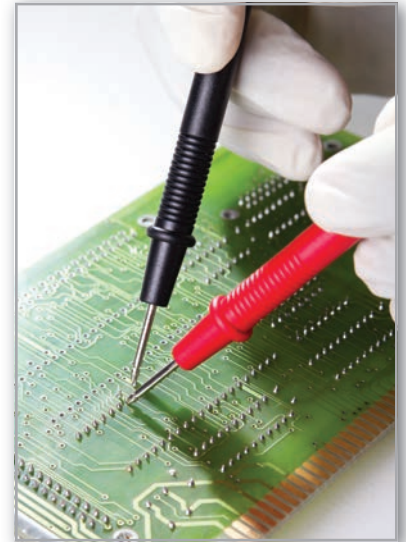
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1.1 Introduction to Systems of Linear Equations

-  Recognize a linear equation in n variables.
-  Find a parametric representation of a solution set.
-  Determine whether a system of linear equations is consistent or inconsistent.
-  Use back-substitution and Gaussian elimination to solve a system of linear equations.

LINEAR EQUATIONS IN n VARIABLES

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, you will find examples and exercises requiring a knowledge of calculus, and these are marked in the text.

Early in your study of linear algebra, you will discover that many of the solution methods involve multiple arithmetic steps, so it is essential that you check your work. Use software or a calculator to check your work and perform routine computations.

Although you will be familiar with some material in this chapter, you should carefully study the methods presented. This will cultivate and clarify your intuition for the more abstract material that follows.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** x and y . Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** x , y , and z . A linear equation in n variables is defined below.

Definition of a Linear Equation in n Variables

A **linear equation in n variables** $x_1, x_2, x_3, \dots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \dots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient**, and x_1 is the **leading variable**.

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

EXAMPLE 1

Linear and Nonlinear Equations

Each equation is linear.

a. $3x + 2y = 7$ b. $\frac{1}{2}x + y - \pi z = \sqrt{2}$ c. $(\sin \pi)x_1 - 4x_2 = e^2$

Each equation is not linear.

a. $xy + z = 2$ b. $e^x - 2y = 4$ c. $\sin x_1 + 2x_2 - 3x_3 = 0$ 

SOLUTIONS AND SOLUTION SETS

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, \dots, s_n$ that satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \dots, \quad x_n = s_n$$

into the equation. For example, $x_1 = 2$ and $x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is its **solution set**, and when you have found this set, you have **solved** the equation. To describe the entire solution set of a linear equation, use a **parametric representation**, as illustrated in Examples 2 and 3.

EXAMPLE 2

Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION

To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. Solving for x_1 in terms of x_2 , you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free**, which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2 . To represent the infinitely many solutions of this equation, it is convenient to introduce a third variable t called a **parameter**. By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \quad x_2 = t, \quad t \text{ is any real number.}$$

To obtain particular solutions, assign values to the parameter t . For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$. 

To parametrically represent the solution set of the linear equation in Example 2 another way, you could have chosen x_1 to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \quad x_2 = 2 - \frac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, when an equation has more than one free variable, choose the variables that occur last in the equation to be the free variables.

EXAMPLE 3

Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

SOLUTION

Choosing y and z to be the free variables, solve for x to obtain

$$\begin{aligned} 3x &= 3 - 2y + z \\ x &= 1 - \frac{2}{3}y + \frac{1}{3}z. \end{aligned}$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \frac{2}{3}s + \frac{1}{3}t, \quad y = s, \quad z = t$$

where s and t are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \quad \text{and} \quad x = 1, y = 1, z = 2.$$


SYSTEMS OF LINEAR EQUATIONS

A **system of m linear equations in n variables** is a set of m equations, each of which is linear in the same n variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.$$

A system of linear equations is also called a **linear system**. A **solution** of a linear system is a sequence of numbers $s_1, s_2, s_3, \dots, s_n$ that is a solution of each equation in the system. For example, the system

$$3x_1 + 2x_2 = 3$$

$$-x_1 + x_2 = 4$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy *both* equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

REMARK

The double-subscript notation indicates a_{ij} is the coefficient of x_j in the i th equation.

DISCOVERY

1. Graph the two lines

$$3x - y = 1$$

$$2x - y = 0$$

in the xy -plane. Where do they intersect? How many solutions does this system of linear equations have?

2. Repeat this analysis for the pairs of lines

$$3x - y = 1$$

$$3x - y = 0$$

$$\text{and } \begin{matrix} 3x - y = 1 \\ 6x - 2y = 2. \end{matrix}$$

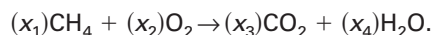
3. What basic types of solution sets are possible for a system of two linear equations in two variables?

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.



LINEAR ALGEBRA APPLIED

In a chemical reaction, atoms reorganize in one or more substances. For example, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form



A chemical reaction can neither create nor destroy atoms. So, all of the atoms represented on the left side of the arrow must also be on the right side of the arrow. This is called *balancing* the chemical equation. In the above example, chemists can use a system of linear equations to find values of x_1 , x_2 , x_3 , and x_4 that will balance the chemical equation.

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It is possible for a system of linear equations to have exactly one solution, infinitely many solutions, or no solution. A system of linear equations is **consistent** when it has at least one solution and **inconsistent** when it has no solution.

EXAMPLE 4**Systems of Two Equations in Two Variables**

Solve and graph each system of linear equations.

a. $x + y = 3$
 $x - y = -1$

b. $x + y = 3$
 $2x + 2y = 6$

c. $x + y = 3$
 $x + y = 1$

SOLUTION

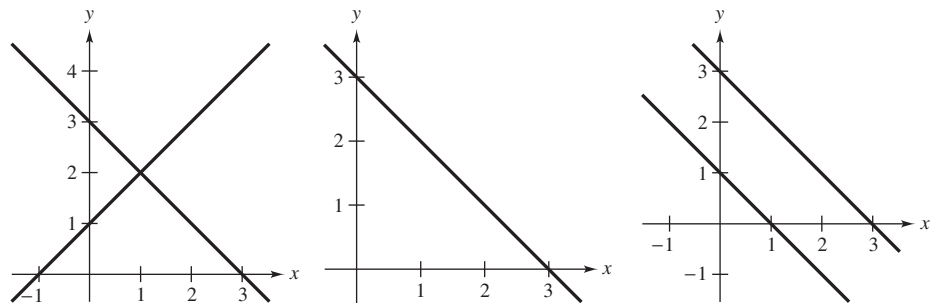
a. This system has exactly one solution, $x = 1$ and $y = 2$. One way to obtain the solution is to add the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is two *intersecting* lines, as shown in Figure 1.1(a).

b. This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

The graph of this system is two *coincident* lines, as shown in Figure 1.1(b).

c. This system has no solution because the sum of two numbers cannot be 3 and 1 simultaneously. The graph of this system is two *parallel* lines, as shown in Figure 1.1(c).



a. Two intersecting lines:

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

b. Two coincident lines:

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

c. Two parallel lines:

$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

Figure 1.1

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

Number of Solutions of a System of Linear Equations

For a system of linear equations, precisely one of the statements below is true.

1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{array}{rcl} x - 2y + 3z & = & 9 \\ -x + 3y & = & -4 \\ 2x - 5y + 5z & = & 17 \end{array} \qquad \begin{array}{rcl} x - 2y + 3z & = & 9 \\ y + 3z & = & 5 \\ z & = & 2 \end{array}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form**, which means that it has a “stair-step” pattern with leading coefficients of 1. To solve such a system, use **back-substitution**.

EXAMPLE 5

Using Back-Substitution in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{array}{rcl} x - 2y & = & 5 \\ y & = & -2 \end{array} \qquad \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \end{array}$$

SOLUTION

From Equation 2, you know that $y = -2$. By substituting this value of y into Equation 1, you obtain

$$\begin{array}{rcl} x - 2(-2) & = & 5 \\ x & = & 1. \end{array} \qquad \begin{array}{l} \text{Substitute } -2 \text{ for } y. \\ \text{Solve for } x. \end{array}$$

The system has exactly one solution: $x = 1$ and $y = -2$. 

The term *back-substitution* implies that you work *backwards*. For instance, in Example 5, the second equation gives you the value of y . Then you substitute that value into the first equation to solve for x . Example 6 further demonstrates this procedure.

EXAMPLE 6

Using Back-Substitution in Row-Echelon Form

Solve the system.

$$\begin{array}{rcl} x - 2y + 3z & = & 9 \\ y + 3z & = & 5 \\ z & = & 2 \end{array} \qquad \begin{array}{l} \text{Equation 1} \\ \text{Equation 2} \\ \text{Equation 3} \end{array}$$

SOLUTION

From Equation 3, you know the value of z . To solve for y , substitute $z = 2$ into Equation 2 to obtain

$$\begin{array}{rcl} y + 3(2) & = & 5 \\ y & = & -1. \end{array} \qquad \begin{array}{l} \text{Substitute 2 for } z. \\ \text{Solve for } y. \end{array}$$

Then, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{array}{rcl} x - 2(-1) + 3(2) & = & 9 \\ x & = & 1. \end{array} \qquad \begin{array}{l} \text{Substitute } -1 \text{ for } y \text{ and } 2 \text{ for } z. \\ \text{Solve for } x. \end{array}$$

The solution is $x = 1$, $y = -1$, and $z = 2$. 

Two systems of linear equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first rewrite it as an *equivalent* system that is in row-echelon form using the operations listed on the next page.



Carl Friedrich Gauss (1777–1855)

German mathematician Carl Friedrich Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. Gauss used a form of what is now known as Gaussian elimination in his research. Although this method was named in his honor, the Chinese used an almost identical method some 2000 years prior to Gauss.

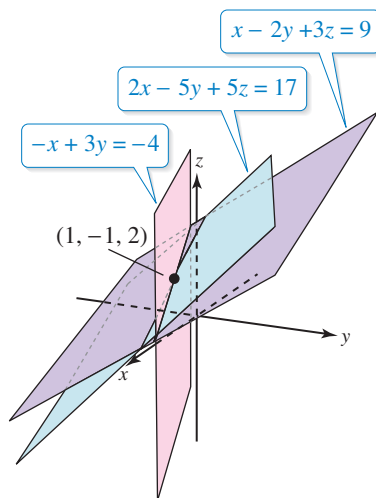


Figure 1.2

Operations That Produce Equivalent Systems

Each of these operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, using one of the three basic operations to obtain each system. This process is called **Gaussian elimination**, after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7

Using Elimination to Rewrite a System in Row-Echelon Form

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

Although there are several ways to begin, you want to use a systematic procedure that can be applied to larger systems. Work from the upper left corner of the system, saving the x at the upper left and eliminating the other x -terms from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

← Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

← Adding -2 times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the first x from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

← Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

← Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad z = 2.$$

Each of the three equations in Example 7 represents a plane in a three-dimensional coordinate system. The unique solution of the system is the point $(x, y, z) = (1, -1, 2)$, so the three planes intersect at this point, as shown in Figure 1.2.

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Many steps are often required to solve a system of linear equations, so it is very easy to make arithmetic errors. You should develop the habit of *checking your solution by substituting it into each equation in the original system*. For instance, in Example 7, check the solution $x = 1$, $y = -1$, and $z = 2$ as shown below.

$$\begin{array}{ll} \text{Equation 1:} & (1) - 2(-1) + 3(2) = 9 \\ \text{Equation 2:} & -(1) + 3(-1) = -4 \\ \text{Equation 3:} & 2(1) - 5(-1) + 5(2) = 17 \end{array} \quad \begin{array}{l} \text{Substitute the solution} \\ \text{into each equation of the} \\ \text{original system.} \end{array}$$

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage of the Gaussian elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 8

An Inconsistent System

Solve the system.

$$\begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 1 \\ 2x_1 - x_2 - 2x_3 & = & 2 \\ x_1 + 2x_2 - 3x_3 & = & -1 \end{array}$$

SOLUTION

$$\begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 1 \\ 5x_2 - 4x_3 & = & 0 \\ x_1 + 2x_2 - 3x_3 & = & -1 \end{array} \quad \begin{array}{l} \text{Adding } -2 \text{ times the first} \\ \text{equation to the second equation} \\ \text{produces a new second equation.} \end{array}$$

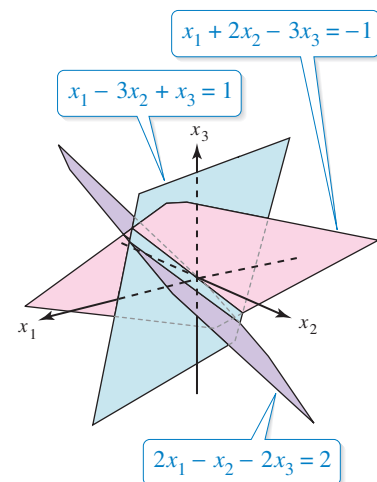
$$\begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 1 \\ 5x_2 - 4x_3 & = & 0 \\ 5x_2 - 4x_3 & = & -2 \end{array} \quad \begin{array}{l} \text{Adding } -1 \text{ times the first} \\ \text{equation to the third equation} \\ \text{produces a new third equation.} \end{array}$$

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.)

$$\begin{array}{rcl} x_1 - 3x_2 + x_3 & = & 1 \\ 5x_2 - 4x_3 & = & 0 \\ 0 & = & -2 \end{array} \quad \begin{array}{l} \text{Subtracting the second equation} \\ \text{from the third equation produces} \\ \text{a new third equation.} \end{array}$$

The statement $0 = -2$ is false, so this system has no solution. Moreover, this system is equivalent to the original system, so the original system also has no solution. 

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown at the right.



This section ends with an example of a system of linear equations that has infinitely many solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

EXAMPLE 9**A System with Infinitely Many Solutions**

Solve the system.

$$\begin{array}{rcl} x_2 - x_3 & = & 0 \\ x_1 & - & 3x_3 = -1 \\ -x_1 + 3x_2 & = & 1 \end{array}$$

SOLUTION

Begin by rewriting the system in row-echelon form, as shown below.

$$\begin{array}{rcl} x_1 & - & 3x_3 = -1 \\ x_2 - x_3 & = & 0 \\ -x_1 + 3x_2 & = & 1 \end{array} \quad \begin{array}{l} \leftarrow \text{Interchange the first} \\ \leftarrow \text{two equations.} \end{array}$$

$$\begin{array}{rcl} x_1 & - & 3x_3 = -1 \\ x_2 - x_3 & = & 0 \\ 3x_2 - 3x_3 & = & 0 \end{array} \quad \begin{array}{l} \leftarrow \text{Adding the first equation to the} \\ \leftarrow \text{third equation produces a new} \\ \leftarrow \text{third equation.} \end{array}$$

$$\begin{array}{rcl} x_1 & - & 3x_3 = -1 \\ x_2 - x_3 & = & 0 \\ 0 & = & 0 \end{array} \quad \begin{array}{l} \leftarrow \text{Adding } -3 \text{ times the second} \\ \leftarrow \text{equation to the third equation} \\ \leftarrow \text{eliminates the third equation.} \end{array}$$

The third equation is unnecessary, so omit it to obtain the system shown below.

$$\begin{array}{rcl} x_1 & - & 3x_3 = -1 \\ x_2 - x_3 & = & 0 \end{array}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t . Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \quad x_2 = t, \quad x_3 = t, \quad t \text{ is any real number.}$$

DISCOVERY

1. Graph the two lines represented by the system of equations.

$$\begin{array}{rcl} x - 2y & = & 1 \\ -2x + 3y & = & -3 \end{array}$$

2. Use Gaussian elimination to solve this system as shown below.

$$\begin{array}{rcl} x - 2y & = & 1 \\ -1y & = & -1 \end{array}$$

$$\begin{array}{rcl} x - 2y & = & 1 \\ y & = & 1 \end{array}$$

$$\begin{array}{rcl} x & = & 3 \\ y & = & 1 \end{array}$$

Graph the system of equations you obtain at each step of this process. What do you observe about the lines?

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

REMARK

You are asked to repeat this graphical analysis for other systems in Exercises 91 and 92.

1.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.

**Linear Equations** In Exercises 1–6, determine whether the equation is linear in the variables x and y .

1. $2x - 3y = 4$
2. $3x - 4xy = 0$
3. $\frac{3}{y} + \frac{2}{x} - 1 = 0$
4. $x^2 + y^2 = 4$
5. $2 \sin x - y = 14$
6. $(\cos 3)x + y = -16$

Parametric Representation In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$
8. $3x - \frac{1}{2}y = 9$
9. $x + y + z = 1$
10. $12x_1 + 24x_2 - 36x_3 = 12$

Graphical Analysis In Exercises 11–24, graph the system of linear equations. Solve the system and interpret your answer.

11. $\begin{cases} 2x + y = 4 \\ x - y = 2 \end{cases}$
12. $\begin{cases} x + 3y = 2 \\ -x + 2y = 3 \end{cases}$
13. $\begin{cases} -x + y = 1 \\ 3x - 3y = 4 \end{cases}$
14. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = 1 \\ -2x + \frac{4}{3}y = -4 \end{cases}$
15. $\begin{cases} 3x - 5y = 7 \\ 2x + y = 9 \end{cases}$
16. $\begin{cases} -x + 3y = 17 \\ 4x + 3y = 7 \end{cases}$
17. $\begin{cases} 2x - y = 5 \\ 5x - y = 11 \end{cases}$
18. $\begin{cases} x - 5y = 21 \\ 6x + 5y = 21 \end{cases}$
19. $\begin{cases} \frac{x+3}{4} + \frac{y-1}{3} = 1 \\ 2x - y = 12 \end{cases}$
20. $\begin{cases} \frac{x-1}{2} + \frac{y+2}{3} = 4 \\ x - 2y = 5 \end{cases}$
21. $\begin{cases} 0.05x - 0.03y = 0.07 \\ 0.07x + 0.02y = 0.16 \end{cases}$
22. $\begin{cases} 0.2x - 0.5y = -27.8 \\ 0.3x - 0.4y = 68.7 \end{cases}$
23. $\begin{cases} \frac{x}{4} + \frac{y}{6} = 1 \\ x - y = 3 \end{cases}$
24. $\begin{cases} \frac{2x}{3} + \frac{y}{6} = \frac{2}{3} \\ 4x + y = 4 \end{cases}$

Back-Substitution In Exercises 25–30, use back-substitution to solve the system.

25. $\begin{cases} x_1 - x_2 = 2 \\ x_2 = 3 \end{cases}$
26. $\begin{cases} 2x_1 - 4x_2 = 6 \\ 3x_2 = 9 \end{cases}$
27. $\begin{cases} -x + y - z = 0 \\ 2y + z = 3 \\ \frac{1}{2}z = 0 \end{cases}$
28. $\begin{cases} x - y = 5 \\ 3y + z = 11 \\ 4z = 8 \end{cases}$
29. $\begin{cases} 5x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + x_2 = 0 \end{cases}$
30. $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = 0 \end{cases}$

Graphical Analysis In Exercises 31–36, complete parts (a)–(e) for the system of equations.

- (a) Use a graphing utility to graph the system.
- (b) Use the graph to determine whether the system is consistent or inconsistent.
- (c) If the system is consistent, approximate the solution.
- (d) Solve the system algebraically.
- (e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

31. $\begin{cases} -3x - y = 3 \\ 6x + 2y = 1 \end{cases}$
32. $\begin{cases} 4x - 5y = 3 \\ -8x + 10y = 14 \end{cases}$
33. $\begin{cases} 2x - 8y = 3 \\ \frac{1}{2}x + y = 0 \end{cases}$
34. $\begin{cases} 9x - 4y = 5 \\ \frac{1}{2}x + \frac{1}{3}y = 0 \end{cases}$
35. $\begin{cases} 4x - 8y = 9 \\ 0.8x - 1.6y = 1.8 \end{cases}$
36. $\begin{cases} -14.7x + 2.1y = 1.05 \\ 44.1x - 6.3y = -3.15 \end{cases}$

System of Linear Equations In Exercises 37–56, solve the system of linear equations.

37. $\begin{cases} x_1 - x_2 = 0 \\ 3x_1 - 2x_2 = -1 \end{cases}$
38. $\begin{cases} 3x + 2y = 2 \\ 6x + 4y = 14 \end{cases}$
39. $\begin{cases} 3u + v = 240 \\ u + 3v = 240 \end{cases}$
40. $\begin{cases} x_1 - 2x_2 = 0 \\ 6x_1 + 2x_2 = 0 \end{cases}$
41. $\begin{cases} 9x - 3y = -1 \\ \frac{1}{5}x + \frac{2}{5}y = -\frac{1}{3} \end{cases}$
42. $\begin{cases} \frac{2}{3}x_1 + \frac{1}{6}x_2 = 0 \\ 4x_1 + x_2 = 0 \end{cases}$
43. $\begin{cases} \frac{x-2}{4} + \frac{y-1}{3} = 2 \\ x - 3y = 20 \end{cases}$
44. $\begin{cases} \frac{x_1+4}{3} + \frac{x_2+1}{2} = 1 \\ 3x_1 - x_2 = -2 \end{cases}$
45. $\begin{cases} 0.02x_1 - 0.05x_2 = -0.19 \\ 0.03x_1 + 0.04x_2 = 0.52 \end{cases}$
46. $\begin{cases} 0.05x_1 - 0.03x_2 = 0.21 \\ 0.07x_1 + 0.02x_2 = 0.17 \end{cases}$
47. $\begin{cases} x - y - z = 0 \\ x + 2y - z = 6 \\ 2x - z = 5 \end{cases}$
48. $\begin{cases} x + y + z = 2 \\ -x + 3y + 2z = 8 \\ 4x + y = 4 \end{cases}$
49. $\begin{cases} 3x_1 - 2x_2 + 4x_3 = 1 \\ x_1 + x_2 - 2x_3 = 3 \\ 2x_1 - 3x_2 + 6x_3 = 8 \end{cases}$

The symbol  indicates an exercise in which you are instructed to use a graphing utility or software program.

$$\begin{aligned} 50. \quad & 5x_1 - 3x_2 + 2x_3 = 3 \\ & 2x_1 + 4x_2 - x_3 = 7 \\ & x_1 - 11x_2 + 4x_3 = 3 \end{aligned}$$

$$\begin{aligned} 51. \quad & 2x_1 + x_2 - 3x_3 = 4 \\ & 4x_1 + 2x_3 = 10 \\ & -2x_1 + 3x_2 - 13x_3 = -8 \end{aligned}$$

$$\begin{aligned} 52. \quad & x_1 + 4x_3 = 13 \\ & 4x_1 - 2x_2 + x_3 = 7 \\ & 2x_1 - 2x_2 - 7x_3 = -19 \end{aligned}$$

$$\begin{aligned} 53. \quad & x - 3y + 2z = 18 \\ & 5x - 15y + 10z = 18 \end{aligned}$$

$$\begin{aligned} 54. \quad & x_1 - 2x_2 + 5x_3 = 2 \\ & 3x_1 + 2x_2 - x_3 = -2 \end{aligned}$$

$$\begin{aligned} 55. \quad & x + y + z + w = 6 \\ & 2x + 3y - w = 0 \\ & -3x + 4y + z + 2w = 4 \\ & x + 2y - z + w = 0 \end{aligned}$$

$$\begin{aligned} 56. \quad & -x_1 + 2x_4 = 1 \\ & 4x_2 - x_3 - x_4 = 2 \\ & x_2 - x_4 = 0 \\ & 3x_1 - 2x_2 + 3x_3 = 4 \end{aligned}$$



System of Linear Equations In Exercises 57–62, use a software program or a graphing utility to solve the system of linear equations.

$$\begin{aligned} 57. \quad & 123.5x + 61.3y - 32.4z = -262.74 \\ & 54.7x - 45.6y + 98.2z = 197.4 \\ & 42.4x - 89.3y + 12.9z = 33.66 \end{aligned}$$

$$\begin{aligned} 58. \quad & 120.2x + 62.4y - 36.5z = 258.64 \\ & 56.8x - 42.8y + 27.3z = -71.44 \\ & 88.1x + 72.5y - 28.5z = 225.88 \end{aligned}$$

$$\begin{aligned} 59. \quad & x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 = 1.1 \\ & 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 = 1.2 \\ & 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 = 1.3 \\ & 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 = 1.4 \end{aligned}$$

$$\begin{aligned} 60. \quad & 0.1x - 2.5y + 1.2z - 0.75w = 108 \\ & 2.4x + 1.5y - 1.8z + 0.25w = -81 \\ & 0.4x - 3.2y + 1.6z - 1.4w = 148.8 \\ & 1.6x + 1.2y - 3.2z + 0.6w = -143.2 \end{aligned}$$

$$\begin{aligned} 61. \quad & \frac{1}{2}x_1 - \frac{3}{7}x_2 + \frac{2}{9}x_3 = \frac{349}{630} \\ & \frac{2}{3}x_1 + \frac{4}{9}x_2 - \frac{2}{5}x_3 = -\frac{19}{45} \\ & \frac{4}{5}x_1 - \frac{1}{8}x_2 + \frac{4}{3}x_3 = \frac{139}{150} \end{aligned}$$

$$\begin{aligned} 62. \quad & \frac{1}{8}x - \frac{1}{7}y + \frac{1}{6}z - \frac{1}{5}w = 1 \\ & \frac{1}{7}x + \frac{1}{6}y - \frac{1}{5}z + \frac{1}{4}w = 1 \\ & \frac{1}{6}x - \frac{1}{5}y + \frac{1}{4}z - \frac{1}{3}w = 1 \\ & \frac{1}{5}x + \frac{1}{4}y - \frac{1}{3}z + \frac{1}{2}w = 1 \end{aligned}$$

Number of Solutions In Exercises 63–66, state why the system of equations must have at least one solution. Then solve the system and determine whether it has exactly one solution or infinitely many solutions.

$$\begin{aligned} 63. \quad & 4x + 3y + 17z = 0 \\ & 5x + 4y + 22z = 0 \\ & 4x + 2y + 19z = 0 \end{aligned} \quad \begin{aligned} 64. \quad & 2x + 3y = 0 \\ & 4x + 3y - z = 0 \\ & 8x + 3y + 3z = 0 \end{aligned}$$

$$\begin{aligned} 65. \quad & 5x + 5y - z = 0 \\ & 10x + 5y + 2z = 0 \\ & 5x + 15y - 9z = 0 \end{aligned} \quad \begin{aligned} 66. \quad & 16x + 3y + z = 0 \\ & 16x + 2y - z = 0 \end{aligned}$$

67. Nutrition One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 227 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 578 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

68. Airplane Speed Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Two hours after the first plane departs, the planes are 3200 kilometers apart. Find the airspeed of each plane.

True or False? In Exercises 69 and 70, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 69.** (a) A system of one linear equation in two variables is always consistent.
 (b) A system of two linear equations in three variables is always consistent.
 (c) If a linear system is consistent, then it has infinitely many solutions.
- 70.** (a) A linear system can have exactly two solutions.
 (b) Two systems of linear equations are equivalent when they have the same solution set.
 (c) A system of three linear equations in two variables is always inconsistent.

71. Find a system of two equations in two variables, x_1 and x_2 , that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where t is any real number. Then show that the solutions to the system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \quad \text{and} \quad x_2 = t.$$

The symbol  indicates that electronic data sets for these exercises are available at LarsonLinearAlgebra.com. The data sets are compatible with MATLAB, Mathematica, Maple, TI-83 Plus, TI-84 Plus, TI-89, and Voyage 200.

72. Find a system of two equations in three variables, x_1 , x_2 , and x_3 , that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t$$

where s and t are any real numbers. Then show that the solutions to the system can also be written as

$$x_1 = 3 + s - t, \quad x_2 = s, \quad \text{and} \quad x_3 = t.$$

Substitution In Exercises 73–76, solve the system of equations by first letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.

73. $\frac{12}{x} - \frac{12}{y} = 7$

74. $\frac{3}{x} + \frac{2}{y} = -1$

$$\frac{3}{x} + \frac{4}{y} = 0$$

$$\frac{2}{x} - \frac{3}{y} = -\frac{17}{6}$$

75. $\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 4$

76. $\frac{2}{x} + \frac{1}{y} - \frac{2}{z} = 5$

$$\frac{4}{x} + \frac{2}{z} = 10$$

$$\frac{3}{x} - \frac{4}{y} = -1$$

$$-\frac{2}{x} + \frac{3}{y} - \frac{13}{z} = -8$$

$$\frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 0$$

Trigonometric Coefficients In Exercises 77 and 78, solve the system of linear equations for x and y .

77. $(\cos \theta)x + (\sin \theta)y = 1$
 $(-\sin \theta)x + (\cos \theta)y = 0$

78. $(\cos \theta)x + (\sin \theta)y = 1$
 $(-\sin \theta)x + (\cos \theta)y = 1$

Coefficient Design In Exercises 79–84, determine the value(s) of k such that the system of linear equations has the indicated number of solutions.

79. No solution

$$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

80. Exactly one solution

$$\begin{aligned} x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

81. Exactly one solution

$$\begin{aligned} kx + 2ky + 3kz &= 4k \\ x + y + z &= 0 \\ 2x - y + z &= 1 \end{aligned}$$

82. No solution

$$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 8z &= 4 \end{aligned}$$

83. Infinitely many solutions

$$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

84. Infinitely many solutions

$$\begin{aligned} kx + y &= 16 \\ 3x - 4y &= -64 \end{aligned}$$

85. Determine the values of k such that the system of linear equations does not have a unique solution.

$$\begin{aligned} x + y + kz &= 3 \\ x + ky + z &= 2 \\ kx + y + z &= 1 \end{aligned}$$

86. **CAPSTONE** Find values of a , b , and c such that the system of linear equations has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution. Explain.

$$\begin{aligned} x + 5y + z &= 0 \\ x + 6y - z &= 0 \\ 2x + ay + bz &= c \end{aligned}$$

87. **Writing** Consider the system of linear equations in x and y .

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \\ a_3x + b_3y &= c_3 \end{aligned}$$

Describe the graphs of these three equations in the xy -plane when the system has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution.

88. **Writing** Explain why the system of linear equations in Exercise 87 must be consistent when the constant terms c_1 , c_2 , and c_3 are all zero.

89. Show that if $ax^2 + bx + c = 0$ for all x , then $a = b = c = 0$.

90. Consider the system of linear equations in x and y .

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

Under what conditions will the system have exactly one solution?

Discovery In Exercises 91 and 92, sketch the lines represented by the system of equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about the lines?

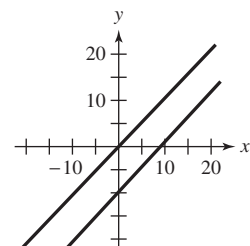
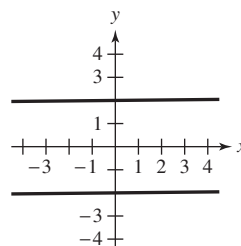
91. $x - 4y = -3$
 $5x - 6y = 13$

92. $2x - 3y = 7$
 $-4x + 6y = -14$

Writing In Exercises 93 and 94, the graphs of the two equations appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.

93. $100y - x = 200$
 $99y - x = -198$

94. $21x - 20y = 0$
 $13x - 12y = 120$



1.2 Gaussian Elimination and Gauss-Jordan Elimination

- Determine the size of a matrix and write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

MATRICES

Section 1.1 introduced Gaussian elimination as a procedure for solving a system of linear equations. In this section, you will study this procedure more thoroughly, beginning with some definitions. The first is the definition of a **matrix**.

REMARK

The plural of matrix is *matrices*. When each entry of a matrix is a *real* number, the matrix is a **real matrix**. Unless stated otherwise, assume all matrices in this text are real matrices.

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ (read “ m by n ”) matrix is a rectangular array

$$\begin{array}{rcccl}
 & \text{Column 1} & \text{Column 2} & \text{Column 3} & \dots & \text{Column } n \\
 \text{Row 1} & a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
 \text{Row 2} & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\
 \text{Row 3} & a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\
 \vdots & \vdots & \vdots & \vdots & & \vdots \\
 \text{Row } m & a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn}
 \end{array}$$

in which each **entry**, a_{ij} , of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

The entry a_{ij} is located in the i th row and the j th column. The index i is called the **row subscript** because it identifies the row in which the entry lies, and the index j is called the **column subscript** because it identifies the column in which the entry lies.

A matrix with m rows and n columns is of **size** $m \times n$. When $m = n$, the matrix is **square** of **order** n and the entries $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are the **main diagonal** entries.

EXAMPLE 1

Sizes of Matrices

Each matrix has the indicated size.

a. $[2]$ Size: 1×1 b. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ Size: 2×2 c. $\begin{bmatrix} e & 2 & -7 \\ \pi & \sqrt{2} & 4 \end{bmatrix}$ Size: 2×3

REMARK

Begin by aligning the variables in the equations vertically. Use 0 to show coefficients of zero in the matrix. Note the fourth column of constant terms in the augmented matrix.

One common use of matrices is to represent systems of linear equations. The matrix derived from the coefficients and constant terms of a system of linear equations is the **augmented matrix** of the system. The matrix containing only the coefficients of the system is the **coefficient matrix** of the system. Here is an example.

System	Augmented Matrix	Coefficient Matrix
$ \begin{array}{rcl} x - 4y + 3z & = & 5 \\ -x + 3y - z & = & -3 \\ 2x & - & 4z = 6 \end{array} $	$ \begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix} $	$ \begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} $

ELEMENTARY ROW OPERATIONS

In the previous section, you studied three operations that produce equivalent systems of linear equations.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations**. An elementary row operation on an augmented matrix produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a finite sequence of elementary row operations.

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Although elementary row operations are relatively simple to perform, they can involve a lot of arithmetic, so it is easy to make a mistake. Noting the elementary row operations performed in each step can make checking your work easier.

Solving some systems involves many steps, so it is helpful to use a shorthand method of notation to keep track of each elementary row operation you perform. The next example introduces this notation.

TECHNOLOGY

Many graphing utilities and software programs can perform elementary row operations on matrices. If you use a graphing utility, you may see something similar to the screen below for Example 2(c). The **Technology Guide** at *CengageBrain.com* can help you use technology to perform elementary row operations.

A

```

[[1 2 -4 3 ]
 [0 3 -2 -1 ]
 [2 1 5 -2 ]]

mRAdd(-2,A,1,3)

[[1 2 -4 3 ]
 [0 3 -2 -1 ]
 [0 -3 13 -8 ]]
  
```

EXAMPLE 2

Elementary Row Operations

- a. Interchange the first and second rows.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$

- b. Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$(\frac{1}{2})R_1 \rightarrow R_1$

- c. Add -2 times the first row to the third row to produce a new third row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$

Notice that adding -2 times row 1 to row 3 does not change row 1.



In Example 7 in Section 1.1, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 3**Using Elementary Row Operations to Solve a System****Linear System**

$$\begin{aligned}x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17\end{aligned}$$

Add the first equation to the second equation.

$$\begin{aligned}x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17\end{aligned}$$

Add -2 times the first equation to the third equation.

$$\begin{aligned}x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1\end{aligned}$$

Add the second equation to the third equation.

$$\begin{aligned}x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4\end{aligned}$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{aligned}x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2\end{aligned}$$

Associated Augmented Matrix

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17\end{array}\right]$$

Add the first row to the second row to produce a new second row.

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17\end{array}\right] \quad R_2 + R_1 \rightarrow R_2$$

Add -2 times the first row to the third row to produce a new third row.

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1\end{array}\right] \quad R_3 + (-2)R_1 \rightarrow R_3$$

Add the second row to the third row to produce a new third row.

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4\end{array}\right] \quad R_3 + R_2 \rightarrow R_3$$

Multiply the third row by $\frac{1}{2}$ to produce a new third row.

$$\left[\begin{array}{ccc|c}1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2\end{array}\right] \quad \left(\frac{1}{2}\right)R_3 \rightarrow R_3$$

Use back-substitution to find the solution, as in Example 6 in Section 1.1. The solution is $x = 1$, $y = -1$, and $z = 2$.

REMARK

The term *echelon* refers to the stair-step pattern formed by the nonzero elements of the matrix.

The last matrix in Example 3 is in **row-echelon** form. To be in this form, a matrix must have the properties listed below.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the properties below.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

EXAMPLE 4**Row-Echelon Form**

Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is also in reduced row-echelon form.

a.
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

e.
$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

f.
$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

SOLUTION

The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because the row of all zeros does not occur at the bottom of the matrix. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not 1. 

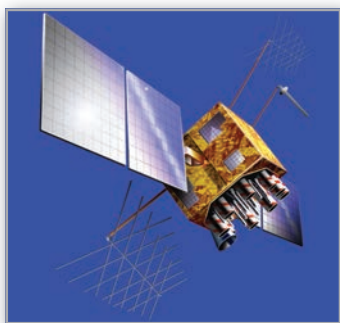
Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 4(e), multiplying the second row in the matrix by $\frac{1}{2}$ changes the matrix to row-echelon form.

The procedure for using Gaussian elimination with back-substitution is summarized below.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form, and use back-substitution to find the solution.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which you perform the elementary row operations is important. Operate from *left to right by columns*, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

**LINEAR ALGEBRA APPLIED**

The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Today, GPS technology is used in a wide variety of civilian applications, such as package delivery, farming, mining, surveying, construction, banking, weather forecasting, and disaster relief. A GPS receiver works by using satellite readings to calculate its location. In three dimensions, the receiver uses signals from at least four satellites to “trilaterate” its position. In a simplified mathematical model, a system of three linear equations in four unknowns (three dimensions and time) is used to determine the coordinates of the receiver as functions of time.

edobric/Shutterstock.com

TECHNOLOGY

Use a graphing utility or a software program to find the row-echelon forms of the matrices in Examples 4(b) and 4(e) and the reduced row-echelon forms of the matrices in Examples 4(a), 4(b), 4(c), and 4(e). The

Technology Guide at *CengageBrain.com* can help you use technology to find the row-echelon and reduced row-echelon forms of a matrix. Similar exercises and projects are also available on the website.

EXAMPLE 5

Gaussian Elimination with Back-Substitution

Solve the system.

$$\begin{array}{rclcl} x_2 + x_3 - 2x_4 & = & -3 \\ x_1 + 2x_2 - x_3 & = & 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 & = & -2 \\ x_1 - 4x_2 - 7x_3 - x_4 & = & -19 \end{array}$$

SOLUTION

The augmented matrix for this system is

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}.$$

Obtain a leading 1 in the upper left corner and zeros elsewhere in the first column.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← Interchange the first two rows. $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

← Adding -2 times the first row to the third row produces a new third row. $R_3 + (-2)R_1 \rightarrow R_3$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & -6 & -6 & -1 & -21 \end{bmatrix}$$

← Adding -1 times the first row to the fourth row produces a new fourth row. $R_4 + (-1)R_1 \rightarrow R_4$

Now that the first column is in the desired form, change the second column as shown below.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & -13 & -39 \end{bmatrix} \quad \leftarrow \quad \begin{array}{l} \text{Adding 6 times the} \\ \text{second row to the fourth} \\ \text{row produces a new} \\ \text{fourth row.} \end{array} \quad R_4 + (6)R_2 \rightarrow R_4$$

To write the third and fourth columns in proper form, multiply the third row by $\frac{1}{3}$ and the fourth row by $-\frac{1}{13}$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 2 \\ 0 & 1 & 1 & -2 & -3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} \text{Multiplying the third} \\ \text{row by } \frac{1}{3} \text{ and the fourth} \\ \text{row by } -\frac{1}{13} \text{ produces new} \\ \text{third and fourth rows.} \end{array} \quad \begin{array}{l} (\frac{1}{3})R_3 \rightarrow R_3 \\ (-\frac{1}{13})R_4 \rightarrow R_4 \end{array}$$

The matrix is now in row-echelon form, and the corresponding system is shown below.

$$\begin{array}{rcl} x_1 + 2x_2 - x_3 & = & 2 \\ x_2 + x_3 - 2x_4 & = & -3 \\ x_3 - x_4 & = & -2 \\ x_4 & = & 3 \end{array}$$

Use back-substitution to find that the solution is $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, and $x_4 = 3$.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then it is unnecessary to continue the process. Simply conclude that the system has no solution, or is *inconsistent*.

EXAMPLE 6**A System with No Solution**

Solve the system.

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 4 \\x_1 + x_3 &= 6 \\2x_1 - 3x_2 + 5x_3 &= 4 \\3x_1 + 2x_2 - x_3 &= 1\end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\left[\begin{array}{cccc}1 & -1 & 2 & 4 \\1 & 0 & 1 & 6 \\2 & -3 & 5 & 4 \\3 & 2 & -1 & 1\end{array}\right]$$

Apply Gaussian elimination to the augmented matrix.

$$\left[\begin{array}{cccc}1 & -1 & 2 & 4 \\0 & 1 & -1 & 2 \\2 & -3 & 5 & 4 \\3 & 2 & -1 & 1\end{array}\right] \quad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{cccc}1 & -1 & 2 & 4 \\0 & 1 & -1 & 2 \\0 & -1 & 1 & -4 \\3 & 2 & -1 & 1\end{array}\right] \quad R_3 + (-2)R_1 \rightarrow R_3$$

$$\left[\begin{array}{cccc}1 & -1 & 2 & 4 \\0 & 1 & -1 & 2 \\0 & -1 & 1 & -4 \\0 & 5 & -7 & -11\end{array}\right] \quad R_4 + (-3)R_1 \rightarrow R_4$$

$$\left[\begin{array}{cccc}1 & -1 & 2 & 4 \\0 & 1 & -1 & 2 \\0 & 0 & 0 & -2 \\0 & 5 & -7 & -11\end{array}\right] \quad R_3 + R_2 \rightarrow R_3$$

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is *inconsistent*. To see why this is true, convert back to a system of linear equations.

$$\begin{aligned}x_1 - x_2 + 2x_3 &= 4 \\x_2 - x_3 &= 2 \\0 &= -2 \\5x_2 - 7x_3 &= -11\end{aligned}$$

The third equation is not possible, so the system has no solution.



GAUSS-JORDAN ELIMINATION

With Gaussian elimination, you apply elementary row operations to a matrix to obtain a (row-equivalent) row-echelon form. A second method of elimination, called **Gauss-Jordan elimination** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until a *reduced* row-echelon form is obtained. Example 7 demonstrates this procedure.

EXAMPLE 7

Gauss-Jordan Elimination

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Use Gauss-Jordan elimination to solve the system.

$$\begin{aligned}x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17\end{aligned}$$

SOLUTION

In Example 3, you used Gaussian elimination to obtain the row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Now, apply elementary row operations until you obtain zeros above each of the leading 1's, as shown below.

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + (2)R_2 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_2 + (-3)R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + (-9)R_3 \rightarrow R_1$$

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{aligned}x &= 1 \\ y &= -1 \\ z &= 2.\end{aligned}$$



The elimination procedures described in this section can sometimes result in fractional coefficients. For example, in the elimination procedure for the system

$$\begin{aligned}2x - 5y + 5z &= 14 \\ 3x - 2y + 3z &= 9 \\ -3x + 4y &= -18\end{aligned}$$

REMARK

No matter which elementary row operations or order you use, the reduced row-echelon form of a matrix is the same.

you may be inclined to first multiply Row 1 by $\frac{1}{2}$ to produce a leading 1, which will result in working with fractional coefficients. Sometimes, judiciously choosing which elementary row operations you apply, and the order in which you apply them, enables you to avoid fractions.

DISCOVERY

1. Without performing any row operations, explain why the system of linear equations below is consistent.

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 0 \\ -5x_1 + 6x_2 - 17x_3 &= 0 \\ 7x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}$$

2. The system below has more variables than equations. Why does it have an infinite number of solutions?

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 + 2x_4 &= 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 &= 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 &= 0 \end{aligned}$$

The next example demonstrates how Gauss-Jordan elimination can be used to solve a system with infinitely many solutions.

EXAMPLE 8

A System with Infinitely Many Solutions

Solve the system of linear equations.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

SOLUTION

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right].$$

Using a graphing utility, a software program, or Gauss-Jordan elimination, verify that the reduced row-echelon form of the matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array} \right].$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1. \end{aligned}$$

Now, using the parameter t to represent x_3 , you have

$$x_1 = 2 - 5t, \quad x_2 = -1 + 3t, \quad x_3 = t, \quad t \text{ is any real number.}$$

Note in Example 8 that the arbitrary parameter t represents the *nonleading* variable x_3 . The variables x_1 and x_2 are written as functions of t .

You have looked at two elimination methods for solving a system of linear equations. Which is better? To some degree the answer depends on personal preference. In real-life applications of linear algebra, systems of linear equations are usually solved by computer. Most software uses a form of Gaussian elimination, with special emphasis on ways to reduce rounding errors and minimize storage of data. The examples and exercises in this text focus on the underlying concepts, so you should know both elimination methods.

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations in which each of the constant terms is zero are called **homogeneous**. A homogeneous system of m equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

REMARK

A homogeneous system of three equations in the three variables x_1 , x_2 , and x_3 has the trivial solution $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$.

A homogeneous system must have at least one solution. Specifically, if all variables in a homogeneous system have the value zero, then each of the equations is satisfied. Such a solution is **trivial** (or obvious).

EXAMPLE 9

Solving a Homogeneous System of Linear Equations

Solve the system of linear equations.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

SOLUTION

Applying Gauss-Jordan elimination to the augmented matrix

$$\left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{array} \right]$$

yields the matrices shown below.

$$\begin{aligned} \left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array} \right] & \quad R_2 + (-2)R_1 \rightarrow R_2 \\ \left[\begin{array}{cccc} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] & \quad \left(\frac{1}{3}\right)R_2 \rightarrow R_2 \\ \left[\begin{array}{cccc} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] & \quad R_1 + R_2 \rightarrow R_1 \end{aligned}$$

The system of equations corresponding to this matrix is

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0. \end{aligned}$$

Using the parameter $t = x_3$, the solution set is $x_1 = -2t$, $x_2 = t$, and $x_3 = t$, where t is any real number. This system has infinitely many solutions, one of which is the trivial solution ($t = 0$). 

As illustrated in Example 9, a homogeneous system with fewer equations than variables has infinitely many solutions.

THEOREM 1.1 The Number of Solutions of a Homogeneous System

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

To prove Theorem 1.1, use the procedure in Example 9, but for a general matrix.

1.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Matrix Size In Exercises 1–6, determine the size of the matrix.

1. $\begin{bmatrix} 1 & 2 & -4 \\ 3 & -4 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & -1 & 1 \\ -6 & 2 & 0 & 1 \end{bmatrix}$

4. $[-1]$

5. $\begin{bmatrix} 8 & 6 & 4 & 1 & 3 \\ 2 & 1 & -7 & 4 & 1 \\ 1 & 1 & -1 & 2 & 1 \\ 1 & -1 & 2 & 0 & 0 \end{bmatrix}$

6. $[1 \ 2 \ 3 \ 4 \ -10]$

Elementary Row Operations In Exercises 7–10, identify the elementary row operation(s) being performed to obtain the new row-equivalent matrix.

Original Matrix

New Row-Equivalent Matrix

7. $\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$

$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$

Original Matrix

New Row-Equivalent Matrix

8. $\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$

Original Matrix

New Row-Equivalent Matrix

9. $\begin{bmatrix} 0 & -1 & -7 & 7 \\ -1 & 5 & -8 & 7 \\ 3 & -2 & 1 & 2 \end{bmatrix}$

$\begin{bmatrix} -1 & 5 & -8 & 7 \\ 0 & -1 & -7 & 7 \\ 0 & 13 & -23 & 23 \end{bmatrix}$

Original Matrix

New Row-Equivalent Matrix

10. $\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$

$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Augmented Matrix In Exercises 11–18, find the solution set of the system of linear equations represented by the augmented matrix.

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 3 & -1 & 1 & 5 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 & 4 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Row-Echelon Form In Exercises 19–24, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

19. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$

21. $\begin{bmatrix} -2 & 0 & 1 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

System of Linear Equations In Exercises 25–38, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

25. $x + 3y = 11$

26. $2x + 6y = 16$

$3x + y = 9$

$-2x - 6y = -16$

27. $-x + 2y = 1.5$

$2x - 4y = 3$

28. $2x - y = -0.1$

$3x + 2y = 1.6$

29. $-3x + 5y = -22$

$3x + 4y = 4$

$4x - 8y = 32$

30. $x + 2y = 0$

$x + y = 6$

$3x - 2y = 8$

31. $x_1 - 3x_3 = -2$
 $3x_1 + x_2 - 2x_3 = 5$
 $2x_1 + 2x_2 + x_3 = 4$
32. $3x_1 - 2x_2 + 3x_3 = 22$
 $3x_2 - x_3 = 24$
 $6x_1 - 7x_2 = -22$
33. $2x_1 + 3x_3 = 3$
 $4x_1 - 3x_2 + 7x_3 = 5$
 $8x_1 - 9x_2 + 15x_3 = 10$
34. $x_1 + x_2 - 5x_3 = 3$
 $x_1 - 2x_3 = 1$
 $2x_1 - x_2 - x_3 = 0$
35. $4x + 12y - 7z - 20w = 22$
 $3x + 9y - 5z - 28w = 30$
36. $x + 2y + z = 8$
 $-3x - 6y - 3z = -21$
37. $3x + 3y + 12z = 6$
 $x + y + 4z = 2$
 $2x + 5y + 20z = 10$
 $-x + 2y + 8z = 4$
38. $2x + y - z + 2w = -6$
 $3x + 4y + w = 1$
 $x + 5y + 2z + 6w = -3$
 $5x + 2y - z - w = 3$



System of Linear Equations In Exercises 39–42, use a software program or a graphing utility to solve the system of linear equations.

39. $x_1 - 2x_2 + 5x_3 - 3x_4 = 23.6$
 $x_1 + 4x_2 - 7x_3 - 2x_4 = 45.7$
 $3x_1 - 5x_2 + 7x_3 + 4x_4 = 29.9$
40. $x_1 + x_2 - 2x_3 + 3x_4 + 2x_5 = 9$
 $3x_1 + 3x_2 - x_3 + x_4 + x_5 = 5$
 $2x_1 + 2x_2 - x_3 + x_4 - 2x_5 = 1$
 $4x_1 + 4x_2 + x_3 - 3x_5 = 4$
 $8x_1 + 5x_2 - 2x_3 - x_4 + 2x_5 = 3$
41. $x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 6$
 $3x_1 - 2x_2 + 4x_3 + 4x_4 + 12x_5 = 14$
 $x_2 - x_3 - x_4 - 3x_5 = -3$
 $2x_1 - 2x_2 + 4x_3 + 5x_4 + 15x_5 = 10$
 $2x_1 - 2x_2 + 4x_3 + 4x_4 + 13x_5 = 13$
42. $x_1 + 2x_2 - 2x_3 + 2x_4 - x_5 + 3x_6 = 0$
 $2x_1 - x_2 + 3x_3 + x_4 - 3x_5 + 2x_6 = 17$
 $x_1 + 3x_2 - 2x_3 + x_4 - 2x_5 - 3x_6 = -5$
 $3x_1 - 2x_2 + x_3 - x_4 + 3x_5 - 2x_6 = -1$
 $-x_1 - 2x_2 + x_3 + 2x_4 - 2x_5 + 3x_6 = 10$
 $x_1 - 3x_2 + x_3 + 3x_4 - 2x_5 + x_6 = 11$

Homogeneous System In Exercises 43–46, solve the homogeneous linear system corresponding to the given coefficient matrix.

43. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
44. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$
45. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
46. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

47. **Finance** A small software corporation borrowed \$500,000 to expand its software line. The corporation borrowed some of the money at 3%, some at 4%, and some at 5%. Use a system of equations to determine how much was borrowed at each rate when the annual interest was \$20,500 and the amount borrowed at 4% was $2\frac{1}{2}$ times the amount borrowed at 3%. Solve the system using matrices.

48. **Tips** A food server examines the amount of money earned in tips after working an 8-hour shift. The server has a total of \$95 in denominations of \$1, \$5, \$10, and \$20 bills. The total number of paper bills is 26. The number of \$5 bills is 4 times the number of \$10 bills, and the number of \$1 bills is 1 less than twice the number of \$5 bills. Write a system of linear equations to represent the situation. Then use matrices to find the number of each denomination.

Matrix Representation In Exercises 49 and 50, assume that the matrix is the *augmented* matrix of a system of linear equations, and (a) determine the number of equations and the number of variables, and (b) find the value(s) of k such that the system is consistent. Then assume that the matrix is the *coefficient* matrix of a *homogeneous* system of linear equations, and repeat parts (a) and (b).

49. $A = \begin{bmatrix} 1 & k & 2 \\ -3 & 4 & 1 \end{bmatrix}$
50. $A = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix}$

Coefficient Design In Exercises 51 and 52, find values of a , b , and c (if possible) such that the system of linear equations has (a) a unique solution, (b) no solution, and (c) infinitely many solutions.

51. $x + y = 2$
 $y + z = 2$
 $x + z = 2$
 $ax + by + cz = 0$
52. $x + y = 0$
 $y + z = 0$
 $x + z = 0$
 $ax + by + cz = 0$

53. The system below has one solution: $x = 1$, $y = -1$, and $z = 2$.

$$4x - 2y + 5z = 16 \quad \text{Equation 1}$$

$$x + y = 0 \quad \text{Equation 2}$$

$$-x - 3y + 2z = 6 \quad \text{Equation 3}$$

Solve the systems provided by (a) Equations 1 and 2, (b) Equations 1 and 3, and (c) Equations 2 and 3. (d) How many solutions does each of these systems have?

54. Assume the system below has a unique solution.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \text{Equation 1}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \text{Equation 2}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \text{Equation 3}$$

Does the system composed of Equations 1 and 2 have a unique solution, no solution, or infinitely many solutions?

Row Equivalence In Exercises 55 and 56, find the reduced row-echelon matrix that is row-equivalent to the given matrix.

55. $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ 56. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

57. **Writing** Describe all possible 2×2 reduced row-echelon matrices. Support your answer with examples.
58. **Writing** Describe all possible 3×3 reduced row-echelon matrices. Support your answer with examples.

True or False? In Exercises 59 and 60, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

59. (a) A 6×3 matrix has six rows.
 (b) Every matrix is row-equivalent to a matrix in row-echelon form.
 (c) If the row-echelon form of the augmented matrix of a system of linear equations contains the row $[1 \ 0 \ 0 \ 0 \ 0]$, then the original system is inconsistent.
 (d) A homogeneous system of four linear equations in six variables has infinitely many solutions.
60. (a) A 4×7 matrix has four columns.
 (b) Every matrix has a unique reduced row-echelon form.
 (c) A homogeneous system of four linear equations in four variables is always consistent.
 (d) Multiplying a row of a matrix by a constant is one of the elementary row operations.

61. **Writing** Is it possible for a system of linear equations with fewer equations than variables to have no solution? If so, give an example.

62. **Writing** Does a matrix have a unique row-echelon form? Illustrate your answer with examples.

Row Equivalence In Exercises 63 and 64, determine conditions on a , b , c , and d such that the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will be row-equivalent to the given matrix.

63. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

64. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Homogeneous System In Exercises 65 and 66, find all values of λ (the Greek letter lambda) for which the homogeneous linear system has nontrivial solutions.

65. $(\lambda - 2)x + y = 0$
 $x + (\lambda - 2)y = 0$

66. $(2\lambda + 9)x - 5y = 0$
 $x - \lambda y = 0$

67. The augmented matrix represents a system of linear equations that has been reduced using Gauss-Jordan elimination. Write a system of equations with nonzero coefficients that the reduced matrix could represent.

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are many correct answers.

68. **CAPSTONE** In your own words, describe the difference between a matrix in row-echelon form and a matrix in reduced row-echelon form. Include an example of each to support your explanation.

69. **Writing** Consider the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Perform the sequence of row operations.

- (a) Add (-1) times the second row to the first row.
 (b) Add 1 times the first row to the second row.
 (c) Add (-1) times the second row to the first row.
 (d) Multiply the first row by (-1) .

What happened to the original matrix? Describe, in general, how to interchange two rows of a matrix using only the second and third elementary row operations.

70. **Writing** Describe the row-echelon form of an augmented matrix that corresponds to a linear system that (a) is inconsistent, and (b) has infinitely many solutions.

1.3 Applications of Systems of Linear Equations

- Set up and solve a system of equations to fit a polynomial function to a set of data points.
- Set up and solve a system of equations to represent a network.

Systems of linear equations arise in a wide variety of applications. In this section you will look at two applications, and you will see more in subsequent chapters. The first application shows how to fit a polynomial function to a set of data points in the plane. The second application focuses on networks and Kirchhoff's Laws for electricity.

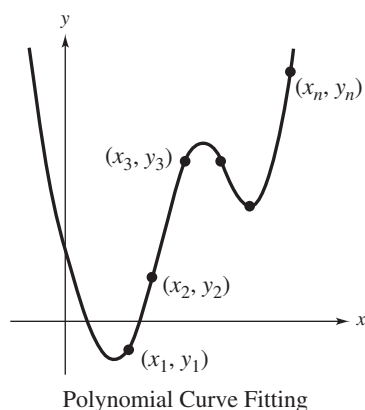


Figure 1.3

POLYNOMIAL CURVE FITTING

Consider n points in the xy -plane

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

that represent a collection of data, and you want to find a polynomial function of degree $n - 1$

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}$$

whose graph passes through the points. This procedure is called **polynomial curve fitting**. When all x -coordinates of the points are distinct, there is precisely one polynomial function of degree $n - 1$ (or less) that fits the n points, as shown in Figure 1.3.

To solve for the n coefficients of $p(x)$, substitute each of the n points into the polynomial function and obtain n linear equations in n variables $a_0, a_1, a_2, \dots, a_{n-1}$.

$$a_0 + a_1x_1 + a_2x_1^2 + \dots + a_{n-1}x_1^{n-1} = y_1$$

$$a_0 + a_1x_2 + a_2x_2^2 + \dots + a_{n-1}x_2^{n-1} = y_2$$

$$\vdots$$

$$a_0 + a_1x_n + a_2x_n^2 + \dots + a_{n-1}x_n^{n-1} = y_n$$

Example 1 demonstrates this procedure with a second-degree polynomial.

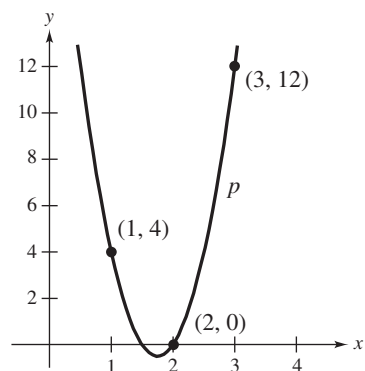


Figure 1.4

EXAMPLE 1

Polynomial Curve Fitting

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points $(1, 4)$, $(2, 0)$, and $(3, 12)$.

SOLUTION

Substituting $x = 1, 2$, and 3 into $p(x)$ and equating the results to the respective y -values produces the system of linear equations in the variables a_0, a_1 , and a_2 shown below.

$$p(1) = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4$$

$$p(2) = a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0$$

$$p(3) = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12$$

The solution of this system is

$$a_0 = 24, a_1 = -28, \text{ and } a_2 = 8$$

so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$

Figure 1.4 shows the graph of p .

EXAMPLE 2**Polynomial Curve Fitting**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find a polynomial that fits the points

$$(-2, 3), (-1, 5), (0, 1), (1, 4), \text{ and } (2, 10).$$

SOLUTION

You are given five points, so choose a fourth-degree polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4.$$

Substituting the points into $p(x)$ produces the system of linear equations shown below.

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\ a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\ a_0 &= 1 \\ a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\ a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10 \end{aligned}$$

The solution of these equations is

$$a_0 = 1, \quad a_1 = -\frac{5}{4}, \quad a_2 = \frac{101}{24}, \quad a_3 = \frac{3}{4}, \quad a_4 = -\frac{17}{24}$$

which means the polynomial function is

$$p(x) = 1 - \frac{5}{4}x + \frac{101}{24}x^2 + \frac{3}{4}x^3 - \frac{17}{24}x^4.$$

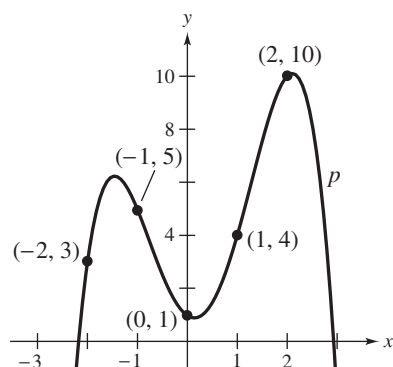


Figure 1.5

Figure 1.5 shows the graph of p .

The system of linear equations in Example 2 is relatively easy to solve because the x -values are small. For a set of points with large x -values, it is usually best to *translate* the values before attempting the curve-fitting procedure. The next example demonstrates this approach.

EXAMPLE 3**Translating Large x -Values Before Curve Fitting**

Find a polynomial that fits the points

$$\begin{aligned} &\underbrace{(x_1, y_1)}_{(2011, 3)}, \quad \underbrace{(x_2, y_2)}_{(2012, 5)}, \quad \underbrace{(x_3, y_3)}_{(2013, 1)}, \quad \underbrace{(x_4, y_4)}_{(2014, 4)}, \quad \underbrace{(x_5, y_5)}_{(2015, 10)}. \end{aligned}$$

SOLUTION

The given x -values are large, so use the translation

$$z = x - 2013$$

to obtain

$$\begin{aligned} &\underbrace{(z_1, y_1)}_{(-2, 3)}, \quad \underbrace{(z_2, y_2)}_{(-1, 5)}, \quad \underbrace{(z_3, y_3)}_{(0, 1)}, \quad \underbrace{(z_4, y_4)}_{(1, 4)}, \quad \underbrace{(z_5, y_5)}_{(2, 10)}. \end{aligned}$$

This is the same set of points as in Example 2. So, the polynomial that fits these points is

$$p(z) = 1 - \frac{5}{4}z + \frac{101}{24}z^2 + \frac{3}{4}z^3 - \frac{17}{24}z^4.$$

Letting $z = x - 2013$, you have

$$p(x) = 1 - \frac{5}{4}(x - 2013) + \frac{101}{24}(x - 2013)^2 + \frac{3}{4}(x - 2013)^3 - \frac{17}{24}(x - 2013)^4.$$

EXAMPLE 4**An Application of Curve Fitting**

Find a polynomial that relates the periods of the three planets that are closest to the Sun to their mean distances from the Sun, as shown in the table. Then use the polynomial to calculate the period of Mars, and compare it to the value shown in the table. (The mean distances are in astronomical units, and the periods are in years.)

<i>Planet</i>	<i>Mercury</i>	<i>Venus</i>	<i>Earth</i>	<i>Mars</i>
<i>Mean Distance</i>	0.387	0.723	1.000	1.524
<i>Period</i>	0.241	0.615	1.000	1.881

SOLUTION

Begin by fitting a quadratic polynomial function

$$p(x) = a_0 + a_1x + a_2x^2$$

to the points

$$(0.387, 0.241), (0.723, 0.615), \text{ and } (1, 1).$$

The system of linear equations obtained by substituting these points into $p(x)$ is

$$a_0 + 0.387a_1 + (0.387)^2a_2 = 0.241$$

$$a_0 + 0.723a_1 + (0.723)^2a_2 = 0.615$$

$$a_0 + a_1 + a_2 = 1.$$

The approximate solution of the system is

$$a_0 \approx -0.0634, \quad a_1 \approx 0.6119, \quad a_2 \approx 0.4515$$

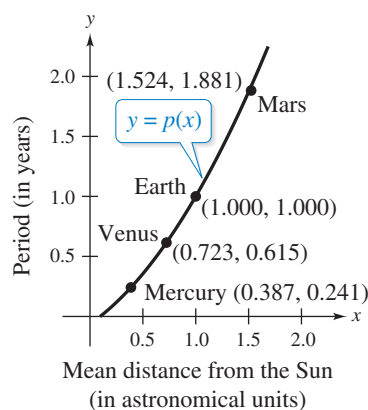
which means that an approximation of the polynomial function is

$$p(x) = -0.0634 + 0.6119x + 0.4515x^2.$$

Using $p(x)$ to evaluate the period of Mars produces

$$p(1.524) \approx 1.918 \text{ years.}$$

Note that the period of Mars is shown in the table as 1.881 years. The figure below provides a graphical comparison of the polynomial function to the values shown in the table.



As illustrated in Example 4, a polynomial that fits some of the points in a data set is not necessarily an accurate model for other points in the data set. Generally, the farther the other points are from those used to fit the polynomial, the worse the fit. For instance, the mean distance of Jupiter from the Sun is 5.203 astronomical units. Using $p(x)$ in Example 4 to approximate the period gives 15.343 years—a poor estimate of Jupiter’s actual period of 11.862 years.

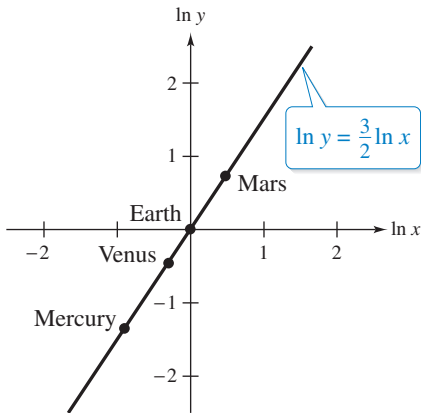
The problem of curve fitting can be difficult. Types of functions other than polynomial functions may provide better fits. For instance, look again at the curve-fitting problem in Example 4. Taking the natural logarithms of the distances and periods produces the results shown in the table.

Planet	Mercury	Venus	Earth	Mars
Mean Distance, x	0.387	0.723	1.000	1.524
$\ln x$	−0.949	−0.324	0.0	0.421
Period, y	0.241	0.615	1.000	1.881
$\ln y$	−1.423	−0.486	0.0	0.632

Now, fitting a polynomial to the logarithms of the distances and periods produces the *linear relationship*

$$\ln y = \frac{3}{2} \ln x$$

which is shown graphically below.



From $\ln y = \frac{3}{2} \ln x$, it follows that $y = x^{3/2}$, or $y^2 = x^3$. In other words, the square of the period (in years) of each planet is equal to the cube of its mean distance (in astronomical units) from the Sun. Johannes Kepler first discovered this relationship in 1619.



LINEAR ALGEBRA APPLIED

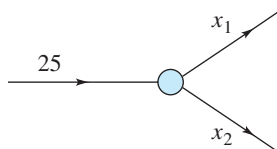
Researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection used a system of linear equations to model the traffic flow at the intersection. To help formulate the system of equations, “operators” stationed themselves at various locations along the intersection and counted the numbers of vehicles that passed them. *(Source: Acoustical Noise Analysis in Road Intersections: A Case Study, Guarnaccia, Claudio, Recent Advances in Acoustics & Music, Proceedings of the 11th WSEAS International Conference on Acoustics & Music: Theory & Applications)*

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NETWORK ANALYSIS

Networks composed of branches and junctions are used as models in such fields as economics, traffic analysis, and electrical engineering. In a network model, you assume that the total flow into a junction is equal to the total flow out of the junction. For example, the junction shown below has 25 units flowing into it, so there must be 25 units flowing out of it. You can represent this with the linear equation

$$x_1 + x_2 = 25.$$



Each junction in a network gives rise to a linear equation, so you can analyze the flow through a network composed of several junctions by solving a system of linear equations. Example 5 illustrates this procedure.

EXAMPLE 5

Analysis of a Network

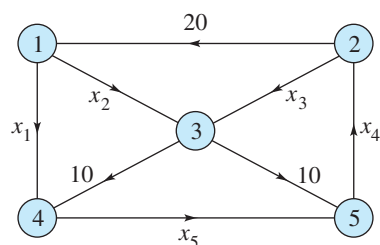


Figure 1.6

Set up a system of linear equations to represent the network shown in Figure 1.6. Then solve the system.

SOLUTION

Each of the network's five junctions gives rise to a linear equation, as shown below.

$$\begin{array}{rcll} x_1 + x_2 & = & 20 & \text{Junction 1} \\ x_3 - x_4 & = & -20 & \text{Junction 2} \\ x_2 + x_3 & = & 20 & \text{Junction 3} \\ x_1 & - & x_5 = -10 & \text{Junction 4} \\ & - & x_4 + x_5 = -10 & \text{Junction 5} \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array} \right].$$

Gauss-Jordan elimination produces the matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

From the matrix above,

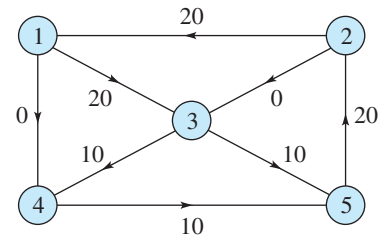
$$x_1 - x_5 = -10, \quad x_2 + x_5 = 30, \quad x_3 - x_5 = -10, \quad \text{and} \quad x_4 - x_5 = 10.$$

Letting $t = x_5$, you have

$$x_1 = t - 10, \quad x_2 = -t + 30, \quad x_3 = t - 10, \quad x_4 = t + 10, \quad x_5 = t$$

where t is any real number, so this system has infinitely many solutions.

In Example 5, if you could control the amount of flow along the branch labeled x_5 , then you could also control the flow represented by each of the other variables. For example, letting $t = 10$ results in the flows shown in the figure at the right. (Verify this.)



REMARK

A closed path is a sequence of branches such that the beginning point of the first branch coincides with the ending point of the last branch.

You may be able to see how the type of network analysis demonstrated in Example 5 could be used in problems dealing with the flow of traffic through the streets of a city or the flow of water through an irrigation system.

An electrical network is another type of network where analysis is commonly applied. An analysis of such a system uses two properties of electrical networks known as **Kirchhoff's Laws**.

1. All the current flowing into a junction must flow out of it.
2. The sum of the products IR (I is current and R is resistance) around a closed path is equal to the total voltage in the path.

In an electrical network, current is measured in amperes, or amps (A), resistance is measured in ohms (Ω , the Greek letter omega), and the product of current and resistance is measured in volts (V). The symbol $\text{---}||\text{---}$ represents a battery. The larger vertical bar denotes where the current flows out of the terminal. The symbol $\sim\sim\sim$ denotes resistance. An arrow in the branch shows the direction of the current.

EXAMPLE 6

Analysis of an Electrical Network

Determine the currents I_1 , I_2 , and I_3 for the electrical network shown in Figure 1.7.

SOLUTION

Applying Kirchhoff's first law to either junction produces

$$I_1 + I_3 = I_2 \quad \text{Junction 1 or Junction 2}$$

and applying Kirchhoff's second law to the two paths produces

$$R_1 I_1 + R_2 I_2 = 3I_1 + 2I_2 = 7 \quad \text{Path 1}$$

$$R_2 I_2 + R_3 I_3 = 2I_2 + 4I_3 = 8. \quad \text{Path 2}$$

So, you have the system of three linear equations in the variables I_1 , I_2 , and I_3 shown below.

$$I_1 - I_2 + I_3 = 0$$

$$3I_1 + 2I_2 = 7$$

$$2I_2 + 4I_3 = 8$$

Applying Gauss-Jordan elimination to the augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 0 & 7 \\ 0 & 2 & 4 & 8 \end{bmatrix}$$

produces the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $I_1 = 1$ amp, $I_2 = 2$ amps, and $I_3 = 1$ amp.

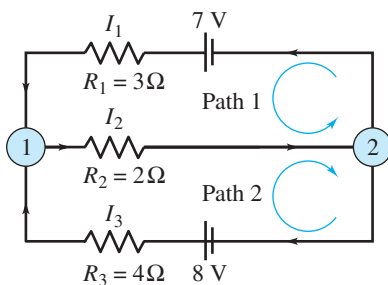
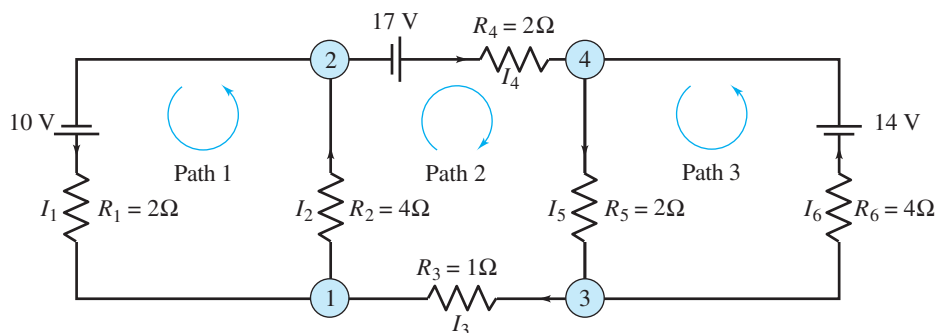


Figure 1.7

EXAMPLE 7**Analysis of an Electrical Network**

Determine the currents I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 for the electrical network shown below.

**SOLUTION**

Applying Kirchhoff's first law to the four junctions produces

$$\begin{aligned} I_1 + I_3 &= I_2 && \text{Junction 1} \\ I_1 + I_4 &= I_2 && \text{Junction 2} \\ I_3 + I_6 &= I_5 && \text{Junction 3} \\ I_4 + I_6 &= I_5 && \text{Junction 4} \end{aligned}$$

and applying Kirchhoff's second law to the three paths produces

$$\begin{aligned} 2I_1 + 4I_2 &= 10 && \text{Path 1} \\ 4I_2 + I_3 + 2I_4 + 2I_5 &= 17 && \text{Path 2} \\ 2I_5 + 4I_6 &= 14. && \text{Path 3} \end{aligned}$$

You now have the system of seven linear equations in the variables I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 shown below.

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ I_1 - I_2 &+ I_4 &= 0 \\ &I_3 - I_5 + I_6 &= 0 \\ &I_4 - I_5 + I_6 &= 0 \\ 2I_1 + 4I_2 &= 10 \\ 4I_2 + I_3 + 2I_4 + 2I_5 &= 17 \\ &2I_5 + 4I_6 &= 14 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{ccccccc} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 & 10 \\ 0 & 4 & 1 & 2 & 2 & 0 & 17 \\ 0 & 0 & 0 & 0 & 2 & 4 & 14 \end{array} \right].$$

Using a graphing utility, a software program, or Gauss-Jordan elimination, solve this system to obtain

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 1, \quad I_4 = 1, \quad I_5 = 3, \quad \text{and} \quad I_6 = 2.$$

So, $I_1 = 1$ amp, $I_2 = 2$ amps, $I_3 = 1$ amp, $I_4 = 1$ amp, $I_5 = 3$ amps, and $I_6 = 2$ amps.



1.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Polynomial Curve Fitting In Exercises 1–12, (a) determine the polynomial function whose graph passes through the points, and (b) sketch the graph of the polynomial function, showing the points.

- (2, 5), (3, 2), (4, 5)
- (0, 0), (2, -2), (4, 0)
- (2, 4), (3, 6), (5, 10)
- (2, 4), (3, 4), (4, 4)
- (-1, 3), (0, 0), (1, 1), (4, 58)
- (0, 42), (1, 0), (2, -40), (3, -72)
- (-2, 28), (-1, 0), (0, -6), (1, -8), (2, 0)
- (-4, 18), (0, 1), (4, 0), (6, 28), (8, 135)
- (2013, 5), (2014, 7), (2015, 12)
- (2012, 150), (2013, 180), (2014, 240), (2015, 360)
- (0.072, 0.203), (0.120, 0.238), (0.148, 0.284)
- (1, 1), (1.189, 1.587), (1.316, 2.080), (1.414, 2.520)

13. Use $\sin 0 = 0$, $\sin \frac{\pi}{2} = 1$, and $\sin \pi = 0$ to estimate $\sin \frac{\pi}{3}$.

14. Use $\log_2 1 = 0$, $\log_2 2 = 1$, and $\log_2 4 = 2$ to estimate $\log_2 3$.

Equation of a Circle In Exercises 15 and 16, find an equation of the circle that passes through the points.

- (1, 3), (-2, 6), (4, 2)
- (-5, 1), (-3, 2), (-1, 1)

17. **Population** The U.S. census lists the population of the United States as 249 million in 1990, 282 million in 2000, and 309 million in 2010. Fit a second-degree polynomial passing through these three points and use it to predict the populations in 2020 and 2030. (Source: U.S. Census Bureau)



18. **Population** The table shows the U.S. populations for the years 1970, 1980, 1990, and 2000. (Source: U.S. Census Bureau)

Year	1970	1980	1990	2000
Population (in millions)	205	227	249	282

- Find a cubic polynomial that fits the data and use it to estimate the population in 2010.
- The actual population in 2010 was 309 million. How does your estimate compare?



19. **Net Profit** The table shows the net profits (in millions of dollars) for Microsoft from 2007 through 2014. (Source: Microsoft Corp.)

Year	2007	2008	2009	2010
Net Profit	14,065	17,681	14,569	18,760

Year	2011	2012	2013	2014
Net Profit	23,150	23,171	22,453	22,074

- Set up a system of equations to fit the data for the years 2007, 2008, 2009, and 2010 to a cubic model.
- Solve the system. Does the solution produce a reasonable model for determining net profits after 2010? Explain.



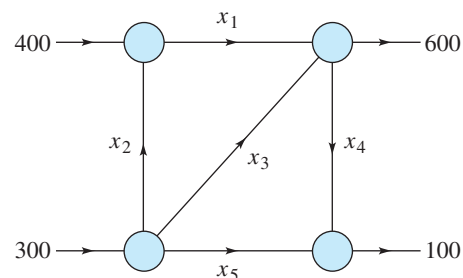
20. **Sales** The table shows the sales (in billions of dollars) for Wal-Mart stores from 2006 through 2013. (Source: Wal-Mart Stores, Inc.)

Year	2006	2007	2008	2009
Sales	348.7	378.8	405.6	408.2

Year	2010	2011	2012	2013
Sales	421.8	447.0	469.2	476.2

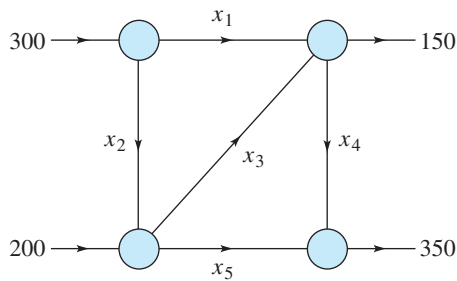
- Set up a system of equations to fit the data for the years 2006, 2007, 2008, 2009, and 2010 to a quartic model.
- Solve the system. Does the solution produce a reasonable model for determining sales after 2010? Explain.

21. **Network Analysis** The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

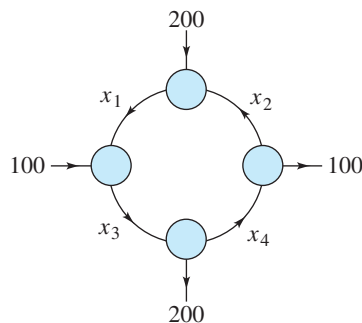


- Solve this system for x_i , $i = 1, 2, \dots, 5$.
- Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.
- Find the traffic flow when $x_3 = x_5 = 100$.

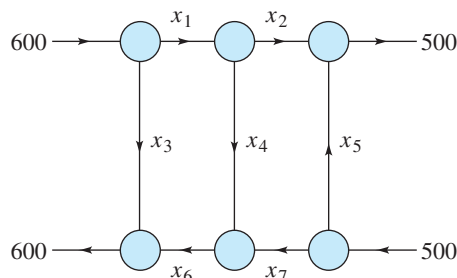
- 22. Network Analysis** The figure shows the flow of traffic (in vehicles per hour) through a network of streets.



- (a) Solve this system for x_i , $i = 1, 2, \dots, 5$.
 (b) Find the traffic flow when $x_2 = 200$ and $x_3 = 50$.
 (c) Find the traffic flow when $x_2 = 150$ and $x_3 = 0$.
- 23. Network Analysis** The figure shows the flow of traffic (in vehicles per hour) through a network of streets.

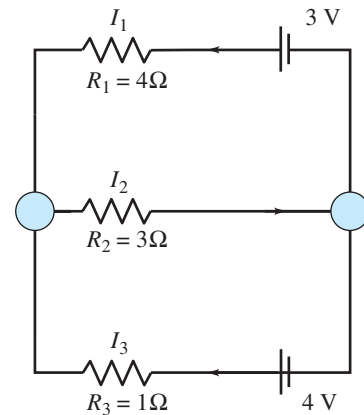


- (a) Solve this system for x_i , $i = 1, 2, 3, 4$.
 (b) Find the traffic flow when $x_4 = 0$.
 (c) Find the traffic flow when $x_4 = 100$.
 (d) Find the traffic flow when $x_1 = 2x_2$.
- 24. Network Analysis** Water is flowing through a network of pipes (in thousands of cubic meters per hour), as shown in the figure.

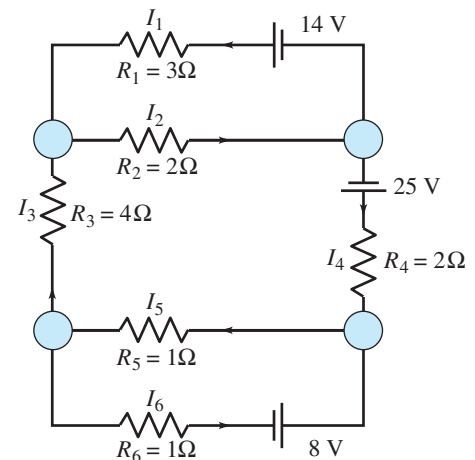


- (a) Solve this system for the water flow represented by x_i , $i = 1, 2, \dots, 7$.
 (b) Find the water flow when $x_1 = x_2 = 100$.
 (c) Find the water flow when $x_6 = x_7 = 0$.
 (d) Find the water flow when $x_5 = 1000$ and $x_6 = 0$.

- 25. Network Analysis** Determine the currents I_1 , I_2 , and I_3 for the electrical network shown in the figure.

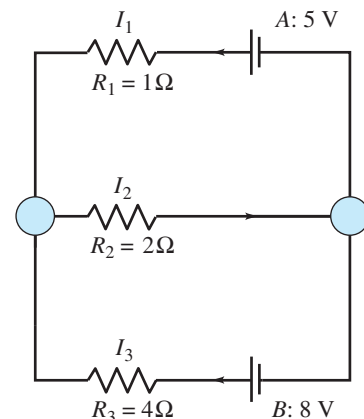


- 26. Network Analysis** Determine the currents I_1 , I_2 , I_3 , I_4 , I_5 , and I_6 for the electrical network shown in the figure.



- 27. Network Analysis**

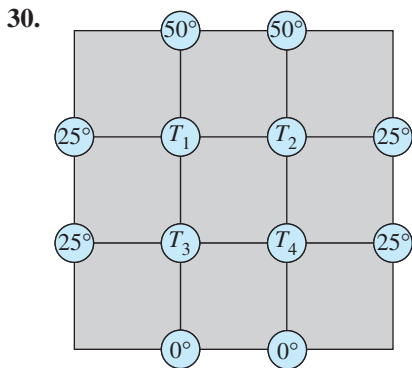
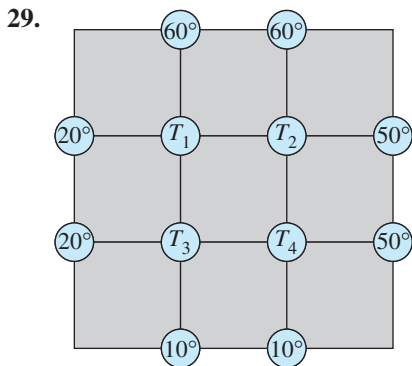
- (a) Determine the currents I_1 , I_2 , and I_3 for the electrical network shown in the figure.
 (b) How is the result affected when A is changed to 2 volts and B is changed to 6 volts?



28. CAPSTONE

- Explain how to use systems of linear equations for polynomial curve fitting.
- Explain how to use systems of linear equations to perform network analysis.

Temperature In Exercises 29 and 30, the figure shows the boundary temperatures (in degrees Celsius) of an insulated thin metal plate. The steady-state temperature at an interior junction is approximately equal to the mean of the temperatures at the four surrounding junctions. Use a system of linear equations to approximate the interior temperatures T_1, T_2, T_3 , and T_4 .



Partial Fraction Decomposition In Exercises 31–34, use a system of equations to find the partial fraction decomposition of the rational expression. Solve the system using matrices.

$$31. \frac{4x^2}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$32. \frac{3x^2 - 7x - 12}{(x+4)(x-4)^2} = \frac{A}{x+4} + \frac{B}{x-4} + \frac{C}{(x-4)^2}$$

$$33. \frac{3x^2 - 3x - 2}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$34. \frac{20 - x^2}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Calculus In Exercises 35 and 36, find the values of x , y , and λ that satisfy the system of equations. Such systems arise in certain problems of calculus, and λ is called the **Lagrange multiplier**.

$$35. \begin{aligned} 2x + \lambda &= 0 \\ 2y + \lambda &= 0 \\ x + y - 4 &= 0 \end{aligned}$$

$$36. \begin{aligned} 2y + 2\lambda + 2 &= 0 \\ 2x + \lambda + 1 &= 0 \\ 2x + y - 100 &= 0 \end{aligned}$$

37. **Calculus** The graph of a parabola passes through the points $(0, 1)$ and $(\frac{1}{2}, \frac{1}{2})$ and has a horizontal tangent line at $(\frac{1}{2}, \frac{1}{2})$. Find an equation for the parabola and sketch its graph.

38. **Calculus** The graph of a cubic polynomial function has horizontal tangent lines at $(1, -2)$ and $(-1, 2)$. Find an equation for the function and sketch its graph.

39. **Guided Proof** Prove that if a polynomial function $p(x) = a_0 + a_1x + a_2x^2$ is zero for $x = -1, x = 0$, and $x = 1$, then $a_0 = a_1 = a_2 = 0$.

Getting Started: Write a system of linear equations and solve the system for a_0, a_1 , and a_2 .

(i) Substitute $x = -1, 0$, and 1 into $p(x)$.

(ii) Set each result equal to 0.

(iii) Solve the resulting system of linear equations in the variables a_0, a_1 , and a_2 .

40. **Proof** Generalizing the statement in Exercise 39, if a polynomial function

$$p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$$

is zero for more than $n - 1$ x -values, then

$$a_0 = a_1 = \cdots = a_{n-1} = 0.$$

Use this result to prove that there is at most one polynomial function of degree $n - 1$ (or less) whose graph passes through n points in the plane with distinct x -coordinates.

41. (a) The graph of a function f passes through the points $(0, 1)$, $(2, \frac{1}{3})$, and $(4, \frac{1}{5})$. Find a quadratic function whose graph passes through these points.

(b) Find a polynomial function p of degree 2 or less that passes through the points $(0, 1)$, $(2, 3)$, and $(4, 5)$. Then sketch the graph of $y = 1/p(x)$ and compare this graph with the graph of the polynomial function found in part (a).

42. **Writing** Try to find a polynomial to fit the data shown in the table. What happens, and why?

x	1	2	3	3	4
y	1	1	2	3	4

1 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Linear Equations In Exercises 1–6, determine whether the equation is linear in the variables x and y .

1. $2x - y^2 = 4$
2. $2xy - 6y = 0$
3. $(\cot 5)x - y = 3$
4. $e^{-2}x + 5y = 8$
5. $\frac{2}{x} + 4y = 3$
6. $\frac{x}{2} - \frac{y}{4} = 0$

Parametric Representation In Exercises 7 and 8, find a parametric representation of the solution set of the linear equation.

7. $-3x + 4y - 2z = 1$
8. $3x_1 + 2x_2 - 4x_3 = 0$

System of Linear Equations In Exercises 9–20, solve the system of linear equations.

9. $x + y = 2$
 $3x - y = 0$
10. $x + y = -1$
 $3x + 2y = 0$
11. $3y = 2x$
 $y = x + 4$
12. $x = y + 3$
 $4x = y + 10$
13. $y + x = 0$
 $2x + y = 0$
14. $y = 5x$
 $y = -x$
15. $x - y = 9$
 $-x + y = 1$
16. $40x_1 + 30x_2 = 24$
 $20x_1 + 15x_2 = -14$
17. $\frac{1}{2}x - \frac{1}{3}y = 0$
 $3x + 2(y + 5) = 10$
18. $\frac{1}{3}x + \frac{4}{7}y = 3$
 $2x + 3y = 15$
19. $0.2x_1 + 0.3x_2 = 0.14$
 $0.4x_1 + 0.5x_2 = 0.20$
20. $0.2x - 0.1y = 0.07$
 $0.4x - 0.5y = -0.01$

Matrix Size In Exercises 21 and 22, determine the size of the matrix.

21. $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 1 \end{bmatrix}$
22. $\begin{bmatrix} 2 & 1 \\ -4 & -1 \\ 0 & 5 \end{bmatrix}$

Augmented Matrix In Exercises 23–26, find the solution set of the system of linear equations represented by the augmented matrix.

23. $\begin{bmatrix} 1 & 2 & -5 \\ 2 & 1 & 5 \end{bmatrix}$
24. $\begin{bmatrix} -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
25. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
26. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row-Echelon Form In Exercises 27–30, determine whether the matrix is in row-echelon form. If it is, determine whether it is also in reduced row-echelon form.

27. $\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
28. $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
29. $\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
30. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

System of Linear Equations In Exercises 31–40, solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination.

31. $-x + y + 2z = 1$
 $2x + 3y + z = -2$
 $5x + 4y + 2z = 4$
32. $4x + 2y + z = 18$
 $4x - 2y - 2z = 28$
 $2x - 3y + 2z = -8$
33. $2x + 3y + 3z = 3$
 $6x + 6y + 12z = 13$
 $12x + 9y - z = 2$
34. $2x + y + 2z = 4$
 $2x + 2y = 5$
 $2x - y + 6z = 2$
35. $x - 2y + z = -6$
 $2x - 3y = -7$
 $-x + 3y - 3z = 11$
36. $2x + 6z = -9$
 $3x - 2y + 11z = -16$
 $3x - y + 7z = -11$
37. $x + 2y + 6z = 1$
 $2x + 5y + 15z = 4$
 $3x + y + 3z = -6$
38. $2x_1 + 5x_2 - 19x_3 = 34$
 $3x_1 + 8x_2 - 31x_3 = 54$
39. $2x_1 + x_2 + x_3 + 2x_4 = -1$
 $5x_1 - 2x_2 + x_3 - 3x_4 = 0$
 $-x_1 + 3x_2 + 2x_3 + 2x_4 = 1$
 $3x_1 + 2x_2 + 3x_3 - 5x_4 = 12$
40. $x_1 + 5x_2 + 3x_3 = 14$
 $4x_2 + 2x_3 + 5x_4 = 3$
 $3x_3 + 8x_4 + 6x_5 = 16$
 $2x_1 + 4x_2 - 2x_5 = 0$
 $2x_1 - x_3 = 0$



System of Linear Equations In Exercises 41–46, use a software program or a graphing utility to solve the system of linear equations.

41. $x_1 + x_2 + x_3 = 15.4$

$$x_1 - x_2 - 2x_3 = 27.9$$

$$3x_1 - 2x_2 + x_3 = 76.9$$

42. $1.1x_1 + 2.3x_2 + 3.4x_3 = 0$

$$1.1x_1 - 2.2x_2 - 4.4x_3 = 0$$

$$-1.7x_1 + 3.4x_2 + 6.8x_3 = 1$$

43. $3x + 3y + 12z = 6$

$$x + y + 4z = 2$$

$$2x + 5y + 20z = 10$$

$$-x + 2y + 8z = 4$$

44. $x + 2y + z + 3w = 0$

$$x - y + w = 0$$

$$5y - z + 2w = 0$$

45. $2x + 10y + 2z = 6$

$$x + 5y + 2z = 6$$

$$x + 5y + z = 3$$

$$-3x - 15y + 3z = -9$$

46. $2x + y - z + 2w = -6$

$$3x + 4y + w = 1$$

$$x + 5y + 2z + 6w = -3$$

$$5x + 2y - z - w = 3$$

Homogeneous System In Exercises 47–50, solve the homogeneous system of linear equations.

47. $x_1 - 2x_2 - 8x_3 = 0$

$$3x_1 + 2x_2 = 0$$

48. $2x_1 + 4x_2 - 7x_3 = 0$

$$x_1 - 3x_2 + 9x_3 = 0$$

49. $-2x_1 + 7x_2 - 3x_3 = 0$

$$4x_1 - 12x_2 + 5x_3 = 0$$

$$12x_2 + 7x_3 = 0$$

50. $x_1 + 3x_2 + 5x_3 = 0$

$$x_1 + 4x_2 + \frac{1}{2}x_3 = 0$$

51. Determine the values of k such that the system of linear equations is inconsistent.

$$kx + y = 0$$

$$x + ky = 1$$

52. Determine the values of k such that the system of linear equations has exactly one solution.

$$x - y + 2z = 0$$

$$-x + y - z = 0$$

$$x + ky + z = 0$$

53. Find values of a and b such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.

$$x + 2y = 3$$

$$ax + by = -9$$

54. Find (if possible) values of a , b , and c such that the system of linear equations has (a) no solution, (b) exactly one solution, and (c) infinitely many solutions.

$$2x - y + z = a$$

$$x + y + 2z = b$$

$$3y + 3z = c$$

55. **Writing** Describe a method for showing that two matrices are row-equivalent. Are the two matrices below row-equivalent?

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \\ 5 & 5 & 10 \end{bmatrix}$$

56. **Writing** Describe all possible 2×3 reduced row-echelon matrices. Support your answer with examples.

57. Let $n \geq 3$. Find the reduced row-echelon form of the $n \times n$ matrix.

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & 2n \\ 2n+1 & 2n+2 & 2n+3 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^2-n+1 & n^2-n+2 & n^2-n+3 & \cdots & n^2 \end{bmatrix}$$

58. Find all values of λ for which the homogeneous system of linear equations has nontrivial solutions.

$$(\lambda + 2)x_1 - 2x_2 + 3x_3 = 0$$

$$-2x_1 + (\lambda - 1)x_2 + 6x_3 = 0$$

$$x_1 + 2x_2 + \lambda x_3 = 0$$

True or False? In Exercises 59 and 60, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

59. (a) There is only one way to parametrically represent the solution set of a linear equation.

(b) A consistent system of linear equations can have infinitely many solutions.

60. (a) A homogeneous system of linear equations must have at least one solution.

(b) A system of linear equations with fewer equations than variables always has at least one solution.

61. **Sports** In Super Bowl I, on January 15, 1967, the Green Bay Packers defeated the Kansas City Chiefs by a score of 35 to 10. The total points scored came from a combination of touchdowns, extra-point kicks, and field goals, worth 6, 1, and 3 points, respectively. The numbers of touchdowns and extra-point kicks were equal. There were six times as many touchdowns as field goals. Find the numbers of touchdowns, extra-point kicks, and field goals scored. (Source: National Football League)

- 62. Agriculture** A mixture of 6 gallons of chemical A, 8 gallons of chemical B, and 13 gallons of chemical C is required to kill a destructive crop insect. Commercial spray X contains 1, 2, and 2 parts, respectively, of these chemicals. Commercial spray Y contains only chemical C. Commercial spray Z contains chemicals A, B, and C in equal amounts. How much of each type of commercial spray is needed to get the desired mixture?

Partial Fraction Decomposition In Exercises 63 and 64, use a system of equations to find the partial fraction decomposition of the rational expression. Solve the system using matrices.

63.
$$\frac{8x^2}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

64.
$$\frac{3x^2 + 3x - 2}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2}$$

Polynomial Curve Fitting In Exercises 65 and 66, (a) determine the polynomial function whose graph passes through the points, and (b) sketch the graph of the polynomial function, showing the points.

65. (2, 5), (3, 0), (4, 20)

66. (-1, -1), (0, 0), (1, 1), (2, 4)

- 67. Sales** A company has sales (measured in millions) of \$50, \$60, and \$75 during three consecutive years. Find a quadratic function that fits the data, and use it to predict the sales during the fourth year.

- 68.** The polynomial function

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

is zero when $x = 1, 2, 3$, and 4 . What are the values of a_0, a_1, a_2 , and a_3 ?

- 69. Deer Population** A wildlife management team studied the population of deer in one small tract of a wildlife preserve. The table shows the population and the number of years since the study began.

Year	0	4	80
Population	80	68	30

- (a) Set up a system of equations to fit the data to a quadratic function.

- (b) Solve the system.

-  (c) Use a graphing utility to fit the data to a quadratic model.

- (d) Compare the quadratic function in part (b) with the model in part (c).

- (e) Cite the statement from the text that verifies your results.

- 70. Vertical Motion** An object moving vertically is at the given heights at the specified times. Find the position equation

$$s = \frac{1}{2}at^2 + v_0t + s_0$$

for the object.

- (a) At $t = 0$ seconds, $s = 160$ feet

At $t = 1$ second, $s = 96$ feet

At $t = 2$ seconds, $s = 0$ feet

- (b) At $t = 1$ second, $s = 134$ feet

At $t = 2$ seconds, $s = 86$ feet

At $t = 3$ seconds, $s = 6$ feet

- (c) At $t = 1$ second, $s = 184$ feet

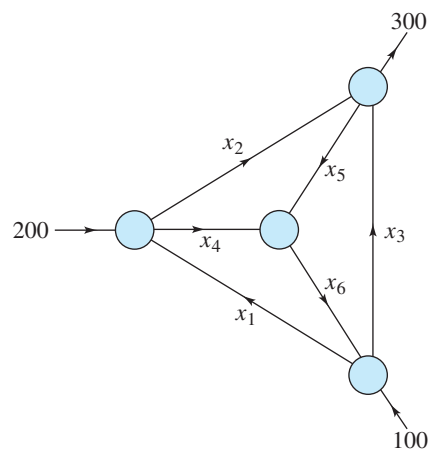
At $t = 2$ seconds, $s = 116$ feet

At $t = 3$ seconds, $s = 16$ feet

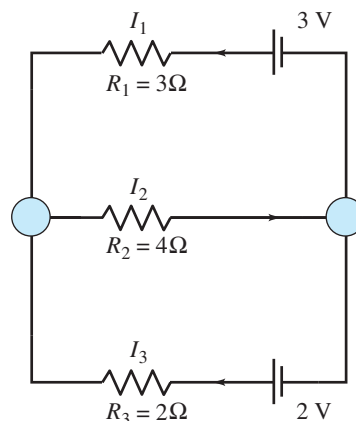
- 71. Network Analysis** The figure shows the flow through a network.

- (a) Solve the system for x_i , $i = 1, 2, \dots, 6$.

- (b) Find the flow when $x_3 = 100$, $x_5 = 50$, and $x_6 = 50$.



- 72. Network Analysis** Determine the currents I_1, I_2 , and I_3 for the electrical network shown in the figure.



1 Projects

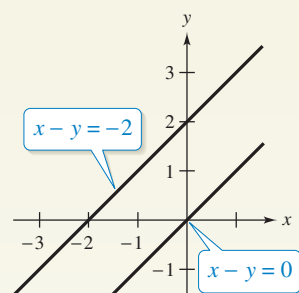
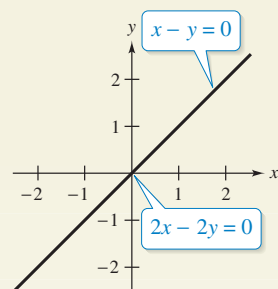
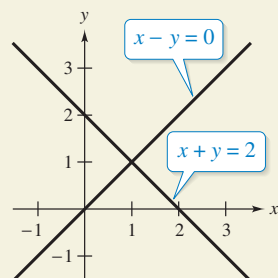
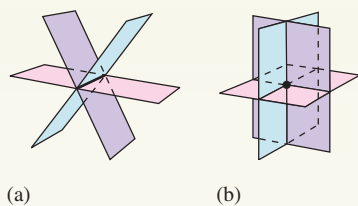
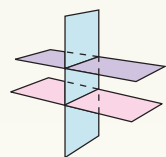


Figure 1.8



(a)

(b)



(c)

Figure 1.9

1 Graphing Linear Equations

You saw in Section 1.1 that you can represent a system of two linear equations in two variables x and y geometrically as two lines in the plane. These lines can intersect at a point, coincide, or be parallel, as shown in Figure 1.8.

1. Consider the system below, where a and b are constants.

$$2x - y = 3$$

$$ax + by = 6$$

- (a) Find values of a and b for which the resulting system has a unique solution.
 - (b) Find values of a and b for which the resulting system has infinitely many solutions.
 - (c) Find values of a and b for which the resulting system has no solution.
 - (d) Graph the lines for each of the systems in parts (a), (b), and (c).
2. Now consider a system of three linear equations in x , y , and z . Each equation represents a plane in the three-dimensional coordinate system.
 - (a) Find an example of a system represented by three planes intersecting in a line, as shown in Figure 1.9(a).
 - (b) Find an example of a system represented by three planes intersecting at a point, as shown in Figure 1.9(b).
 - (c) Find an example of a system represented by three planes with no common intersection, as shown in Figure 1.9(c).
 - (d) Are there other configurations of three planes in addition to those given in Figure 1.9? Explain.

2 Underdetermined and Overdetermined Systems

The system of linear equations below is **underdetermined** because there are more variables than equations.

$$x_1 + 2x_2 - 3x_3 = 4$$

$$2x_1 - x_2 + 4x_3 = -3$$

Similarly, the system below is **overdetermined** because there are more equations than variables.

$$x_1 + 3x_2 = 5$$

$$2x_1 - 2x_2 = -3$$

$$-x_1 + 7x_2 = 0$$

Explore whether the number of variables and the number of equations have any bearing on the consistency of a system of linear equations. For Exercises 1–4, if an answer is yes, give an example. Otherwise, explain why the answer is no.

1. Can you find a consistent underdetermined linear system?
2. Can you find a consistent overdetermined linear system?
3. Can you find an inconsistent underdetermined linear system?
4. Can you find an inconsistent overdetermined linear system?
5. Explain why you would expect an overdetermined linear system to be inconsistent. Must this always be the case?
6. Explain why you would expect an underdetermined linear system to have infinitely many solutions. Must this always be the case?

2 Matrices

- 2.1 Operations with Matrices
- 2.2 Properties of Matrix Operations
- 2.3 The Inverse of a Matrix
- 2.4 Elementary Matrices
- 2.5 Markov Chains
- 2.6 More Applications of Matrix Operations



Data Encryption (p. 94)



Computational Fluid Dynamics (p. 79)



Beam Deflection (p. 64)



Information Retrieval (p. 58)



Flight Crew Scheduling (p. 47)

2.1 Operations with Matrices

-  Determine whether two matrices are equal.
-  Add and subtract matrices and multiply a matrix by a scalar.
-  Multiply two matrices.
-  Use matrices to solve a system of linear equations.
-  Partition a matrix and write a linear combination of column vectors.

EQUALITY OF MATRICES

In Section 1.2, you used matrices to solve systems of linear equations. This chapter introduces some fundamentals of matrix theory and further applications of matrices.

It is standard mathematical convention to represent matrices in any one of the three ways listed below.

1. An uppercase letter such as A , B , or C
2. A representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$
3. A rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

As mentioned in Chapter 1, the matrices in this text are primarily *real matrices*. That is, their entries are real numbers.

Two matrices are *equal* when their corresponding entries are equal.

Definition of Equality of Matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same size ($m \times n$) and $a_{ij} = b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

EXAMPLE 1

Equality of Matrices

Consider the four matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad C = [1 \quad 3], \quad \text{and} \quad D = \begin{bmatrix} 1 & 2 \\ x & 4 \end{bmatrix}.$$

Matrices A and B are **not** equal because they are of different sizes. Similarly, B and C are not equal. Matrices A and D are equal if and only if $x = 3$. 

A matrix that has only one column, such as matrix B in Example 1, is a **column matrix** or **column vector**. Similarly, a matrix that has only one row, such as matrix C in Example 1, is a **row matrix** or **row vector**. Boldface lowercase letters often designate column matrices and row matrices. For instance, matrix A in Example 1 can be partitioned into the two column matrices $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ as shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & | & 2 \\ 3 & | & 4 \end{bmatrix} = [\mathbf{a}_1 \mid \mathbf{a}_2]$$

REMARK

The phrase “if and only if” means the statement is true in both directions. For example, “ p if and only if q ” means that p implies q and q implies p .

MATRIX ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION

To **add** two matrices (of the same size), add their corresponding entries.

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix $A + B = [a_{ij} + b_{ij}]$.

The sum of two matrices of different sizes is undefined.

EXAMPLE 2

Addition of Matrices

$$\text{a. } \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{d. } \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} \text{ is undefined.}$$

REMARK

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For example, factoring the scalar $\frac{1}{2}$ out of the matrix below gives

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}.$$

When working with matrices, real numbers are referred to as **scalars**. To multiply a matrix A by a scalar c , multiply each entry in A by c .

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix $cA = [ca_{ij}]$.

You can use $-A$ to represent the scalar product $(-1)A$. If A and B are of the same size, then $A - B$ represents the sum of A and $(-1)B$. That is, $A - B = A + (-1)B$.

EXAMPLE 3

Scalar Multiplication and Matrix Subtraction

For the matrices A and B , find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

SOLUTION

$$\text{a. } 3A = 3 \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

$$\text{b. } -B = (-1) \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

$$\text{c. } 3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

MATRIX MULTIPLICATION

Another basic matrix operation is **matrix multiplication**. To see the usefulness of this operation, consider the application below, in which matrices are helpful for organizing information.

A football stadium has three concession areas, located in the south, north, and west stands. The top-selling items are peanuts, hot dogs, and soda. Sales for one day are given in the first matrix below, and the prices (in dollars) of the three items are given in the second matrix.

<i>Numbers of Items Sold</i>					
	Peanuts	Hot Dogs	Sodas	Selling Price	
South Stand	120	250	305	2.00	Peanuts
North Stand	207	140	419	3.00	Hot Dogs
West Stand	29	120	190	2.75	Soda

To calculate the total sales of the three top-selling items at the south stand, multiply each entry in the first row of the matrix on the left by the corresponding entry in the price column matrix on the right and add the results. The south stand sales are

$$(120)(2.00) + (250)(3.00) + (305)(2.75) = \$1828.75 \quad \text{South stand sales}$$

Similarly, the sales for the other two stands are shown below.

$$(207)(2.00) + (140)(3.00) + (419)(2.75) = \$1986.25 \quad \text{North stand sales}$$

$$(29)(2.00) + (120)(3.00) + (190)(2.75) = \$940.50 \quad \text{West stand sales}$$

The preceding computations are examples of matrix multiplication. You can write the product of the 3×3 matrix indicating the number of items sold and the 3×1 matrix indicating the selling prices as shown below.

$$\begin{bmatrix} 120 & 250 & 305 \\ 207 & 140 & 419 \\ 29 & 120 & 190 \end{bmatrix} \begin{bmatrix} 2.00 \\ 3.00 \\ 2.75 \end{bmatrix} = \begin{bmatrix} 1828.75 \\ 1986.25 \\ 940.50 \end{bmatrix}$$

The product of these matrices is the 3×1 matrix giving the total sales for each of the three stands.

The definition of the product of two matrices shown below is based on the ideas just developed. Although at first glance this definition may seem unusual, you will see that it has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$\begin{aligned} c_{ij} &= \sum_{k=1}^n a_{ik}b_{kj} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} \end{aligned}$$

This definition means that to find the entry in the i th row and the j th column of the product AB , multiply the entries in the i th row of A by the corresponding entries in the j th column of B and then add the results. The next example illustrates this process.

EXAMPLE 4**Finding the Product of Two Matrices**

Find the product AB , where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

SOLUTION

First, note that the product AB is defined because A has size 3×2 and B has size 2×2 . Moreover, the product AB has size 3×2 , and will take the form

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

To find c_{11} (the entry in the first row and first column of the product), multiply corresponding entries in the first row of A and the first column of B . That is,

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

$c_{11} = (-1)(-3) + (3)(-4) = -9$

Similarly, to find c_{12} , multiply corresponding entries in the first row of A and the second column of B to obtain

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

$c_{12} = (-1)(2) + (3)(1) = 1$

Continuing this pattern produces the results shown below.

$$\begin{aligned} c_{21} &= (4)(-3) + (-2)(-4) = -4 \\ c_{22} &= (4)(2) + (-2)(1) = 6 \\ c_{31} &= (5)(-3) + (0)(-4) = -15 \\ c_{32} &= (5)(2) + (0)(1) = 10 \end{aligned}$$

The product is

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}.$$

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is,

$$\begin{array}{ccc} A & B & = & AB. \\ m \times n & n \times p & & m \times p \end{array}$$

↑ ↑ ↑ ↑
Equal
Size of AB

So, the product BA is not defined for matrices such as A and B in Example 4.



Arthur Cayley
(1821–1895)

British mathematician Arthur Cayley is credited with giving an abstract definition of a matrix. Cayley was a Cambridge University graduate and a lawyer by profession. He began his groundbreaking work on matrices as he studied the theory of transformations. Cayley also was instrumental in the development of determinants (discussed in Chapter 3). Cayley and two American mathematicians, Benjamin Peirce (1809–1880) and his son, Charles S. Peirce (1839–1914), are credited with developing “matrix algebra.”



The general pattern for matrix multiplication is shown below. To obtain the element in the i th row and the j th column of the product AB , use the i th row of A and the j th column of B .

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = c_{ij}$

DISCOVERY

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

1. Find $A + B$ and $B + A$. Is matrix addition commutative?
2. Find AB and BA . Is matrix multiplication commutative?

EXAMPLE 5

Matrix Multiplication

See LarsonLinearAlgebra.com for an interactive version of this type of example.

$$\text{a. } \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & -1 \\ -3 & 6 & 6 \end{bmatrix}$$

$2 \times 3 \qquad 3 \times 3 \qquad 2 \times 3$

$$\text{b. } \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

$$\text{c. } \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

$$\text{d. } \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$1 \times 3 \qquad 3 \times 1 \qquad 1 \times 1$

$$\text{e. } \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$$

$3 \times 1 \qquad 1 \times 3 \qquad 3 \times 3$

Note the difference between the two products in parts (d) and (e) of Example 5. In general, matrix multiplication is not commutative. It is usually not true that the product AB is equal to the product BA . (See Section 2.2 for further discussion of the noncommutativity of matrix multiplication.)

SYSTEMS OF LINEAR EQUATIONS

One practical application of matrix multiplication is representing a system of linear equations. Note how the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written as the matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix of the system, and $\mathbf{x}$ and $\mathbf{b}$ are column matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$A \quad \mathbf{x} = \mathbf{b}$

EXAMPLE 6

Solving a System of Linear Equations

Solve the matrix equation $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

SOLUTION

As a system of linear equations, $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 - 2x_3 &= 0. \end{aligned}$$

Using Gauss-Jordan elimination on the augmented matrix of this system, you obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & 0 \end{array} \right].$$

So, the system has infinitely many solutions. Here a convenient choice of a parameter is $x_3 = 7t$, and you can write the solution set as

$$x_1 = t, \quad x_2 = 4t, \quad x_3 = 7t, \quad t \text{ is any real number.}$$

In matrix terminology, you have found that the matrix equation

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has infinitely many solutions represented by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 4t \\ 7t \end{bmatrix} = t \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad t \text{ is any scalar.}$$

That is, any scalar multiple of the column matrix on the right is a solution. Here are some sample solutions:

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -1 \\ -4 \\ -7 \end{bmatrix}.$$

TECHNOLOGY

Many graphing utilities and software programs can perform matrix addition, scalar multiplication, and matrix multiplication. When you use a graphing utility to check one of the solutions in Example 6, you may see something similar to the screen below.

```

[A]      [[1 -2 1]]
[B]      [[1]]
          [[4]]
[A]*[B]   [[0]]
          [[0]]
  
```

The **Technology Guide** at CengageBrain.com can help you use technology to perform matrix operations.

PARTITIONED MATRICES

The system $A\mathbf{x} = \mathbf{b}$ can be represented in a more convenient way by partitioning the matrices A and $\mathbf{x}$ in the manner shown below. If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

are the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$, then

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{b}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \mathbf{b}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{b}.$$

In other words,

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are the columns of the matrix A . The expression

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

is called a **linear combination** of the column matrices $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ with **coefficients** $x_1, x_2, \dots, x_n$.

Linear Combinations of Column Vectors

The matrix product $A\mathbf{x}$ is a linear combination of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ that form the coefficient matrix A .

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Furthermore, the system

$$A\mathbf{x} = \mathbf{b}$$

is consistent if and only if $\mathbf{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

EXAMPLE 7**Solving a System of Linear Equations**

The linear system

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 0 \\4x_1 + 5x_2 + 6x_3 &= 3 \\7x_1 + 8x_2 + 9x_3 &= 6\end{aligned}$$

can be rewritten as a matrix equation $A\mathbf{x} = \mathbf{b}$, as shown below.

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

Using Gaussian elimination, you can show that this system has infinitely many solutions, one of which is $x_1 = 1$, $x_2 = 1$, $x_3 = -1$.

$$1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

That is, $\mathbf{b}$ can be expressed as a linear combination of the columns of A . This representation of one column vector in terms of others is a fundamental theme of linear algebra. 

Just as you partition A into columns and $\mathbf{x}$ into rows, it is often useful to consider an $m \times n$ matrix partitioned into smaller matrices. For example, you can partition the matrix below as shown.

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{bmatrix} \quad \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{array} \right]$$

You can also partition the matrix into column matrices

$$\left[\begin{array}{c|c|c|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{array} \right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

or row matrices

$$\left[\begin{array}{cccc} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -1 & -2 & 2 & 1 \end{array} \right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}.$$

**LINEAR ALGEBRA APPLIED**

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of a matrix with 837 rows and more than 12,750,000 columns. To solve this application of *linear programming*, researchers partitioned the problem into smaller pieces and solved it on a computer. (Source: *Very Large-Scale Linear Programming. A Case Study in Combining Interior Point and Simplex Methods*, Bixby, Robert E., et al., *Operations Research*, 40, no. 5)

Andres/Shutterstock.com

2.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Equality of Matrices In Exercises 1–4, find x and y .

$$1. \begin{bmatrix} x & -2 \\ 7 & y \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & 22 \end{bmatrix}$$

$$2. \begin{bmatrix} -5 & x \\ y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$$

$$3. \begin{bmatrix} 16 & 4 & 5 & 4 \\ -3 & 13 & 15 & 6 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ -3 & 13 & & 15 & 3x \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$$

$$4. \begin{bmatrix} x+2 & 8 & -3 \\ 1 & 2y & 2x \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & 11 \end{bmatrix}$$

Operations with Matrices In Exercises 5–10, find, if possible, (a) $A + B$, (b) $A - B$, (c) $2A$, (d) $2A - B$, and (e) $B + \frac{1}{2}A$.

$$5. A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 6 & -1 \\ 2 & 4 \\ -3 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ -1 & 5 \\ 1 & 10 \end{bmatrix}$$

$$7. A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$$

$$8. A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 1 \\ 5 & 4 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$9. A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}, B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$$

$$10. A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$$

11. Find (a) c_{21} and (b) c_{13} , where $C = 2A - 3B$,

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & 2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 2 & -7 \\ 0 & -5 & 1 \end{bmatrix}.$$

12. Find (a) c_{23} and (b) c_{32} , where $C = 5A + 2B$,

$$A = \begin{bmatrix} 4 & 11 & -9 \\ 0 & 3 & 2 \\ -3 & 1 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 5 \\ -4 & 6 & 11 \\ -6 & 4 & 9 \end{bmatrix}.$$

13. Solve for x , y , and z in the matrix equation

$$4 \begin{bmatrix} x & y \\ z & -1 \end{bmatrix} = 2 \begin{bmatrix} y & z \\ -x & 1 \end{bmatrix} + 2 \begin{bmatrix} 4 & x \\ 5 & -x \end{bmatrix}.$$

14. Solve for x , y , z , and w in the matrix equation

$$\begin{bmatrix} w & x \\ y & x \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix} + 2 \begin{bmatrix} y & w \\ z & x \end{bmatrix}.$$

Finding Products of Two Matrices In Exercises 15–28, find, if possible, (a) AB and (b) BA .

$$15. A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$$

$$16. A = \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 2 & -2 \end{bmatrix}$$

$$17. A = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & -2 \\ 2 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -4 & 1 & 3 \\ -4 & -1 & -2 \end{bmatrix}$$

$$18. A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$$

$$19. A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$$

$$20. A = \begin{bmatrix} 3 & 2 & 1 \\ -3 & 0 & 4 \\ 4 & -2 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & -2 \end{bmatrix}$$

$$21. A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$22. A = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 3 & 2 \end{bmatrix}$$

$$23. A = \begin{bmatrix} -1 & 3 \\ 4 & -5 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & 7 \end{bmatrix}$$

$$24. A = \begin{bmatrix} 2 & -3 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$25. A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & -1 & -2 \\ -2 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 4 & 0 & 1 & 3 \\ -1 & 2 & -3 & -1 \\ -2 & 1 & 4 & 3 \end{bmatrix}$$

$$27. A = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 6 \end{bmatrix}, B = \begin{bmatrix} 10 & 12 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 6 & 13 & 8 & -17 & 20 \end{bmatrix}, B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$$

Matrix Size In Exercises 29–36, let A, B, C, D , and E be matrices with the sizes shown below.

$$A: 3 \times 4 \quad B: 3 \times 4 \quad C: 4 \times 2 \quad D: 4 \times 2 \quad E: 4 \times 3$$

If defined, determine the size of the matrix. If not defined, explain why.

29. $A + B$

30. $C + E$

31. $\frac{1}{2}D$

32. $-4A$

33. AC

34. BE

35. $E - 2A$

36. $2D + C$

Solving a Matrix Equation In Exercises 37 and 38, solve the matrix equation $Ax = 0$.

37. $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving a System of Linear Equations In Exercises 39–48, write the system of linear equations in the form $Ax = b$ and solve this matrix equation for x .

39. $-x_1 + x_2 = 4$

40. $2x_1 + 3x_2 = 5$

$-2x_1 + x_2 = 0$

$x_1 + 4x_2 = 10$

41. $-2x_1 - 3x_2 = -4$

42. $-4x_1 + 9x_2 = -13$

$6x_1 + x_2 = -36$

$x_1 - 3x_2 = 12$

43. $x_1 - 2x_2 + 3x_3 = 9$

$-x_1 + 3x_2 - x_3 = -6$

$2x_1 - 5x_2 + 5x_3 = 17$

44. $x_1 + x_2 - 3x_3 = -1$

$-x_1 + 2x_2 = 1$

$x_1 - x_2 + x_3 = 2$

45. $x_1 - 5x_2 + 2x_3 = -20$

$-3x_1 + x_2 - x_3 = 8$

$-2x_2 + 5x_3 = -16$

46. $x_1 - x_2 + 4x_3 = 17$

$x_1 + 3x_2 = -11$

$-6x_2 + 5x_3 = 40$

47. $2x_1 - x_2 + x_4 = 3$

$3x_2 - x_3 - x_4 = -3$

$x_1 + x_3 - 3x_4 = -4$

$x_1 + x_2 + 2x_3 = 0$

48. $x_1 + x_2 = 0$

$x_2 + x_3 = 0$

$x_3 + x_4 = 0$

$x_4 + x_5 = 0$

$-x_1 + x_2 - x_3 + x_4 - x_5 = 5$

Writing a Linear Combination In Exercises 49–52, write the column matrix b as a linear combination of the columns of A .

49. $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

50. $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

51. $A = \begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

52. $A = \begin{bmatrix} -3 & 5 \\ 3 & 4 \\ 4 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} -22 \\ 4 \\ 32 \end{bmatrix}$

Solving a Matrix Equation In Exercises 53 and 54, solve for A .

53. $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

54. $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Solving a Matrix Equation In Exercises 55 and 56, solve the matrix equation for a, b, c , and d .

55. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 19 & 2 \end{bmatrix}$

56. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 17 \\ 4 & -1 \end{bmatrix}$

Diagonal Matrix In Exercises 57 and 58, find the product AA for the diagonal matrix. A square matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

is a diagonal matrix when all entries that are not on the main diagonal are zero.

57. $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

58. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Finding Products of Diagonal Matrices In Exercises 59 and 60, find the products AB and BA for the diagonal matrices.

59. $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 0 \\ 0 & 4 \end{bmatrix}$

60. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$

- 61. Guided Proof** Prove that if A and B are diagonal matrices (of the same size), then $AB = BA$.

Getting Started: To prove that the matrices AB and BA are equal, you need to show that their corresponding entries are equal.

- (i) Begin your proof by letting $A = [a_{ij}]$ and $B = [b_{ij}]$ be two diagonal $n \times n$ matrices.
 (ii) The ij th entry of the product AB is

$$c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}.$$

- (iii) Evaluate the entries c_{ij} for the two cases $i \neq j$ and $i = j$.
 (iv) Repeat this analysis for the product BA .

- 62. Writing** Let A and B be 3×3 matrices, where A is diagonal.

- (a) Describe the product AB . Illustrate your answer with examples.
 (b) Describe the product BA . Illustrate your answer with examples.
 (c) How do the results in parts (a) and (b) change when the diagonal entries of A are all equal?

Trace of a Matrix In Exercises 63–66, find the trace of the matrix. The trace of an $n \times n$ matrix A is the sum of the main diagonal entries. That is, $\text{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

63.
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ 3 & 1 & 3 \end{bmatrix}$$

64.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

65.
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}$$

66.
$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 0 & 6 & 1 \\ 3 & 6 & 2 & 1 \\ 2 & 1 & 1 & -3 \end{bmatrix}$$

- 67. Proof** Prove that each statement is true when A and B are square matrices of order n and c is a scalar.

- (a) $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$
 (b) $\text{Tr}(cA) = c\text{Tr}(A)$

- 68. Proof** Prove that if A and B are square matrices of order n , then $\text{Tr}(AB) = \text{Tr}(BA)$.

- 69.** Find conditions on w , x , y , and z such that $AB = BA$ for the matrices below.

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

- 70.** Verify $AB = BA$ for the matrices below.

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}$$

- 71.** Show that the matrix equation has no solution.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- 72.** Show that no 2×2 matrices A and B exist that satisfy the matrix equation

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- 73. Exploration** Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

- (a) Find A^2 , A^3 , and A^4 . (Note: $A^2 = AA$, $A^3 = AAA = A^2A$, and so on.) Identify any similarities with i^2 , i^3 , and i^4 .
 (b) Find and identify B^2 .

- 74. Guided Proof** Prove that if the product AB is a square matrix, then the product BA is defined.

Getting Started: To prove that the product BA is defined, you need to show that the number of columns of B equals the number of rows of A .

- (i) Begin your proof by noting that the number of columns of A equals the number of rows of B .
 (ii) Then assume that A has size $m \times n$ and B has size $n \times p$.
 (iii) Use the hypothesis that the product AB is a square matrix.

- 75. Proof** Prove that if both products AB and BA are defined, then AB and BA are square matrices.

- 76.** Let A and B be matrices such that the product AB is defined. Show that if A has two identical rows, then the corresponding two rows of AB are also identical.

- 77.** Let A and B be $n \times n$ matrices. Show that if the i th row of A has all zero entries, then the i th row of AB will have all zero entries. Give an example using 2×2 matrices to show that the converse is not true.

78. CAPSTONE Let matrices A and B be of sizes 3×2 and 2×2 , respectively. Answer each question and explain your answers.

- (a) Is it possible that $A = B$?
 (b) Is $A + B$ defined?
 (c) Is AB defined? If so, is it possible that $AB = BA$?

- 79. Agriculture** A fruit grower raises two crops, apples and peaches. The grower ships each of these crops to three different outlets. In the matrix

$$A = \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix}$$

a_{ij} represents the number of units of crop i that the grower ships to outlet j . The matrix

$$B = [\$3.50 \quad \$6.00]$$

represents the profit per unit. Find the product BA and state what each entry of the matrix represents.

80. **Manufacturing** A corporation has three factories, each of which manufactures acoustic guitars and electric guitars. In the matrix

$$A = \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix}$$

a_{ij} represents the number of guitars of type i produced at factory j in one day. Find the production levels when production increases by 20%.

81. **Politics** In the matrix

$$P = \begin{matrix} & \begin{matrix} \text{From} \\ \text{R} & \text{D} & \text{I} \end{matrix} \\ \begin{matrix} \text{R} \\ \text{D} \\ \text{I} \end{matrix} & \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \end{matrix} \left. \begin{matrix} \text{R} \\ \text{D} \\ \text{I} \end{matrix} \right\} \text{To}$$

each entry p_{ij} ($i \neq j$) represents the proportion of the voting population that changes from party j to party i , and p_{ii} represents the proportion that remains loyal to party i from one election to the next. Find and interpret the product of P with itself.

82. **Population** The matrices show the numbers of people (in thousands) who lived in each region of the United States in 2010 and 2013. The regional populations are separated into three age categories. (Source: U.S. Census Bureau)

	2010		
	0–17	18–64	65+
Northeast	12,306	35,240	7830
Midwest	16,095	41,830	9051
South	27,799	72,075	14,985
Mountain	5698	13,717	2710
Pacific	12,222	31,867	5901

	2013		
	0–17	18–64	65+
Northeast	12,026	35,471	8446
Midwest	15,772	41,985	9791
South	27,954	73,703	16,727
Mountain	5710	14,067	3104
Pacific	12,124	32,614	6636

- The total population in 2010 was approximately 309 million and the total population in 2013 was about 316 million. Rewrite the matrices to give the information as percents of the total population.
- Write a matrix that gives the changes in the percents of the population in each region and age group from 2010 to 2013.
- Based on the result of part (b), which age group(s) show relative growth from 2010 to 2013?

Block Multiplication In Exercises 83 and 84, perform the block multiplication of matrices A and B . If matrices A and B are each partitioned into four submatrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

then you can block multiply A and B , provided the sizes of the submatrices are such that the matrix multiplications and additions are defined.

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$83. A = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \end{array} \right], \quad B = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 3 \end{array} \right]$$

$$84. A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right], \quad B = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right]$$

True or False? In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- For the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix.
 - The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination of the columns of A , where the coefficients of the linear combination are a solution of the system.
86. (a) If A is an $m \times n$ matrix and B is an $n \times r$ matrix, then the product AB is an $m \times r$ matrix.
- (b) The matrix equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix and $\mathbf{x}$ and $\mathbf{b}$ are column matrices, can be used to represent a system of linear equations.
87. The columns of matrix T show the coordinates of the vertices of a triangle. Matrix A is a transformation matrix.
- $$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \end{bmatrix}$$
- Find AT and AAT . Then sketch the original triangle and the two transformed triangles. What transformation does A represent?
 - A triangle is determined by AAT . Describe the transformation process that produces the triangle determined by AT and then the triangle determined by T .

2.2 Properties of Matrix Operations

-  Use the properties of matrix addition, scalar multiplication, and zero matrices.
-  Use the properties of matrix multiplication and the identity matrix.
-  Find the transpose of a matrix.

ALGEBRA OF MATRICES

In Section 2.1, you concentrated on the mechanics of the three basic matrix operations: matrix addition, scalar multiplication, and matrix multiplication. This section begins to develop the **algebra of matrices**. You will see that this algebra shares many (but not all) of the properties of the algebra of real numbers. Theorem 2.1 lists several properties of matrix addition and scalar multiplication.

THEOREM 2.1 Properties of Matrix Addition and Scalar Multiplication

If A , B , and C are $m \times n$ matrices, and c and d are scalars, then the properties below are true.

- | | |
|--------------------------------|--|
| 1. $A + B = B + A$ | Commutative property of addition |
| 2. $A + (B + C) = (A + B) + C$ | Associative property of addition |
| 3. $(cd)A = c(dA)$ | Associative property of multiplication |
| 4. $1A = A$ | Multiplicative identity |
| 5. $c(A + B) = cA + cB$ | Distributive property |
| 6. $(c + d)A = cA + dA$ | Distributive property |

PROOF

The proofs of these six properties follow directly from the definitions of matrix addition, scalar multiplication, and the corresponding properties of real numbers. For example, to prove the commutative property of *matrix addition*, let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then, using the commutative property of *addition of real numbers*, write

$$A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A.$$

Similarly, to prove Property 5, use the distributive property (for real numbers) of multiplication over addition to write

$$c(A + B) = [c(a_{ij} + b_{ij})] = [ca_{ij} + cb_{ij}] = cA + cB.$$

The proofs of the remaining four properties are left as exercises. (See Exercises 61–64.)

The preceding section defined matrix addition as the sum of *two* matrices, making it a binary operation. The associative property of matrix addition now allows you to write expressions such as $A + B + C$ as $(A + B) + C$ or as $A + (B + C)$. This same reasoning applies to sums of four or more matrices.

EXAMPLE 1

Addition of More than Two Matrices

To obtain the sum of four matrices, add corresponding entries as shown below.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

One important property of the addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c . For matrices, a similar property holds. Specifically, if A is an $m \times n$ matrix and O_{mn} is the $m \times n$ matrix consisting entirely of zeros, then $A + O_{mn} = A$. The matrix O_{mn} is a **zero matrix**, and it is the **additive identity** for the set of all $m \times n$ matrices. For example, the matrix below is the additive identity for the set of all 2×3 matrices.

$$O_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

When the size of the matrix is understood, you may denote a zero matrix simply by O or $\mathbf{0}$.

The properties of zero matrices listed below are relatively easy to prove, and their proofs are left as an exercise. (See Exercise 65.)

REMARK

Property 2 can be described by saying that matrix $-A$ is the **additive inverse** of A .

THEOREM 2.2 Properties of Zero Matrices

If A is an $m \times n$ matrix and c is a scalar, then the properties below are true.

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then $c = 0$ or $A = O_{mn}$.

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the two solutions below.

Real Numbers
(Solve for x .)

$$\begin{aligned} x + a &= b \\ x + a + (-a) &= b + (-a) \\ x + 0 &= b - a \\ x &= b - a \end{aligned}$$

$m \times n$ Matrices
(Solve for X .)

$$\begin{aligned} X + A &= B \\ X + A + (-A) &= B + (-A) \\ X + O &= B - A \\ X &= B - A \end{aligned}$$

Example 2 demonstrates the process of solving a matrix equation.

EXAMPLE 2

Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

SOLUTION

Begin by solving the equation for X to obtain

$$\begin{aligned} 3X &= B - A \\ X &= \frac{1}{3}(B - A). \end{aligned}$$

Now, using the matrices A and B , you have

$$\begin{aligned} X &= \frac{1}{3} \left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \right) \\ &= \frac{1}{3} \begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}. \end{aligned}$$



REMARK

Note that no commutative property of matrix multiplication is listed in Theorem 2.3. The product AB may not be equal to the product BA , as illustrated in Example 4 on the next page.

PROPERTIES OF MATRIX MULTIPLICATION

The next theorem extends the algebra of matrices to include some useful properties of matrix multiplication. The proof of Property 2 is below. The proofs of the remaining properties are left as an exercise. (See Exercise 66.)

THEOREM 2.3 Properties of Matrix Multiplication

If A , B , and C are matrices (with sizes such that the matrix products are defined), and c is a scalar, then the properties below are true.

1. $A(BC) = (AB)C$ Associative property of multiplication
2. $A(B + C) = AB + AC$ Distributive property
3. $(A + B)C = AC + BC$ Distributive property
4. $c(AB) = (cA)B = A(cB)$

PROOF

To prove Property 2, show that the corresponding entries of matrices $A(B + C)$ and $AB + AC$ are equal. Assume A has size $m \times n$, B has size $n \times p$, and C has size $n \times p$. Using the definition of matrix multiplication, the entry in the i th row and j th column of $A(B + C)$ is $a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{in}(b_{nj} + c_{nj})$. Moreover, the entry in the i th row and j th column of $AB + AC$ is

$$(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}).$$

By distributing and regrouping, you can see that these two ij th entries are equal. So,

$$A(B + C) = AB + AC.$$

The associative property of matrix multiplication permits you to write such matrix products as ABC without ambiguity, as demonstrated in Example 3.

EXAMPLE 3**Matrix Multiplication Is Associative**

Find the matrix product ABC by grouping the factors first as $(AB)C$ and then as $A(BC)$. Show that you obtain the same result from both processes.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION

Grouping the factors as $(AB)C$, you have

$$\begin{aligned} (AB)C &= \left(\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix}. \end{aligned}$$

Grouping the factors as $A(BC)$, you obtain the same result.

$$\begin{aligned} A(BC) &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 4 \\ 13 & 14 \end{bmatrix} \end{aligned}$$

The next example shows that even when both products AB and BA are defined, they may not be equal.

EXAMPLE 4**Noncommutativity of Matrix Multiplication**

Show that AB and BA are not equal for the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

SOLUTION

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$AB \neq BA$$

Do not conclude from Example 4 that the matrix products AB and BA are *never* equal. Sometimes they are equal. For example, find AB and BA for the matrices below.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 4 \\ 2 & -2 \end{bmatrix}$$

You will see that the two products are equal. The point is that although AB and BA are sometimes equal, AB and BA are usually not equal.

Another important quality of matrix algebra is that it does not have a general cancellation property for matrix multiplication. That is, when $AC = BC$, it is not necessarily true that $A = B$. Example 5 demonstrates this. (In the next section you will see that, for some special types of matrices, cancellation is valid.)

EXAMPLE 5**An Example in Which Cancellation Is Not Valid**

Show that $AC = BC$.

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

SOLUTION

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}, \quad BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$AC = BC, \text{ even though } A \neq B.$$

You will now look at a special type of *square* matrix that has 1's on the main diagonal and 0's elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$n \times n$

For instance, for $n = 1, 2$, and 3 ,

$$I_1 = [1], \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When the order of the matrix is understood to be n , you may denote I_n simply as I .

As stated in Theorem 2.4 on the next page, the matrix I_n serves as the **identity** for matrix multiplication; it is the **identity matrix of order n** . The proof of this theorem is left as an exercise. (See Exercise 67.)

REMARK

Note that if A is a *square* matrix of order n , then $AI_n = I_nA = A$.

THEOREM 2.4 Properties of the Identity Matrix

If A is a matrix of size $m \times n$, then the properties below are true.

1. $AI_n = A$
2. $I_mA = A$

EXAMPLE 6**Multiplication by an Identity Matrix**

$$\text{a. } \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{b. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

For repeated multiplication of *square* matrices, use the same exponential notation used with real numbers. That is, $A^1 = A$, $A^2 = AA$, and for a positive integer k , A^k is

$$A^k = \underbrace{AA \cdots A}_{k \text{ factors}}$$

It is convenient also to define $A^0 = I_n$ (where A is a square matrix of order n). These definitions allow you to establish the properties (1) $A^jA^k = A^{j+k}$ and (2) $(A^j)^k = A^{jk}$, where j and k are nonnegative integers.

EXAMPLE 7**Repeated Multiplication of a Square Matrix**

For the matrix $A = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$,

$$A^3 = \left(\begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} \right) \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 3 & -6 \end{bmatrix}.$$

In Section 1.1, you saw that a system of linear equations has exactly one solution, infinitely many solutions, or no solution. You can use matrix algebra to prove this.

THEOREM 2.5 Number of Solutions of a Linear System

For a system of linear equations, precisely one of the statements below is true.

1. The system has exactly one solution.
2. The system has infinitely many solutions.
3. The system has no solution.

PROOF

Represent the system by the matrix equation $A\mathbf{x} = \mathbf{b}$. If the system has exactly one solution or no solution, then there is nothing to prove. So, assume that the system has at least two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. If you show that this assumption implies that the system has infinitely many solutions, then the proof will be complete. When $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, you have $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$ and $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. This implies that the (nonzero) column matrix $\mathbf{x}_h = \mathbf{x}_1 - \mathbf{x}_2$ is a solution of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$. So, for any scalar c ,

$$A(\mathbf{x}_1 + c\mathbf{x}_h) = A\mathbf{x}_1 + A(c\mathbf{x}_h) = \mathbf{b} + c(A\mathbf{x}_h) = \mathbf{b} + c\mathbf{0} = \mathbf{b}.$$

Then $\mathbf{x}_1 + c\mathbf{x}_h$ is a solution of $A\mathbf{x} = \mathbf{b}$ for any scalar c . There are infinitely many possible values of c and each value produces a different solution, so the system has infinitely many solutions.

DISCOVERY

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and

$$B = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}.$$

1. Find $(AB)^T$, $A^T B^T$, and $B^T A^T$.
2. Make a conjecture about the transpose of a product of two square matrices.
3. Select two other square matrices to check your conjecture.

THE TRANSPOSE OF A MATRIX

The **transpose** of a matrix is formed by writing its rows as columns. For example, if A is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Size: $m \times n$

then the transpose, denoted by A^T , is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

REMARK

Note that the square matrix in part (c) is equal to its transpose. Such a matrix is **symmetric**. A matrix A is symmetric when $A = A^T$. From this definition it should be clear that a symmetric matrix must be square. Also, if $A = [a_{ij}]$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all $i \neq j$.

EXAMPLE 8

Transposes of Matrices

Find the transpose of each matrix.

a. $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ b. $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ c. $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $D = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$

SOLUTION

a. $A^T = [2 \quad 8]$ b. $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

c. $C^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ d. $D^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$

THEOREM 2.6 Properties of Transposes

If A and B are matrices (with sizes such that the matrix operations are defined) and c is a scalar, then the properties below are true.

1. $(A^T)^T = A$ Transpose of a transpose
2. $(A + B)^T = A^T + B^T$ Transpose of a sum
3. $(cA)^T = c(A^T)$ Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$ Transpose of a product

PROOF

The transpose operation interchanges rows and columns, so Property 1 seems to make sense. To prove Property 1, let A be an $m \times n$ matrix. Observe that A^T has size $n \times m$ and $(A^T)^T$ has size $m \times n$, the same as A . To show that $(A^T)^T = A$, you must show that the ij th entries are the same. Let a_{ij} be the ij th entry of A . Then a_{ij} is the j th entry of A^T , and the ij th entry of $(A^T)^T$. This proves Property 1. The proofs of the remaining properties are left as an exercise. (See Exercise 68.)

REMARK

Remember that you *reverse the order* of multiplication when forming the transpose of a product. That is, the transpose of AB is $(AB)^T = B^T A^T$ and is usually *not* equal to $A^T B^T$.

Properties 2 and 4 can be generalized to cover sums or products of any finite number of matrices. For instance, the transpose of the sum of three matrices is $(A + B + C)^T = A^T + B^T + C^T$ and the transpose of the product of three matrices is $(ABC)^T = C^T B^T A^T$.

EXAMPLE 9**Finding the Transpose of a Product**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Show that $(AB)^T$ and $B^T A^T$ are equal.

$$A = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}$$

SOLUTION

$$AB = \begin{bmatrix} 2 & 1 & -2 \\ -1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

EXAMPLE 10**The Product of a Matrix and Its Transpose**

For the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}$, find the product AA^T and show that it is symmetric.

SOLUTION

$$AA^T = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -6 & -5 \\ -6 & 4 & 2 \\ -5 & 2 & 5 \end{bmatrix}$$

It follows that $AA^T = (AA^T)^T$, so AA^T is symmetric.

REMARK

The property demonstrated in Example 10 is true in general. That is, for any matrix A , the matrix AA^T is symmetric. The matrix $A^T A$ is also symmetric. You are asked to prove these properties in Exercise 69.

**LINEAR ALGEBRA APPLIED**

Information retrieval systems such as Internet search engines make use of matrix theory and linear algebra to keep track of information. To illustrate, consider a simplified example. You could represent the occurrences of m available keywords in a database of n documents with A , an $m \times n$ matrix in which an entry is 1 when the keyword occurs in the document and 0 when it does not occur in the document. You could represent a search with the $m \times 1$ column matrix $\mathbf{x}$, in which a 1 entry represents a keyword you are searching and 0 represents a keyword you are not searching. Then, the $n \times 1$ matrix product $A^T \mathbf{x}$ would represent the number of keywords in your search that occur in each of the n documents. For a discussion on the PageRank algorithm that is used in Google's search engine, see Section 2.5 (page 86).

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2.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Evaluating an Expression In Exercises 1–6, evaluate the expression.

- $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$
- $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$
- $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$
- $\frac{1}{2}([5 \ -2 \ 4 \ 0] + [14 \ 6 \ -18 \ 9])$
- $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$
- $-\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$

Operations with Matrices In Exercises 7–12, perform the operations, given $a = 3$, $b = -4$, and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- $aA + bB$
- $A + B$
- $ab(B)$
- $(a + b)B$
- $(a - b)(A - B)$
- $(ab)O$
- Solve for X in the equation, given
 $A = \begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}$.
 (a) $3X + 2A = B$ (b) $2A - 5B = 3X$
 (c) $X - 3A + 2B = O$ (d) $6X - 4A - 3B = O$
- Solve for X in the equation, given
 $A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$.
 (a) $X = 3A - 2B$ (b) $2X = 2A - B$
 (c) $2X + 3A = B$ (d) $2A + 4B = -2X$

Operations with Matrices In Exercises 15–22, perform the operations, given $c = -2$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

- $c(BA)$
- $c(CB)$
- $B(CA)$
- $C(BC)$
- $(B + C)A$
- $B(C + O)$
- $cB(C + C)$
- $B(cA)$

Associativity of Matrix Multiplication In Exercises 23 and 24, find the matrix product ABC by (a) grouping the factors as $(AB)C$, and (b) grouping the factors as $A(BC)$. Show that you obtain the same result from both processes.

$$23. A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$24. A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -5 & 0 \\ -2 & 3 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} -3 & 4 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

Noncommutativity of Matrix Multiplication In Exercises 25 and 26, show that AB and BA are not equal for the given matrices.

$$25. A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$26. A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Equal Matrix Products In Exercises 27 and 28, show that $AC = BC$, even though $A \neq B$.

$$27. A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

Zero Matrix Product In Exercises 29 and 30, show that $AB = O$, even though $A \neq O$ and $B \neq O$.

$$29. A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$30. A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Operations with Matrices In Exercises 31–36, perform the operations when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$

- IA
- AI
- $A(I + A)$
- $A + IA$
- A^2
- A^4

Writing In Exercises 37 and 38, explain why the formula is *not* valid for matrices. Illustrate your argument with examples.

37. $(A + B)(A - B) = A^2 - B^2$

38. $(A + B)(A + B) = A^2 + 2AB + B^2$

Finding the Transpose of a Matrix In Exercises 39 and 40, find the transpose of the matrix.

39. $D = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & -1 \end{bmatrix}$ 40. $D = \begin{bmatrix} 6 & -7 & 19 \\ -7 & 0 & 23 \\ 19 & 23 & -32 \end{bmatrix}$

Finding the Transpose of a Product of Two Matrices In Exercises 41–44, verify that $(AB)^T = B^T A^T$.

41. $A = \begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 0 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$

42. $A = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$

43. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$

44. $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}$

Multiplication with the Transpose of a Matrix In Exercises 45–48, find (a) $A^T A$ and (b) AA^T . Show that each of these products is symmetric.

45. $A = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ 46. $A = \begin{bmatrix} 1 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix}$

 47. $A = \begin{bmatrix} 0 & -4 & 3 & 2 \\ 8 & 4 & 0 & 1 \\ -2 & 3 & 5 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$

 48. $A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$

Finding a Power of a Matrix In Exercises 49–52, find the power of A for the matrix

$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

49. A^{16}

50. A^{17}

51. A^{19}

52. A^{20}

Finding an n th Root of a Matrix In Exercises 53 and 54, find the n th root of the matrix B . An n th root of a matrix B is a matrix A such that $A^n = B$.

53. $B = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, $n = 2$

54. $B = \begin{bmatrix} 8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 27 \end{bmatrix}$, $n = 3$

True or False? In Exercises 55 and 56, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

55. (a) Matrix addition is commutative.

(b) The transpose of the product of two matrices equals the product of their transposes; that is, $(AB)^T = A^T B^T$.

(c) For any matrix C the matrix CC^T is symmetric.

56. (a) Matrix multiplication is commutative.

(b) If the matrices A , B , and C satisfy $AB = AC$, then $B = C$.

(c) The transpose of the sum of two matrices equals the sum of their transposes.

57. Consider the matrices below.

$X = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $Z = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $W = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find scalars a and b such that $Z = aX + bY$.

(b) Show that there do not exist scalars a and b such that $W = aX + bY$.

(c) Show that if $aX + bY + cW = O$, then $a = 0$, $b = 0$, and $c = 0$.

(d) Find scalars a , b , and c , not all equal to zero, such that $aX + bY + cZ = O$.

58. CAPSTONE In the matrix equation

$aX + A(bB) = b(AB + IB)$

X , A , B , and I are square matrices, and a and b are nonzero scalars. Justify each step in the solution below.

$aX + (Ab)B = b(AB + B)$

$aX + bAB = bAB + bB$

$aX + bAB + (-bAB) = bAB + bB + (-bAB)$

$aX = bAB + bB + (-bAB)$

$aX = bAB + (-bAB) + bB$

$aX = bB$

$X = \frac{b}{a}B$

Polynomial Function In Exercises 59 and 60, find $f(A)$ using the definition below.

If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial function, then for a square matrix A ,

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

59. $f(x) = 2 - 5x + x^2$, $A = \begin{bmatrix} 2 & 0 \\ 4 & 5 \end{bmatrix}$

60. $f(x) = -10 + 5x - 2x^2 + x^3$, $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

61. **Guided Proof** Prove the associative property of matrix addition: $A + (B + C) = (A + B) + C$.

Getting Started: To prove that $A + (B + C)$ and $(A + B) + C$ are equal, show that their corresponding entries are equal.

- Begin your proof by letting A , B , and C be $m \times n$ matrices.
- Observe that the ij th entry of $B + C$ is $b_{ij} + c_{ij}$.
- Furthermore, the ij th entry of $A + (B + C)$ is $a_{ij} + (b_{ij} + c_{ij})$.
- Determine the ij th entry of $(A + B) + C$.

62. **Proof** Prove the associative property of multiplication: $(cd)A + c(dA)$.

63. **Proof** Prove that the scalar 1 is the identity for scalar multiplication: $1A = A$.

64. **Proof** Prove the distributive property:

$$(c + d)A = cA + dA.$$

65. **Proof** Prove Theorem 2.2.

66. **Proof** Complete the proof of Theorem 2.3.

- Prove the associative property of multiplication:

$$A(BC) = (AB)C.$$

- Prove the distributive property:

$$(A + B)C = AC + BC.$$

- Prove the property:

$$c(AB) = (cA)B = A(cB).$$

67. **Proof** Prove Theorem 2.4.

68. **Proof** Prove Properties 2, 3, and 4 of Theorem 2.6.

69. **Guided Proof** Prove that if A is an $m \times n$ matrix, then AA^T and A^TA are symmetric matrices.

Getting Started: To prove that AA^T is symmetric, you need to show that it is equal to its transpose, $(AA^T)^T = AA^T$.

- Begin your proof with the left-hand matrix expression $(AA^T)^T$.
- Use the properties of the transpose operation to show that $(AA^T)^T$ can be simplified to equal the right-hand expression, AA^T .
- Repeat this analysis for the product A^TA .

70. **Proof** Let A and B be two $n \times n$ symmetric matrices.

- Give an example to show that the product AB is not necessarily symmetric.
- Prove that the product AB is symmetric if and only if $AB = BA$.

Symmetric and Skew-Symmetric Matrices In Exercises 71–74, determine whether the matrix is symmetric, skew-symmetric, or neither. A square matrix is skew-symmetric when $A^T = -A$.

71. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

72. $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

73. $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

74. $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

75. **Proof** Prove that the main diagonal of a skew-symmetric matrix consists entirely of zeros.

76. **Proof** Prove that if A and B are $n \times n$ skew-symmetric matrices, then $A + B$ is skew-symmetric.

77. **Proof** Let A be a square matrix of order n .

- Show that $\frac{1}{2}(A + A^T)$ is symmetric.
- Show that $\frac{1}{2}(A - A^T)$ is skew-symmetric.
- Prove that A can be written as the sum of a symmetric matrix B and a skew-symmetric matrix C , $A = B + C$.
- Write the matrix below as the sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -3 & 6 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

78. **Proof** Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew-symmetric.



79. Consider matrices of the form

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

- Write a 2×2 matrix and a 3×3 matrix in the form of A .
- Use a graphing utility to raise each of the matrices to higher powers. Describe the result.
- Use the result of part (b) to make a conjecture about powers of A when A is a 4×4 matrix. Use a graphing utility to test your conjecture.
- Use the results of parts (b) and (c) to make a conjecture about powers of A when A is an $n \times n$ matrix.

2.3 The Inverse of a Matrix

-  Find the inverse of a matrix (if it exists).
-  Use properties of inverse matrices.
-  Use an inverse matrix to solve a system of linear equations.

MATRICES AND THEIR INVERSES

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for x , multiply both sides of the equation by a^{-1} (provided $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is the *multiplicative inverse* of a because $a^{-1}a = 1$ (the identity element for multiplication). The definition of the multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n . The matrix B is the (multiplicative) **inverse** of A . A matrix that does not have an inverse is **noninvertible** (or **singular**).

Nonsquare matrices do not have inverses. To see this, note that if A is of size $m \times n$ and B is of size $n \times m$ (where $m \neq n$), then the products AB and BA are of different sizes and cannot be equal to each other. Not all square matrices have inverses. (See Example 4.) The next theorem, however, states that if a matrix *does* have an inverse, then that inverse is unique.

THEOREM 2.7 Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1} .

PROOF

If A is invertible, then it has at least one inverse B such that

$$AB = I = BA.$$

Assume that A has another inverse C such that

$$AC = I = CA.$$

Demonstrate that B and C are equal, as shown on the next page.

$$\begin{aligned}
 AB &= I \\
 C(AB) &= CI \\
 (CA)B &= C \\
 IB &= C \\
 B &= C
 \end{aligned}$$

Consequently $B = C$, and it follows that the inverse of a matrix is unique. 

The inverse A^{-1} of an invertible matrix A is unique, so you can call it *the* inverse of A and write $AA^{-1} = A^{-1}A = I$.

EXAMPLE 1

The Inverse of a Matrix

Show that B is the inverse of A , where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

REMARK

Recall that it is not always true that $AB = BA$, even when both products are defined. If A and B are both square matrices and $AB = I_n$, however, then it can be shown that $BA = I_n$. (The proof of this is omitted.) So, in Example 1, you need only check that $AB = I_2$.

SOLUTION

Using the definition of an inverse matrix, show that B is the inverse of A by showing that $AB = I = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 & 2 - 2 \\ -1 + 1 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$


The next example shows how to use a system of equations to find the inverse of a matrix.

EXAMPLE 2

Finding the Inverse of a Matrix

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

SOLUTION

To find the inverse of A , solve the matrix equation $AX = I$ for X .

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, you obtain two systems of linear equations.

$$\begin{aligned}
 x_{11} + 4x_{21} &= 1 & x_{12} + 4x_{22} &= 0 \\
 -x_{11} - 3x_{21} &= 0 & -x_{12} - 3x_{22} &= 1
 \end{aligned}$$

Solving the first system, you find that $x_{11} = -3$ and $x_{21} = 1$. Similarly, solving the second system, you find that $x_{12} = -4$ and $x_{22} = 1$. So, the inverse of A is

$$X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

Use matrix multiplication to check this result. 

Generalizing the method used to solve Example 2 provides a convenient method for finding an inverse. Note that the two systems of linear equations

$$\begin{array}{rcl} x_{11} + 4x_{21} & = & 1 \\ -x_{11} - 3x_{21} & = & 0 \end{array} \quad \begin{array}{rcl} x_{12} + 4x_{22} & = & 0 \\ -x_{12} - 3x_{22} & = & 1 \end{array}$$

have the *same coefficient matrix*. Rather than solve the two systems represented by

$$\begin{bmatrix} 1 & 4 & 1 \\ -1 & -3 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

separately, solve them simultaneously by **adjoining** the identity matrix to the coefficient matrix to obtain

$$\begin{bmatrix} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}.$$

By applying Gauss-Jordan elimination to this matrix, solve *both* systems with a single elimination process, as shown below.

$$\begin{array}{l} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \quad \begin{array}{l} R_2 + R_1 \rightarrow R_2 \\ R_1 + (-4)R_2 \rightarrow R_1 \end{array}$$

Applying Gauss-Jordan elimination to the “doubly augmented” matrix $[A \quad I]$, you obtain the matrix $[I \quad A^{-1}]$.

$$\begin{array}{cc} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \textcolor{red}{A} & \textcolor{red}{I} & & \textcolor{red}{I} \quad \textcolor{red}{A^{-1}} \end{array}$$

This procedure (or algorithm) works for an arbitrary $n \times n$ matrix. If A cannot be row reduced to I_n , then A is noninvertible (or singular). This procedure will be formally justified in the next section, after introducing the concept of an elementary matrix. For now, a summary of the algorithm is shown below.

Finding the Inverse of a Matrix by Gauss-Jordan Elimination

Let A be a square matrix of order n .

1. Write the $n \times 2n$ matrix that consists of A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \quad I]$. This process is called **adjoining** matrix I to matrix A .
2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A \quad I]$. The result will be the matrix $[I \quad A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.



LINEAR ALGEBRA APPLIED

Recall Hooke's law, which states that for relatively small deformations of an elastic object, the amount of deflection is directly proportional to the force causing the deformation. In a simply supported elastic beam subjected to multiple forces, deflection $\mathbf{d}$ is related to force $\mathbf{w}$ by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where F is a *flexibility matrix* whose entries depend on the material of the beam. The inverse of the flexibility matrix, F^{-1} , is the *stiffness matrix*. In Exercises 61 and 62, you are asked to find the stiffness matrix F^{-1} and the force matrix $\mathbf{w}$ for a given set of flexibility and deflection matrices.

nostal6ie/Shutterstock.com

EXAMPLE 3**Finding the Inverse of a Matrix**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

SOLUTION

Begin by adjoining the identity matrix to A to form the matrix

$$[A \quad I] = \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{bmatrix}.$$

Use elementary row operations to obtain the form

$$[I \quad A^{-1}]$$

as shown below.

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} \quad R_2 + (-1)R_1 \rightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{bmatrix} \quad R_3 + (6)R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{bmatrix} \quad R_3 + (4)R_2 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{bmatrix} \quad (-1)R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{bmatrix} \quad R_2 + R_3 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{bmatrix} \quad R_1 + R_2 \rightarrow R_1$$

The matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Confirm this by showing that

$$AA^{-1} = I = A^{-1}A.$$

TECHNOLOGY

Many graphing utilities and software programs can find the inverse of a square matrix. When you use a graphing utility, you may see something similar to the screen below for Example 3. The **Technology Guide** at CengageBrain.com can help you use technology to find the inverse of a matrix.

The process shown in Example 3 applies to any $n \times n$ matrix A and will find the inverse of A , if it exists. When A has no inverse, the process will also tell you that. The next example applies the process to a singular matrix (one that has no inverse).

EXAMPLE 4**A Singular Matrix**

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION

Adjoin the identity matrix to A to form

$$[A \quad I] = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{bmatrix}$$

and apply Gauss-Jordan elimination to obtain

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{bmatrix}.$$

Note that the “ A portion” of the matrix has a row of zeros. So it is not possible to rewrite the matrix $[A \quad I]$ in the form $[I \quad A^{-1}]$. This means that A has no inverse, or is noninvertible (or singular). 

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of size 3×3 or greater. For 2×2 matrices, however, you can use a formula for the inverse rather than Gauss-Jordan elimination.

If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Moreover, if $ad - bc \neq 0$, then the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

REMARK

The denominator $ad - bc$ is called the **determinant** of A . You will study determinants in detail in Chapter 3.

EXAMPLE 5**Finding Inverses of 2×2 Matrices**

If possible, find the inverse of each matrix.

$$\text{a. } A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \quad \text{b. } B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$$

SOLUTION

a. For the matrix A , apply the formula for the inverse of a 2×2 matrix to obtain $ad - bc = (3)(2) - (-1)(-2) = 4$. This quantity is not zero, so A is invertible. Form the inverse by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$, as shown below.

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

b. For the matrix B , you have $ad - bc = (3)(2) - (-1)(-6) = 0$, which means that B is noninvertible. 

PROPERTIES OF INVERSES

Theorem 2.8 below lists important properties of inverse matrices.

THEOREM 2.8 Properties of Inverse Matrices

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1} , A^k , cA , and A^T are invertible and the statements below are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

PROOF

The key to the proofs of Properties 1, 3, and 4 is the fact that the inverse of a matrix is unique (Theorem 2.7). That is, if $BC = CB = I$, then C is the inverse of B .

Property 1 states that the inverse of A^{-1} is A itself. To prove this, observe that $A^{-1}A = AA^{-1} = I$, which means that A is the inverse of A^{-1} . Thus, $A = (A^{-1})^{-1}$.

Similarly, Property 3 states that $\frac{1}{c}A^{-1}$ is the inverse of (cA) , $c \neq 0$. To prove this, use the properties of scalar multiplication given in Theorems 2.1 and 2.3.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = \left(c\frac{1}{c}\right)AA^{-1} = (1)I = I$$

$$\left(\frac{1}{c}A^{-1}\right)(cA) = \left(\frac{1}{c}c\right)A^{-1}A = (1)I = I$$

So $\frac{1}{c}A^{-1}$ is the inverse of (cA) , which implies that $(cA)^{-1} = \frac{1}{c}A^{-1}$. Properties 2 and 4 are left for you to prove. (See Exercises 63 and 64.)

For nonsingular matrices, the exponential notation used for repeated multiplication of *square* matrices can be extended to include exponents that are negative integers. This may be done by defining A^{-k} to be

$$A^{-k} = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k.$$

Using this convention you can show that the properties $A^jA^k = A^{j+k}$ and $(A^j)^k = A^{jk}$ are true for any integers j and k .

DISCOVERY

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$.

1. Find $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$.
2. Make a conjecture about the inverse of a product of two nonsingular matrices. Then select two other nonsingular matrices of the same order and see whether your conjecture holds.

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

EXAMPLE 6**The Inverse of the Square of a Matrix**

Compute A^{-2} two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION

One way to find A^{-2} is to find $(A^2)^{-1}$ by squaring the matrix A to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a 2×2 matrix to obtain

$$(A^2)^{-1} = \frac{1}{4} \begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find A^{-2} is to find $(A^{-1})^2$ by finding A^{-1}

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Note that both methods produce the same result. 

The next theorem gives a formula for computing the inverse of a product of two matrices.

THEOREM 2.9 The Inverse of a Product

If A and B are invertible matrices of order n , then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

PROOF

To show that $B^{-1}A^{-1}$ is the inverse of AB , you need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I.$$

In a similar way, $(B^{-1}A^{-1})(AB) = I$. So, AB is invertible and its inverse is $B^{-1}A^{-1}$. 

Theorem 2.9 states that the inverse of a product of two invertible matrices is the product of their inverses taken in the *reverse* order. This can be generalized to include the product of more than two invertible matrices:

$$(A_1A_2A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}.$$

(See Example 4 in Appendix.)

EXAMPLE 7**Finding the Inverse of a Matrix Product**

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

using the fact that A^{-1} and B^{-1} are

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

REMARK

Note that you *reverse the order* of multiplication to find the inverse of AB . That is, $(AB)^{-1} = B^{-1}A^{-1}$, and the inverse of AB is usually *not* equal to $A^{-1}B^{-1}$.

SOLUTION

Using Theorem 2.9 produces

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ &= \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}. \end{aligned}$$

One important property in the algebra of real numbers is the cancellation property. That is, if $ac = bc$ ($c \neq 0$), then $a = b$. *Invertible* matrices have similar cancellation properties.

THEOREM 2.10 Cancellation Properties

If C is an invertible matrix, then the properties below are true.

1. If $AC = BC$, then $A = B$. Right cancellation property
2. If $CA = CB$, then $A = B$. Left cancellation property

PROOF

To prove Property 1, use the fact that C is invertible and write

$$\begin{aligned} AC &= BC \\ (AC)C^{-1} &= (BC)C^{-1} \\ A(CC^{-1}) &= B(CC^{-1}) \\ AI &= BI \\ A &= B. \end{aligned}$$

The second property can be proved in a similar way. (See Exercise 65.)

Be sure to remember that Theorem 2.10 can be applied only when C is an *invertible* matrix. If C is not invertible, then cancellation is not usually valid. For instance, Example 5 in Section 2.2 gives an example of a matrix equation $AC = BC$ in which $A \neq B$, because C is not invertible in the example.

SYSTEMS OF EQUATIONS

For *square* systems of equations (those having the same number of equations as variables), you can use the theorem below to determine whether the system has a unique solution.

THEOREM 2.11 Systems of Equations with Unique Solutions

If A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

PROOF

The matrix A is nonsingular, so the steps shown below are valid.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

This solution is unique because if $\mathbf{x}_1$ and $\mathbf{x}_2$ were two solutions, then you could apply the cancellation property to the equation $A\mathbf{x}_1 = \mathbf{b} = A\mathbf{x}_2$ to conclude that $\mathbf{x}_1 = \mathbf{x}_2$. 

One use of Theorem 2.11 is in solving *several* systems that all have the same coefficient matrix A . You could find the inverse matrix once and then solve each system by computing the product $A^{-1}\mathbf{b}$.

EXAMPLE 8

Solving Systems of Equations Using an Inverse Matrix

Use an inverse matrix to solve each system.

$$\begin{array}{lll} \text{a. } 2x + 3y + z = -1 & \text{b. } 2x + 3y + z = 4 & \text{c. } 2x + 3y + z = 0 \\ 3x + 3y + z = 1 & 3x + 3y + z = 8 & 3x + 3y + z = 0 \\ 2x + 4y + z = -2 & 2x + 4y + z = 5 & 2x + 4y + z = 0 \end{array}$$

SOLUTION

First note that the coefficient matrix for each system is $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$.

Using Gauss-Jordan elimination, $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$.

$$\text{a. } \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \begin{array}{l} \text{The solution is } x = 2, \\ y = -1, \text{ and } z = -2. \end{array}$$

$$\text{b. } \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix} \quad \begin{array}{l} \text{The solution is } x = 4, \\ y = 1, \text{ and } z = -7. \end{array}$$

$$\text{c. } \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{The solution is trivial:} \\ x = 0, y = 0, \text{ and } z = 0. \end{array}$$


2.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



The Inverse of a Matrix In Exercises 1–6, show that B is the inverse of A .

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

5. $A = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \frac{1}{3} \begin{bmatrix} -4 & -5 & 3 \\ -4 & -8 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

Finding the Inverse of a Matrix In Exercises 7–30, find the inverse of the matrix (if it exists).

7. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

8. $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

11. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

12. $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ 7 & 16 & -21 \end{bmatrix}$

16. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

18. $\begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

20. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

21. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

22. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

25. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

27. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

28. $\begin{bmatrix} 4 & 8 & -7 & 14 \\ 2 & 5 & -4 & 6 \\ 0 & 2 & 1 & -7 \\ 3 & 6 & -5 & 10 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$

30. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Finding the Inverse of a 2×2 Matrix In Exercises 31–36, use the formula on page 66 to find the inverse of the 2×2 matrix (if it exists).

31. $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

33. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$

34. $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

35. $\begin{bmatrix} \frac{7}{2} & -\frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}$

36. $\begin{bmatrix} -\frac{1}{4} & \frac{9}{4} \\ \frac{5}{3} & \frac{8}{9} \end{bmatrix}$

Finding the Inverse of the Square of a Matrix In Exercises 37–40, compute A^{-2} two different ways and show that the results are equal.

37. $A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$

38. $A = \begin{bmatrix} 2 & 7 \\ -5 & 6 \end{bmatrix}$

39. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

40. $A = \begin{bmatrix} 6 & 0 & 4 \\ -2 & 7 & -1 \\ 3 & 1 & 2 \end{bmatrix}$

Finding the Inverses of Products and Transposes In Exercises 41–44, use the inverse matrices to find (a) $(AB)^{-1}$, (b) $(A^T)^{-1}$, and (c) $(2A)^{-1}$.

41. $A^{-1} = \begin{bmatrix} 2 & 5 \\ -7 & 6 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 7 & -3 \\ 2 & 0 \end{bmatrix}$

42. $A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$

43. $A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{4} \\ \frac{3}{2} & \frac{1}{2} & -2 \\ \frac{1}{4} & 1 & \frac{1}{2} \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 2 & 4 & \frac{5}{2} \\ -\frac{3}{4} & 2 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 2 \end{bmatrix}$

44. $A^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 6 & 5 & -3 \\ -2 & 4 & -1 \\ 1 & 3 & 4 \end{bmatrix}$

Solving a System of Equations Using an Inverse

In Exercises 45–48, use an inverse matrix to solve each system of linear equations.

45. (a) $x + 2y = -1$
 $x - 2y = 3$
 (b) $x + 2y = 10$
 $x - 2y = -6$
46. (a) $2x - y = -3$
 $2x + y = 7$
 (b) $2x - y = -1$
 $2x + y = -3$
47. (a) $x_1 + 2x_2 + x_3 = 2$
 $x_1 + 2x_2 - x_3 = 4$
 $x_1 - 2x_2 + x_3 = -2$
 (b) $x_1 + 2x_2 + x_3 = 1$
 $x_1 + 2x_2 - x_3 = 3$
 $x_1 - 2x_2 + x_3 = -3$
48. (a) $x_1 + x_2 - 2x_3 = 0$
 $x_1 - 2x_2 + x_3 = 0$
 $x_1 - x_2 - x_3 = -1$
 (b) $x_1 + x_2 - 2x_3 = -1$
 $x_1 - 2x_2 + x_3 = 2$
 $x_1 - x_2 - x_3 = 0$



Solving a System of Equations Using an Inverse

In Exercises 49–52, use a software program or a graphing utility to solve the system of linear equations using an inverse matrix.

49. $x_1 + 2x_2 - x_3 + 3x_4 - x_5 = -3$
 $x_1 - 3x_2 + x_3 + 2x_4 - x_5 = -3$
 $2x_1 + x_2 + x_3 - 3x_4 + x_5 = 6$
 $x_1 - x_2 + 2x_3 + x_4 - x_5 = 2$
 $2x_1 + x_2 - x_3 + 2x_4 + x_5 = -3$
50. $x_1 + x_2 - x_3 + 3x_4 - x_5 = 3$
 $2x_1 + x_2 + x_3 + x_4 + x_5 = 4$
 $x_1 + x_2 - x_3 + 2x_4 - x_5 = 3$
 $2x_1 + x_2 + 4x_3 + x_4 - x_5 = -1$
 $3x_1 + x_2 + x_3 - 2x_4 + x_5 = 5$
51. $2x_1 - 3x_2 + x_3 - 2x_4 + x_5 - 4x_6 = 20$
 $3x_1 + x_2 - 4x_3 + x_4 - x_5 + 2x_6 = -16$
 $4x_1 + x_2 - 3x_3 + 4x_4 - x_5 + 2x_6 = -12$
 $-5x_1 - x_2 + 4x_3 + 2x_4 - 5x_5 + 3x_6 = -2$
 $x_1 + x_2 - 3x_3 + 4x_4 - 3x_5 + x_6 = -15$
 $3x_1 - x_2 + 2x_3 - 3x_4 + 2x_5 - 6x_6 = 25$
52. $4x_1 - 2x_2 + 4x_3 + 2x_4 - 5x_5 - x_6 = 1$
 $3x_1 + 6x_2 - 5x_3 - 6x_4 + 3x_5 + 3x_6 = -11$
 $2x_1 - 3x_2 + x_3 + 3x_4 - x_5 - 2x_6 = 0$
 $-x_1 + 4x_2 - 4x_3 - 6x_4 + 2x_5 + 4x_6 = -9$
 $3x_1 - x_2 + 5x_3 + 2x_4 - 3x_5 - 5x_6 = 1$
 $-2x_1 + 3x_2 - 4x_3 - 6x_4 + x_5 + 2x_6 = -12$

Matrix Equal to Its Own Inverse

In Exercises 53 and 54, find x such that the matrix is equal to its own inverse.

53. $A = \begin{bmatrix} 3 & x \\ -2 & -3 \end{bmatrix}$ 54. $A = \begin{bmatrix} 2 & x \\ -1 & -2 \end{bmatrix}$

Singular Matrix

In Exercises 55 and 56, find x such that the matrix is singular.

55. $A = \begin{bmatrix} 4 & x \\ -2 & -3 \end{bmatrix}$ 56. $A = \begin{bmatrix} x & 2 \\ -3 & 4 \end{bmatrix}$

Solving a Matrix Equation

In Exercises 57 and 58, find A .

57. $(2A)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 58. $(4A)^{-1} = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$

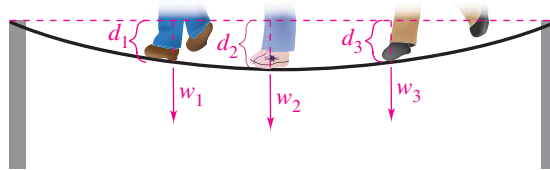
Finding the Inverse of a Matrix

In Exercises 59 and 60, show that the matrix is invertible and find its inverse.

59. $A = \begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix}$ 60. $A = \begin{bmatrix} \sec \theta & \tan \theta \\ \tan \theta & \sec \theta \end{bmatrix}$

Beam Deflection

In Exercises 61 and 62, forces w_1 , w_2 , and w_3 (in pounds) act on a simply supported elastic beam, resulting in deflections d_1 , d_2 , and d_3 (in inches) in the beam (see figure).



Use the matrix equation $\mathbf{d} = F\mathbf{w}$, where

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

and F is the 3×3 flexibility matrix for the beam, to find the stiffness matrix F^{-1} and the force matrix $\mathbf{w}$. The entries of F are measured in inches per pound.

61. $F = \begin{bmatrix} 0.008 & 0.004 & 0.003 \\ 0.004 & 0.006 & 0.004 \\ 0.003 & 0.004 & 0.008 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 0.585 \\ 0.640 \\ 0.835 \end{bmatrix}$
62. $F = \begin{bmatrix} 0.017 & 0.010 & 0.008 \\ 0.010 & 0.012 & 0.010 \\ 0.008 & 0.010 & 0.017 \end{bmatrix}$, $\mathbf{d} = \begin{bmatrix} 0 \\ 0.15 \\ 0 \end{bmatrix}$

Proof

Prove Property 2 of Theorem 2.8: If A is an invertible matrix and k is a positive integer, then

$$(A^k)^{-1} = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k$$

Proof

Prove Property 4 of Theorem 2.8: If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$.

Proof

Prove Property 2 of Theorem 2.10: If C is an invertible matrix such that $CA = CB$, then $A = B$.

Proof

Prove that if $A^2 = A$, then

$$I - 2A = (I - 2A)^{-1}.$$

- 67. Guided Proof** Prove that the inverse of a symmetric nonsingular matrix is symmetric.

Getting Started: To prove that the inverse of A is symmetric, you need to show that $(A^{-1})^T = A^{-1}$.

- (i) Let A be a symmetric, nonsingular matrix.
- (ii) This means that $A^T = A$ and A^{-1} exists.
- (iii) Use the properties of the transpose to show that $(A^{-1})^T$ is equal to A^{-1} .

- 68. Proof** Prove that if A , B , and C are square matrices and $ABC = I$, then B is invertible and $B^{-1} = CA$.

- 69. Proof** Prove that if A is invertible and $AB = O$, then $B = O$.

- 70. Guided Proof** Prove that if $A^2 = A$, then either A is singular or $A = I$.

Getting Started: You must show that either A is singular or A equals the identity matrix.

- (i) Begin your proof by observing that A is either singular or nonsingular.
- (ii) If A is singular, then you are done.
- (iii) If A is nonsingular, then use the inverse matrix A^{-1} and the hypothesis $A^2 = A$ to show that $A = I$.

True or False? In Exercises 71 and 72, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 71.** (a) If the matrices A , B , and C satisfy $BA = CA$ and A is invertible, then $B = C$.
 (b) The inverse of the product of two matrices is the product of their inverses; that is, $(AB)^{-1} = A^{-1}B^{-1}$.
 (c) If A can be row reduced to the identity matrix, then A is nonsingular.
- 72.** (a) The inverse of the inverse of a nonsingular matrix A , $(A^{-1})^{-1}$, is equal to A itself.
 (b) The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible when $ab - dc \neq 0$.
 (c) If A is a square matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution.
- 73. Writing** Is the sum of two invertible matrices invertible? Explain why or why not. Illustrate your conclusion with appropriate examples.
- 74. Writing** Under what conditions will the diagonal matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

be invertible? Assume that A is invertible and find its inverse.

- 75.** Use the result of Exercise 74 to find A^{-1} for each matrix.

(a) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

76. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

- (a) Show that $A^2 - 2A + 5I = O$, where I is the identity matrix of order 2.
- (b) Show that $A^{-1} = \frac{1}{5}(2I - A)$.
- (c) Show that for any square matrix satisfying $A^2 - 2A + 5I = O$, the inverse of A is $A^{-1} = \frac{1}{5}(2I - A)$.

- 77. Proof** Let $\mathbf{u}$ be an $n \times 1$ column matrix satisfying $\mathbf{u}^T\mathbf{u} = I_1$. The $n \times n$ matrix $H = I_n - 2\mathbf{u}\mathbf{u}^T$ is called a **Householder matrix**.

- (a) Prove that H is symmetric and nonsingular.
- (b) Let $\mathbf{u} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$. Show that $\mathbf{u}^T\mathbf{u} = I_1$ and find the Householder matrix H .

- 78. Proof** Let A and B be $n \times n$ matrices. Prove that if the matrix $I - AB$ is nonsingular, then so is $I - BA$.

- 79.** Let A , D , and P be $n \times n$ matrices satisfying $AP = PD$. Assume that P is nonsingular and solve this equation for A . Must it be true that $A = D$?

- 80.** Find an example of a singular 2×2 matrix satisfying $A^2 = A$.

- 81. Writing** Explain how to determine whether the inverse of a matrix exists. If so, explain how to find the inverse.

- 82. CAPSTONE** As mentioned on page 66, if A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then A is invertible if and only if $ad - bc \neq 0$. Verify that the inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- 83. Writing** Explain in your own words how to write a system of three linear equations in three variables as a matrix equation, $A\mathbf{x} = \mathbf{b}$, as well as how to solve the system using an inverse matrix.

2.4 Elementary Matrices

-  Factor a matrix into a product of elementary matrices.
-  Find and use an LU -factorization of a matrix to solve a system of linear equations.

ELEMENTARY MATRICES AND ELEMENTARY ROW OPERATIONS

Section 1.2 introduced the three elementary row operations for matrices listed below.

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

In this section, you will see how to use matrix multiplication to perform these operations.

REMARK

The identity matrix I_n is elementary by this definition because it can be obtained from itself by multiplying any one of its rows by 1.

Definition of an Elementary Matrix

An $n \times n$ matrix is an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation.

EXAMPLE 1

Elementary Matrices and Nonelementary Matrices

Which of the matrices below are elementary? For those that are, describe the corresponding elementary row operation.

a.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

e.
$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

f.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

SOLUTION

- a. This matrix *is* elementary. To obtain it from I_3 , multiply the second row of I_3 by 3.
- b. This matrix is *not* elementary because it is not square.
- c. This matrix is *not* elementary because to obtain it from I_3 , you must multiply the third row of I_3 by 0 (row multiplication must be by a nonzero constant).
- d. This matrix *is* elementary. To obtain it from I_3 , interchange the second and third rows of I_3 .
- e. This matrix *is* elementary. To obtain it from I_2 , multiply the first row of I_2 by 2 and add the result to the second row.
- f. This matrix is *not* elementary because it requires two elementary row operations to obtain from I_3 .

Elementary matrices are useful because they enable you to use matrix multiplication to perform elementary row operations, as demonstrated in Example 2.

EXAMPLE 2**Elementary Matrices and Elementary Row Operations**

- a. In the matrix product below, E is the elementary matrix in which the first two rows of I_3 are interchanged.

$$\begin{matrix} E & A \end{matrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

Note that the first two rows of A are interchanged when multiplying *on the left* by E .

- b. In the matrix product below, E is the elementary matrix in which the second row of I_3 is multiplied by $\frac{1}{2}$.

$$\begin{matrix} E & A \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

Note that the second row of A is multiplied by $\frac{1}{2}$ when multiplying *on the left* by E .

- c. In the matrix product below, E is the elementary matrix in which 2 times the first row of I_3 is added to the second row.

$$\begin{matrix} E & A \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{bmatrix}$$

Note that 2 times the first row of A is added to the second row when multiplying *on the left* by E .

REMARK

Be sure to remember in Theorem 2.12 to multiply A *on the left* by the elementary matrix E . This text does not consider right multiplication by elementary matrices, which involves column operations.

Notice from Example 2(b) that you can use matrix multiplication to perform elementary row operations on *nonsquare* matrices. If the size of A is $n \times p$, then E must have order n .

In each of the three products in Example 2, you are able to perform elementary row operations by multiplying *on the left* by an elementary matrix. The next theorem, stated without proof, generalizes this property of elementary matrices.

THEOREM 2.12 Representing Elementary Row Operations

Let E be the elementary matrix obtained by performing an elementary row operation on I_m . If that same elementary row operation is performed on an $m \times n$ matrix A , then the resulting matrix is the product EA .

Most applications of elementary row operations require a sequence of operations. For instance, Gaussian elimination usually requires several elementary row operations to row reduce a matrix. This translates into multiplication on the left by several elementary matrices. The order of multiplication is important; the elementary matrix immediately to the left of A corresponds to the row operation performed first. Example 3 demonstrates this process.

EXAMPLE 3**Using Elementary Matrices**

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

SOLUTION

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_3 \rightarrow R_3$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

The three elementary matrices E_1 , E_2 , and E_3 can be used to perform the same elimination.

REMARK

The procedure demonstrated in Example 3 is primarily of theoretical interest. In other words, this procedure is not a practical method for performing Gaussian elimination.

$$\begin{aligned} B = E_3 E_2 E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{aligned}$$

The two matrices in Example 3

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

are row-equivalent because you can obtain B by performing a sequence of row operations on A . That is, $B = E_3 E_2 E_1 A$.

The definition of row-equivalent matrices is restated below using elementary matrices.

Definition of Row Equivalence

Let A and B be $m \times n$ matrices. Matrix B is **row-equivalent** to A when there exists a finite number of elementary matrices $E_1, E_2, \dots, E_k$ such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A.$$

You know from Section 2.3 that not all square matrices are invertible. Every elementary matrix, however, is invertible. Moreover, the inverse of an elementary matrix is itself an elementary matrix.

THEOREM 2.13 Elementary Matrices Are Invertible

If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

The inverse of an elementary matrix E is the elementary matrix that converts E back to I_n . For instance, the inverses of the three elementary matrices in Example 3 are shown below.

REMARK

E_2^{-1} is as shown because to convert E_2 back to I_3 , in E_2 you would add 2 times row 1 to row 3.

Elementary Matrix		Inverse Matrix	
$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$
$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$	$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$R_3 + (2)R_1 \rightarrow R_3$
$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$(\frac{1}{2})R_3 \rightarrow R_3$	$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$(2)R_3 \rightarrow R_3$

Use matrix multiplication to check these results.

The next theorem states that every invertible matrix can be written as the product of elementary matrices.

THEOREM 2.14 A Property of Invertible Matrices

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

PROOF

The phrase “if and only if” means that there are actually two parts to the theorem. On the one hand, you have to show that *if* A is invertible, *then* it can be written as the product of elementary matrices. Then you have to show that *if* A can be written as the product of elementary matrices, *then* A is invertible.

To prove the theorem in one direction, assume A is invertible. From Theorem 2.11 you know that the system of linear equations represented by $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. But this implies that the augmented matrix $[A \quad \mathbf{0}]$ can be rewritten in the form $[I \quad \mathbf{0}]$ (using elementary row operations corresponding to $E_1, E_2, \dots$, and E_k). So, $E_k \cdot \dots \cdot E_2 E_1 A = I$ and it follows that $A = E_1^{-1} E_2^{-1} \cdot \dots \cdot E_k^{-1}$. A can be written as the product of elementary matrices.

To prove the theorem in the other direction, assume A is the product of elementary matrices. Every elementary matrix is invertible and the product of invertible matrices is invertible, so it follows that A is invertible. This completes the proof. 

Example 4 illustrates the first part of this proof.

EXAMPLE 4**Writing a Matrix as the Product of Elementary Matrices**

Find a sequence of elementary matrices whose product is the nonsingular matrix

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}.$$

SOLUTION

Begin by finding a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$	$(-1)R_1 \rightarrow R_1$	$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$	$R_2 + (-3)R_1 \rightarrow R_2$	$E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	$(\frac{1}{2})R_2 \rightarrow R_2$	$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_1 + (-2)R_2 \rightarrow R_1$	$E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Now, from the matrix product $E_4 E_3 E_2 E_1 A = I$, solve for A to obtain $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$. This implies that A is a product of elementary matrices.

$$A = \overset{E_1^{-1}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}} \overset{E_2^{-1}}{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}} \overset{E_3^{-1}}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}} \overset{E_4^{-1}}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

In Section 2.3, you learned a process for finding the inverse of a nonsingular matrix A . There, you used Gauss-Jordan elimination to reduce the augmented matrix $[A \ I]$ to $[I \ A^{-1}]$. You can now use Theorem 2.14 to justify this procedure. Specifically, the proof of Theorem 2.14 allows you to write the product

$$I = E_k \cdot \cdots \cdot E_2 E_1 A.$$

Multiplying both sides of this equation (on the right) by A^{-1} , $A^{-1} = E_k \cdot \cdots \cdot E_2 E_1 I$. In other words, a sequence of elementary matrices that reduces A to the identity I also reduces the identity I to A^{-1} . Applying the corresponding sequence of elementary row operations to the matrices A and I simultaneously, you have

$$E_k \cdot \cdots \cdot E_2 E_1 [A \ I] = [I \ A^{-1}].$$

Of course, if A is singular, then no such sequence is possible.

The next theorem ties together some important relationships between $n \times n$ matrices and systems of linear equations. The essential parts of this theorem have already been proved (see Theorems 2.11 and 2.14); it is left to you to fill in the other parts of the proof.

THEOREM 2.15 Equivalent Conditions

If A is an $n \times n$ matrix, then the statements below are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n .
5. A can be written as the product of elementary matrices.

THE LU -FACTORIZATION

At the heart of the most efficient and modern algorithms for solving linear systems $A\mathbf{x} = \mathbf{b}$ is the LU -factorization, in which the square matrix A is expressed as a product, $A = LU$. In this product, the square matrix L is **lower triangular**, which means all the entries above the main diagonal are zero. The square matrix U is **upper triangular**, which means all the entries below the main diagonal are zero.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3×3 lower triangular matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

3×3 upper triangular matrix

Definition of LU -Factorization

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U , then $A = LU$ is an LU -factorization of A .

EXAMPLE 5

LU -Factorizations

$$\text{a. } \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = LU$$

is an LU -factorization of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

as the product of the lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the upper triangular matrix

$$U = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}.$$

$$\text{b. } A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

is an LU -factorization of the matrix A .



LINEAR ALGEBRA APPLIED

Computational fluid dynamics (CFD) is the computer-based simulation of such real-life phenomena as fluid flow, heat transfer, and chemical reactions. Solving the conservation of energy, mass, and momentum equations involved in a CFD analysis can involve large systems of linear equations. So, for efficiency in computing, CFD analyses often use matrix partitioning and LU -factorization in their algorithms. Aerospace companies such as Boeing and Airbus have used CFD analysis in aircraft design. For instance, engineers at Boeing used CFD analysis to simulate airflow around a virtual model of their 787 aircraft to help produce a faster and more efficient design than those of earlier Boeing aircraft.

Goncharuk/Shutterstock.com

If a square matrix A row reduces to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is relatively easy to find an LU -factorization of the matrix A . All you need to do is keep track of the individual row operations, as shown in the next example.

EXAMPLE 6**Finding an LU -Factorization of a Matrix**

Find an LU -factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix}$.

SOLUTION

Begin by row reducing A to upper triangular form while keeping track of the elementary matrices used for each row operation.

Matrix	Elementary Row Operation	Elementary Matrix
$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$	$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$	$R_3 + (4)R_2 \rightarrow R_3$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

The reduced matrix above is an upper triangular matrix U , and it follows that $E_2E_1A = U$, or $A = E_1^{-1}E_2^{-1}U$. The product of the lower triangular matrices

$$E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

is a lower triangular matrix L , so the factorization $A = LU$ is complete. Notice that this is the same LU -factorization as in Example 5(b). 

If A row reduces to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then A has an LU -factorization.

$$\begin{aligned} E_k \cdot \cdots \cdot E_2 E_1 A &= U \\ A &= E_1^{-1} E_2^{-1} \cdot \cdots \cdot E_k^{-1} U = LU \end{aligned}$$

Here L is the product of the inverses of the elementary matrices used in the row reduction.

Note that the multipliers in Example 6 are -2 and 4 , which are the negatives of the corresponding entries in L . This is true in general. If U can be obtained from A using only the elementary row operation of adding a multiple of one row to another row below it, then the matrix L is lower triangular (with 1 's along the diagonal), and the negative of each multiplier is in the same position as that of the corresponding zero in U below the main diagonal.

Once you have obtained an LU -factorization of a matrix A , you can then solve the system of n linear equations in n variables $A\mathbf{x} = \mathbf{b}$ very efficiently in two steps.

1. Write $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
2. Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

The column matrix $\mathbf{x}$ is the solution of the original system because $A\mathbf{x} = LU\mathbf{x} = L\mathbf{y} = \mathbf{b}$.

The second step is just back-substitution, because the matrix U is upper triangular. The first step is similar, except that it starts at the top of the matrix, because L is lower triangular. For this reason, the first step is often called **forward substitution**.

EXAMPLE 7**Solving a Linear System Using LU-Factorization**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Solve the linear system.

$$\begin{aligned}x_1 - 3x_2 &= -5 \\x_2 + 3x_3 &= -1 \\2x_1 - 10x_2 + 2x_3 &= -20\end{aligned}$$

SOLUTION

You obtained an LU -factorization of the coefficient matrix A in Example 6.

$$\begin{aligned}A &= \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \\&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}\end{aligned}$$

First, let $\mathbf{y} = U\mathbf{x}$ and solve the system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix}$$

Solve this system using forward substitution. Starting with the first equation, you have $y_1 = -5$. The second equation gives $y_2 = -1$. Finally, from the third equation,

$$\begin{aligned}2y_1 - 4y_2 + y_3 &= -20 \\y_3 &= -20 - 2y_1 + 4y_2 \\y_3 &= -20 - 2(-5) + 4(-1) \\y_3 &= -14.\end{aligned}$$

The solution of $L\mathbf{y} = \mathbf{b}$ is

$$\mathbf{y} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}.$$

Now solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ using back-substitution.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

From the bottom equation, $x_3 = -1$. Then, the second equation gives

$$x_2 + 3(-1) = -1$$

or $x_2 = 2$. Finally, the first equation gives

$$x_1 - 3(2) = -5$$

or $x_1 = 1$. So, the solution of the original system of equations is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

2.4 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Elementary Matrices In Exercises 1–8, determine whether the matrix is elementary. If it is, state the elementary row operation used to produce it.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$

Finding an Elementary Matrix In Exercises 9–12, let A , B , and C be

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & -3 \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}.$$

9. Find an elementary matrix E such that $EA = B$.
10. Find an elementary matrix E such that $EA = C$.
11. Find an elementary matrix E such that $EB = A$.
12. Find an elementary matrix E such that $EC = A$.

Finding a Sequence of Elementary Matrices In Exercises 13–18, find a sequence of elementary matrices that can be used to write the matrix in row-echelon form.

13. $\begin{bmatrix} 0 & 1 & 7 \\ 5 & 10 & -5 \end{bmatrix}$

14. $\begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 4 & 8 & -4 \\ -6 & 12 & 8 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix}$

17. $\begin{bmatrix} -2 & 1 & 0 \\ 3 & -4 & 0 \\ 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -6 & 0 & 2 \\ 0 & -3 & 3 & 9 \\ 2 & 5 & -1 & 1 \\ 4 & 8 & -5 & 1 \end{bmatrix}$

Finding the Inverse of an Elementary Matrix In Exercises 19–24, find the inverse of the elementary matrix.

19. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

20. $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

23. $\begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 $k \neq 0$

24. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Finding the Inverse of a Matrix In Exercises 25–28, find the inverse of the matrix using elementary matrices.

25. $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

26. $\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

27. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Finding a Sequence of Elementary Matrices In Exercises 29–36, find a sequence of elementary matrices whose product is the given nonsingular matrix.

29. $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

31. $\begin{bmatrix} 4 & -1 \\ 3 & -1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

34. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

36. $\begin{bmatrix} 4 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 1 & 0 & 0 & -2 \end{bmatrix}$

37. **Writing** Is the product of two elementary matrices always elementary? Explain.
38. **Writing** E is the elementary matrix obtained by interchanging two rows in I_n . A is an $n \times n$ matrix.
 - (a) How will EA compare with A ? (b) Find E^2 .
39. Use elementary matrices to find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}, \quad c \neq 0.$$

40. Use elementary matrices to find the inverse of

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix},$$

$c \neq 0$.

True or False? In Exercises 41 and 42, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

41. (a) The identity matrix is an elementary matrix.
 (b) If E is an elementary matrix, then $2E$ is an elementary matrix.
 (c) The inverse of an elementary matrix is an elementary matrix.
42. (a) The zero matrix is an elementary matrix.
 (b) A square matrix is nonsingular when it can be written as the product of elementary matrices.
 (c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

Finding an LU -Factorization of a Matrix In Exercises 43–46, find an LU -factorization of the matrix.

43. $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ 44. $\begin{bmatrix} -2 & 1 \\ -6 & 4 \end{bmatrix}$

45. $\begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$ 46. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix}$

Solving a Linear System Using LU -Factorization In Exercises 47 and 48, use an LU -factorization of the coefficient matrix to solve the linear system.

47.
$$\begin{aligned} 2x + y &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$

48.
$$\begin{aligned} 2x_1 &= 4 \\ -2x_1 + x_2 - x_3 &= -4 \\ 6x_1 + 2x_2 + x_3 &= 15 \\ -x_4 &= -1 \end{aligned}$$

Idempotent Matrices In Exercises 49–52, determine whether the matrix is idempotent. A square matrix A is idempotent when $A^2 = A$.

49. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ 50. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

51. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ 52. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

53. Determine a and b such that A is idempotent.

$$A = \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}$$

54. **Guided Proof** Prove that A is idempotent if and only if A^T is idempotent.

Getting Started: The phrase “if and only if” means that you have to prove two statements:

1. If A is idempotent, then A^T is idempotent.
2. If A^T is idempotent, then A is idempotent.
 - (i) Begin your proof of the first statement by assuming that A is idempotent.
 - (ii) This means that $A^2 = A$.
 - (iii) Use the properties of the transpose to show that A^T is idempotent.
 - (iv) Begin your proof of the second statement by assuming that A^T is idempotent.

55. **Proof** Prove that if A is an $n \times n$ matrix that is idempotent and invertible, then $A = I_n$.

56. **Proof** Prove that if A and B are idempotent and $AB = BA$, then AB is idempotent.

57. **Guided Proof** Prove that if A is row-equivalent to B and B is row-equivalent to C , then A is row-equivalent to C .

Getting Started: To prove that A is row-equivalent to C , you have to find elementary matrices $E_1, E_2, \dots, E_k$ such that $A = E_k \cdot \dots \cdot E_2 E_1 C$.

- (i) Begin by observing that A is row-equivalent to B and B is row-equivalent to C .
- (ii) This means that there exist elementary matrices $F_1, F_2, \dots, F_n$ and $G_1, G_2, \dots, G_m$ such that $A = F_n \cdot \dots \cdot F_2 F_1 B$ and $B = G_m \cdot \dots \cdot G_2 G_1 C$.
- (iii) Combine the matrix equations from step (ii).

58. **Proof** Prove that if A is row-equivalent to B , then B is row-equivalent to A .

59. **Proof** Let A be a nonsingular matrix. Prove that if B is row-equivalent to A , then B is also nonsingular.

60. CAPSTONE

- (a) Explain how to find an elementary matrix.
- (b) Explain how to use elementary matrices to find an LU -factorization of a matrix.
- (c) Explain how to use LU -factorization to solve a linear system.

61. Show that the matrix below does not have an LU -factorization.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2.5 Markov Chains

-  Use a stochastic matrix to find the n th state matrix of a Markov chain.
-  Find the steady state matrix of a Markov chain.
-  Find the steady state matrix of an absorbing Markov chain.

STOCHASTIC MATRICES AND MARKOV CHAINS

Many types of applications involve a finite set of *states* $\{S_1, S_2, \dots, S_n\}$ of a population. For instance, residents of a city may live downtown or in the suburbs. Voters may vote Democrat, Republican, or Independent. Soft drink consumers may buy Coca-Cola, Pepsi Cola, or another brand.

The probability that a member of a population will change from the j th state to the i th state is represented by a number p_{ij} , where $0 \leq p_{ij} \leq 1$. A probability of $p_{ij} = 0$ means that the member is certain *not* to change from the j th state to the i th state, whereas a probability of $p_{ij} = 1$ means that the member is certain to change from the j th state to the i th state.

$$P = \begin{array}{c} \text{To} \left\{ \begin{array}{c} S_1 \\ S_2 \\ \vdots \\ S_n \end{array} \right. \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{array}{c} \text{From} \\ S_1 \quad S_2 \quad \cdots \quad S_n \end{array} \end{array}$$

P is called the **matrix of transition probabilities** because it gives the probabilities of each possible type of transition (or change) within the population.

At each transition, each member in a given state must either stay in that state or change to another state. For probabilities, this means that the sum of the entries in any column of P is 1. For instance, in the first column,

$$p_{11} + p_{21} + \cdots + p_{n1} = 1.$$

Such a matrix is called **stochastic** (the term “stochastic” means “regarding conjecture”). That is, an $n \times n$ matrix P is a **stochastic matrix** when each entry is a number between 0 and 1 inclusive, and the sum of the entries in each column of P is 1.

EXAMPLE 1

Examples of Stochastic Matrices and Nonstochastic Matrices

The matrices in parts (a), (b), and (c) are stochastic, but the matrices in parts (d), (e), and (f) are not.

$$\begin{array}{lll} \text{a. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \text{b. } \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{4} & \frac{13}{60} & 0 & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} & \text{c. } \begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix} \\ \text{d. } \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} \end{bmatrix} & \text{e. } \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} & \text{f. } \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.5 \\ 0.3 & 0.4 & 0.5 & 0.6 \\ 0.4 & 0.5 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

Example 2 describes the use of a stochastic matrix to measure consumer preferences.

EXAMPLE 2

A Consumer Preference Model

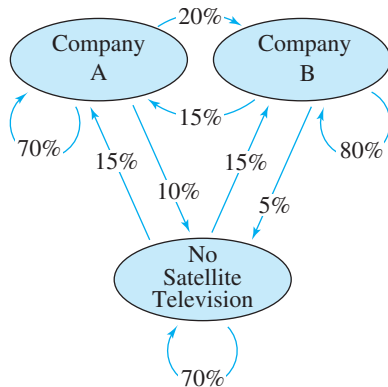


Figure 2.1

SOLUTION

The matrix of transition probabilities is

$$P = \begin{matrix} & \begin{matrix} \text{From} \\ \text{A} & \text{B} & \text{None} \end{matrix} \\ \begin{matrix} \text{To} \\ \text{A} \\ \text{B} \\ \text{None} \end{matrix} & \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \end{matrix}$$

and the initial **state matrix** representing the portions of the total population in the three states is

$$X_0 = \begin{bmatrix} 0.1500 \\ 0.2000 \\ 0.6500 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix}$$

To find the state matrix representing the portions of the population in the three states in one year, multiply P by X_0 to obtain

$$X_1 = PX_0 = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} 0.1500 \\ 0.2000 \\ 0.6500 \end{bmatrix} = \begin{bmatrix} 0.2325 \\ 0.2875 \\ 0.4800 \end{bmatrix}.$$

REMARK

Always assume that the matrix P of transition probabilities in a Markov chain remains constant between states.

In one year, Company A will have $0.2325(100,000) = 23,250$ subscribers and Company B will have $0.2875(100,000) = 28,750$ subscribers.

A **Markov chain**, named after Russian mathematician Andrey Andreyevich Markov (1856–1922), is a sequence $\{X_n\}$ of state matrices that are related by the equation $X_{k+1} = PX_k$, where P is a stochastic matrix. For instance, consider the consumer preference model discussed in Example 2. To find the state matrix representing the portions of the population in each state in three years, repeatedly multiply the initial state matrix X_0 by the matrix of transition probabilities P .

$$X_1 = PX_0$$

$$X_2 = PX_1 = P \cdot PX_0 = P^2X_0$$

$$X_3 = PX_2 = P \cdot P^2X_0 = P^3X_0$$

In general, the n th state matrix of a Markov chain is P^nX_0 , as summarized below.

n th State Matrix of a Markov Chain

The n th state matrix of a Markov chain for which P is the matrix of transition probabilities and X_0 is the initial state matrix is

$$X_n = P^nX_0.$$

Example 3 uses the model discussed in Example 2 to demonstrate this process.

EXAMPLE 3**A Consumer Preference Model**

Assuming the matrix of transition probabilities from Example 2 remains the same year after year, find the number of subscribers each satellite television company will have after (a) 3 years, (b) 5 years, (c) 10 years, and (d) 15 years.

SOLUTION

- a. To find the numbers of subscribers after 3 years, first find X_3 .

$$X_3 = P^3 X_0 \approx \begin{bmatrix} 0.3028 \\ 0.3904 \\ 0.3068 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 3 years}$$

After 3 years, Company A will have about $0.3028(100,000) = 30,280$ subscribers and Company B will have about $0.3904(100,000) = 39,040$ subscribers.

- b. To find the numbers of subscribers after 5 years, first find X_5 .

$$X_5 = P^5 X_0 \approx \begin{bmatrix} 0.3241 \\ 0.4381 \\ 0.2378 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 5 years}$$

After 5 years, Company A will have about $0.3241(100,000) = 32,410$ subscribers and Company B will have about $0.4381(100,000) = 43,810$ subscribers.

- c. To find the numbers of subscribers after 10 years, first find X_{10} .

$$X_{10} = P^{10} X_0 \approx \begin{bmatrix} 0.3329 \\ 0.4715 \\ 0.1957 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 10 years}$$

After 10 years, Company A will have about $0.3329(100,000) = 33,290$ subscribers and Company B will have about $0.4715(100,000) = 47,150$ subscribers.

- d. To find the numbers of subscribers after 15 years, first find X_{15} .

$$X_{15} = P^{15} X_0 \approx \begin{bmatrix} 0.3333 \\ 0.4756 \\ 0.1911 \end{bmatrix} \begin{matrix} \text{A} \\ \text{B} \\ \text{None} \end{matrix} \quad \text{After 15 years}$$

After 15 years, Company A will have about $0.3333(100,000) = 33,330$ subscribers and Company B will have about $0.4756(100,000) = 47,560$ subscribers. 

**LINEAR ALGEBRA APPLIED**

Google's PageRank algorithm makes use of Markov chains. For a search set that contains n web pages, define an $n \times n$ matrix A such that $a_{ij} = 1$ when page j references page i and $a_{ij} = 0$ otherwise. Adjust A to account for web pages without external references, scale each column of A so that A is stochastic, and call this matrix B . Then define

$$M = pB + \frac{1-p}{n}E$$

where p is the probability that a user follows a link on a page, $1 - p$ is the probability that the user goes to any page at random, and E is an $n \times n$ matrix whose entries are all 1. The Markov chain whose matrix of transition probabilities is M converges to a unique *steady state matrix*, which gives an estimate of page ranks. Section 10.3 discusses a method that can be used to estimate the steady state matrix.

d8nn/Shutterstock.com

STEADY STATE MATRIX OF A MARKOV CHAIN

In Example 3, notice that there is little difference between the numbers of subscribers after 10 years and after 15 years. If you continue the process shown in this example, then the state matrix X_n eventually reaches a **steady state**. That is, as long as the matrix P does not change, the matrix product $P^n X$ approaches a limit $\bar{X}$. In Example 3, the limit is the *steady state matrix*

$$\bar{X} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{21} \\ \frac{4}{21} \end{bmatrix} \approx \begin{bmatrix} 0.3333 \\ 0.4762 \\ 0.1905 \end{bmatrix}.$$

A
B
None

Steady state matrix

Check to see that $P\bar{X} = \bar{X}$, as shown below.

$$P\bar{X} = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{10}{21} \\ \frac{4}{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{21} \\ \frac{4}{21} \end{bmatrix} = \bar{X}$$

In Example 5, you will verify the above result by finding the steady state matrix $\bar{X}$.

The matrix of transition probabilities P used above is an example of a *regular* stochastic matrix. A stochastic matrix P is **regular** when some power of P has only positive entries.

EXAMPLE 4

Regular Stochastic Matrices

a. The stochastic matrix

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}$$

is regular because P^1 has only positive entries.

b. The stochastic matrix

$$P = \begin{bmatrix} 0.50 & 1.00 \\ 0.50 & 0 \end{bmatrix}$$

is regular because

$$P^2 = \begin{bmatrix} 0.75 & 0.50 \\ 0.25 & 0.50 \end{bmatrix}$$

has only positive entries.

c. The stochastic matrix

$$P = \begin{bmatrix} \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$$

is not regular because every power of P has two zeros in its second column. (Verify this.)

When P is a regular stochastic matrix, the corresponding **regular Markov chain**

$$PX_0, P^2X_0, P^3X_0, \dots$$

approaches a unique **steady state matrix** $\bar{X}$. You are asked to prove this in Exercise 56.

REMARK

For a regular stochastic matrix P , the sequence of successive powers

$$P, P^2, P^3, \dots$$

approaches a *stable matrix* $\bar{P}$.

The entries in each column of $\bar{P}$ are equal to the corresponding entries in the steady state matrix $\bar{X}$. You are asked to show this in Exercise 55.

EXAMPLE 5**Finding a Steady State Matrix**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the steady state matrix $\bar{X}$ of the Markov chain whose matrix of transition probabilities is the regular matrix

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}.$$

SOLUTION

Note that P is the matrix of transition probabilities that you found in Example 2 and whose steady state matrix $\bar{X}$ you verified at the top of page 87. To find $\bar{X}$, begin by

letting $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then use the matrix equation $P\bar{X} = \bar{X}$ to obtain

$$\begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{aligned} 0.70x_1 + 0.15x_2 + 0.15x_3 &= x_1 \\ 0.20x_1 + 0.80x_2 + 0.15x_3 &= x_2 \\ 0.10x_1 + 0.05x_2 + 0.70x_3 &= x_3. \end{aligned}$$

Use these equations and the fact that $x_1 + x_2 + x_3 = 1$ to write the system of linear equations below.

$$\begin{aligned} -0.30x_1 + 0.15x_2 + 0.15x_3 &= 0 \\ 0.20x_1 - 0.20x_2 + 0.15x_3 &= 0 \\ 0.10x_1 + 0.05x_2 - 0.30x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

Use any appropriate method to verify that the solution of this system is

$$x_1 = \frac{1}{3}, \quad x_2 = \frac{10}{21}, \quad \text{and} \quad x_3 = \frac{4}{21}.$$

So the steady state matrix is

$$\bar{X} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{21} \\ \frac{4}{21} \end{bmatrix} \approx \begin{bmatrix} 0.3333 \\ 0.4762 \\ 0.1905 \end{bmatrix}.$$

Check that $P\bar{X} = \bar{X}$.

**REMARK**

Recall from Example 2 that the state matrix consists of entries that are portions of the whole. So it should make sense that

$$x_1 + x_2 + x_3 = 1.$$

REMARK

If P is *not* regular, then the corresponding Markov chain may or may not have a unique steady state matrix.

A summary for finding the steady state matrix $\bar{X}$ of a Markov chain is below.

Finding the Steady State Matrix of a Markov Chain

1. Check to see that the matrix of transition probabilities P is a regular matrix.
2. Solve the system of linear equations obtained from the matrix equation $P\bar{X} = \bar{X}$ along with the equation $x_1 + x_2 + \cdots + x_n = 1$.
3. Check the solution found in Step 2 in the matrix equation $P\bar{X} = \bar{X}$.

REMARK

In Exercise 50, you will investigate another type of Markov chain, one with *reflecting boundaries*.

ABSORBING MARKOV CHAINS

The Markov chain discussed in Examples 3 and 5 is *regular*. Other types of Markov chains can be used to model real-life situations. One of these includes *absorbing* Markov chains.

Consider a Markov chain with n different states $\{S_1, S_2, \dots, S_n\}$. The i th state S_i is an **absorbing state** when, in the matrix of transition probabilities P , $p_{ii} = 1$. That is, the entry on the main diagonal of P is 1 and all other entries in the i th column of P are 0. An **absorbing Markov chain** has the two properties listed below.

1. The Markov chain has at least one absorbing state.
2. It is possible for a member of the population to move from any nonabsorbing state to an absorbing state in a finite number of transitions.

EXAMPLE 6**Absorbing and Nonabsorbing Markov Chains**

- a. For the matrix

$$P = \begin{array}{c} \text{From} \\ \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix} \begin{array}{l} S_1 \\ S_2 \\ S_3 \end{array} \end{array} \left. \vphantom{\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix}} \right\} \text{To}$$

the second state, represented by the second column, is absorbing. Moreover, the corresponding Markov chain is also absorbing because it is possible to move from S_1 to S_2 in two transitions, and it is possible to move from S_3 to S_2 in one transition. (See Figure 2.2.)

- b. For the matrix

$$P = \begin{array}{c} \text{From} \\ \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix} \begin{array}{l} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \end{array} \left. \vphantom{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix}} \right\} \text{To}$$

the second state is absorbing. However, the corresponding Markov chain is *not* absorbing because there is no way to move from state S_3 or state S_4 to state S_2 . (See Figure 2.3.)

- c. The matrix

$$P = \begin{array}{c} \text{From} \\ \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix} \begin{array}{l} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \end{array} \left. \vphantom{\begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix}} \right\} \text{To}$$

has two absorbing states: S_2 and S_4 . Moreover, the corresponding Markov chain is also absorbing because it is possible to move from either of the nonabsorbing states, S_1 or S_3 , to either of the absorbing states in one step. (See Figure 2.4.)

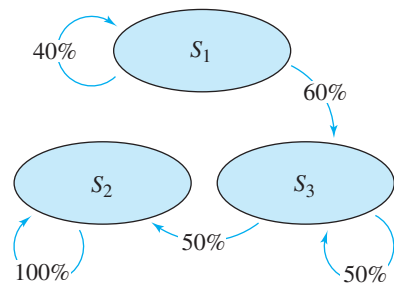


Figure 2.2

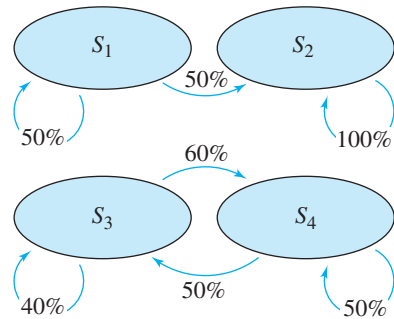


Figure 2.3

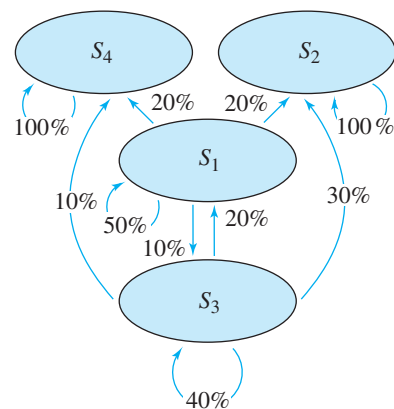


Figure 2.4

It is possible for some absorbing Markov chains to have a unique steady state matrix. Other absorbing Markov chains have an infinite number of steady state matrices. Example 7 demonstrates.

EXAMPLE 7**Finding Steady State Matrices of Absorbing Markov Chains**

Find the steady state matrix $\bar{X}$ of each absorbing Markov chain with matrix of transition probabilities P .

$$\text{a. } P = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix} \quad \text{b. } P = \begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix}$$

SOLUTION

a. Use the matrix equation $P\bar{X} = \bar{X}$, or

$$\begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0.6 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 = 1$ to write the system of linear equations

$$\begin{aligned} -0.6x_1 &= 0 \\ 0.5x_3 &= 0 \\ 0.6x_1 - 0.5x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1. \end{aligned}$$

The solution of this system is $x_1 = 0$, $x_2 = 1$, and $x_3 = 0$, so the steady state matrix is $\bar{X} = [0 \ 1 \ 0]^T$. Note that $\bar{X}$ coincides with the second column of the matrix of transition probabilities P .

b. Use the matrix equation $P\bar{X} = \bar{X}$, or

$$\begin{bmatrix} 0.5 & 0 & 0.2 & 0 \\ 0.2 & 1 & 0.3 & 0 \\ 0.1 & 0 & 0.4 & 0 \\ 0.2 & 0 & 0.1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

along with the equation $x_1 + x_2 + x_3 + x_4 = 1$ to write the system of linear equations

$$\begin{aligned} -0.5x_1 + 0.2x_3 &= 0 \\ 0.2x_1 + 0.3x_3 &= 0 \\ 0.1x_1 - 0.6x_3 &= 0 \\ 0.2x_1 + 0.1x_3 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 1. \end{aligned}$$

The solution of this system is $x_1 = 0$, $x_2 = 1 - t$, $x_3 = 0$, and $x_4 = t$, where t is any real number such that $0 \leq t \leq 1$. So, the steady state matrix is $\bar{X} = [0 \ 1 - t \ 0 \ t]^T$. The Markov chain has an infinite number of steady state matrices. 

REMARK

Note that the steady state matrix for an absorbing Markov chain has nonzero values only in the absorbing state(s). These states *absorb* the population.

In general, a regular Markov chain or an absorbing Markov chain with one absorbing state has a unique steady state matrix regardless of the initial state matrix. Further, an absorbing Markov chain with two or more absorbing states has an infinite number of steady state matrices, which depend on the initial state matrix. In Exercise 49, you are asked to show this dependence for the Markov chain whose matrix of transition probabilities is given in Example 7(b).

2.5 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Stochastic Matrices In Exercises 1–6, determine whether the matrix is stochastic.

1. $\begin{bmatrix} \frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{7}{5} \end{bmatrix}$

2. $\begin{bmatrix} 1 + \sqrt{2} & 1 - \sqrt{2} \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$

3. $\begin{bmatrix} 0.\overline{3} & 0.1\overline{6} & 0.25 \\ 0.\overline{3} & 0.\overline{6} & 0.25 \\ 0.\overline{3} & 0.1\overline{6} & 0.5 \end{bmatrix}$

4. $\begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.1 & 0.2 & 0.7 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} \frac{1}{2} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{4}{15} \\ \frac{1}{6} & \frac{2}{9} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

7. **Airplane Allocation** An airline has 30 airplanes in Los Angeles, 12 airplanes in St. Louis, and 8 airplanes in Dallas. During an eight-hour period, 20% of the planes in Los Angeles fly to St. Louis and 10% fly to Dallas. Of the planes in St. Louis, 25% fly to Los Angeles and 50% fly to Dallas. Of the planes in Dallas, 12.5% fly to Los Angeles and 50% fly to St. Louis. How many planes are in each city after 8 hours?

8. **Chemistry** In a chemistry experiment, a test tube contains 10,000 molecules of a compound. Initially, 20% of the molecules are in a gas state, 60% are in a liquid state, and 20% are in a solid state. After introducing a catalyst, 40% of the gas molecules change to liquid, 30% of the liquid molecules change to solid, and 50% of the solid molecules change to liquid. How many molecules are in each state after introducing the catalyst?

Finding State Matrices In Exercises 9 and 10, use the matrix of transition probabilities P and initial state matrix X_0 to find the state matrices X_1 , X_2 , and X_3 .

9. $P = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.8 \end{bmatrix}$

10. $P = \begin{bmatrix} 0.6 & 0.2 & 0 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.1 & 0.9 \end{bmatrix}, \quad X_0 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

11. **Purchase of a Product** The market research department at a manufacturing plant determines that 20% of the people who purchase the plant's product during any month will not purchase it the next month. On the other hand, 30% of the people who do not purchase the product during any month will purchase it the next month. In a population of 1000 people, 100 people purchased the product this month. How many will purchase the product (a) next month and (b) in 2 months?

12. **Spread of a Virus** A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% probability that an infected mouse will overcome the virus, and during the same week there is a 10% probability that a noninfected mouse will become infected. Three hundred mice are currently infected with the virus. How many will be infected (a) next week and (b) in 3 weeks?

13. **Television Watching** A college dormitory houses 200 students. Those who watch an hour or more of television on any day always watch for less than an hour the next day. One-fourth of those who watch television for less than an hour one day will watch an hour or more the next day. Half of the students watched television for an hour or more today. How many will watch television for an hour or more (a) tomorrow, (b) in 2 days, and (c) in 30 days?

14. **Sports Activities** Students in a gym class have a choice of swimming or playing basketball each day. Thirty percent of the students who swim one day will swim the next day. Sixty percent of the students who play basketball one day will play basketball the next day. Today, 100 students swam and 150 students played basketball. How many students will swim (a) tomorrow, (b) in two days, and (c) in four days?

15. **Smokers and Nonsmokers** In a population of 10,000, there are 5000 nonsmokers, 2500 smokers of one pack or less per day, and 2500 smokers of more than one pack per day. During any month, there is a 5% probability that a nonsmoker will begin smoking a pack or less per day, and a 2% probability that a nonsmoker will begin smoking more than a pack per day. For smokers who smoke a pack or less per day, there is a 10% probability of quitting and a 10% probability of increasing to more than a pack per day. For smokers who smoke more than a pack per day, there is a 5% probability of quitting and a 10% probability of dropping to a pack or less per day. How many people will be in each group (a) in 1 month, (b) in 2 months, and (c) in 1 year?

- 16. Consumer Preference** In a population of 100,000 consumers, there are 20,000 users of Brand A, 30,000 users of Brand B, and 50,000 who use neither brand. During any month, a Brand A user has a 20% probability of switching to Brand B and a 5% probability of not using either brand. A Brand B user has a 15% probability of switching to Brand A and a 10% probability of not using either brand. A nonuser has a 10% probability of purchasing Brand A and a 15% probability of purchasing Brand B. How many people will be in each group (a) in 1 month, (b) in 2 months, and (c) in 18 months?

Regular and Steady State Matrices In Exercises 17–30, determine whether the stochastic matrix P is regular. Then find the steady state matrix $\bar{X}$ of the Markov chain with matrix of transition probabilities P .

$$17. P = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}$$

$$18. P = \begin{bmatrix} 0 & 0.3 \\ 1 & 0.7 \end{bmatrix}$$

$$19. P = \begin{bmatrix} 1 & 0.75 \\ 0 & 0.25 \end{bmatrix}$$

$$20. P = \begin{bmatrix} 0.2 & 0 \\ 0.8 & 1 \end{bmatrix}$$

$$21. P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

$$22. P = \begin{bmatrix} \frac{2}{5} & \frac{7}{10} \\ \frac{3}{5} & \frac{3}{10} \end{bmatrix}$$

$$23. P = \begin{bmatrix} \frac{2}{5} & \frac{3}{10} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{2}{5} & \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

$$24. P = \begin{bmatrix} \frac{2}{9} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{4}{9} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

$$25. P = \begin{bmatrix} 1 & 0 & 0.15 \\ 0 & 1 & 0.10 \\ 0 & 0 & 0.75 \end{bmatrix}$$

$$26. P = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix}$$

$$27. P = \begin{bmatrix} 0.22 & 0.20 & 0.65 \\ 0.62 & 0.60 & 0.15 \\ 0.16 & 0.20 & 0.20 \end{bmatrix}$$

$$28. P = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$$

$$29. P = \begin{bmatrix} \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \end{bmatrix}$$

$$30. P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- 31.** (a) Find the steady state matrix $\bar{X}$ using the matrix of transition probabilities P in Exercise 9.
(b) Find the steady state matrix $\bar{X}$ using the matrix of transition probabilities P in Exercise 10.
- 32.** Find the steady state matrix for each stochastic matrix in Exercises 1–6.
- 33. Fundraising** A nonprofit organization collects contributions from members of a community. During any year, 40% of those who make contributions will not contribute the next year. On the other hand, 10% of those who do not make contributions will contribute the next year. Find and interpret the steady state matrix for this situation.
- 34. Grade Distribution** In a college class, 70% of the students who receive an “A” on one assignment will receive an “A” on the next assignment. On the other hand, 10% of the students who do not receive an “A” on one assignment will receive an “A” on the next assignment. Find and interpret the steady state matrix for this situation.
- 35. Stock Sales and Purchases** Eight hundred fifty stockholders invest in one of three stocks. During any month, 25% of Stock A holders move their investment to Stock B and 10% to Stock C. Of Stock B holders, 10% move their investment to Stock A. Of Stock C holders, 15% move their investment to Stock A and 5% to Stock B. Find and interpret the steady state matrix for this situation.
- 36. Customer Preference** Two movie theatres that show several different movies each night compete for the same audience. Of the people who attend Theatre A one night, 10% will attend again the next night and 5% will attend Theatre B the next night. Of the people who attend Theatre B one night, 8% will attend again the next night and 6% will attend Theatre A the next night. Of the people who attend neither theatre one night, 3% will attend Theatre A the next night and 4% will attend Theatre B the next night. Find and interpret the steady state matrix for this situation.

Absorbing Markov Chains In Exercises 37–40, determine whether the Markov chain with matrix of transition probabilities P is absorbing. Explain.

$$37. P = \begin{bmatrix} 0.8 & 0.3 & 0 \\ 0.2 & 0.1 & 0 \\ 0 & 0.6 & 1 \end{bmatrix}$$

$$38. P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3 & 0.9 \\ 0 & 0.7 & 0.1 \end{bmatrix}$$

$$39. P = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 & 0 \\ \frac{1}{5} & \frac{3}{5} & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$40. P = \begin{bmatrix} 0.3 & 0.7 & 0.2 & 0 \\ 0.2 & 0.1 & 0.1 & 0 \\ 0.1 & 0.1 & 0.1 & 0 \\ 0.4 & 0.1 & 0.6 & 1 \end{bmatrix}$$

Finding a Steady State Matrix In Exercises 41–44, find the steady state matrix $\bar{X}$ of the absorbing Markov chain with matrix of transition probabilities P .

$$41. P = \begin{bmatrix} 0.6 & 0 & 0.3 \\ 0.2 & 1 & 0.6 \\ 0.2 & 0 & 0.1 \end{bmatrix} \quad 42. P = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.7 & 0 & 1 \end{bmatrix}$$

$$43. P = \begin{bmatrix} 1 & 0.2 & 0.1 & 0.3 \\ 0 & 0.3 & 0.6 & 0.3 \\ 0 & 0.1 & 0.2 & 0.2 \\ 0 & 0.4 & 0.1 & 0.2 \end{bmatrix}$$

$$44. P = \begin{bmatrix} 0.7 & 0 & 0.2 & 0.1 \\ 0.1 & 1 & 0.5 & 0.6 \\ 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 \end{bmatrix}$$

45. **Epidemic Model** In a population of 200,000 people, 40,000 are infected with a virus. After a person becomes infected and then recovers, the person is immune (cannot become infected again). Of the people who are infected, 5% will die each year and the others will recover. Of the people who have never been infected, 25% will become infected each year. How many people will be infected in 4 years?

46. **Chess Tournament** Two people are engaged in a chess tournament. Each starts with two playing chips. After each game, the loser must give the winner one chip. Player 2 is more advanced than Player 1 and has a 70% chance of winning each game. The tournament is over when one player obtains all four chips. What is the probability that Player 1 will win the tournament?

47. Explain how you can determine the steady state matrix $\bar{X}$ of an absorbing Markov chain by inspection.

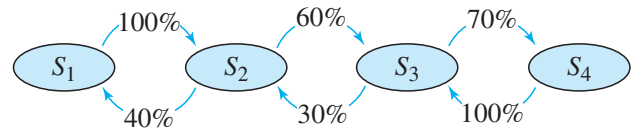
48. CAPSTONE

- Explain how to find the n th state matrix of a Markov chain.
- Explain how to find the steady state matrix of a Markov chain.
- What is a regular Markov chain?
- What is an absorbing Markov chain?
- How is an absorbing Markov chain different than a regular Markov chain?

49. Consider the Markov chain whose matrix of transition probabilities P is given in Example 7(b). Show that the steady state matrix $\bar{X}$ depends on the initial state matrix X_0 by finding $\bar{X}$ for each X_0 .

$$(a) X_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} \quad (b) X_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.40 \\ 0.10 \end{bmatrix}$$

50. **Markov Chain with Reflecting Boundaries** The figure below illustrates an example of a Markov chain with reflecting boundaries.



- Explain why it is appropriate to say that this type of Markov chain has *reflecting boundaries*.
- Use the figure to write the matrix of transition probabilities P for the Markov chain.
- Find P^{30} and P^{31} . Find several other high even powers $2n$ and odd powers $2n + 1$ of P . What do you observe?
- Find the steady state matrix $\bar{X}$ of the Markov chain. How are the entries in the columns of P^{2n} and P^{2n+1} related to the entries in $\bar{X}$?

Nonabsorbing Markov Chain In Exercises 51 and 52, consider the matrix P in Example 6(b).

- Is it possible to find a steady state matrix $\bar{X}$ for the corresponding Markov chain? If so, find a steady state matrix. If not, explain why.
- Create a new matrix P' by changing the second column of P to $[0.6 \ 0.4 \ 0 \ 0]^T$, resulting in a second state that is no longer absorbing. Determine whether each matrix X below can be a steady state matrix for the Markov chain corresponding to P' . Explain.

$$(a) X = \begin{bmatrix} \frac{6}{11} \\ \frac{5}{11} \\ 0 \\ 0 \end{bmatrix} \quad (b) X = \begin{bmatrix} 0 \\ 0 \\ \frac{5}{11} \\ \frac{6}{11} \end{bmatrix}$$

- Proof** Prove that the product of two 2×2 stochastic matrices is stochastic.
- Proof** Let P be a 2×2 stochastic matrix. Prove that there exists a 2×1 state matrix X with nonnegative entries such that $PX = X$.
- In Example 5, show that for the regular stochastic matrix P , the sequence of successive powers $P, P^2, P^3, \dots$

approaches a stable matrix $\bar{P}$, where the entries in each column of $\bar{P}$ are equal to the corresponding entries in the steady state matrix $\bar{X}$. Repeat for several other regular stochastic matrices P and corresponding steady state matrices $\bar{X}$.

- Proof** Prove that when P is a regular stochastic matrix, the corresponding regular Markov chain $PX_0, P^2X_0, P^3X_0, \dots$ approaches a unique steady state matrix $\bar{X}$.

2.6 More Applications of Matrix Operations

-  Use matrix multiplication to encode and decode messages.
-  Use matrix algebra to analyze an economic system (Leontief input-output model).
-  Find the least squares regression line for a set of data.

CRYPTOGRAPHY

A **cryptogram** is a message written according to a secret code (the Greek word *kryptos* means “hidden”). One method of using matrix multiplication to **encode** and **decode** messages is introduced below.

To begin, assign a number to each letter in the alphabet (with 0 assigned to a blank space), as shown.

0 = _	9 = I	18 = R
1 = A	10 = J	19 = S
2 = B	11 = K	20 = T
3 = C	12 = L	21 = U
4 = D	13 = M	22 = V
5 = E	14 = N	23 = W
6 = F	15 = O	24 = X
7 = G	16 = P	25 = Y
8 = H	17 = Q	26 = Z

Then convert the message to numbers and partition it into **uncoded row matrices**, each having n entries, as demonstrated in Example 1.

EXAMPLE 1

Forming Uncoded Row Matrices

Write the uncoded row matrices of size 1×3 for the message MEET ME MONDAY.

SOLUTION

Partitioning the message (including blank spaces, but ignoring punctuation) into groups of three produces the uncoded row matrices shown below.

$$\begin{array}{ccccccc} [13 & 5 & 5] & [20 & 0 & 13] & [5 & 0 & 13] & [15 & 14 & 4] & [1 & 25 & 0] \\ M & E & E & T & _ & M & E & _ & M & O & N & D & A & Y & _ \end{array}$$

Note the use of a blank space to fill out the last uncoded row matrix.



LINEAR ALGEBRA APPLIED

Information security is of the utmost importance when conducting business online. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, Social Security numbers, bank account details, or sensitive company information, then the effects can be damaging. To protect the confidentiality and integrity of such information, Internet security can include the use of data *encryption*, the process of encoding information so that the only way to decode it, apart from an “exhaustion attack,” is to use a *key*. Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.

Andrea Danti/Shutterstock.com

To **encode** a message, choose an $n \times n$ invertible matrix A and multiply the uncoded row matrices (on the right) by A to obtain **coded row matrices**. Example 2 demonstrates this process.

EXAMPLE 2

Encoding a Message

Use the invertible matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to encode the message MEET ME MONDAY.

SOLUTION

Obtain the coded row matrices by multiplying each of the uncoded row matrices found in Example 1 by the matrix A , as shown below.

Uncoded Row Matrix	Encoding Matrix A	Coded Row Matrix
$[13 \quad 5 \quad 5]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [13 \quad -26 \quad 21]$
$[20 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [33 \quad -53 \quad -12]$
$[5 \quad 0 \quad 13]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [18 \quad -23 \quad -42]$
$[15 \quad 14 \quad 4]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [5 \quad -20 \quad 56]$
$[1 \quad 25 \quad 0]$	$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$	$= [-24 \quad 23 \quad 77]$

The sequence of coded row matrices is

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

Finally, removing the matrix notation produces the cryptogram

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77.$$

For those who do not know the encoding matrix A , decoding the cryptogram found in Example 2 is difficult. But for an authorized receiver who knows the encoding matrix A , decoding is relatively simple. The receiver just needs to multiply the coded row matrices by A^{-1} to retrieve the uncoded row matrices. In other words, if

$$X = [x_1 \quad x_2 \quad \dots \quad x_n]$$

is an uncoded $1 \times n$ matrix, then $Y = XA$ is the corresponding encoded matrix. The receiver of the encoded matrix can decode Y by multiplying on the right by A^{-1} to obtain

$$YA^{-1} = (XA)A^{-1} = X.$$

Example 3 demonstrates this procedure.

EXAMPLE 3**Decoding a Message**

Use the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

to decode the cryptogram

$$13 \quad -26 \quad 21 \quad 33 \quad -53 \quad -12 \quad 18 \quad -23 \quad -42 \quad 5 \quad -20 \quad 56 \quad -24 \quad 23 \quad 77.$$

SOLUTION

Begin by using Gauss-Jordan elimination to find A^{-1} .

$$\begin{array}{c} [A \quad I] \\ \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & 3 & 0 & 1 & 0 \\ 1 & -1 & -4 & 0 & 0 & 1 \end{array} \right] \end{array} \rightarrow \begin{array}{c} [I \quad A^{-1}] \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -10 & -8 \\ 0 & 1 & 0 & -1 & -6 & -5 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array} \right] \end{array}$$

Now, to decode the message, partition the message into groups of three to form the coded row matrices

$$[13 \quad -26 \quad 21][33 \quad -53 \quad -12][18 \quad -23 \quad -42][5 \quad -20 \quad 56][-24 \quad 23 \quad 77].$$

To obtain the decoded row matrices, multiply each coded row matrix by A^{-1} (on the right).

Coded Row Matrix	Decoding Matrix A^{-1}	Decoded Row Matrix
$[13 \quad -26 \quad 21]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$= [13 \quad 5 \quad 5]$
$[33 \quad -53 \quad -12]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$= [20 \quad 0 \quad 13]$
$[18 \quad -23 \quad -42]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$= [5 \quad 0 \quad 13]$
$[5 \quad -20 \quad 56]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$= [15 \quad 14 \quad 4]$
$[-24 \quad 23 \quad 77]$	$\begin{bmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{bmatrix}$	$= [1 \quad 25 \quad 0]$

The sequence of decoded row matrices is

$$[13 \quad 5 \quad 5][20 \quad 0 \quad 13][5 \quad 0 \quad 13][15 \quad 14 \quad 4][1 \quad 25 \quad 0]$$

and the message is

$$\begin{array}{cccccccccccccccc} 13 & 5 & 5 & 20 & 0 & 13 & 5 & 0 & 13 & 15 & 14 & 4 & 1 & 25 & 0. \\ M & E & E & T & _ & M & E & _ & M & O & N & D & A & Y & _ \end{array}$$



LEONTIEF INPUT-OUTPUT MODELS

In 1936, American economist Wassily W. Leontief (1906–1999) published a model concerning the input and output of an economic system. In 1973, Leontief received a Nobel prize for his work in economics. A brief discussion of Leontief's model follows.

Consider an economic system that has n different industries $I_1, I_2, \dots, I_n$, each having **input** needs (raw materials, utilities, etc.) and an **output** (finished product). In producing each unit of output, an industry may use the outputs of other industries, including itself. For example, an electric utility uses outputs from other industries, such as coal and water, and also uses its own electricity.

Let d_{ij} be the amount of output the j th industry needs from the i th industry to produce one unit of output per year. The matrix of these coefficients is the **input-output matrix**.

$$D = \begin{array}{c} \text{User (Output)} \\ \begin{array}{cccc} I_1 & I_2 & \cdots & I_n \end{array} \\ \left[\begin{array}{cccc} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{array} \right] \begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array} \end{array} \left. \vphantom{\begin{array}{c} \text{User (Output)} \\ \begin{array}{cccc} I_1 & I_2 & \cdots & I_n \end{array} \\ \left[\begin{array}{cccc} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{array} \right] \begin{array}{c} I_1 \\ I_2 \\ \vdots \\ I_n \end{array} \right\} \text{Supplier (Input)}$$

To understand how to use this matrix, consider $d_{12} = 0.4$. This means that for Industry 2 to produce one unit of its product, it must use 0.4 unit of Industry 1's product. If $d_{33} = 0.2$, then Industry 3 needs 0.2 unit of its own product to produce one unit. For this model to work, the values of d_{ij} must satisfy $0 \leq d_{ij} \leq 1$ and the sum of the entries in any column must be less than or equal to 1.

EXAMPLE 4

Forming an Input-Output Matrix

Consider a simple economic system consisting of three industries: electricity, water, and coal. Production, or output, of one unit of electricity requires 0.5 unit of itself, 0.25 unit of water, and 0.25 unit of coal. Production of one unit of water requires 0.1 unit of electricity, 0.6 unit of itself, and 0 units of coal. Production of one unit of coal requires 0.2 unit of electricity, 0.15 unit of water, and 0.5 unit of itself. Find the input-output matrix for this system.

SOLUTION

The column entries show the amounts each industry requires from the others, and from itself, to produce one unit of output.

$$\begin{array}{c} \text{User (Output)} \\ \begin{array}{ccc} \text{E} & \text{W} & \text{C} \end{array} \\ \left[\begin{array}{ccc} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{array} \right] \begin{array}{c} \text{E} \\ \text{W} \\ \text{C} \end{array} \end{array} \left. \vphantom{\begin{array}{c} \text{User (Output)} \\ \begin{array}{ccc} \text{E} & \text{W} & \text{C} \end{array} \\ \left[\begin{array}{ccc} 0.5 & 0.1 & 0.2 \\ 0.25 & 0.6 & 0.15 \\ 0.25 & 0 & 0.5 \end{array} \right] \begin{array}{c} \text{E} \\ \text{W} \\ \text{C} \end{array} \right\} \text{Supplier (Input)}$$

The row entries show the amounts each industry supplies to the others, and to itself, for that industry to produce one unit of output. For instance, the electricity industry supplies 0.5 unit to itself, 0.1 unit to water, and 0.2 unit to coal.

To develop the Leontief input-output model further, let the total output of the i th industry be denoted by x_i . If the economic system is **closed** (that is, the economic system sells its products only to industries within the system, as in the example above), then the total output of the i th industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n. \quad \text{Closed system}$$

On the other hand, if the industries within the system sell products to nonproducing groups (such as governments or charitable organizations) outside the system, then the system is **open** and the total output of the i th industry is

$$x_i = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n + e_i \quad \text{Open system}$$

where e_i represents the external demand for the i th industry's product. The system of n linear equations below represents the collection of total outputs for an open system.

$$\begin{aligned} x_1 &= d_{11}x_1 + d_{12}x_2 + \cdots + d_{1n}x_n + e_1 \\ x_2 &= d_{21}x_1 + d_{22}x_2 + \cdots + d_{2n}x_n + e_2 \\ &\vdots \\ x_n &= d_{n1}x_1 + d_{n2}x_2 + \cdots + d_{nn}x_n + e_n \end{aligned}$$

The matrix form of this system is $X = DX + E$, where X is the **output matrix** and E is the **external demand matrix**.

EXAMPLE 5

Solving for the Output Matrix of an Open System

See LarsonLinearAlgebra.com for an interactive version of this type of example.

An economic system composed of three industries has the input-output matrix shown below.

$$D = \begin{array}{c} \text{User (Output)} \\ \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} 0.1 & 0.43 & 0 \\ 0.15 & 0 & 0.37 \\ 0.23 & 0.03 & 0.02 \end{bmatrix} \end{array} \left. \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \right\} \text{Supplier (Input)}$$

Find the output matrix X when the external demands are

$$E = \begin{bmatrix} 20,000 \\ 30,000 \\ 25,000 \end{bmatrix} \left. \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \right\}$$

SOLUTION

Letting I be the identity matrix, write the equation $X = DX + E$ as $IX - DX = E$, which means that $(I - D)X = E$. Using the matrix D above produces

$$I - D = \begin{bmatrix} 0.9 & -0.43 & 0 \\ -0.15 & 1 & -0.37 \\ -0.23 & -0.03 & 0.98 \end{bmatrix}.$$

Using Gauss-Jordan elimination,

$$(I - D)^{-1} \approx \begin{bmatrix} 1.25 & 0.55 & 0.21 \\ 0.30 & 1.14 & 0.43 \\ 0.30 & 0.16 & 1.08 \end{bmatrix}.$$

So, the output matrix is

$$X = (I - D)^{-1}E \approx \begin{bmatrix} 1.25 & 0.55 & 0.21 \\ 0.30 & 1.14 & 0.43 \\ 0.30 & 0.16 & 1.08 \end{bmatrix} \begin{bmatrix} 20,000 \\ 30,000 \\ 25,000 \end{bmatrix} = \begin{bmatrix} 46,750 \\ 50,950 \\ 37,800 \end{bmatrix} \left. \begin{array}{c} \text{A} \\ \text{B} \\ \text{C} \end{array} \right\}$$

To produce the given external demands, the outputs of the three industries must be approximately 46,750 units for industry A, 50,950 units for industry B, and 37,800 units for industry C.

REMARK

The economic systems described in Examples 4 and 5 are, of course, simple ones. In the real world, an economic system would include many industries or industrial groups. A detailed analysis using the Leontief input-output model could easily require an input-output matrix greater than 100×100 in size. Clearly, this type of analysis would require the aid of a computer.

LEAST SQUARES REGRESSION ANALYSIS

You will now look at a procedure used in statistics to develop linear models. The next example demonstrates a visual method for approximating a line of best fit for a set of data points.

EXAMPLE 6

A Visual Straight-Line Approximation

Determine a line that appears to best fit the points (1, 1), (2, 2), (3, 4), (4, 4), and (5, 6).

SOLUTION

Plot the points, as shown in Figure 2.5. It appears that a good choice would be the line whose slope is 1 and whose y-intercept is 0.5. The equation of this line is

$$y = 0.5 + x.$$

An examination of the line in Figure 2.5 reveals that you can improve the fit by rotating the line counterclockwise slightly, as shown in Figure 2.6. It seems clear that this line, whose equation is $y = 1.2x$, fits the points better than the original line.

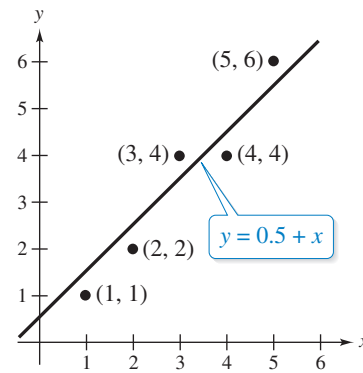
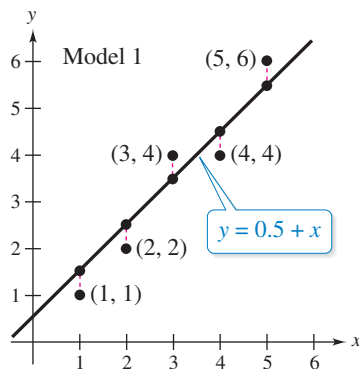


Figure 2.5

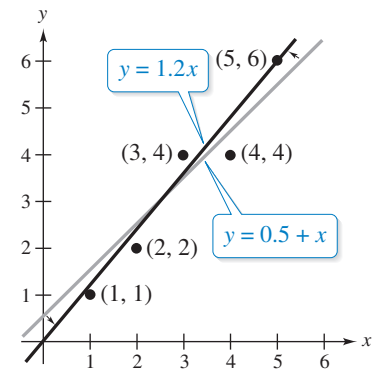


Figure 2.6

One way of measuring how well a function $y = f(x)$ fits a set of points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

is to compute the differences between the values from the function $f(x_i)$ and the actual values y_i . These values are shown in Figure 2.7. By squaring the differences and summing the results, you obtain a measure of error called the **sum of squared error**. The table shows the sums of squared errors for the two linear models.

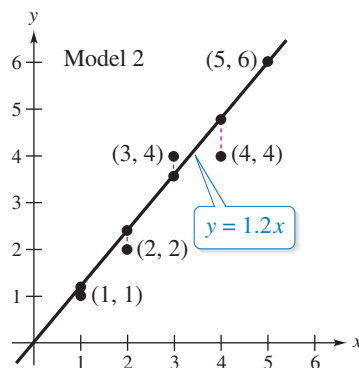


Figure 2.7

Model 1: $f(x) = 0.5 + x$				Model 2: $f(x) = 1.2x$			
x_i	y_i	$f(x_i)$	$[y_i - f(x_i)]^2$	x_i	y_i	$f(x_i)$	$[y_i - f(x_i)]^2$
1	1	1.5	$(-0.5)^2$	1	1	1.2	$(-0.2)^2$
2	2	2.5	$(-0.5)^2$	2	2	2.4	$(-0.4)^2$
3	4	3.5	$(+0.5)^2$	3	4	3.6	$(+0.4)^2$
4	4	4.5	$(-0.5)^2$	4	4	4.8	$(-0.8)^2$
5	6	5.5	$(+0.5)^2$	5	6	6.0	$(0.0)^2$
Sum			1.25	Sum			1.00

The sums of squared errors confirm that the second model fits the points better than the first model.

Of all possible linear models for a given set of points, the model that has the best fit is the one that minimizes the sum of squared error. This model is the **least squares regression line**, and the procedure for finding it is the **method of least squares**.

Definition of Least Squares Regression Line

For a set of points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

the **least squares regression line** is the linear function

$$f(x) = a_0 + a_1x$$

that minimizes the sum of squared error

$$[y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \dots + [y_n - f(x_n)]^2.$$

To find the least squares regression line for a set of points, begin by forming the system of linear equations

$$\begin{aligned} y_1 &= f(x_1) + [y_1 - f(x_1)] \\ y_2 &= f(x_2) + [y_2 - f(x_2)] \\ &\vdots \\ y_n &= f(x_n) + [y_n - f(x_n)] \end{aligned}$$

where the right-hand term

$$[y_i - f(x_i)]$$

of each equation is the error in the approximation of y_i by $f(x_i)$. Then write this error as

$$e_i = y_i - f(x_i)$$

and write the system of equations in the form

$$\begin{aligned} y_1 &= (a_0 + a_1x_1) + e_1 \\ y_2 &= (a_0 + a_1x_2) + e_2 \\ &\vdots \\ y_n &= (a_0 + a_1x_n) + e_n. \end{aligned}$$

Now, if you define Y , X , A , and E as

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

then the n linear equations may be replaced by the matrix equation

$$Y = XA + E.$$

Note that the matrix X has a column of 1's (corresponding to a_0) and a column containing the x_i 's. This matrix equation can be used to determine the coefficients of the least squares regression line, as shown on the next page.

REMARK

You will learn more about this procedure in Section 5.4.

Matrix Form for Linear Regression

For the regression model $Y = XA + E$, the coefficients of the least squares regression line are given by the matrix equation

$$A = (X^T X)^{-1} X^T Y$$

and the sum of squared error is $E^T E$.

Example 7 demonstrates the use of this procedure to find the least squares regression line for the set of points from Example 6.

EXAMPLE 7**Finding the Least Squares Regression Line**

Find the least squares regression line for the points $(1, 1)$, $(2, 2)$, $(3, 4)$, $(4, 4)$, and $(5, 6)$.

SOLUTION

The matrices X and Y are

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix}.$$

This means that

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$$

and

$$X^T Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 17 \\ 63 \end{bmatrix}.$$

Now, using $(X^T X)^{-1}$ to find the coefficient matrix A , you have

$$\begin{aligned} A &= (X^T X)^{-1} X^T Y \\ &= \frac{1}{50} \begin{bmatrix} 55 & -15 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} 17 \\ 63 \end{bmatrix} \\ &= \begin{bmatrix} -0.2 \\ 1.2 \end{bmatrix}. \end{aligned}$$

So, the least squares regression line is

$$y = -0.2 + 1.2x$$

as shown in Figure 2.8. The sum of squared error for this line is 0.8 (verify this), which means that this line fits the data better than either of the two experimental linear models determined earlier.

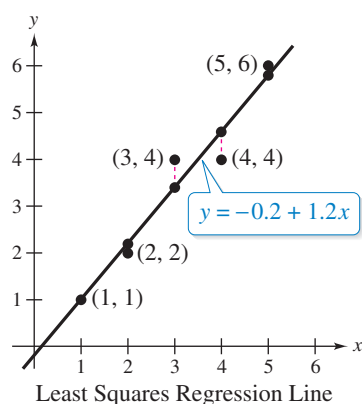


Figure 2.8

2.6 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Encoding a Message In Exercises 1 and 2, write the uncoded row matrices for the message. Then encode the message using the matrix A .

1. **Message:** SELL CONSOLIDATED

Row Matrix Size: 1×3

Encoding Matrix: $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

2. **Message:** HELP IS COMING

Row Matrix Size: 1×4

Encoding Matrix: $A = \begin{bmatrix} -2 & 3 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 2 \\ 3 & 1 & -2 & -4 \end{bmatrix}$

Decoding a Message In Exercises 3–6, use A^{-1} to decode the cryptogram.

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$,

11 21 64 112 25 50 29 53 23 46 40 75 55 92

4. $A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$,

85 120 6 8 10 15 84 117 42 56 90 125 60 80
30 45 19 26

5. $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$,

13 19 10 -1 -33 -77 3 -2 -14 4 1 -9 -5
-25 -47 4 1 -9

6. $A = \begin{bmatrix} 3 & -4 & 2 \\ 0 & 2 & 1 \\ 4 & -5 & 3 \end{bmatrix}$,

112 -140 83 19 -25 13 72 -76 61 95 -118
71 20 21 38 35 -23 36 42 -48 32

7. **Decoding a Message** The cryptogram below was encoded with a 2×2 matrix. The last word of the message is __RON. What is the message?

8 21 -15 -10 -13 -13 5 10 5 25 5 19 -1 6
20 40 -18 -18 1 16

8. **Decoding a Message** The cryptogram below was encoded with a 2×2 matrix. The last word of the message is __SUE. What is the message?

5 2 25 11 -2 -7 -15 -15 32 14 -8 -13 38
19 -19 -19 37 16

9. **Decoding a Message** Use a software program or a graphing utility to decode the cryptogram.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

38 -14 29 56 -15 62 17 3 38 18 20 76 18 -5
21 29 -7 32 32 9 77 36 -8 48 33 -5 51 41
3 79 12 1 26 58 -22 49 63 -19 69 28 8 67 31
-11 27 41 -18 28

10. **Decoding a Message** A code breaker intercepted the encoded message below.

45 -35 38 -30 18 -18 35 -30 81 -60 42
-28 75 -55 2 -2 22 -21 15 -10

Let the inverse of the encoding matrix be

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}.$$

(a) You know that $[45 \ -35]A^{-1} = [10 \ 15]$ and $[38 \ -30]A^{-1} = [8 \ 14]$. Write and solve two systems of equations to find w , x , y , and z .

(b) Decode the message.

11. **Industrial System** A system composed of two industries, coal and steel, has the input requirements below.

(a) To produce \$1.00 worth of output, the coal industry requires \$0.10 of its one product and \$0.80 of steel.

(b) To produce \$1.00 worth of output, the steel industry requires \$0.10 of its own product and \$0.20 of coal.

Find D , the input-output matrix for this system. Then solve for the output matrix X in the equation $X = DX + E$, where E is the external demand matrix

$$E = \begin{bmatrix} 10,000 \\ 20,000 \end{bmatrix}.$$

12. **Industrial System** An industrial system has two industries with the input requirements below.

(a) To produce \$1.00 worth of output, Industry A requires \$0.30 of its own product and \$0.40 of Industry B's product.

(b) To produce \$1.00 worth of output, Industry B requires \$0.20 of its own product and \$0.40 of Industry A's product.

Find D , the input-output matrix for this system. Then solve for the output matrix X in the equation $X = DX + E$, where E is the external demand matrix

$$E = \begin{bmatrix} 10,000 \\ 20,000 \end{bmatrix}.$$

-  **13. Solving for the Output Matrix** A small community includes a farmer, a baker, and a grocer and has the input-output matrix D and external demand matrix E below.

$$D = \begin{bmatrix} 0.4 & 0.5 & 0.5 \\ 0.3 & 0.0 & 0.3 \\ 0.2 & 0.2 & 0.0 \end{bmatrix} \begin{matrix} \text{Farmer} \\ \text{Baker} \\ \text{Grocer} \end{matrix} \quad \text{and} \quad E = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}$$

Solve for the output matrix X in the equation $X = DX + E$.

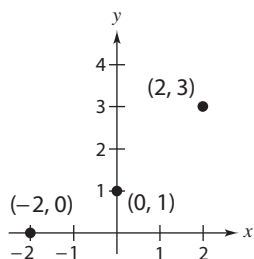
-  **14. Solving for the Output Matrix** An industrial system with three industries has the input-output matrix D and external demand matrix E below.

$$D = \begin{bmatrix} 0.2 & 0.4 & 0.4 \\ 0.4 & 0.2 & 0.2 \\ 0.0 & 0.2 & 0.2 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 5000 \\ 2000 \\ 8000 \end{bmatrix}$$

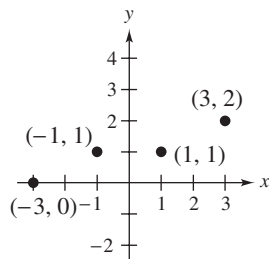
Solve for the output matrix X in the equation $X = DX + E$.

Least Squares Regression Analysis In Exercises 15–18, (a) sketch the line that appears to be the best fit for the given points, (b) find the least squares regression line, and (c) determine the sum of squared error.

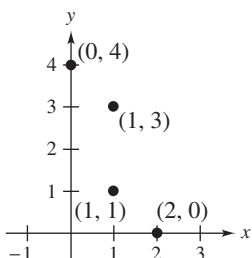
15.



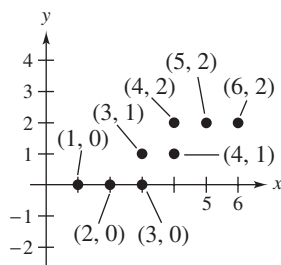
16.



17.



18.



Finding the Least Squares Regression Line In Exercises 19–26, find the least squares regression line.

19. $(0, 0)$, $(1, 1)$, $(2, 4)$
20. $(1, 0)$, $(3, 3)$, $(5, 6)$
21. $(-2, 0)$, $(-1, 1)$, $(0, 1)$, $(1, 2)$
22. $(-4, -1)$, $(-2, 0)$, $(2, 4)$, $(4, 5)$
23. $(-5, 1)$, $(1, 3)$, $(2, 3)$, $(2, 5)$
24. $(-3, 4)$, $(-1, 2)$, $(1, 1)$, $(3, 0)$
25. $(-5, 10)$, $(-1, 8)$, $(3, 6)$, $(7, 4)$, $(5, 5)$
26. $(0, 6)$, $(4, 3)$, $(5, 0)$, $(8, -4)$, $(10, -5)$

- 27. Demand** A hardware retailer wants to know the demand for a rechargeable power drill as a function of price. The ordered pairs $(25, 82)$, $(30, 75)$, $(35, 67)$, and $(40, 55)$ represent the price x (in dollars) and the corresponding monthly sales y .

- (a) Find the least squares regression line for the data.
- (b) Estimate the demand when the price is \$32.95.

- 28. Wind Energy Consumption** The table shows the wind energy consumptions y (in quadrillions of Btus, or British thermal units) in the United States from 2009 through 2013. Find the least squares regression line for the data. Let t represent the year, with $t = 9$ corresponding to 2009. Use the linear regression capabilities of a graphing utility to check your work. (Source: U.S. Energy Information Administration)

Year	2009	2010	2011	2012	2013
Consumption, y	0.72	0.92	1.17	1.34	1.60



- 29. Wildlife** A wildlife management team studied the reproduction rates of deer in three tracts of a wildlife preserve. The team recorded the number of females x in each tract and the percent of females y in each tract that had offspring the following year. The table shows the results.

Number, x	100	120	140
Percent, y	75	68	55

- (a) Find the least squares regression line for the data.
- (b) Use a graphing utility to graph the model and the data in the same viewing window.
- (c) Use the model to create a table of estimated values for y . Compare the estimated values with the actual data.
- (d) Use the model to estimate the percent of females that had offspring when there were 170 females.
- (e) Use the model to estimate the number of females when 40% of the females had offspring.

30. CAPSTONE

- (a) Explain how to use matrix multiplication to encode and decode messages.
- (b) Explain how to use a Leontief input-output model to analyze an economic system.
- (c) Explain how to use matrices to find the least squares regression line for a set of data.

- 31.** Use your school's library, the Internet, or some other reference source to derive the matrix form for linear regression given at the top of page 101.

2 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Operations with Matrices In Exercises 1–6, perform the matrix operations.

$$1. \begin{bmatrix} 2 & 1 & 0 \\ 0 & 5 & -4 \end{bmatrix} - 3 \begin{bmatrix} 5 & 3 & -6 \\ 0 & -2 & 5 \end{bmatrix}$$

$$2. -2 \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} + 8 \begin{bmatrix} 7 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$$

$$3. \begin{bmatrix} 1 & 2 \\ 5 & -4 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$$

$$4. \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$$

$$5. \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & -3 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$6. \begin{bmatrix} 2 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & 4 \end{bmatrix}$$

Solving a System of Linear Equations In Exercises 7–10, write the system of linear equations in the form $Ax = b$. Then use Gaussian elimination to solve this matrix equation for x .

$$7. \begin{aligned} 2x_1 + x_2 &= -8 \\ x_1 + 4x_2 &= -4 \end{aligned} \quad 8. \begin{aligned} 2x_1 - x_2 &= 5 \\ 3x_1 + 2x_2 &= -4 \end{aligned}$$

$$9. \begin{aligned} -3x_1 - x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 - 5x_3 &= -3 \\ x_1 - 2x_2 + 3x_3 &= 1 \end{aligned}$$

$$10. \begin{aligned} 2x_1 + 3x_2 + x_3 &= 10 \\ 2x_1 - 3x_2 - 3x_3 &= 22 \\ 4x_1 - 2x_2 + 3x_3 &= -2 \end{aligned}$$

Finding and Multiplying with a Transpose In Exercises 11–14, find A^T , $A^T A$, and AA^T .

$$11. A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \quad 12. A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$13. A = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \quad 14. A = [1 \quad -2 \quad -3]$$

Finding the Inverse of a Matrix In Exercises 15–18, find the inverse of the matrix (if it exists).

$$15. \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} \quad 16. \begin{bmatrix} 4 & -1 \\ -8 & 2 \end{bmatrix}$$

$$17. \begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & 0 & 3 \end{bmatrix} \quad 18. \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the Inverse of a Matrix In Exercises 19–26, use an inverse matrix to solve each system of linear equations or matrix equation.

$$19. \begin{aligned} 5x_1 + 4x_2 &= 2 \\ -x_1 + x_2 &= -22 \end{aligned} \quad 20. \begin{aligned} 3x_1 + 2x_2 &= 1 \\ x_1 + 4x_2 &= -3 \end{aligned}$$

$$21. \begin{aligned} -x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + 3x_2 + x_3 &= -2 \\ 5x_1 + 4x_2 + 2x_3 &= 4 \end{aligned}$$

$$22. \begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ x_1 - x_2 + x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 2 \end{aligned}$$

$$23. \begin{bmatrix} 5 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ -6 \end{bmatrix}$$

$$24. \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$25. \begin{bmatrix} 0 & 1 & -2 \\ -1 & 3 & 1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$26. \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 4 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -7 \end{bmatrix}$$

Solving a Matrix Equation In Exercises 27 and 28, find A .

$$27. (3A)^{-1} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \quad 28. (2A)^{-1} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Nonsingular Matrix In Exercises 29 and 30, find x such that the matrix A is nonsingular.

$$29. A = \begin{bmatrix} 3 & 1 \\ x & -1 \end{bmatrix} \quad 30. A = \begin{bmatrix} 2 & x \\ 1 & 4 \end{bmatrix}$$

Finding the Inverse of an Elementary Matrix In Exercises 31 and 32, find the inverse of the elementary matrix.

$$31. \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 32. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finding a Sequence of Elementary Matrices In Exercises 33–36, find a sequence of elementary matrices whose product is the given nonsingular matrix.

$$33. \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad 34. \begin{bmatrix} -3 & 13 \\ 1 & -4 \end{bmatrix}$$

$$35. \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} \quad 36. \begin{bmatrix} 3 & 0 & 6 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

37. Find two 2×2 matrices A such that $A^2 = I$.
38. Find two 2×2 matrices A such that $A^2 = O$.
39. Find three 2×2 idempotent matrices. (Recall that a square matrix A is *idempotent* when $A^2 = A$.)
40. Find 2×2 matrices A and B such that $AB = O$ but $BA \neq O$.
41. Consider the matrices below.

$$X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad Y = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \quad Z = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \quad W = \begin{bmatrix} 3 \\ 2 \\ -4 \\ -1 \end{bmatrix}$$

- (a) Find scalars a , b , and c such that $W = aX + bY + cZ$.
- (b) Show that there do not exist scalars a and b such that $Z = aX + bY$.
- (c) Show that if $aX + bY + cZ = O$, then $a = b = c = 0$.
42. **Proof** Let A , B , and $A + B$ be nonsingular matrices. Prove that $A^{-1} + B^{-1}$ is nonsingular by showing that $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$.

Finding an LU-Factorization of a Matrix In Exercises 43–46, find an *LU*-factorization of the matrix.

43. $\begin{bmatrix} 2 & 5 \\ 6 & 14 \end{bmatrix}$ 44. $\begin{bmatrix} -3 & 1 \\ 12 & 0 \end{bmatrix}$

45. $\begin{bmatrix} 4 & 1 & 0 \\ 0 & 3 & -7 \\ -16 & 11 & 1 \end{bmatrix}$ 46. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solving a Linear System Using LU-Factorization In Exercises 47 and 48, use an *LU*-factorization of the coefficient matrix to solve the linear system.

47.
$$\begin{aligned} x + z &= 3 \\ 2x + y + 2z &= 7 \\ 3x + 2y + 6z &= 8 \end{aligned}$$

48.
$$\begin{aligned} 2x_1 + x_2 + x_3 - x_4 &= 7 \\ 3x_2 + x_3 - x_4 &= -3 \\ -2x_3 &= 2 \\ 2x_1 + x_2 + x_3 - 2x_4 &= 8 \end{aligned}$$

49. **Manufacturing** A company manufactures tables and chairs at two locations. Matrix C gives the costs of manufacturing at each location.

$$C = \begin{bmatrix} \text{Location 1} & \text{Location 2} \\ 627 & 681 \\ 135 & 150 \end{bmatrix} \begin{matrix} \text{Tables} \\ \text{Chairs} \end{matrix}$$

- (a) Labor accounts for $\frac{2}{3}$ of the cost. Determine the matrix L that gives the labor costs at each location.
- (b) Find the matrix M that gives material costs at each location. (Assume there are only labor and material costs.)

50. **Manufacturing** A corporation has four factories, each of which manufactures sport utility vehicles and pickup trucks. In the matrix

$$A = \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix}$$

a_{ij} represents the number of vehicles of type i produced at factory j in one day. Find the production levels when production increases by 10%.

51. **Gasoline Sales** Matrix A shows the numbers of gallons of 87-octane, 89-octane, and 93-octane gasoline sold at a convenience store over a weekend.

$$A = \begin{bmatrix} \text{Octane} \\ 87 & 89 & 93 \\ 580 & 840 & 320 \\ 560 & 420 & 160 \\ 860 & 1020 & 540 \end{bmatrix} \begin{matrix} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{matrix}$$

Matrix B gives the selling prices (in dollars per gallon) and the profits (in dollars per gallon) for the three grades of gasoline.

$$B = \begin{bmatrix} \text{Selling Price} & \text{Profit} \\ b_{11} & 0.05 \\ b_{21} & 0.08 \\ b_{31} & 0.10 \end{bmatrix} \begin{matrix} 87 \\ 89 \\ 93 \end{matrix} \text{ Octane}$$

- (a) Find AB and interpret the result.
- (b) Find the convenience store's profit from gasoline sales for the weekend.

52. **Final Grades** Two midterms and a final exam determine the final grade in a course at a liberal arts college. The matrices below show the grades for six students and two possible grading systems.

$$A = \begin{bmatrix} \text{Midterm 1} & \text{Midterm 2} & \text{Final Exam} \\ 78 & 82 & 80 \\ 84 & 88 & 85 \\ 92 & 93 & 90 \\ 88 & 86 & 90 \\ 74 & 78 & 80 \\ 96 & 95 & 98 \end{bmatrix} \begin{matrix} \text{Student 1} \\ \text{Student 2} \\ \text{Student 3} \\ \text{Student 4} \\ \text{Student 5} \\ \text{Student 6} \end{matrix}$$

$$B = \begin{bmatrix} \text{Grading System 1} & \text{Grading System 2} \\ 0.25 & 0.20 \\ 0.25 & 0.20 \\ 0.50 & 0.60 \end{bmatrix} \begin{matrix} \text{Midterm 1} \\ \text{Midterm 2} \\ \text{Final Exam} \end{matrix}$$

- (a) Describe the grading systems in matrix B .
- (b) Compute the numerical grades for the six students (to the nearest whole number) using the two grading systems.
- (c) How many students received an "A" in each grading system? (Assume 90 or greater is an "A.")

Polynomial Function In Exercises 53 and 54, find $f(A)$ using the definition below.

If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial function, then for a square matrix A ,

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n.$$

$$53. f(x) = 6 - 7x + x^2, \quad A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$54. f(x) = 2 - 3x + x^3, \quad A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Stochastic Matrices In Exercises 55–58, determine whether the matrix is stochastic.

$$55. \begin{bmatrix} \frac{12}{25} & \frac{2}{25} \\ \frac{13}{25} & \frac{23}{25} \end{bmatrix}$$

$$56. \begin{bmatrix} 0.3 & 0.7 \\ 0 & 1 \end{bmatrix}$$

$$57. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.1 \\ 0 & 0.1 & 0.5 \end{bmatrix}$$

$$58. \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.4 & 0.5 \\ 0.5 & 0.2 & 0.4 \end{bmatrix}$$

Finding State Matrices In Exercises 59–62, use the matrix of transition probabilities P and initial state matrix X_0 to find the state matrices X_1 , X_2 , and X_3 .

$$59. P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}, \quad X_0 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$60. P = \begin{bmatrix} 0.23 & 0.45 \\ 0.77 & 0.55 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.65 \\ 0.35 \end{bmatrix}$$

$$61. P = \begin{bmatrix} 0.50 & 0.25 & 0 \\ 0.25 & 0.70 & 0.15 \\ 0.25 & 0.05 & 0.85 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

$$62. P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}, \quad X_0 = \begin{bmatrix} \frac{2}{9} \\ \frac{4}{9} \\ \frac{1}{3} \end{bmatrix}$$

63. Caribbean Cruise Three hundred people go on a Caribbean cruise. When the ship stops at a port, each person has a choice of going on shore or not. Seventy percent of the people who go on shore one day will not go on shore the next day. Sixty percent of the people who do not go on shore one day will go on shore the next day. Today, 200 people go on shore. How many people will go on shore (a) tomorrow and (b) the day after tomorrow?

64. Population Migration A country has three regions. Each year, 10% of the residents of Region 1 move to Region 2 and 5% move to Region 3, 15% of the residents of Region 2 move to Region 1 and 5% move to Region 3, and 10% of the residents of Region 3 move to Region 1 and 10% move to Region 2. This year, each region has a population of 100,000. Find the populations of each region (a) in 1 year and (b) in 3 years.

Regular and Steady State Matrix In Exercises 65–68, determine whether the stochastic matrix P is regular. Then find the steady state matrix $\bar{X}$ of the Markov chain with matrix of transition probabilities P .

$$65. P = \begin{bmatrix} 0.8 & 0.5 \\ 0.2 & 0.5 \end{bmatrix}$$

$$66. P = \begin{bmatrix} 1 & \frac{4}{7} \\ 0 & \frac{3}{7} \end{bmatrix}$$

$$67. P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{2} & \frac{5}{6} & 1 \end{bmatrix}$$

$$68. P = \begin{bmatrix} 0 & 0 & 0.2 \\ 0.5 & 0.9 & 0 \\ 0.5 & 0.1 & 0.8 \end{bmatrix}$$

69. Sales Promotion As a promotional feature, a store conducts a weekly raffle. During any week, 40% of the customers who turn in one or more tickets do not bother to turn in tickets the following week. On the other hand, 30% of the customers who do not turn in tickets will turn in one or more tickets the following week. Find and interpret the steady matrix for this situation.

70. Classified Documents A courtroom has 2000 documents, of which 1250 are classified. Each week, 10% of the classified documents become declassified and 20% are shredded. Also, 20% of the unclassified documents become classified and 5% are shredded. Find and interpret the steady state matrix for this situation.

Absorbing Markov Chains In Exercises 71 and 72, determine whether the Markov chain with matrix of transition probabilities P is absorbing. Explain.

$$71. P = \begin{bmatrix} 0 & 0.4 & 0.1 \\ 0.7 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.5 \end{bmatrix}$$

$$72. P = \begin{bmatrix} 1 & 0 & 0.38 \\ 0 & 0.30 & 0 \\ 0 & 0.70 & 0.62 \end{bmatrix}$$

True or False? In Exercises 73–76, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

73. (a) Addition of matrices is not commutative.
(b) The transpose of the sum of matrices is equal to the sum of the transposes of the matrices.
74. (a) If an $n \times n$ matrix A is not symmetric, then A^TA is not symmetric.
(b) If A and B are nonsingular $n \times n$ matrices, then $A + B$ is a nonsingular matrix.
75. (a) A stochastic matrix can have negative entries.
(b) A Markov chain that is not regular can have a unique steady state matrix.
76. (a) A regular stochastic matrix can have entries of 0.
(b) The steady state matrix of an absorbing Markov chain always depends on the initial state matrix.

Encoding a Message In Exercises 77 and 78, write the uncoded row matrices for the message. Then encode the message using the matrix A .

77. **Message:** ONE IF BY LAND

Row Matrix Size: 1×2

Encoding Matrix: $A = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$

78. **Message:** BEAM ME UP SCOTTY

Row Matrix Size: 1×3

Encoding Matrix: $A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ -2 & -1 & -3 \end{bmatrix}$

Decoding a Message In Exercises 79–82, use A^{-1} to decode the cryptogram.

79. $A = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$,
 -45 34 36 -24 -43 37 -23 22 -37 29 57 -38
 -39 31

80. $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$,
 11 52 -8 -9 -13 -39 5 20 12 56 5 20 -2 7
 9 41 25 100

81. $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$
 -2 2 5 39 -53 -72 -6 -9 93 4 -12 27 31
 -49 -16 19 -24 -46 -8 -7 99

82. $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 2 & -4 \end{bmatrix}$
 66 27 -31 37 5 -9 61 46 -73 46 -14 9 94
 21 -49 32 -4 12 66 31 -53 47 33 -67 32 19
 -56 43 -9 -20 68 23 -34

83. **Industrial System** An industrial system has two industries with the input requirements below.

(a) To produce \$1.00 worth of output, Industry A requires \$0.20 of its own product and \$0.30 of Industry B's product.

(b) To produce \$1.00 worth of output, Industry B requires \$0.10 of its own product and \$0.50 of Industry A's product.

Find D , the input-output matrix for this system. Then solve for the output matrix X in the equation $X = DX + E$, where E is the external demand matrix

$$E = \begin{bmatrix} 40,000 \\ 80,000 \end{bmatrix}.$$

84. **Solving for the Output Matrix** An industrial system with three industries has the input-output matrix D and external demand matrix E below.

$$D = \begin{bmatrix} 0.1 & 0.3 & 0.2 \\ 0.0 & 0.2 & 0.3 \\ 0.4 & 0.1 & 0.1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 3000 \\ 3500 \\ 8500 \end{bmatrix}$$

Solve for the output matrix X in the equation $X = DX + E$.

Finding the Least Squares Regression Line In Exercises 85–88, find the least squares regression line.

85. (1, 5), (2, 4), (3, 2)

86. (2, 1), (3, 3), (4, 2), (5, 4), (6, 4)

87. (1, 1), (1, 3), (1, 2), (1, 4), (2, 5)

88. (-2, 4), (-1, 2), (0, 1), (1, -2), (2, -3)

89. **Cellular Phone Subscribers** The table shows the numbers of cellular phone subscribers y (in millions) in the United States from 2008 through 2013. (Source: CTIA—The Wireless Association)

Year	2008	2009	2010	2011	2012	2013
Number, y	270	286	296	316	326	336

- Find the least squares regression line for the data. Let x represent the year, with $x = 8$ corresponding to 2008.
- Use the linear regression capabilities of a graphing utility to find a linear model for the data. How does this model compare with the model obtained in part (a)?
- Use the linear model to create a table of estimated values for y . Compare the estimated values with the actual data.

90. **Major League Baseball Salaries** The table shows the average salaries y (in millions of dollars) of Major League Baseball players on opening day of baseball season from 2008 through 2013. (Source: Major League Baseball)

Year	2008	2009	2010	2011	2012	2013
Salary, y	2.93	3.00	3.01	3.10	3.21	3.39

- Find the least squares regression line for the data. Let x represent the year, with $x = 8$ corresponding to 2008.
- Use the linear regression capabilities of a graphing utility to find a linear model for the data. How does this model compare with the model obtained in part (a)?
- Use the linear model to create a table of estimated values for y . Compare the estimated values with the actual data.

2 Projects

	Test 1	Test 2
Anna	84	96
Bruce	56	72
Chris	78	83
David	82	91



1 Exploring Matrix Multiplication

The table shows the first two test scores for Anna, Bruce, Chris, and David. Use the table to create a matrix M to represent the data. Input M into a software program or a graphing utility and use it to answer the questions below.

- Which test was more difficult? Which was easier? Explain.
- How would you rank the performances of the four students?
- Describe the meanings of the matrix products $M \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $M \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Describe the meanings of the matrix products $[1 \ 0 \ 0 \ 0]M$ and $[0 \ 0 \ 1 \ 0]M$.
- Describe the meanings of the matrix products $M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{2}M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Describe the meanings of the matrix products $[1 \ 1 \ 1 \ 1]M$ and $\frac{1}{4}[1 \ 1 \ 1 \ 1]M$.
- Describe the meaning of the matrix product $[1 \ 1 \ 1 \ 1]M \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Use matrix multiplication to find the combined overall average score on both tests.
- How could you use matrix multiplication to scale the scores on test 1 by a factor of 1.1?

2 Nilpotent Matrices

Let A be a nonzero square matrix. Is it possible that a positive integer k exists such that $A^k = O$? For example, find A^3 for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A square matrix A is **nilpotent of index k** when $A \neq O$, $A^2 \neq O$, $\dots$, $A^{k-1} \neq O$, but $A^k = O$. In this project you will explore nilpotent matrices.

- The matrix in the example above is nilpotent. What is its index?

-  2. Use a software program or a graphing utility to determine which matrices below are nilpotent and find their indices.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

- Find 3×3 nilpotent matrices of indices 2 and 3.
- Find 4×4 nilpotent matrices of indices 2, 3, and 4.
- Find a nilpotent matrix of index 5.
- Are nilpotent matrices invertible? Prove your answer.
- When A is nilpotent, what can you say about A^T ? Prove your answer.
- Show that if A is nilpotent, then $I - A$ is invertible.